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# Institut de Science Financière et d'Assurances

## *Les Cahiers de Recherche de l'ISFA*

ON FINITE-TIME RUIN PROBABILITIES FOR CLASSICAL RISK MODELS

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# On Finite-Time Ruin Probabilities for Classical Risk Models

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This paper is concerned with the problem of ruin in the classical compound binomial and compound Poisson risk models. Our primary purpose is to extend to those models an exact formula derived by Picard and Lefèvre [24] for the probability of (non-)ruin within finite time. First, a standard method based on the ballot theorem and an argument of Seal-type provides an initial (known) formula for that probability. Then, a concept of pseudo-distributions for the cumulated claim amounts, combined with some simple implications of the ballot theorem, leads to the desired formula. Two expressions for the (non-)ruin probability over an infinite horizon are also deduced as corollaries. Finally, an illustration within the framework of Solvency II is briefly presented.

**Keywords:** ruin probability, finite and infinite horizon, compound binomial model, compound Poisson model, ballot theorem, pseudo-distributions, Solvency II, Value-at-Risk.

## 1 Introduction

The two classical models in risk theory are *the compound binomial model*, which is a discrete-time process where the claim amounts are usually assumed to be integer-valued random variables, and *the compound Poisson model*, which is a continuous-time analogue of this process where the claim amounts are generally assumed to have an absolutely continuous distribution. Although the continuous-time version is quite popular, a discrete-time version can be appropriate for some applications and provides a better intuitive understanding.

The present paper deals with the problem of evaluating, for both risk models, the probability of (non-)ruin within finite time. This problem has received much attention in the literature, and different algorithms have been proposed, with their own advantages and drawbacks; see, e.g., the analysis by Dickson [11] and the references therein.

Recently, Picard and Lefèvre [24] derived an elegant explicit formula, called P.L. formula below, for the finite-time non-ruin probability in a compound Poisson model where the claim amounts are integer-valued. Such a case is important because in practice a

discretization of the claim amounts is often required for numerical calculations (e.g., de Vylder and Goovaerts [9]). The importance of the P.L. formula has been pointed out by De Vylder [6], [7] and Ignatov et al. [17]. A refined discussion of its merits in comparison with other available formulas is given in Rullière and Loisel [28]. Besides, De Vylder and Goovaerts [10] proved that a similar formula holds too for a compound Poisson model with continuous claim amounts; see also Ignatov and Kaishev [18].

Several methods are possible to establish the P.L. formula. In Picard and Lefèvre [24] and De Vylder and Goovaerts [10], the first step consists in stipulating for a total claim amount until some time  $t$  and conditioning on the last claim instant and amount before  $t$ . This yields an integral equation which is then solved by the former authors using the generalized Appell polynomials (in an extended framework), and by the latter authors using the convolution products and the Laplace transforms. On another hand, Rullière and Loisel [28] showed that the P.L. formula can be linked to the well-known ballot theorem and a Seal-type formula (see, e.g., Seal [29] and Gerber [12]).

The primary aim of the present paper is to establish that the formula derived by Picard and Lefèvre [24] can be generalized to the classical compound binomial and compound Poisson risk models. As in its original version, this extended formula will be given explicitly in terms of the pseudo-distributions of some cumulated claim amounts. The method of proof will enable us to provide a unified approach to the ruin problem over a finite horizon. To begin with, we obtain a first formula for the finite-time probability of non-ruin by applying a standard argument, based on the ballot theorem when the initial reserves are null and on a Seal-type formula when the reserves are positive. The result here is not new but this will put us in a position to develop an alternative argument. Succinctly, we first introduce a concept of pseudo-distributions for the cumulated claim amounts, and we show how to compute numerically these pseudo-laws using simple recursions. This mathematical tool, combined with some simple implications of the ballot theorem, will then enable us to derive the desired formula of P.L.-type. In addition, we will also deduce from the Seal and P.L.-types formulas two expressions for the (non-)ruin probability over an infinite horizon. Finally, a brief illustration is presented that points out how finite-time ruin probabilities could be of interest within the framework of Solvency II.

## 2 Compound binomial risk model

This Section is concerned with a discrete-time risk model equivalent to the compound binomial model, but given under a less traditional formulation where more than one claim can arise per time period. We mention that such a model has been investigated by Gerber [13], De Vylder and Goovaerts [9], Shiu [31], Willmot [35], Dickson [11], De Vylder and Marceau [8], Cheng et al. [4], Li and Garrido [21] and Picard et al. [25].

Let us consider an insurance company that evaluates its reserves at periodic times fixed in advance. Claims arise at any date but they can be registered only at the end of the period during which they occur. The successive claim amounts per period,  $X_i$  for  $i = 1, 2, \dots$ , are non-negative integer-valued independent identically distributed (i.i.d.) random variables; let  $\{p_n, n \in \mathbb{N}\}$  be their common probability mass function ( $\mathbb{N} \equiv \{0, 1, 2, \dots\}$ ). At the beginning of each period, the company receives a constant premium,

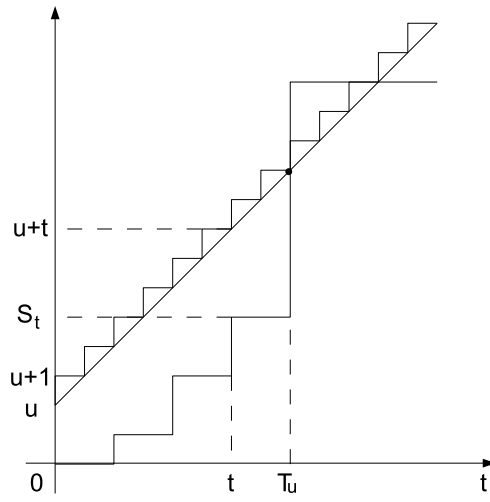


Figure 1: Cumulated premium and claim amounts.

assumed to be equal to 1, as a prepayment to cover the risk during that period (see the remark below). The initial reserves are of amount  $u \in \mathbb{N}$ .

Take the length of a period as the time unit, and denote by  $t \in \mathbb{N}$  the new time scale. During any time interval  $(0, t]$ , the cumulated claim amount is given by  $S_t = X_1 + \dots + X_t$ , and the total premium income is equal to  $t$ . Thus, the reserves  $R_t$  of the company at time  $t$  are given by

$$R_t = u + t - S_t, \quad t \geq 1. \quad (2.1)$$

Note that at time  $t + 0$ , the company will receive an additional premium of 1 but this is a prepayment for the next period  $(t, t + 1]$ . A typical situation is represented in Figure 1.

Ruin will occur at the first time  $T_u$  when the reserves become negative or null. In other words,  $T_u \geq 1$  necessarily and for  $t \geq 1$ ,

$$T_u \geq t + 1 \quad \text{means} \quad S_i < u + i \quad \text{for} \quad 1 \leq i \leq t. \quad (2.2)$$

This definition of ruin is that adopted by Gerber [13] rather than Shiu [31] where ruin occurs when the reserves become negative. The exact distribution of  $T_u$  is the topic of investigation hereafter.

Let us mention that if the premium per time unit is equal to a positive constant  $c \neq 1$ , so that the reserves are  $R_t = u + ct - S_t$ , the ruin problem is different from the actual one. Indeed, the monetary unit might be adapted by dividing by  $c$ , but this would give claim amounts that are no longer integer-valued. Nevertheless, the ruin probability for this case can be evaluated by following the method developed in Picard et al. [25].

