



Soundness of the System of Semantic Trees for Classical Logic based on Fitting and Smullyan

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Soundness (and Consistency) of the System of Semantic Trees for Classical Logic

Soundness (coherence) and Consistency (non-contradiction) are related though Soundness is more general:

When we say in metalogic that we prove that a given proof system is “sound” we mean that with this system we cannot prove any *formula it should not*. For example; if our Tree system for classical logic were *not sound* then we would be able to have a closed tree for formula such as say $A \rightarrow B$. *If the system is inconsistent or contradictory, then we would be able to have a closed tree for $A \wedge \neg A$.* So inconsistency is a special case of unsoundness: a system where one is able to prove not only contingencies but also contradictions. Since Soundness is more general we prove soundness

To prove this we need some previous work:

Definition 1 [Satisfiable]:

Let us consider a set S of signed formulae such as $\mathbf{T} A \wedge B$, $\mathbf{F} \neg A \vee C$, $\mathbf{F} \neg A \wedge D$, $\mathbf{T} A$. We say that S is satisfiable in the model \mathbf{M} if we can find a valuation such that:

for very $\mathbf{X}A$ is in S , (where \mathbf{X} signalises that the formula is \mathbf{T} - or \mathbf{F} -signed), $v(A)=1$ in \mathbf{M} , in other words A is true in the model \mathbf{M} . – where:

$$v(\mathbf{T}A) = v(A) = 1$$

$$v(\mathbf{F}A) = v(\neg A) = 1$$

(In our example one valuation would be:

$$v(\mathbf{T} A \wedge B) = v(A \wedge B) = 1, \text{ and this means that } v(A) = 1, \text{ and } v(B) = 1,$$

$$v(\mathbf{F} \neg A \vee C) = v(\neg(\neg A \vee C)) = 1, \text{ and this means that } v(\neg\neg(A)) = 1, \text{ AND } v(\neg C) = 1, \text{ and this means } v(A) = 1 \text{ and } v(\neg C) = 0$$

$$v(\mathbf{F} \neg A \wedge D) = v(\neg(\neg A \wedge D)) = 1, \text{ and this means that } v(\neg\neg(A)) = 1 \text{ OR } v(D) = \text{is whatever, say } 0)$$

- We say that a branch of a tree is satisfiable if the set of labelled signed formulae on it is satisfiable in at least one model
- We say that a tree (with all of his branches) is satisfiable if some branch of it is satisfiable

Soundness lemma 1 (SL1):

A closed tree(a tree where all the branches are closed) is not satisfiable

PROOF:

- Suppose that we had a tree that was both closed and satisfiable.
- Since it is satisfiable, some branch of it is. Let S be the set of formulae on that branch and let it be satisfiable in the model \mathbf{M} by means of some valuation

- Since the tree is closed (all of its branches are closed) then in *every* branch we must find at least one atomic formula A that is labelled as **TA** AND as **FA**. But then both $v(A)=1$ and $v(\neg A)=1$ must be the case in M but this is not possible.

Corollary of SL1: If one branch is satisfiable the whole tree is *open*

Soundness lemma 2 (SL2):

If (a section of) a tree is satisfiable and a branch of that (section of) dialogue is extended by appropriate particle rules, the result is another satisfiable (section of) a tree.

(Obviously this assumes that the formula that triggers the extension is not atomic (WHY?)).

PROOF:

Let D be a (section of a) satisfiable dialogue and let B be the branch that is extended.

The proof requires several steps. We begin with two main steps:

By hypothesis at least one branch is satisfiable, now this branch could be B or could be B^* .

I) if the satisfiable branch is B^* the extension of B will leave B^* unchanged, thus after the particle rule has been applied to B , D will still be satisfiable (because B^* is).

II) if the satisfiable branch is B and it is satisfiable in the model M the proof is by cases.

That is, by the consideration of all the ways to extend the branch B by the application of the corresponding particle rule to a labelled and signed formula at the end of that branch. Namely by the application of a **F**- and a **T**-rule.

1) Let us start with $FA \rightarrow B$. If we apply the correspondent rule we will produce the branch $B1$ containing the formulae:

TA
FB

Since B is by hypothesis satisfiable in M and $F(A \rightarrow B)$ is on B we have that $v(F(A \rightarrow B))=1$, that is $v(\neg(A \rightarrow B))=1$ is in M . But then by definition of truth in a model we have that $v(A)=1$ is AND $v(B)=0$. But the latter is what we have in the extended section of the branch. Indeed:

$v(TA) = v(A)=1$ in M . and
 $v(FB) = (v(\neg(B))=1 \text{ (or: } v(B)=0) \text{ in } M$.

It follows then that $B1$, that contains **TA** and **FB** is satisfiable in the same model with the same valuation

The other cases without branching are similar.

Exercise: complete the other non branching cases (**T**-conjunction, **F**-disjunction, **T**-negation; **F**-negation)

2) Let us assume now that it is $\mathbf{T}(A \rightarrow B)$ that will produce an extension of **B**.

If we apply the correspondent particle rule and the shifting rule we will produce two branches **B1** and **B2** containing respectively the formulae:

FA

TB

Since **B** is by hypothesis satisfiable in **M** and $\mathbf{T}(A \rightarrow B)$ is on **B** we have that $v(\mathbf{T}(A \rightarrow B)) = v(A \rightarrow B) = 1$. But then by definition of truth in a model we have $v(A) = 0$ OR $v(B) = 1$. But the latter is what we have in the extended section of the branch. Indeed:

$$v(\mathbf{FA}) = v(\neg A) = 1 \quad (v(A) = 0) \quad \text{OR} \quad v(\mathbf{TB}) = v(B) = 1$$

If the left case holds then **FA** is satisfiable, if the right case holds then **TB** is satisfiable. Either way, at least one of the extensions **B1** or **B2** of **B** is satisfiable. Thus at least one branch is satisfiable, so the tree is itself satisfiable.

The other cases with branching are similar and are left as exercise

Soundness theorem:

If a tree for **FA** closes, **A** is (classically) valid.

PROOF:

Assume that there is a closed tree for **F A**, but **A** is not (classically) valid. We show that from this a contradiction follows.

Since there is a closed tree **D** for **FA** then it starts with **FA**. Let us call the first section of the tree **D₀** that consists in the thesis ${}_0\mathbf{FA}$. The following sections of the tree **D** are constructed by extending **D₀**.

Since we assumed that **A** is not (classically) valid, there is some model **M** where **A** is not true. Accordingly the set (of one member) $\{{}_0\mathbf{FA}\}$ is satisfiable in the model **M**. Thus **D₀** is satisfiable, since the set of formulae on its only branch is satisfiable

Since **D₀** is satisfiable by lemma SL2 so any tree we get that starts with **D₀** and results by extending **D₀** is satisfiable

It follows then that **D** is satisfiable.

D is closed by hypothesis, and this is impossible by SL1.

Quod erat demonstrandum

EXERCISE

1 Take the following tree rules for the logical constant “tonk”

$$\mathbf{P} \text{ Atonk} B$$

$$\mathbf{P} A$$

$$\mathbf{P} B$$

$$\mathbf{O} \text{ Atonk} B$$

$$\mathbf{O} A$$

$$\mathbf{O} B$$

- 1.1) Prove that the rules will yield a closed tableau for a formula and its negation
- 1.2) Prove that these rules are unsound in relation to classical logic.