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Soundness (and Consistency) of the System of Semantic Trees for Classical Logic
Soundness (coherence) and Consistency (non-contradiction) are related though Soundness is more general:
When we say in metalogic that we prove that a given proof system is "sound" we mean that with this system we cannot prove any *formula it should not*. For example; if our Tree system for classical logic were *not sound* then we would be able to have a closed tree for formula such as say $A \rightarrow B$. *If the system is inconsistent or contradictory , then we would be able to have a closed tree for* $A \wedge \neg A$. So inconsistency is a special case of unsoundness: a system where one is able to prove not only contingencies but also contradictions. Since Soundness is more general we prove soundness

To prove this we need some previous work:

Definition 1 [Satisfiable]:

Let us consider a set ***S*** of signed formulae such as **T** $A \wedge B$, **F** $\neg A \vee C$, **F** $\neg A \wedge D$, **T**A. We say that ***S*** is satisfiable in the model ***M*** if we can find a valuation such that:

> for very **X**A is in S, (where **X** signalises that the formula is **T**- or **F**-signed), $v(A)$=1 in ***M***, in other words A is true in the model ***M***. – where:
> v (**T**A)= v (A)=1
> v (**F**A)= v ($\neg A$) =1

(In our example one valuation would be:
v (**T** $A \wedge B$)= v ($A \wedge B$)=1, *and this means that* v (A)=1, and v (B)= 1,
v (**F** $\neg A \vee C$)= v $\neg(\neg A \vee C)$=1, *and this means that* v $\neg\neg(A)$=1, AND v ($\neg C$)=1 , and this means $v(A)$=1 and v ($\neg C$)=0
v (**F** $\neg A \wedge D$)= v $\neg(\neg A \wedge D)$=1, *and this means that* v $\neg\neg(A)$=1 OR v (D)= is whatever, say 0)

- We say that a branch of a tree is satisfiable if the set of labelled signed formulae on it is satisfiable in at least one model
- We say that a tree (with all of his branches) is satisfiable if some branch of it is satisfiable

Soundness lemma 1 (SL1):

A closed tree(a tree where all the branches are closed) is not satisfiable

PROOF:
- Suppose that we had a tree that was both closed and satisfiable.
- Since it is satisfiable, some branch of it is. Let S be the set of formulae on that branch and let it be satisfiable in the model ***M*** by means of some valuation

- Since the tree is closed (all of its branches are closed) then in *every* branch we must find at least one atomic formula A that is labelled as **T**A AND as **F**A. But then both $v\,(A)=1$ and $v\,\neg A=1$ must be the case in $\boldsymbol{M}$ but this is not possible.

Corollary of SL1: If one branch is satisfiable the whole tree is o*pen*

Soundness lemma 2 (SL2):

If (a section of) a tree is satisfiable and a branch of that (section of) dialogue is extended by appropriate particle rules , the result is another satisfiable (section of) a tree.

(Obviously this assumes that the formula that triggers the extension is not atomic (WHY?)).

PROOF:

Let $\boldsymbol{D}$ be a (section of a) satisfiable dialogue and let **B** be the branch that is extended.

The proof requires several steps. We begin with two main steps:
By hypothesis at least one branch is satisfiable, now this branch could be **B** or could be **B***.

I) if the satisfiable branch is **B*** the extension of **B** will leave **B*** unchanged, thus after the particle rule has been applied to **B**, $\boldsymbol{D}$ will still be satisfiable (because **B*** is) .

II) if the satisfiable branch is **B** and it satisfiable in the model $\boldsymbol{M}$the proof is by cases.

That is, by the consideration of all the ways to extend the branch **B** by the application of the corresponding particle rule to a labelled and signed formula at the end of that branch. Namely by the application of a **F-**and a **T-rule**.

1) Let us start with **F**$A\to B$). If we apply the correspondent rule we will produce the branch **B1** containing the formulae:

> **T**A
> **F**B

Since **B** is by hypothesis satisfiable in $\boldsymbol{M}$ and **F**$(A\to B)$ is on **B** we have that $v(\mathbf{F}(A\to B))=1$, that is $v(\neg(A\to B))=1$ is in $\boldsymbol{M}$. But then by definition of truth in a model we have that $v(A)=1$ is AND $v(B)=0$. But the latter is what we have in the extended section of the branch. Indeed:

> $v(\mathbf{T}A)= v(A)=1$ in $\boldsymbol{M}$. and
> $v\,(\mathbf{F}(B)= (v\,(\neg(B)=1$ (or: $v(B)=0$) in $\boldsymbol{M}$.

It follows then that **B1**, that contains **T**A and **F**B is satisfiable in the same model with the same valuation

The other cases without branching are similar.

Exercise: complete the other non branching cases (**T**-conjunction, **F**-disjunction, **T**-negation; **F**- negation)

2) Let us assume now that it is **T**($A \rightarrow B$) that will produce and extension of **B**.
If we apply the correspondent particle rule and the shifting rule we will produce two branches **B1** and **B2** containing respectively the formulae:

F*A* **T***B*

Since **B** is by hypothesis satisfiable in ***M*** and **T**($A \rightarrow B$) is on **B** we have that v(**T**($A \rightarrow B$))= $v(A \rightarrow B)$=1. But then by definition of truth in a model we have $v(A)$=0 OR $v(B)$=1. But the latter is what we have in the extended section of the branch. Indeed:

v(**F***A*)=$v(\neg A)$=1 ($v(A)$=0) OR v(**T***B*)= $v(B)$=1

If the left case holds then **F***A* is satisfiable, if the right case holds then **T***B* is satisfiable. Either way, at least one of the extensions **B1** or **B2** of **B** is satisfiable. Thus at least one branch is satisfiable, so the tree is itself satisfiable.

The other cases with branching are similar and are left as exercise

Soundness theorem:

If a tree for **F***A* closes, *A* is (classically) valid.

PROOF:
Assume that there is closed tree for **F** *A*, but *A* is not (classically) valid. We show that from this a contradiction follows.

Since there is A closed tree ***D*** for **F***A* then it starts with **F***A***.** Let us call the first section of the tree $\boldsymbol{D_0}$ that consists in the thesis ${}_0$**F***A*. The following sections of the tree ***D*** are constructed by extending $\boldsymbol{D_0}$**.**

Since we assumed that *A* is not (classically) valid, there is some model ***M*** where *A* is not true. Accordingly the set (of one member) {${}_0$**F***A*} is satisfiable in the model *M*. Thus $\boldsymbol{D_0}$ is satisfiable, since the set of formulae on its only branch is satisfiable

Since $\boldsymbol{D_0}$ is satisfiable by lemma SL2 so any tree we get that starts with $\boldsymbol{D_0}$ and results by extending $\boldsymbol{D_0}$ is satisfiable

It follows then that ***D*** is satisfiable.

D is closed by hypothesis, and this is impossible by SL1.

Quod erat demonstrandum

EXERCISE

1 Take the following tree rules for the logical constant "tonk"

P *A*tonk*B*

P *A*
P *B*

O *A*tonk*B*

O *A*
O *B*

1.1) Prove that the rules will yield a closed tableau for a formula and its negation
1.2) Prove that these rules are unsound in relation to classical logic.