## 2.1 A standard approach

**Basic case**  $u = 0$ . Ballot-type problems are among the oldest questions discussed in probability. Their applications are numerous, in risk theory as here and in many other fields such as queueing, reliability, sequential analysis and random graphs. Much on this

can be found in the book by Takács [34]. A well-known generalization of the original ballot theorem due to Takács [33] is as follows.

**Lemma 2.1** (*Ballot theorem, discrete case*). *Let  $X_i, i \geq 1$ , be a sequence of non-negative integer-valued i.i.d. random variables. Consider the associated partial sums process  $S_i = X_1 + \dots + X_i, i \geq 1$ . Then, for any integers  $t \geq 1$  and  $0 \leq n \leq t - 1$ ,*

$$P(S_i < i \text{ for } 1 \leq i < t, \text{ and } S_t = n) = \frac{t-n}{t} p_n^{*t}, \quad (2.3)$$

where  $\{p_n^{*t}, n \in \mathbb{N}\}$  denotes the probability mass function of  $S_t$  (i.e. the  $t$ -th convolution of the law of  $X_1$ ).

Graphically, the left-hand side of (2.3) represents the probability that the partial sums process reaches the level  $n$  at time  $t$  without meeting or crossing earlier the diagonal line of slope 1 (which is an upper boundary); see Figure 2.

Note that formula (2.3) remains true for exchangeable sequences. Many proofs of this result have been proposed, including methods based on a combinatorial argument (e.g., Takács [33]), a stopping time theorem for backward martingales (e.g., Grimmett and Stirzaker [16]) or a queueing theoretic construction (Konstantopoulos [20]). To get a self-contained presentation, we choose to give below an elementary proof in the i.i.d. case, by applying a simple trick that will be useful too later.

*Proof of (2.3).* Let us make a rotation of  $180^\circ$  in Figure 2, and consider the partial sums trajectories in reversed time. Since the increments of the process are i.i.d., each sample path keeps the same probability after rotation. Note also that the reversed partial sums process starts at level  $t - n$  and is backward shifted in the sense that the first jump  $X_1$  occurs at time 0, the second jump  $X_2$  at time 1, and so on; see Figure 3. Therefore, (2.3) is equivalent to the following identity: for  $t \geq 1$  and  $0 \leq n \leq t - 1$ ,

$$P(t - n + S_i > i \text{ for } 1 \leq i < t, \text{ and } t - n + S_t = t) = \frac{t-n}{t} p_n^{*t}. \quad (2.4)$$

For  $m \geq 1$ , let  $\tau_m$  be the first-meeting time of the process  $m + S_i, i \geq 1$ , with the diagonal line (which is a lower boundary). We observe that the probability in the left-hand side of (2.4) corresponds to  $P(\tau_{t-n} = t)$ . Obviously,  $\tau_m \geq m$  and  $P(\tau_m = m) = p_0^{*m}$ ,  $m \geq 1$ . For  $n \geq 1$ ,  $\tau_m = m + n$  means that firstly,  $S_m$  takes an arbitrary integer value  $j$  say, between 1 and  $n$ , and then, independently, a similar shifted partial sums process starting at level  $j$  meets the diagonal line at time  $n$  (Figure 3); this gives

$$P(\tau_m = m + n) = \sum_{j=1}^n p_j^{*m} P(\tau_j = n). \quad (2.5)$$

We point out that (2.5) provides a recursive formula to calculate the probabilities  $P(\tau_m = m + n)$  for  $m, n \geq 1$ .

On another hand, let us consider the partial sums  $S_m, m \geq 1$ . The assumption of i.i.d. increments implies that, for  $n \geq 1$ ,

$$E(S_m \mid S_{m+n} = n) = \frac{1}{p_n^{*(m+n)}} \sum_{j=1}^n j p_j^{*m} p_{n-j}^{*n}, \quad (2.6)$$

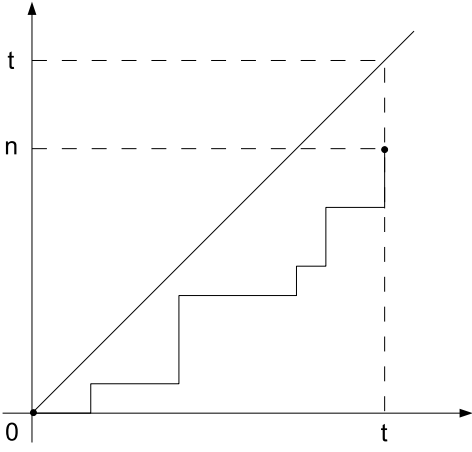


Figure 2: Non-crossing of  $\{S_i\}$  through the diagonal line  $y = x$ .

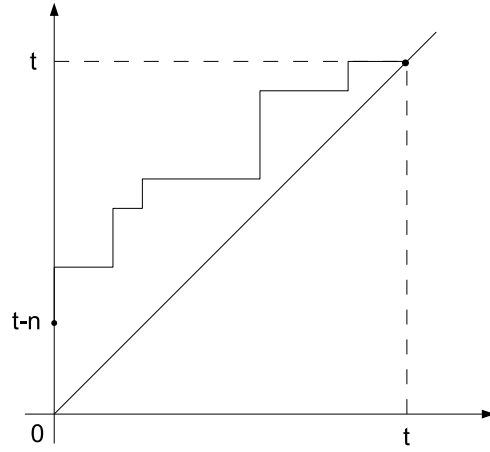


Figure 3: First-crossing of  $\{t - n + S_i\}$  through the diagonal line  $y = x$ .

as well as

$$E(S_m \mid S_{m+n} = n) = m \frac{n}{m+n}. \quad (2.7)$$

Combining (2.6) and (2.7) then yields the identity

$$\frac{m}{m+n} p_n^{*(m+n)} = \sum_{j=1}^n p_j^{*m} \frac{j}{n} p_{n-j}^{*n}. \quad (2.8)$$

Let us now define the quantities  $\theta_m(m+n) \equiv [m/(m+n)] p_n^{*(m+n)}$ , for  $m \geq 1$ ,  $n \in \mathbb{N}$ . Thus,  $\theta_m(m) = p_0^{*m}$  and for  $n \geq 1$ , (2.8) gives

$$\theta_m(m+n) = \sum_{j=1}^n p_j^{*m} \theta_j(n). \quad (2.9)$$

Here too, (2.9) allows us to determine recursively the quantities  $\theta_m(m+n)$ . In fact, the recursive formulas (2.5) and (2.9) are identical, so that  $P(\tau_m = m+n) = \theta_m(m+n)$ . Formula (2.3) then follows by putting  $m = t - n$ .  $\diamond$

Let us turn to the ruin time  $T_0$  for the risk model with  $u = 0$ . Of course,  $T_0 = 1$  when  $X_1 \geq 1$ , and for  $t \geq 1$ , we can write that

$$P(T_0 \geq t+1) = \sum_{n=0}^{t-1} P(S_i < i \text{ for } 1 \leq i \leq t, \text{ and } S_t = n). \quad (2.10)$$

Applying the ballot formula (2.3) to the right-hand side of (2.10) then leads to the following result (e.g., Takács [34]).

