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Souhail Chebbi, Pascal Gourdel, Hakim Hammami. A generalization of Fan's matching theorem. 2006. hal-00756058v2

**HAL Id: hal-00756058**

**<https://shs.hal.science/hal-00756058v2>**

Submitted on 7 Dec 2006 (v2), last revised 6 Feb 2008 (v3)

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## A Generalization of Fan's Matching Theorem

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2006.60



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SCIENTIFIQUE

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<http://mse.univ-paris1.fr/Publicat.htm>

ISSN : 1624-0340

# A Generalization of Fan's Matching Theorem

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2 mai 2006

## Résumé

Nous présentons ici un théorème d'existence d'éléments maximaux pour une correspondance dont les composantes sont hémi-continues supérieurement par rapport à une partie des variables et qui vérifie par rapport aux autres l'une des conditions suivantes : (i) semi-continues inférieurement si l'espace est de dimension finie, (ii) semi-continues inférieurement et à valeurs fermées si l'espace est complet. (iii) à images inverses ouvertes. Ce théorème généralise les résultats de Gale and Mas-Colell (1975-1979), celui de Bergstrom (1975) et étend à un cadre de dimension infinie celui de Gourdel (1995).

**Mots clés:** L-structures, convexité généralisée, L-KKM maps, coercivité, théorèmes de points-fixes.

## Abstract

We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend the Fan's matching theorem.

**Keywords:** L-structures, L-spaces, L-KKM maps, L-coercing family, matching theorems, fixed points.

**JEL Classification:** C02, C60.

**AMS Classification:** Primary 47H04, 47H10, 54H25.

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# A GENERALIZATION OF FAN'S MATCHING THEOREM

S. CHEBBI, P. GOURDEL, AND H. HAMMAMI

**ABSTRACT.** We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend the Fan's matching theorem.

The purpose of this note is to study a coercivity type condition for set-valued maps defined in L-spaces and to extend the non-compactness condition of Fan-KKM's matching theorem.

We recall the L-structure of convexity for topological spaces given in [BCFL], the notion of KKM maps is easily extended to L-spaces. We then introduce the concept of L-coercing family for set-valued maps defined in L-spaces and give some examples. This coercivity type condition is very different from the one defined in topological vector spaces by Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF]. The main result that we obtain is a Fan's type matching theorem concerning the intersection of L-KKM set-valued maps with strongly compactly closed values and admitting a coercing family. A result on fixed point is then deduced. All these results extend classical results obtained in topological vector spaces by Fan in [F2] and Ding and Tan in [DT] as well as results obtained in H-spaces by Bardaro and Ceppitelli in [BC1] and [BC2] or in convex spaces in the sense of Lassonde in [L].

In what follows, the family of all nonempty finite subset of any set  $X$  is denoted by  $\langle X \rangle$ . If  $X$  is a vector space, the convex hull of a subset  $A$  of  $X$  is denoted by  $\text{conv}A$ . Since topological spaces in this paper are not supposed to be Hausssdorf, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausssdorf quasi-compact is called compact. If  $n$  is any integer,  $\Delta_n$  will denote the unit-simplex of  $\mathbb{R}^n$  and for every  $J \subset \{0, 1, \dots, n\}$ ,  $\Delta_J$  denotes the face of  $\Delta_n$  corresponding to  $J$ . Set-valued maps (or multifunctions) will be simply called maps and represented by capital letters  $F, G, Q, S, \Gamma, \dots$ . Functions in the usual sense will be represented by small letters. If  $X$  and  $Y$  are two topological spaces,  $\zeta(X, Y)$  denotes the set of all continuous functions from  $X$  to  $Y$ .

Let  $X$  be a topological space. An *L-structure* (also called *L-convexity*) on  $X$  is given by a nonempty valued map  $\Gamma : \langle X \rangle \rightarrow X$  such that for every  $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ ,

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2000 *Mathematical Subject Classification*: Primary 47H04, 47H10, 54H25.

*Key words and phrases*: L-structures, L-spaces, L-KKM maps, L-coercing family, matching theorems, fixed points.

there exists a continuous function  $f^A : \Delta_n \rightarrow \Gamma(A)$  such that for all  $J \subset \{0, \dots, n\}$ ,  $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$ . Such a pair  $(X, \Gamma)$  is called an *L-space*. A subset  $C \subset X$  is said to be *L-convex* if for every  $A \in \langle C \rangle$ ,  $\Gamma(A) \subset C$ . A subset  $P \subset X$  is said to be *L-quasi-compact* if for every  $A \in \langle X \rangle$ , there is a quasi-compact L-convex set  $D$  such that  $A \cup P \subset D$ . Clearly, if  $C$  is an L-convex subset of an L-space  $(X, \Gamma)$ , then the pair  $(C, \Gamma|_{\langle C \rangle})$  is an L-space.

