Tableaux for Basic Modal Logic

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DEFINITION

- A *label* is finite sequence of positive integers. A *labelled formula* is an expression of the form iφ, where **i** is the label of the formula φ.

- If the label **i** is a sequence of length >1 the positive integers of the sequence will be separated by periods. Thus, if **i** is a label and an **n** is positive integer, then **i.n** is a new label, called *an extension of **i**.* The label **i** is then an *initial segment* of **i.n**.

Restricted Labelling	Unrestricted Labelling
(**V**)□Ai ------------------------------- (**V**)Aj	(**F**)□Ai ------------------------------- (**F**)Aj
(**F**) ◊Ai ------------------------------- (**F**)Aj	(**T**) ◊Ai ------------------------------- (**T**)Aj
K: j must occur already in the branch **Reflexivity**: j=i **Seriality**: j can be new **Symmetry:** Assume that i = k.n, then j= k (k.n. is an immediate extension of k, and k occurs already in the branch). **Transitivity**: i is an initial segment of j (such as j = i.m or j = i.m.n…). **Euclideanity**: If in the branch occur, i, i.n and it is the case that (**V**)□Ai.m(or (**F**) ◊Ai.m), then j = i.n:	j: is new
[i] [i.n] (**F**) ◊Ai.m ------------------------------- (**F**)Ai.n	
[i] [i.n] (**V**)□Ai.m	

------------------------------ (**V**)Ai.n	

Tableaux and as generated by winning strategies

- If ∑ is a set of dialogically signed formulae and *X* is a single dialogically signed formula, we will write ∑, *X* for ∑4{*X*}.
- Observe that the formulae below the line always represent pairs of attack and defence moves. In other words, they represent rounds.
- The vertical bar ‘|’ indicates alternative choices for **O**, **P**'s strategy must have a defence for both possibilities (dialogical games defining two possible plays).
- The rules containing yielding two lines indicate that it is **P** who has the choice – an he thus might need only one of both possible choices.
- Note that the expressions between the symbols "<" and ">", such as <(**P**)?> or <(**O**)?> are moves – more precisely they are attacks – but not formulae (assertions) which could be attacked. These expressions are not actually part or the tableau. They are formulae which are included in the set of formulae. **These expressions are rather part of the algorithmic device to reconstruct the correspondent dialogues.**

Intuitionistic Tableaux

(**O**)-*Cases*	(**P**)-*Cases*
∑, (**O**)A∨B ------------------------------- ∑, <(**P**)?-∨>(**O**)A | ∑, <(**P**)?-∨>(**O**)B	∑, (**P**)A∨B -------------------- $\Sigma_{[O]}$, <(**O**)?-∨>(**P**)A $\Sigma_{[O]}$, <(**O**)?-∨>(**P**)B
∑, (**O**)A∧B --------------------- ∑, <(**P**)?-L>(**O**)A ∑, <(**P**)?-R>(**O**)B	∑, (**P**)A∧B ------------------------------- $\Sigma_{[O]}$, <(**O**)?-L>(**P**)A | $\Sigma_{[O]}$, <(**O**)?-R>(**P**)B
∑,(**O**)A→B ------------------------------- $\Sigma_{[O]}$, (**P**)A ... | <(**P**)A>(**O**)B	∑, (**P**)A→B ------------------- $\Sigma_{[O]}$, (**O**)A; (**P**)B
∑, (**O**)¬A ------------------ $\Sigma_{[O]}$, (**P**)A; —	∑, (**P**)¬A --------------- $\Sigma_{[O]}$, (**O**)A; —
∑, (**O**)∀*x*A --------------------- ∑, <(**P**) ?-∀ *x*/k_i>(**O**)$A_{[x/ki]}$	∑, (**P**)∀*x*A -------------------- $\Sigma_{[O]}$, <(**O**) ?-∀ *x*/k_i >(**P**)$A_{[x/ki]}$

	k_i *is new*
Σ, (**O**)$\exists x$A	Σ, (**P**)$\exists x$A
---------------------	---------------------
Σ, $<$(**P**)?-$\exists$$>$(**O**)$A_{[x/ki]}$	$\Sigma_{[\mathbf{O}]}$, $<$(**O**)?-$\exists$$>$(**P**)$A_{[x/ki]}$
k_i *is new*	

Intuitionistic tableaux are generated with the addition of the set $\Sigma_{[\mathbf{O}]}$. which only contains O-signed formulae): the totality of the previous **P**-formulæ on the same branch of the tree are eliminated.