**Proposition 2.2** (*Takács-type formula*).  $P(T_0 = 1) = P(X_1 \geq 1)$ , and

$$P(T_0 \geq t+1) = \frac{1}{t} \sum_{n=0}^{t-1} (t-n) p_n^{*t}, \quad t \geq 1. \quad (2.11)$$

We notice that the probability of non-ruin before time  $t$  is given explicitly in terms of the distribution of  $S_t$  evaluated at points  $n = 0, \dots, t - 1$ .

**Passage to  $u \geq 0$ .** Looking at Figure 2, let us now suppose that the upper boundary for the partial sums process  $S_i$ ,  $i \geq 1$ , is a straight line of slope 1 beginning at level  $u \in \mathbb{N}$  instead of 0.

**Lemma 2.3** For any integers  $u \geq 0$ ,  $t \geq 1$  and  $u \leq n \leq u + t - 1$ ,

$$P(S_i < u + i \text{ for } 1 \leq i < t, \text{ and } S_t = n) = p_n^{*t} - \sum_{j=u+1}^n \frac{t+u-n}{t+u-j} p_{n-j}^{*(t+u-j)} p_j^{*(j-u)}. \quad (2.12)$$

*Proof.* Operating a rotation of  $180^\circ$  as before, we observe that

$$P(S_i < u + i \text{ for } 1 \leq i < t, \text{ and } S_t = n) = P(\tau_{t+u-n} \geq t \text{ and } S_t = n), \quad (2.13)$$

where  $\tau_{t+u-n}$  denotes again the first-crossing time of the process  $t + u - n + S_i$ ,  $i \geq 1$ , with the (lower) diagonal line.

Now, for any  $m \geq 1$ ,  $P(m + S_t = m + n) = p_n^{*t}$ . On another hand, the process  $m + S_i$ ,  $1 \leq i < t$ , will either cross or not the diagonal line, and if yes, for the first time at some level  $k$  say, between  $m$  and  $t - 1$ . Therefore, we can write that

$$P(m + S_t = m + n) = P(\tau_m \geq t \text{ and } S_t = n) + \sum_{k=m}^{t-1} P(\tau_m = k) P(S_{t-k} = m + n - k). \quad (2.14)$$

Using formula (2.3), we obtain from (2.14) that

$$P(\tau_m \geq t \text{ and } S_t = n) = p_n^{*t} - \sum_{k=m}^{t-1} \frac{m}{k} p_{k-m}^{*k} p_{m+n-k}^{*(t-k)}. \quad (2.15)$$

Putting  $m = t + u - n$  in (2.15) and taking  $j = t + u - k$  as a new index of summation then yields formula (2.12).  $\diamond$

*Remark.* When  $u = 0$ , (2.12) and (2.3) hold true and the following identity then follows: for  $t \geq 1$  and  $0 \leq n \leq t - 1$ ,

$$p_n^{*(t)} = \sum_{j=0}^n \frac{t-n}{t-j} p_{n-j}^{*(t-j)} p_j^{*j}. \quad (2.16)$$

This is precisely a generalization of (2.16) which will be the key result in the analysis of Section 2.3.

Formula (2.16) is easily obtained on basis of a graphical representation. Indeed, consider a line of slope 1 passing at point  $(t, n)$  (of equation  $y = n - t + x$ ). The event  $S_t = n$  implies that the process  $S_i$ ,  $1 \leq i \leq t$ , necessarily crosses this line, for the first



time at some point  $(t - j, j)$  say, with  $0 \leq j \leq n$ . Note that the first-crossing time,  $\inf\{i : S_i = n - t + i\}$ , corresponds to the time  $\tau_{t-n}$  above. Thus, we have

$$P(S_t = n) = \sum_{j=0}^n P(\tau_{t-n} = t - j) p_j^{*j}.$$

which gives (2.16) by virtue of (2.3).  $\diamond$

Let us examine the ruin time  $T_u$  for the risk model with  $u \geq 0$ . Of course,  $T_u = 1$  when  $X_1 \geq u + 1$ , and for  $t \geq 1$ , we get

$$P(T_u \geq t + 1) = \sum_{n=0}^u P(S_t = n) + \sum_{n=u+1}^{u+t-1} P(S_i < u + i \text{ for } 1 \leq i < t, \text{ and } S_t = n). \quad (2.17)$$

Using (2.12) in the right-hand side of (2.17) and permuting the two sums, we then find formula (2.18) below. A formula of this type was already obtained by Seal [29].

**Proposition 2.4** (*Seal-type formula*).  $P(T_u = 1) = P(X_1 \geq u + 1)$ , and

$$P(T_u \geq t + 1) = \sum_{j=0}^{u+t-1} p_j^{*t} - \sum_{j=u+1}^{u+t-1} p_j^{*(j-u)} \left( \sum_{n=j}^{u+t-1} \frac{t + u - n}{t + u - j} p_{n-j}^{*(t+u-j)} \right), \quad t \geq 1. \quad (2.18)$$

Note that formula (2.18) is given explicitly in terms of the distribution of  $S_1, \dots, S_t$  at different points between 0 and  $u + t - 1$ .

## 2.2 An alternative approach

**Pseudo-distributions for  $S_t$ .** To begin with, we are going to point out an underlying polynomial structure in the distribution of  $S_t$ ,  $t \in \mathbb{N}$ . Without real loss of generality, it is assumed that  $0 < p_0 = P(X_1 = 0) < 1$ . Now, among the  $t$  claims with sum equal to  $S_t$ , the number of strictly positive claim amounts is a binomial random variable with exponent  $t$  and parameter  $1 - p_0$ . For  $k \geq 1$ , let  $S_k^c = X_1^c + \dots + X_k^c$  where the random variables  $X_i^c$  are i.i.d. and distributed as  $(X_1 | X_1 \geq 1)$ , i.e.  $P(X_i^c = n) = p_n / (1 - p_0)$ ,  $n \geq 1$ ; put  $S_0^c = 0$ . By the law of total probability, we can then write that

$$P(S_t = n) = p_0^t \sum_{k=0}^n \binom{t}{k} \left( \frac{1 - p_0}{p_0} \right)^k P(S_k^c = n), \quad n \in \mathbb{N}, \quad (2.19)$$

From (2.19), we thus see that the law of  $S_t$  is of the form

$$P(S_t = n) = p_0^t e_n(t), \quad (2.20)$$

where  $e_0(t) = 1$  and for  $n \geq 1$ ,  $e_n(t)$  represents some polynomial of degree  $n$  in  $t$  satisfying  $e_n(0) = 0$ .

So far, we were only concerned with values of  $t \in \mathbb{N}$ . Formally, the right-hand side of (2.19), (2.20) might be evaluated for any value of  $t$ , positive integer or not. So, considering its natural extension on  $\mathbb{R}$ , we introduce the following quantities:

$$p_0^t e_n(t) \equiv \tilde{P}(S_t = n) \text{ say, with } t \in \mathbb{R}. \quad (2.21)$$

Let us underline that the notation  $\tilde{P}(S_t = n)$  in (2.21) has no probabilistic meaning except when  $t \in \mathbb{N}$  as in (2.19). For example, when  $t = -1$ ,  $\tilde{P}(S_t = n) < 0$  for all  $n \geq 1$ . By analogy, however,  $\tilde{P}(S_t = n)$ ,  $n \in \mathbb{N}$ , can be viewed as a pseudo-probability mass function for  $S_t$ .