It should be noticed that the L-convexity is different from the G-convexity defined by Park and Kim in [PK] which assume in addition the following condition:

$$\text{For all } A, B \in \langle X \rangle, A \subset B \text{ implies } \Gamma(A) \subset \Gamma(B).$$

The class of L-spaces contains topological vector spaces and their convex subsets as well as number of spaces with abstract topological convexity (examples of L-space are given in [BCFL]). The notion of L-quasi-compactness generalizes the H-compactness given in [BC1] and the c-compactness defined in [L].

The notion of KKM maps is easily extended to L-spaces as follows:

**Definition 1.** Let  $(X, \Gamma)$  be an L-space and  $Z \subset X$  an arbitrary subset. A map  $F : Z \rightarrow X$  is called *L-KKM* if and only if:

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

We now introduce the notion of coercing family in L-spaces for a given map:

**Definition 2.** Let  $Z$  be an arbitrary set in an L-space  $(X, \Gamma)$ ,  $Y$  a topological space and  $s \in \zeta(X, Y)$ . A family  $\{(C_a, K)\}_{a \in X}$  is said to be *L-coercing* for a map  $F : Z \rightarrow Y$  with respect to  $s$  if and only if:

- (i)  $K$  is a quasi-compact subset of  $Y$ ;
- (ii) For each  $A \in \langle Z \rangle$ , there exists a quasi-compact L-convex set  $D^A$  in  $X$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z;$$

- (iii)  $\left\{ y \in Y : y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K$ .

**Remark 1.** It is easy to see that in the previous definition, condition (ii) implies that for all  $x \in Z$ ,  $C_x$  is an L-quasi-compact subset of  $X$ .

**Definition 3.** If  $X$  is a topological space, a subset  $B$  of  $X$  is called *strongly compactly closed* (open respectively) if for every quasi-compact set  $K$  of  $X$ ,  $B \cap K$  is closed (open, respectively) in  $K$ .

**Remark 2.** Some authors define the notion of compactly-closed when for every compact set  $K$  of  $X$ ,  $B \cap K$  is closed (open, respectively) in  $K$ . It is easy to check that if  $A$  is strongly compactly closed, it is compactly closed.

**Remark 3.** Since condition (iii) can be rewritten as follows:

$$\left\{ z \in X : s(z) \in \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset s^{-1}(K),$$

we can deduce that:

- (1) If  $X$  is quasi-compact, then (iii) is automatically satisfied.
- (2) If the family  $\{(C_a, K)\}_{a \in X}$  is reduced to one element  $\{(C, K)\}$  and the  $L$ -convexity is reduced to the usual convexity of a topological vector space, by putting  $s$  the identity function, condition (iii) of Definition 2 is reduced to the condition used in Theorem 4 of [F2].

For any map  $F : X \rightarrow Y$ , let  $F^* : Y \rightarrow X$  be the “dual” map of  $F$  defined, for all  $y \in Y$ , by  $F^*(y) = X \setminus F^{-1}(y)$ , where  $F^{-1}(y) = \{x \in X : y \in F(x)\}$ .

**Remark 4.** By the previous remark and if we take  $s$  the identity map, we can see that the sens of coercivity of Definition 2 becomes from the fact that condition (iii) is equivalent, for  $F^*$ , to the following one:

$$\forall y \in Y \setminus K, \quad F^*(y) \cap C_y \neq \emptyset,$$

which means that outside of the quasi-compact set  $K$ , we have some control on the values of  $F^*$ .

To see that our coercivity is different from the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], we give the following example:

**Example 1.** For all  $x \in \mathbb{R}$ , let  $x^+ = \max\{0, x\}$ ,  $x^- = \min\{0, x\}$  and let define the map  $F : \mathbb{R} \rightarrow \mathbb{R}$  by  $F(x) = [-x^+, x^+]$ . The family  $\{(C_x, K)\}_{x \in \mathbb{R}}$  defined, for all  $x \in \mathbb{R}$ , by  $C_x = [x^-, x^+]$  and  $K$  is any compact set containing  $\{0\}$  is  $L$ -coercing for  $F$  with respect to the identity function if we reduce the  $L$ -convexity to the usual convexity and we put, for all finite subset  $A$ ,  $D^A = \text{conv}(\{0\} \cup A)$ . This family does not satisfy condition (ii) of Definition 2.1 in [BCF] since if we take  $x > 0$  and  $y < 0$ , we cannot find  $z \in \mathbb{R}$  such that  $[y, 0] \cup [0, x] \subset C_z$ .

