



**HAL**  
open science

# Soundness of the System of Semantic Trees for Classical Logic based on Fitting and Smullyan

Shahid Rahman

► **To cite this version:**

Shahid Rahman. Soundness of the System of Semantic Trees for Classical Logic based on Fitting and Smullyan. Licence. France. 2015. cel-01228875

**HAL Id: cel-01228875**

**<https://shs.hal.science/cel-01228875v1>**

Submitted on 17 Nov 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Shahid Rahman

[shahid.rahman@univ-lille3.fr](mailto:shahid.rahman@univ-lille3.fr)

Univ. Lille; Philosophie, UMR 8163: STL

### Soundness (and Consistency) of the System of Semantic Trees for Classical Logic

Soundness (coherence) and Consistency (non-contradiction) are related though Soundness is more general:

When we say in metalogic that we prove that a given proof system is “sound” we mean that with this system we cannot prove any *formula it should not*. For example; if our Tree system for classical logic were *not sound* then we would be able to have a closed tree for formula such as say  $A \rightarrow B$ . *If the system is inconsistent or contradictory, then we would be able to have a closed tree for  $A \wedge \neg A$ .* So inconsistency is a special case of unsoundness: a system where one is able to prove not only contingencies but also contradictions. Since Soundness is more general we prove soundness

To prove this we need some previous work:

#### Definition 1 [Satisfiable]:

Let us consider a set  $S$  of signed formulae such as  $\mathbf{T} A \wedge B$ ,  $\mathbf{F} \neg A \vee C$ ,  $\mathbf{F} \neg A \wedge D$ ,  $\mathbf{T} A$ . We say that  $S$  is satisfiable in the model  $M$  if we can find a valuation such that:

for every  $\mathbf{X}A$  is in  $S$ , (where  $\mathbf{X}$  signalises that the formula is  $\mathbf{T}$ - or  $\mathbf{F}$ -signed),  $v(A)=1$  in  $M$ , in other words  $A$  is true in the model  $M$ . – where:

$$v(\mathbf{T}A) = v(A) = 1$$

$$v(\mathbf{F}A) = v(\neg A) = 1$$

(In our example one valuation would be:

$$v(\mathbf{T} A \wedge B) = v(A \wedge B) = 1, \text{ and this means that } v(A) = 1, \text{ and } v(B) = 1,$$

$$v(\mathbf{F} \neg A \vee C) = v(\neg(\neg A \vee C)) = 1, \text{ and this means that } v(\neg\neg(A)) = 1, \text{ AND } v(\neg C) = 1, \text{ and this means } v(A) = 1 \text{ and } v(\neg C) = 0$$

$$v(\mathbf{F} \neg A \wedge D) = v(\neg(\neg A \wedge D)) = 1, \text{ and this means that } v(\neg\neg(A)) = 1 \text{ OR } v(D) = \text{whatever, say } 0)$$

- We say that a branch of a tree is satisfiable if the set of labelled signed formulae on it is satisfiable in at least one model
- We say that a tree (with all of his branches) is satisfiable if some branch of it is satisfiable

#### Soundness lemma 1 (SL1):

A closed tree(a tree where all the branches are closed) is not satisfiable

PROOF:

- Suppose that we had a tree that was both closed and satisfiable.
- Since it is satisfiable, some branch of it is. Let  $S$  be the set of formulae on that branch and let it be satisfiable in the model  $M$  by means of some valuation

- Since the tree is closed (all of its branches are closed) then in *every* branch we must find at least one atomic formula  $A$  that is labelled as **TA** AND as **FA**. But then both  $v(A)=1$  and  $v(\neg A)=1$  must be the case in  $M$  but this is not possible.

**Corollary of SL1:** If one branch is satisfiable the whole tree is *open*

**Soundness lemma 2 (SL2):**

If (a section of) a tree is satisfiable and a branch of that (section of) dialogue is extended by appropriate particle rules, the result is another satisfiable (section of) a tree.

(Obviously this assumes that the formula that triggers the extension is not atomic (WHY?)).

PROOF:

Let  $D$  be a (section of a) satisfiable dialogue and let  $B$  be the branch that is extended.

The proof requires several steps. We begin with two main steps:

By hypothesis at least one branch is satisfiable, now this branch could be  $B$  or could be  $B^*$ .

I) if the satisfiable branch is  $B^*$  the extension of  $B$  will leave  $B^*$  unchanged, thus after the particle rule has been applied to  $B$ ,  $D$  will still be satisfiable (because  $B^*$  is).

II) if the satisfiable branch is  $B$  and it is satisfiable in the model  $M$  the proof is by cases.