The process  $S_t$ ,  $t \in \mathbb{N}$ , having independent stationary increments, the convolution property gives, for  $t_1, t_2 \in \mathbb{N}$ ,

$$P(S_{t_1+t_2} = n) = \sum_{i=0}^n P(S_{t_1} = n-i) P(S_{t_2} = i), \quad n \in \mathbb{N}, \quad (2.22)$$

and by inserting (2.20), this yields a formula of binomial type:

$$e_n(t_1 + t_2) = \sum_{i=0}^n e_{n-i}(t_1) e_i(t_2). \quad (2.23)$$

Now, we observe that the identity (2.23) still holds true for any  $t_1, t_2 \in \mathbb{R}$  since both sides are polynomials of degree  $n$  in  $t_1$  (and  $t_2$ ) that are identical for an infinite number of values of  $t_1$  (and  $t_2$ ). Therefore, (2.22) can be generalized to the pseudo-distributions above, giving

$$\tilde{P}(S_{t_1+t_2} = n) = \sum_{i=0}^n \tilde{P}(S_{t_1} = n-i) \tilde{P}(S_{t_2} = i), \quad t_1, t_2 \in \mathbb{R}. \quad (2.24)$$

In the sequel, we will have recourse to the pseudo-probability mass functions of  $S_t$  and  $S_{-t}$  when  $t \in \mathbb{N}$ . These pseudo-distributions can be calculated numerically by recursion. For  $S_t$ , it suffices to write that for  $t \geq 0$ ,

$$P(S_{t+1} = n) = \sum_{i=0}^n p_{n-i} P(S_t = i), \quad n \in \mathbb{N}, \quad (2.25)$$

with  $S_0 = 0$ . A similar procedure can be followed for  $S_{-t}$ . Indeed, by (2.24) we have, for  $t \geq 1$ ,

$$\tilde{P}(S_{-t-1} = n) = \sum_{i=0}^n \tilde{P}(S_{-1} = n-i) \tilde{P}(S_{-t} = i), \quad n \in \mathbb{N}, \quad (2.26)$$

so that it remains to find the pseudo-distribution of  $S_{-1}$ . For this, we recall that  $P(S_0 = n) = \delta_{n,0}$  (the Kronecker delta), so that (2.24) with  $t_1 = -1$  and  $t_2 = 1$  yields the recursion

$$\delta_{n,0} = \sum_{i=0}^n \tilde{P}(S_{-1} = n-i) p_i, \quad n \in \mathbb{N}. \quad (2.27)$$

**Implications for  $T_u$ .** For clarity, we denote  $\tilde{P}(S_t = n) = \tilde{p}_n^{*t}$  when negative values for  $t$  are allowed. We establish below that thanks to the previous pseudo-distributions, formula (2.16) can be generalized to the case where  $u \in \mathbb{N}$ .

**Lemma 2.5** For any integers  $u \geq 0$ ,  $t \geq 1$  and  $u \leq n \leq u + t - 1$ ,

$$p_n^{*t} = \sum_{j=0}^n \frac{t+u-n}{t+u-j} p_{n-j}^{*(t+u-j)} \tilde{p}_j^{*(j-u)}. \quad (2.28)$$

*Proof.* By (2.24) we can write that

$$P(S_t = n) = \sum_{k=0}^n P(S_{t+u} = n-k) \tilde{P}(S_{-u} = k). \quad (2.29)$$

Now, consider the event  $(S_{t+u} = n-k)$  in (2.29). We first draw a straight line of slope

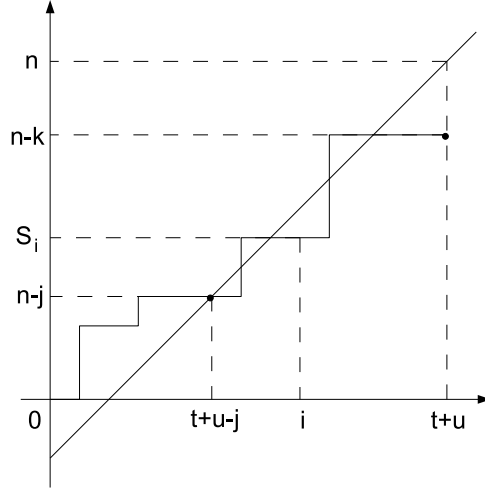


Figure 4: A trajectory  $\{S_i\}$  with respect to the line of equation  $y = -(t+u-n) + x$ .

1 passing at point  $(t+u, n)$  (of equation  $y = -(t+u-n) + x$ ); see Figure 4. Remember that  $t+u-n \geq 1$  by assumption. Thus, to reach the level  $n-k$  at time  $t+u$ , the process  $S_i$ ,  $1 \leq i \leq t+u$ , necessarily crosses that straight line. Suppose that the first-crossing arises at some point  $(t+u-j, n-j)$  say, with  $k \leq j \leq n$ . Since this first-crossing time corresponding to  $\tau_{t+u-n}$ , we obtain, using (2.3) for  $P(\tau_{t+u-n} = n-j)$ , that

$$P(S_{t+u} = n-k) = \sum_{j=k}^n \frac{t+u-n}{t+u-j} p_{n-j}^{*(t+u-j)} p_{j-k}^{*j}. \quad (2.30)$$

Inserting (2.30) in (2.29) then yields, after permutation of the two sums,

$$p_n^{*t} = \sum_{j=0}^n \frac{t+u-n}{t+u-j} p_{n-j}^{*(t+u-j)} \sum_{k=0}^j p_{j-k}^{*j} \tilde{p}_k^{*(-u)}, \quad (2.31)$$

which reduces to (2.28) since by (2.24), the sum  $(\sum_{k=0}^j \dots)$  in (2.31) gives  $\tilde{p}_j^{*(j-u)}$ .  $\diamond$

Note that  $\tilde{p}_j^{*(j-u)}$  in (2.28) is a standard probabilistic convolution when  $j \geq u$  but this is not true for the first  $u$  values of  $j$ . By substituting (2.28) for  $p_n^{*t}$  in (2.12), we are able to deduce a new representation for the non-crossing probability of interest.

**Lemma 2.6** For any integers  $u \geq 0$ ,  $t \geq 1$  and  $u \leq n \leq u + t - 1$ ,

$$P(S_i < u + i \text{ for } 1 \leq i < t, \text{ and } S_t = n) = \sum_{j=0}^u \frac{t + u - n}{t + u - j} p_{n-j}^{*(t+u-j)} \tilde{p}_j^{*(j-u)}. \quad (2.32)$$

A remarkable property of formula (2.32) is that the sum is taken over  $u + 1$  terms only, whatever the value of  $n$ . From that point of view, (2.32) is simpler than (2.12) where the sum contains  $n - u$  terms. Let us also point out that when  $u = 0$ , (2.32) directly reduces to the ballot formula (2.3).

For the non-ruin probability before  $t$ , inserting (2.32) in (2.17) provides the following alternative expression. A result of similar nature was obtained by Picard and Lefèvre [24] for a continuous-time version of the model.