The following example show that our coercivity is very general:

**Example 2.** Let  $X$  be a convex subset of a Hausdorff topological vector space and  $F : X \rightarrow X$  a map. Suppose that  $F$  satisfies the following condition given in Theorem III of [L] or in Lemma 1 of [DT] (in term of the “dual” map): There is a nonempty compact

convex subset  $X_0$  of  $X$  such that the set  $\left\{ y \in X : y \in \bigcap_{x \in \text{conv}(X_0 \cup \{y\})} F(x) \right\}$  is compact.

Then, if we reduce  $\Gamma$  to the usual convexity and  $s$  to the identity map, the family  $\{(C_y, K)\}_{y \in X}$  defined for all  $y \in X$  by:

$$C_y = \text{conv}(X_0 \cup \{y\}) \text{ and } K = \left\{ y \in X : y \in \bigcap_{x \in \text{conv}(X_0 \cup \{y\})} F(x) \right\}$$

is an  $L$ -coercing family for  $F$  with respect to  $s$  in the sense respectively of Definition 2 and Remark 3 by putting for each  $A \in \langle X \rangle$ ,  $D^A = \text{conv}(X_0 \cup A)$ .

The following result is an extension of the Lemma in [F1] to  $L$ -spaces:

**Lemma .** Let  $(X, \Gamma)$  be an  $L$ -space,  $Z$  a nonempty subset of  $X$  and  $F : Z \rightarrow X$  an  $L$ -KKM map with strongly compactly closed values. Suppose that for some  $z \in Z$ ,  $F(z)$  is quasi-compact, then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ .

Proof : The map  $F$  have strongly compactly closed values then, in order to prove that  $\bigcap_{x \in Z} F(x) \neq \emptyset$ , by the quasi-compactness condition, it suffices to prove that for each finite subset  $B$  of  $X$ ,  $\bigcap_{x \in B} F(x) \cap F(z) \neq \emptyset$ . Let  $B = \{x_0, \dots, x_{n-1}\}$  be a finite arbitrary subset of  $X$  and take  $A = \{x_0, \dots, x_n\}$ , where  $x_n = z$ , and  $f : \Delta_n \rightarrow \Gamma(A)$  the corresponding continuous function of  $A$  satisfying for all  $J \subset \{0, \dots, n\}$ ,  $f(\Delta_J) \subset \Gamma(\{x_j \mid j \in J\})$ . Then  $\{f^{-1}(F(x_i) \cap f(\Delta_n)) \mid i = 0, \dots, n\}$  is a family of closed subsets of  $\Delta_n$  such that  $\Delta_J \subset \bigcup_{j \in J} f^{-1}(F(x_j) \cap f(\Delta_n))$ . It follows by Lemma in [KKM] that  $\bigcap_{i=0}^n f^{-1}(F(x_i) \cap f(\Delta_n)) \neq \emptyset$ , consequently  $\bigcap_{x \in X} F(x) \neq \emptyset$ . ■

**Remark 5.** The following example will show that the lemma can not be extended for compactly closed valued correspondence. Let  $Z = X = [0, 1]$  and  $Y = X$  endowed with the trivial topology  $\{\emptyset, X\}$ , and

$$F(x) = \begin{cases} ]0, x] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

It is easy to check that  $F$  is KKM and that  $F(0)$  is compact. Moreover, for this topology, any subset of  $Y$  is compactly closed. Finally, it is obvious that the intersection is empty.

The main result of this paper is the following extension of the well known matching theorem of Fan (Theorem 4 in [F2]):

**Theorem 1.** *Let  $Z$  be an arbitrary set in the  $L$ -space  $(X, \Gamma)$ ,  $Y$  an arbitrary topological space and  $F : Z \rightarrow Y$  a strongly compactly closed maps. Suppose that there is a function  $s \in \zeta(X, Y)$  such that:*

- (1) *The map  $R : Z \rightarrow X$  defined by  $R(x) = s^{-1}(F(x))$  is  $L$ -KKM;*
- (2) *There exists an  $L$ -coercing family  $\{(C_x, K)\}_{x \in X}$  for  $F$  with respect to  $s$ ;*

*Then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ , more precisely  $K \cap (\bigcap_{x \in Z} F(x)) \neq \emptyset$ .*

Proof: The map  $F$  has compactly closed values, then in order to prove that:

$$K \cap \left( \bigcap_{x \in Z} F(x) \right) \neq \emptyset,$$

it suffices to prove that for each finite subset  $A$  of  $Z$ ,  $\bigcap_{x \in A} F(x) \cap K \neq \emptyset$ .