That is, by the consideration of all the ways to extend the branch  $B$  by the application of the corresponding particle rule to a labelled and signed formula at the end of that branch. Namely by the application of a **F**- and a **T**-rule.

1) Let us start with  $\mathbf{FA} \rightarrow B$ ). If we apply the correspondent rule we will produce the branch  $B1$  containing the formulae:

**TA**  
**FB**

Since  $B$  is by hypothesis satisfiable in  $M$  and  $\mathbf{F}(A \rightarrow B)$  is on  $B$  we have that  $v(\mathbf{F}(A \rightarrow B))=1$ , that is  $v(\neg(A \rightarrow B))=1$  is in  $M$ . But then by definition of truth in a model we have that  $v(A)=1$  is AND  $v(B)=0$ . But the latter is what we have in the extended section of the branch. Indeed:

$v(\mathbf{TA})=v(A)=1$  in  $M$ . and  
 $v(\mathbf{FB})=(v(\neg(B))=1$  (or:  $v(B)=0$ ) in  $M$ .

It follows then that  $B1$ , that contains **TA** and **FB** is satisfiable in the same model with the same valuation

The other cases without branching are similar.

Exercise: complete the other non branching cases (**T**-conjunction, **F**-disjunction, **T**-negation; **F**- negation)

2) Let us assume now that it is  $\mathbf{T}(A \rightarrow B)$  that will produce an extension of  $\mathbf{B}$ .

If we apply the correspondent particle rule and the shifting rule we will produce two branches  $\mathbf{B1}$  and  $\mathbf{B2}$  containing respectively the formulae:

$\mathbf{FA}$

$\mathbf{TB}$

Since  $\mathbf{B}$  is by hypothesis satisfiable in  $\mathbf{M}$  and  $\mathbf{T}(A \rightarrow B)$  is on  $\mathbf{B}$  we have that  $v(\mathbf{T}(A \rightarrow B)) = v(A \rightarrow B) = 1$ . But then by definition of truth in a model we have  $v(A) = 0$  OR  $v(B) = 1$ . But the latter is what we have in the extended section of the branch. Indeed:

$$v(\mathbf{FA}) = v(\neg A) = 1 \quad (v(A) = 0) \quad \text{OR} \quad v(\mathbf{TB}) = v(B) = 1$$

If the left case holds then  $\mathbf{FA}$  is satisfiable, if the right case holds then  $\mathbf{TB}$  is satisfiable. Either way, at least one of the extensions  $\mathbf{B1}$  or  $\mathbf{B2}$  of  $\mathbf{B}$  is satisfiable. Thus at least one branch is satisfiable, so the tree is itself satisfiable.

The other cases with branching are similar and are left as exercise

### Soundness theorem:

If a tree for  $\mathbf{FA}$  closes,  $A$  is (classically) valid.

PROOF:

Assume that there is a closed tree for  $\mathbf{FA}$ , but  $A$  is not (classically) valid. We show that from this a contradiction follows.

Since there is a closed tree  $\mathbf{D}$  for  $\mathbf{FA}$  then it starts with  $\mathbf{FA}$ . Let us call the first section of the tree  $\mathbf{D}_0$  that consists in the thesis  ${}_0\mathbf{FA}$ . The following sections of the tree  $\mathbf{D}$  are constructed by extending  $\mathbf{D}_0$ .

Since we assumed that  $A$  is not (classically) valid, there is some model  $\mathbf{M}$  where  $A$  is not true. Accordingly the set (of one member)  $\{{}_0\mathbf{FA}\}$  is satisfiable in the model  $\mathbf{M}$ . Thus  $\mathbf{D}_0$  is satisfiable, since the set of formulae on its only branch is satisfiable

Since  $\mathbf{D}_0$  is satisfiable by lemma SL2 so any tree we get that starts with  $\mathbf{D}_0$  and results by extending  $\mathbf{D}_0$  is satisfiable

It follows then that  $\mathbf{D}$  is satisfiable.

$\mathbf{D}$  is closed by hypothesis, and this is impossible by SL1.

Quod erat demonstrandum

## EXERCISE

1 Take the following tree rules for the logical constant “tonk”

$$\mathbf{P} \text{ Atonk} B$$

-----

$$\mathbf{P} A$$

$$\mathbf{P} B$$

$$\mathbf{O} \text{ Atonk} B$$

-----

$$\mathbf{O} A$$

$$\mathbf{O} B$$

- 1.1) Prove that the rules will yield a closed tableau for a formula and its negation
- 1.2) Prove that these rules are unsound in relation to classical logic.