**Proposition 2.7** (*P.L.-type formula*).  $P(T_u = 1) = P(X_1 \geq u + 1)$ , and

$$P(T_u \geq t + 1) = \sum_{j=0}^u \left( p_j^{*t} + \tilde{p}_j^{*(j-u)} \sum_{n=u+1}^{u+t-1} \frac{t + u - n}{t + u - j} p_{n-j}^{*(t+u-j)} \right), \quad t \geq 1. \quad (2.33)$$

Formula (2.33) is given explicitly in terms of the pseudo-distributions of  $S_{-u}, \dots, S_{t+u}$  evaluated at different points between 0 and  $u + t - 1$ . Comparing the efficacy of (2.18) and (2.33) is rather delicate (see, e.g., Rullière and Loisel [28]). Some numerical experiments show that (2.33) becomes especially fast when  $u$  is small compared to  $t$ .

## 2.3 Infinite-time horizon

Let  $\mu = E(X_1)$  be the expected amount of a claim. Since  $S_t/t \rightarrow \mu$  as  $t \rightarrow \infty$ , ruin will occur a.s. if  $\mu \geq 1$ . Assuming now that  $\mu < 1$ , we want to obtain the probability of ultimate (non-)ruin. Specifically, from (2.18) and (2.33) we are going to establish the two formulas below.

**Corollary 2.8** If  $\mu < 1$ , then

$$P(T_u < \infty) = (1 - \mu) \sum_{j=u+1}^{\infty} p_j^{*(j-u)}, \quad (2.34)$$

$$P(T_u = \infty) = (1 - \mu) \sum_{j=0}^u \tilde{p}_j^{*(j-u)}. \quad (2.35)$$

*Proof.* To begin with, define by  $Z_a^{u+t-j}$  the expectation

$$Z_a^{u+t-j} = E \left[ \left( 1 - \frac{S_{t+u-j}}{t + u - j} \right) I_{(a \leq S_{t+u-j} \leq t+u-j)} \right], \quad (2.36)$$

for any naturals  $a \in [0, t + u - j]$  ( $I_A$  is the indicator of the event  $A$ ). Applying to (2.36) the dominated convergence theorem, we get that

$$Z_a^{t+u-j} \rightarrow 1 - \mu \quad \text{as } t \rightarrow \infty. \quad (2.37)$$

Now, consider the Seal-type formula (2.18). It contains a sum  $(\sum_{n=j}^{t+u+1} \dots)$  that corresponds to  $Z_0^{t+u-j}$ . We then find that

$$\begin{aligned} P(T_u < \infty) &= 1 - \lim_{t \rightarrow \infty} P(T_u \geq t + 1) \\ &= 1 - \lim_{t \rightarrow \infty} P(S_t \leq u + t - 1) + \lim_{t \rightarrow \infty} \sum_{j=u+1}^{u+t-1} p_j^{*(j-u)} Z_0^{t+u-j}, \end{aligned}$$

which reduces to formula (2.34) by virtue of (2.37) and since  $P(S_t \leq u + t - 1) \rightarrow 1$ .

For the P.L.-type formula (2.33), we recognize the inner sum  $(\sum_{n=u+1}^{u+t-1} \dots)$  as  $Z_{u+1-j}^{t+u-j}$ . By (2.37) and since  $P(S_t \leq j) \rightarrow 0$ , we then obtain formula (2.35).  $\diamond$

It can be checked that (2.34) and (2.35) correspond to formulas (40) and (42) obtained by Gerber [13] for the usual version of the model with at most one claim per period. Let us point out that in practice, the sum (2.35) with  $u + 1$  terms will be much easier to handle than the series (2.34).

### 3 Compound Poisson risk model

A continuous-time analogue of the previous model is the compound Poisson risk model. This model plays a prominent role in risk theory and a considerable literature is devoted to its study. The reader is referred to the books by Gerber [12], Grandell [15], Panjer and Willmot [23], Asmussen [2] and Kaas et al. [19]); see also, e.g., Gerber and Shiu [16], Albrecher et al. [1] and Cardoso and Waters [3] for recent results on the ruin problem.

The insurance company is now assumed to be able to follow the evolution of its reserves in continuous-time. The occurrence of claims is described by a Poisson process  $N(t)$ ,  $t \geq 0$ , with rate  $\lambda > 0$ . The successive claim amounts  $X_i$ ,  $i = 1, 2, \dots$ , are positive i.i.d. continuous random variables, with density function  $f(x)$  for  $x > 0$ . The initial reserves are of amount  $u \geq 0$ , and the premium rate is constant and equal to  $c > 0$ . Thus, the reserves at time  $t > 0$  are given by  $R_t = u + ct - S_t$  (as in (2.1)) but with  $S_t = X_1 + \dots + X_{N(t)}$  and  $c > 0$ . Note that the law of  $S_t$  has an atom at 0 with  $P(S_t = 0) = e^{-\lambda t}$ , and is continuous on  $(0, \infty)$ .

The ruin time is the first instant  $T_u > 0$  where  $R_t$  becomes negative or null. In the sequel, we suppose that  $c = 1$ . This assumption, however, is not restrictive for the present model since it comes to change the monetary unit and claim amounts are here continuous. So, for  $t > 0$ ,

$$T_u \geq t \quad \text{means} \quad S_y < u + y \quad \text{for} \quad 0 < y < t. \quad (3.1)$$

To obtain the exact distribution of  $T_u$ , we will adapt the method of proof followed for the discrete-time model.

#### 3.1 A standard approach

**Basic case**  $u = 0$ . The starting point is the following well-known ballot theorem in continuous-time (see, e.g., Takács [34]).

**Lemma 3.1** (*Ballot theorem, continuous case*). Let  $S_y$ ,  $y \geq 0$ , be a compound Poisson process with positive continuous jumps. Then, for any  $t > 0$  and  $0 < x < t$ ,

$$P[S_y < y \text{ for } 0 < y < t, \text{ and } S_t \in (x, x + dx)] = \frac{t-x}{t} f_t(x) dx, \quad (3.2)$$

where  $f_t(\cdot)$  denotes the density of  $S_t$  on  $(0, \infty)$ .

Let us recall that this result can be extended to processes with cyclically interchangeable increments. Various proofs of (3.2) can be found in the references mentioned earlier. We derive below formula (3.2) by arguing exactly as for (2.3).

*Proof of (3.2).* Denote by  $g_t(x)dx$  the probability in the left-hand side of (3.2). Making a rotation of  $180^\circ$ ,  $g_t(x)$  can be viewed as the density of the first-meeting time  $\tau_{t-x}$  of the process  $t-x+S_y$ ,  $y > 0$ , with the diagonal line of slope 1. In other words,

$$\begin{aligned} g_t(x)dx &= P[\tau_{t-x} \in (t, t+dx)] \\ &= P[t-x+S_y > y \text{ for } 0 < y < t, \text{ and } t-x+S_t \in (t, t+dx)]. \end{aligned} \quad (3.3)$$

Considering the possible jumps during the time interval  $(0, t-x)$ , we find that

$$g_t(x) = \int_0^x f_{t-x}(y) g_x(x-y) dy. \quad (3.4)$$

On another hand, since the increments of the process are independent and stationary, the following identity holds:

$$\begin{aligned} E(S_{t-x} | S_t = x) &= (t-x) \frac{x}{t} \\ &= \int_0^x y \frac{f_{t-x}(y) f_x(x-y)}{f_t(x)} dy. \end{aligned} \quad (3.5)$$

Let us define  $\phi_t(x) \equiv [(t-x)/t] f_t(x)$ , for  $0 < x < t$ ; from (3.5) we see that these quantities satisfy the integral equation

$$\phi_t(x) = \int_0^x f_{t-x}(y) \phi_x(x-y) dy. \quad (3.6)$$

Comparing (3.4) and (3.6), we deduce that  $g_t(x) = \phi_t(x)$ , i.e. (3.2).  $\diamond$

Turning to the ruin time  $T_0$ , we can write that

$$P(T_0 \geq t) = e^{-\lambda t} + \int_0^t P[S_y < y \text{ for } 0 < y \leq t, \text{ and } S_t \in (x, x+dx)],$$

and thanks to (3.2), this yields formula (3.7) below, of structure similar to (2.10) (and going back to Cramér [5]).