Let  $A \in \langle Z \rangle$ , by condition (ii) of Definition 2, there exists a quasi-compact  $L$ -convex set  $D^A$  containing  $A$  such that for all  $y \in D^A$ ,  $C_y \cap Z \subset D^A \cap Z$ . Consider now the map  $R^A : D^A \cap Z \rightarrow D^A$  defined by  $R^A(x) = s_0^{-1}(F(x) \cap s(D^A))$  where  $s_0 : D^A \rightarrow s(D^A)$  be the continuous function given by  $s_0(x) = s(x)$  for  $x \in D^A$ . By hypothesis (1) and the  $L$ -convexity of  $D^A$ , it is immediate that  $R^A = R(x) \cap D^A$  is an  $L$ -KKM map. Next, by the continuity of  $s$ ,  $F(x) \cap s(D^A)$  is closed in  $s(D^A)$  then  $R^A(x)$  is closed in  $D^A$ . Since  $(D^A, \Gamma|_{<D^A>})$  is also an  $L$ -space, we deduce by the Lemma that  $\bigcap_{x \in D^A \cap Z} R^A(x) \neq \emptyset$ . Since

for all  $x \in D^A \cap Z$ ,  $s(R^A(x)) \subset F(x) \cap s(D^A)$ , we have:  $\bigcap_{x \in D^A \cap Z} \{F(x) \cap s(D^A)\} \neq \emptyset$ . To

finish the proof, we will show that:  $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x) \cap K$ .

Indeed, it is clear that  $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x)$ , then it only remains to show that:  $\bigcap_{x \in D^A \cap Z} \{F(x) \cap s(D^A)\} \subset K$ . Let  $y \in \bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A))$ , then  $y \in s(D^A)$  which implies that there exists  $z \in s^{-1}(y) \cap D^A$ . By condition (ii) of Definition 2,  $C_z \cap Z \subset D^A \cap Z$ , it follows that  $y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x)$ . Hence, by hypothesis (2),  $y \in K$  and the theorem is proved. ■

**Remark 6.** *Taking a continuous function  $s$  in the previous theorem is inspired by Aubin ([A], p. 400), and has been used by [BC1], [BC2] and [L].*



Now we can use Theorem 1 to prove the following result on fixed points :

**Theorem 2.** *Let  $(X, \Gamma)$  be an L-space,  $Y$  an arbitrary topological space,  $s \in \zeta(X, Y)$  and  $S : X \rightarrow Y$  a map such that:*

- (i) *For each  $x \in X$ ,  $S(x)$  is strongly compactly open in  $Y$ ;*
- (ii) *For each  $y \in Y$ ,  $S^{-1}(y)$  is nonempty and L-convex;*
- (iii) *There exists an L-coercing family  $\{(C_x, K)\}_{x \in X}$  for the map  $Q(x) = Y \setminus S(x)$  with respect to  $s$ .*

*Then, there exists  $x_0 \in X$  such that  $s(x_0) \in S(x_0)$ . In particular, if  $s$  equal to identity map,  $S$  has a fixed point.*

Proof : It follows by (i) that  $Q$  has strongly compactly closed values and by (iii) that  $\{(C_x, K)\}_{x \in X}$  is an L-coercing family for  $Q$ . Since for each  $y \in Y$ ,  $S^{-1}(y)$  is nonempty, then  $\bigcap_{x \in X} Q(x) = \emptyset$ . Now, let  $R : X \rightarrow X$  be the map defined by  $R(x) = s^{-1}(Q(x))$ . We deduce by Theorem 1 that  $R$  is not L-KKM. Let  $A$  be a finite subset of  $X$  and  $x_0 \in \Gamma(A)$  such that  $x_0 \notin \bigcup_{x \in A} R(x)$ , it follows that  $s(x_0) \in \bigcap_{x \in A} S(x)$ . This means that  $A \subset S^{-1}(s(x_0))$ , then by (ii),  $\Gamma(A) \subset S^{-1}(s(x_0))$ . Hence  $s(x_0) \in S(x_0)$ . ■

Though [L] and [BC1], [BC2] use the notion of compactly-closed (without precising the compactness notion), in view of Remark 5, a cautious look at their papers shows that their assumption of compactly-closed corresponds (in our terminology) to strongly compactly closed.

Note that Theorem 1 extends Theorem 1 in [BC1] and Theorem I in [L]. Theorem 2 generalizes Theorem 1 in [BC2] and Theorem 1.1 in [L]. If the L-convexity is reduced to the usual convexity, then by Example 2, Theorem 1 and Theorem 2 extends respectively Theorem III in [L] and Lemma 1 in [DT]. Obviously, if  $X$  is compact, the result of Theorem 1 follows immediately.

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