DEFINITION

Let us look at two examples, namely one for classical logic and one for intuitionistic logic. We use the tree shape of the tableau made popular by Smullyan ([18]):

If Θ is a given set of (**P**-, or **O**-) signed formulae, we say one of the rules ***R*** of above rules of the tableaux-system *applies* to Θ if by appropriate choice of Θ, the collection of signed formulae above the line in rule R becomes Θ.

By an *application of **R** to the **set*** Θ, we mean the replacement of Θ by Θ_1 (or by Θ_1 and Θ_2, if ***R*** is (**P**)$\wedge$, (**O**)$\vee$, or (**O**)$\rightarrow$) where Θ is the set of formulae above the line in rule R (after suitable substitution for $\Sigma_,$ and for the formulae **A** (and **B**)) and Θ_1 (or Θ_1 and Θ_2) is the set of formulae below. This assumes ***R*** to apply to Θ. Otherwise the result is again Θ. For example, by applying rule (**P**)$\rightarrow$ to the set Θ: {(**O**)**A**, (**P**)**B**, (**P**)(**C**$\rightarrow$**D**)} we may get $\Sigma_{[\mathbf{O}]}4\Theta_{1:}$ {(**O**)**A**, (**O**)**C**, **(P)D**)} – notice that (**P**)**B** disappeared because we have $\Sigma_{[\mathbf{O}]}$ and not Σ.

By ***configuration*** we mean a finite collection {Σ_1, Σ_2**, ...,** Σ_n} of sets of signed formulae, where Σ might stand for Σ and/or $\Sigma_{[\mathbf{O}]}$.

By an *application of **R** to the **configuration*** {Σ_1, Σ_2**, ...,** Σ_n} we mean the replacement of this configuration with a new one which is like the first except for containing instead of some Σ_i the result (or results) of applying rue ***R*** to Σ_i.

By a tableau we mean a finite sequence of configurations $\mathfrak{C}_1$, $\mathfrak{C}_2$**, ...,** $\mathfrak{C}_n$ in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.

A set of signed formulae is closed if it contains both (**O**)*a* and (**P**)*a* (for *a* atomic). A configuration {$\sum_1$, ($\sum_2$**, ...,** $\sum_n$} is closed if each $\sum_i$ is closed. A tableau $\mathfrak{C}_1$, $\mathfrak{C}_2$**, ...,** $\mathfrak{C}_n$ is closed if some $\mathfrak{C}_i$ is closed.

By a tableau for a set $\sum$ of signed formulae we mean a tableau $\mathfrak{C}_1$, $\mathfrak{C}_2$**, ...,** $\mathfrak{C}_n$ in which $\mathfrak{C}_1$ is {$\sum$}.

Modal Logic

Restricted Labelling	Unrestricted Labelling
(**O**)□Ai ------------------------------- <(**P**) ?j> (**O**)Aj	(**P**)□Ai ------------------------------- <(**O**) ?j> (**P**)Aj
(**P**) ◊Ai ------------------------------- <(**O**) ? ◊> (**P**)Aj	(**O**) ◊Ai ------------------------------- <(**P**) ? ◊> (**O**)Aj
K: j must occur already in the branch **Reflexivity**: j=i **Seriality**: j can be new **Symmetry:** Assume that i = k.n, then j= k (k.n. is an immediate extension of k, and k occurs already in the branch). **Transitivity**: i is an initial segment of j (such as j = i.m or j = i.m.n…). **Euclideanity**: If in the branch occur, i, i.n and it is the case that (**O**)□Ai.m(or (**P**) ◊Ai.m), then j = i.n: [i] [i.n] (**P**) ◊Ai.m ------------------------------- (**P**)Ai.n [i] [i.n] (**O**)□Ai.m ------------------------------- (**O**)Ai.n	j: is new