**Proposition 3.2** (*Takács-type formula*)

$$P(T_0 \geq t) = e^{-\lambda t} + \frac{1}{t} \int_0^t (t-x) f_t(x) dx, \quad t > 0. \quad (3.7)$$

**Passage to  $u \geq 0$ .** In the framework of the ballot problem, let us now suppose that the upper boundary is a straight line of slope 1 passing at point  $(0, u)$ ,  $u \geq 0$ .

**Lemma 3.3** For any real numbers  $u \geq 0$ ,  $t > 0$  and  $u < x < u + t$ ,

$$(1/dx) P[S_y < u + y \text{ for } 0 < y < t, \text{ and } S_t \in (x, x + dx)] = f_t(x) - e^{-\lambda(t+u-x)} f_{x-u}(x) - \int_u^x \frac{t+u-x}{t+u-z} f_{t+u-z}(x-z) f_{z-u}(z) dz. \quad (3.8)$$

*Proof.* After a rotation of  $180^\circ$ , the probability in the left-hand side of (3.8), denoted by  $h_t(x, u)dx$ , may be represented as

$$h_t(x, u)dx = P[\tau_{t+u-x} > t \text{ and } S_t \in (x, x + dx)]. \quad (3.9)$$

Proceeding as for (2.13), we observe that to reach the level  $x$  at time  $t$ , the process  $t + u - x + S_y$ ,  $0 < y \leq t$ , will either cross or not the diagonal line, and if yes, at some level  $t + u - z$  where  $z \in (u, x]$ . Note that  $t + u - x$  is a possible crossing-level with the probability  $e^{-\lambda(t+u-x)}$ . Thus, we get

$$f_t(x) = h_t(x, u) + e^{-\lambda(t+u-x)} f_{x-u}(x) + \int_u^x g_{t+u-z}(x-z) f_{z-u}(z) dz, \quad (3.10)$$

where  $g_t(x)$  is the density defined through (3.3). By the ballot formula (3.2),

$$g_{t+u-z}(x-z) = \frac{t+u-x}{t+u-z} f_{t+u-z}(x-z), \quad (3.11)$$

so that substituting (3.11) in (3.10) and using (3.9) yields formula (3.8).  $\diamond$

Now, concerning the ruin time  $T_u$ , we have, for  $t > 0$ ,

$$P(T_u \geq t) = e^{-\lambda t} + \int_0^u f_t(z) dz + \int_u^{u+t} P[S_y < u + y \text{ for } 0 < y < t, \text{ and } S_t \in (x, x + dx)]. \quad (3.12)$$

Inserting (3.8) in (3.12) and permuting the two integrals, we then deduce the following result (originating from Prabhu [27]).

**Proposition 3.4** (*Seal-type formula*)

$$P(T_u \geq t) = e^{-\lambda t} + \int_0^{u+t} f_t(z) dz - \int_u^{u+t} e^{-\lambda(t+u-z)} f_{z-u}(z) dz - \int_{z=u}^{u+t} f_{z-u}(z) \left( \int_{x=z}^{u+t} \frac{t+u-x}{t+u-z} f_{t+u-z}(x-z) dx \right) dz, \quad t > 0. \quad (3.13)$$

This formula is similar to (2.18) and shows how to obtain the probability of non-ruin before time  $t$  from the densities of  $S_y$  where  $0 < y < u + t$ .

### 3.2 An alternative approach

**Pseudo-distributions for  $S_t$ .** First, let us consider the law of  $S_t$ ,  $t > 0$ . Apart from an atom at 0, it has a density on  $(0, \infty)$  given by

$$f_t(x) = e^{-\lambda t} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} f^{*i}(x), \quad x > 0, \quad (3.14)$$

$f^{*i}(\cdot)$  being the  $i$ -th convolution of the density of  $X_1$ . Thus, similarly to (2.20), we can write  $f_t(x)$  under the form

$$f_t(x) = e^{-\lambda t} e_x(t), \quad (3.15)$$

where  $e_x(t)$  represents the corresponding power series in  $t$ .

For  $t = 0$ , let  $e_x(0) = 0$ . Now, let us extend the domain of  $t$  to  $\mathbb{R}$  in (3.15). It is directly checked that the series defining  $e_x(t)$  converges absolutely. Therefore, as with (2.21), we may introduce a pseudo-density for  $S_t$  by putting

$$e^{-\lambda t} e_x(t) \equiv \tilde{f}_t(x) \text{ say, with } t \in \mathbb{R}. \quad (3.16)$$

For  $t_1, t_2 > 0$ ,  $S_{t_1+t_2}$  is distributed as  $S_{t_1} + S_{t_2}$ . In terms of the previous power series, this implies that

$$e_x(t_1 + t_2) = e_x(t_1) + e_x(t_2) + \int_0^x e_{x-y}(t_1) e_y(t_2) dy, \quad x > 0. \quad (3.17)$$

Notice that (3.17) is an identity in  $t_1$  (and  $t_2$ ) that remains valid when  $t_1 < 0$  (and  $t_2 < 0$ ). Thus, the convolution property is still true for pseudo-densities, yielding

$$\tilde{f}_{t_1+t_2}(x) = \tilde{f}_{t_1}(x) e^{-\lambda t_2} + \tilde{f}_{t_2}(x) e^{-\lambda t_1} + \int_0^x \tilde{f}_{t_1}(x-y) \tilde{f}_{t_2}(y) dy, \quad t_1, t_2 \in \mathbb{R}. \quad (3.18)$$

These pseudo-densities can be rather easily calculated. Indeed, the number of claims at  $t > 0$  being Poisson distributed, we have for  $f_t(\cdot)$ ,  $t > 0$ , the classical Panjer representation, namely

$$f_t(x) = \lambda t \int_0^x \frac{y}{x} f(y) f_t(x-y) dy, \quad x > 0. \quad (3.19)$$

Furthermore, it is clear that formula (3.19) remains true on  $\mathbb{R}$ , i.e. for  $\tilde{f}_t(\cdot)$ . Now, (3.19) corresponds to a Volterra integral equation of the second kind, and such an equation can be solved numerically by appropriate discretization (see, e. g., Appendix D in Panjer and Willmot [23]).

**Implications for the law of  $T_u$ .** We will first derive a continuous variant of the identity (2.28).

**Lemma 3.5** *For any real numbers  $u \geq 0$ ,  $t > 0$  and  $u \leq x \leq u + t$ ,*

$$f_t(x) = e^{-\lambda(t+u-x)} f_{x-u}(x) + \frac{t+u-x}{t+u} f_{t+u}(x) + \int_0^x \frac{t+u-x}{t+u-y} f_{t+u-y}(x-y) \tilde{f}_{y-u}(y) dy. \quad (3.20)$$



*Proof.* We write  $f_t = f_{t+u-u}$ , so that applying (3.18) gives

$$f_t(x) = f_{t+u}(x) e^{\lambda u} + \tilde{f}_{-u}(x) e^{-\lambda(t+u)} + \int_0^x f_{t+u}(x-y) \tilde{f}_{-u}(y) dy. \quad (3.21)$$

To begin with, let us consider the density  $f_{t+u}(x-y)$  and a straight line of slope 1 passing at point  $(t+u, x)$ . The compound Poisson process will reach the level  $x-y$  at time  $t+u$  necessarily after crossing that straight line, for the first time at some level  $x-z$  where  $z \in [y, x]$  (at time  $t+u-z$ ). Notice that both levels 0 and  $x-y$  are possible first-crossing levels with positive probabilities. Using the density  $g_t(x)$  defined by (3.3), we then get

$$\begin{aligned} f_{t+u}(x-y) &= e^{-\lambda(t+u-x)} f_x(x-y) + g_{t+u-y}(x-y) e^{-\lambda y} \\ &\quad + \int_y^x g_{t+u-z}(x-z) f_z(z-y) dz. \end{aligned} \quad (3.22)$$

Now, in the right-hand side of (3.21), let us first examine the term  $f_{t+u}(x) e^{\lambda u}$ . From (3.22) with  $y = 0$ , we see that

$$f_{t+u}(x) e^{\lambda u} = e^{-\lambda(t-x)} f_x(x) + e^{\lambda u} g_{t+u}(x) + e^{\lambda u} \int_y^x g_{t+u-z}(x-z) f_z(z) dz. \quad (3.23)$$

Then, consider the integral in (3.21). Substituting (3.22) for  $f_{t+u}(x-y)$ , this integral becomes

$$\begin{aligned} \int_0^x f_{t+u}(x-y) \tilde{f}_{-u}(y) dy &= e^{-\lambda(t+u-x)} \int_0^x f_x(x-y) \tilde{f}_{-u}(y) dy \\ &\quad + \int_0^x e^{-\lambda y} g_{t+u-y}(x-y) \tilde{f}_{-u}(y) dy \\ &\quad + \int_0^s g_{t+u-y}(x-y) \left[ \int_0^x f_z(z-y) \tilde{f}_{-u}(z) dz \right] dy. \end{aligned} \quad (3.24)$$

By (3.17), the first integral in the right-hand side of (3.24) can be expressed as

$$\int_0^x f_x(x-y) \tilde{f}_{-u}(y) dy = f_{x-u}(x) - e^{\lambda u} f_x(x) - e^{-\lambda x} \tilde{f}_{-u}(x); \quad (3.25)$$

a similar expression holds for the third integral too. Making these two substitutions, we obtain for (3.24)

$$\begin{aligned} \int_0^x f_{t+u}(x-y) \tilde{f}_{-u}(y) dy &= e^{-\lambda(t+u-x)} [f_{x-u}(x) - e^{\lambda u} f_x(x) - e^{-\lambda x} \tilde{f}_{-u}(x)] \\ &\quad + \int_0^x e^{-\lambda y} g_{t+u-y}(x-y) \tilde{f}_{-u}(y) dy \\ &\quad + \int_0^x g_{t+u-y}(x-y) [f_{y-u}(y) - e^{\lambda u} f_y(y) - e^{-\lambda y} \tilde{f}_{-u}(y)] dy. \end{aligned} \quad (3.26)$$

Finally, inserting (3.23) and (3.26) in (3.21), and using the ballot formula (3.3) leads to the result (3.20).  $\diamond$

Lemma 3.5 allows us to obtain a new representation for the probability (3.8) of non-crossing before time  $t$ .

**Lemma 3.6** For any real numbers  $u \geq 0$ ,  $t > 0$  and  $u \leq x \leq u + t$ ,

$$\begin{aligned} (1/dx) P[S_y < u + y \text{ for } 0 < y < t, \text{ and } S_t \in (x, x + dx)] \\ = \frac{t + u - x}{t + u} f_{t+u}(x) + \int_0^u \frac{t + u - x}{t + u - y} f_{t+u-y}(x - y) \tilde{f}_{y-u}(y) dy. \end{aligned} \quad (3.27)$$

Inserting (3.27) in (3.12), we then deduce the desired formula.

**Proposition 3.7** (P.L.-type formula)

$$\begin{aligned} P(T_u \geq t) = e^{-\lambda t} + \int_0^u f_t(y) dy + \int_u^{t+u} \frac{t + u - y}{t + u} f_{t+u}(y) dy \\ + \int_{y=0}^u \tilde{f}_{y-u}(y) \left( \int_{x=u}^{u+t} \frac{t + u - x}{t + u - y} f_{t+u-y}(x - y) dx \right) dy, \quad t > 0. \end{aligned} \quad (3.28)$$

Note that (3.28) is constructed from the (pseudo-)densities  $\tilde{f}_s(\cdot)$ ,  $-u < s < t + u$ , evaluated at points between 0 and  $t + u$ . The same formula was obtained by De Vylder and Goovaerts [10], using a much less direct method based on the solution of some integro-differential equation.

### 3.3 Infinite-time horizon

Let  $\mu = E(X_1)$  and assume that  $\lambda\mu < 1$ . Then, ruin is not a.s. and the ultimate (non-)ruin probability can be expressed as follows.

**Corollary 3.8** If  $\lambda\mu < 1$ , then

$$P(T_u < \infty) = (1 - \lambda\mu) \int_u^\infty f_{y-u}(y) e^{\lambda(u-y)} dy, \quad (3.29)$$

$$P(T_u = \infty) = (1 - \lambda\mu) \left[ e^{\lambda u} + \int_0^u \tilde{f}_{y-u}(y) e^{\lambda(u-y)} dy \right]. \quad (3.30)$$

*Proof.* We observe that given any non-negative real  $a \in [0, t + u - y]$ , the expectation

$$Z_a^{t+u-y} \equiv E \left[ \left( 1 - \frac{S_{t+u-y}}{t + u - y} \right) I_{(a \leq S_{t+u-y} \leq t+u-y)} \right] \quad (3.31)$$

converges to  $1 - \lambda\mu$  as  $t \rightarrow \infty$ . Applying this result inside (3.13) and (3.28) then yields (3.29) and (3.30), respectively.  $\diamond$

It can be checked that (3.29) is equivalent to the classical Pollaczek-Khinchine (Beekman) formula. Formula (3.30), less standard, is given in De Vylder and Goovaerts [10]; see also Shiu [30] for discrete claim amounts as discussed below.

### 3.4 Discrete claim amounts

Picard and Lefèvre [24] derived a similar formula (recalled in (3.37)) for a compound Poisson risk model where the claim amounts are positive integer-valued instead of continuous as before. The method of proof followed in that paper is primarily of algebraic nature. Let us sketch that the approach developed here, of more probabilistic essence, works well too.

Let  $\{p_t(n), n \in \mathbb{N}\}$  be the probability mass function of  $S_t$  at time  $t > 0$ , i.e.

$$p_t(n) = e^{-\lambda t} \sum_{i=1}^n \frac{(\lambda t)^i}{i!} p_n^{*i}, \quad (3.32)$$

$\{p_n^{*i}, n \in \mathbb{N}\}$  being the  $i$ -th convolution of the law of  $X_1$ . As in (2.20), we write it as

$$p_t(n) = e^{-\lambda t} e_n(t), \quad (3.33)$$

where  $e_0(t) = 1$  and  $e_n(t), n \geq 1$ , is a polynomial of degree  $n$  in  $t$  with  $e_n(0) = 0$ . Then, for all  $t \in \mathbb{R}$ , we may define the pseudo-probability mass function  $\{\tilde{p}_t(n), n \in \mathbb{N}\}$  of  $S_t$  just by natural extension of (3.32), (3.33) to  $\mathbb{R}$ . In practice, these pseudo-distributions are calculated by the Panjer algorithm: for  $t \in \mathbb{R}$ ,

$$\tilde{p}_t(0) = e^{-\lambda t}, \quad \text{and} \quad \tilde{p}_t(n) = \frac{\lambda t}{n} \sum_{i=1}^n i p_i \tilde{p}_t(n-i), \quad n \geq 1. \quad (3.34)$$

The following results can then be established. Suppose (for clarity) that  $u$  is a non-negative integer, and denote by  $[t]$  the integer part of  $t$ .

(Takács-type formula):

$$P(T_0 \geq t) = \frac{1}{t} \sum_{n=0}^{[t]} (t-n) p_t(n), \quad t > 0, \quad (3.35)$$

(Seal-type formula):

$$P(T_u \geq t) = \sum_{j=0}^{u+[t]} p_t(j) - \sum_{j=u+1}^{u+[t]} p_{j-u}(j) \left( \sum_{n=j}^{u+[t]} \frac{t+u-n}{t+u-j} p_{t+u-j}(n-j) \right), \quad t > 0, \quad (3.36)$$

(P.L.-type formula):

$$P(T_u \geq t) = \sum_{j=0}^u \left( p_t(j) + \tilde{p}_{j-u}(j) \sum_{n=u+1}^{u+[t]} \frac{t+u-n}{t+u-j} p_{t+u-j}(n-j) \right), \quad t > 0. \quad (3.37)$$

These formulas are very similar to those derived for the compound binomial model (see (2.11), (2.18) and (2.33)).

### 3.5 A brief illustration

The Solvency II project provides a framework for a future prudential regulation of insurance in the European Union. In its Pillar 1, a main objective is to define the capital levels a company will have to allocate in order to cover all the risks inherent to the business. Two capital requirements are considered, the MCR (Minimum Capital Requirement) and the SCR (Solvency Capital Requirement) which has the key role. To determine the SCR, companies may follow standard approaches or apply internal models (the latter are more time-consuming but are expected to yield lower requirements).

A rather well accepted rule for the SCR consists in fixing the probability of solvency *in any one year*, most likely at the level of 99.5%. In other words, the SCR is such that the one-year 99.5% Value-at-Risk ( $VaR$ ) is controlled. In recent discussions (Quantitative Impact Study 3), it has been suggested that the level of SCR could guarantee the MCR level to be available at anytime within 5 or 10 years, with a sufficiently high probability. Such a proposition would imply to deal with non-ruin probabilities *in continuous-time* over some given period (as investigated in the present paper).

Let us briefly illustrate that a finite-time non-ruin probability could indeed be used as an alternative to a Value-at-Risk. For that, we consider a compound Poisson risk model in which the Poisson parameter is  $\lambda = 1$ , claim amounts are exponentially distributed with mean  $\mu = 1$  and the premium rate is  $c = 1.1$ . First, we calculated the probabilities  $P(R_t \leq 0)$  of getting negative reserves at times  $t = 1, 5, 10$  years, given initial reserves  $u = 0, 1, 2, 5, 10, 20$ , by using the Panjer algorithm; we also calculated the ruin probabilities  $\psi(u, t)$  over the period  $(0, t)$ , for the same values of  $t$  and  $u$ , by applying the P.L.-type formula (3.37), as well as  $\psi(u) \equiv \psi(u, \infty)$  by using the well-known exact formula for the exponential case. The results are presented in Table 1. We observe that as expected, these probabilities can be very different, especially over a long horizon or for small initial reserves. Then, choosing several solvency thresholds 90, 95, 97.5, 99.5%, we proceeded in a similar way to determine the amounts  $u$  of initial reserves that enable to meet these thresholds, the criterion being either the Value-at-Risk  $VaR_t$  or the non-ruin probability  $\phi(u, t)$  with  $t = 1, 5, 10$  years. The results, given in Table 2, show that the differences between required capitals can be important over a long horizon or for low solvency thresholds. Moreover, we note that, for example, an amount  $u = 6.03$  needed to get a 1-year 99.5% Value-at-Risk allows also to get a 5-year non-ruin probability of  $\sim 93\%$ . In practice, however, controlling such a level of finite-time non-ruin probabilities is often found to be more reliable (see Loisel et al. [22]). So, in this framework, the choice of the 1-year 99.5% Value-at-Risk as the solvency criterion of reference could be questioned. A more detailed discussion on this problem is the object of a paper under preparation.

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$u$	$P(R_1 \leq 0)$	$\psi(u, 1)$	$P(R_5 \leq 0)$	$\psi(u, 5)$	$P(R_{10} \leq 0)$	$\psi(u, 10)$	$\psi(u)$
0	0.327	0.463	0.016	0.720	$2.7 \cdot 10^{-4}$	0.785	0.909
1	0.173	0.238	$8.1 \cdot 10^{-3}$	0.512	$1.2 \cdot 10^{-4}$	0.613	0.83
2	0.089	0.120	$3.8 \cdot 10^{-3}$	0.354	$5.7 \cdot 10^{-5}$	0.470	0.758
5	0.011	0.014	$4.0 \cdot 10^{-4}$	0.103	$5.3 \cdot 10^{-6}$	0.191	0.577
10	$2.5 \cdot 10^{-4}$	$3.1 \cdot 10^{-4}$	$7.8 \cdot 10^{-6}$	$9.2 \cdot 10^{-3}$	$1.0 \cdot 10^{-7}$	0.032	0.366
20	$8.6 \cdot 10^{-8}$	$9.9 \cdot 10^{-8}$	$6.0 \cdot 10^{-9}$	$3.3 \cdot 10^{-5}$	$6.0 \cdot 10^{-8}$	$4.0 \cdot 10^{-4}$	0.148

Table 1: Probabilities  $P(R_t \leq 0)$  of getting negative reserves at time  $t$  and ruin probabilities  $\psi(u, t)$  over the period  $(0, t)$ , calculated for several values of  $t$  and  $u$ .

Level	$VaR_1$	$\phi(u, 1)$	$VaR_5$	$\phi(u, 5)$	$VaR_{10}$	$\phi(u, 10)$	$\phi(u)$
99.5%	6.03	6.37	10.33	11.17	13.28	14.50	57.23
97.5%	3.81	4.19	7.01	8.02	9.15	10.62	39.53
95%	2.82	3.24	5.46	6.58	7.18	8.82	31.90
90%	1.81	2.26	3.80	5.06	5.04	6.91	24.28

Table 2: Initial reserves amounts  $u$  needed to satisfy different solvency thresholds, determined on the basis of the value-at-risk levels  $VaR_t$  or the non-ruin probabilities  $\phi(u, t)$ , for several values of  $t$ .

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