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Résumé

Nous présentons ici un théorème d'existence d'éléments maximaux pour une correspondance dont les composantes sont hémi-continues supérieurement par rapport à une partie des variables et qui vérifie par rapport aux autres l'une des conditions suivantes : (i) semi-continues inférieurement si l'espace est de dimension finie, (ii) semi-continues inférieurement et à valeurs fermées si l'espace est complet. (iii) à images inverses ouvertes. Ce théorème généralise les résultats de Gale and Mas-Colell (1975-1979), celui de Bergstrom (1975) et étend à un cadre de dimension infinie celui de Gourdel (1995).

Mots clés: L-structures, convexité généralisée, L-KKM maps, coercivité, théorèmes de points-fixes.

Abstract

We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend the Fan's matching theorem.

Keywords: L-structures, L-spaces, L-KKM maps, L-coercing family, matching theorems, fixed points.

JEL Classification: C02, C60.

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A GENERALIZATION OF FAN'S MATCHING THEOREM

S. CHEBBI, P. GOURDEL, AND H. HAMMAMI

ABSTRACT. We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend the Fan's matching theorem.

The purpose of this note is to study a coercivity type condition for set-valued maps defined in L-spaces and to extend the non-compactness condition of Fan-KKM's matching theorem.

We recall the L-structure of convexity for topological spaces given in [BCFL], the notion of KKM maps is easily extended to L-spaces. We then introduce the concept of L-coercing family for set-valued maps defined in L-spaces and give some examples. This coercivity type condition is very different from the one defined in topological vector spaces by Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF]. The main result that we obtain is a Fan's type matching theorem concerning the intersection of L-KKM set-valued maps with strongly compactly closed values and admitting a coercing family. A result on fixed point is then deduced. All these results extend classical results obtained in topological vector spaces by Fan in [F2] and Ding and Tan in [DT] as well as results obtained in H-spaces by Bardaro and Ceppitelli in [BC1] and [BC2] or in convex spaces in the sense of Lassonde in [L].

In what follows, the family of all nonempty finite subset of any set X is denoted by $\langle X \rangle$. If X is a vector space, the convex hull of a subset A of X is denoted by $\text{conv}A$. Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. If n is any integer, Δ_n will denote the unit-simplex of \mathbb{R}^n and for every $J \subset \{0, 1, \dots, n\}$, Δ_J denotes the face of Δ_n corresponding to J . Set-valued maps (or multifunctions) will be simply called maps and represented by capital letters $F, G, Q, S, \Gamma, \dots$. Functions in the usual sense will be represented by small letters. If X and Y are two topological spaces, $\zeta(X, Y)$ denotes the set of all continuous functions from X to Y .

Let X be a topological space. An *L-structure* (also called *L-convexity*) on X is given by a nonempty valued map $\Gamma : \langle X \rangle \rightarrow X$ such that for every $A = \{x_0, \dots, x_n\} \in \langle X \rangle$,

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Key words and phrases: L-structures, L-spaces, L-KKM maps, L-coercing family, matching theorems, fixed points.

there exists a continuous function $f^A : \Delta_n \rightarrow \Gamma(A)$ such that for all $J \subset \{0, \dots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$. Such a pair (X, Γ) is called an *L-space*. A subset $C \subset X$ is said to be *L-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$. A subset $P \subset X$ is said to be *L-quasi-compact* if for every $A \in \langle X \rangle$, there is a quasi-compact L-convex set D such that $A \cup P \subset D$. Clearly, if C is an L-convex subset of an L-space (X, Γ) , then the pair $(C, \Gamma|_{\langle C \rangle})$ is an L-space.

It should be noticed that the L-convexity is different from the G-convexity defined by Park and Kim in [PK] which assume in addition the following condition:

$$\text{For all } A, B \in \langle X \rangle, A \subset B \text{ implies } \Gamma(A) \subset \Gamma(B).$$

The class of L-spaces contains topological vector spaces and their convex subsets as well as number of spaces with abstract topological convexity (examples of L-space are given in [BCFL]). The notion of L-quasi-compactness generalizes the H-compactness given in [BC1] and the c-compactness defined in [L].

The notion of KKM maps is easily extended to L-spaces as follows:

Definition 1. Let (X, Γ) be an L-space and $Z \subset X$ an arbitrary subset. A map $F : Z \rightarrow X$ is called *L-KKM* if and only if:

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

We now introduce the notion of coercing family in L-spaces for a given map:

Definition 2. Let Z be an arbitrary set in an L-space (X, Γ) , Y a topological space and $s \in \zeta(X, Y)$. A family $\{(C_a, K)\}_{a \in X}$ is said to be *L-coercing* for a map $F : Z \rightarrow Y$ with respect to s if and only if:

- (i) K is a quasi-compact subset of Y ;
- (ii) For each $A \in \langle Z \rangle$, there exists a quasi-compact L-convex set D^A in X containing A such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z;$$

- (iii) $\left\{ y \in Y : y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K$.

Remark 1. It is easy to see that in the previous definition, condition (ii) implies that for all $x \in Z$, C_x is an L-quasi-compact subset of X .

Definition 3. If X is a topological space, a subset B of X is called *strongly compactly closed* (*open respectively*) if for every quasi-compact set K of X , $B \cap K$ is closed (*open, respectively*) in K .

Remark 2. Some authors define the notion of compactly-closed when for every compact set K of X , $B \cap K$ is closed (open, respectively) in K . It is easy to check that if A is strongly compactly closed, it is compactly closed.

Remark 3. Since condition (iii) can be rewritten as follows:

$$\left\{ z \in X : s(z) \in \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset s^{-1}(K),$$

we can deduce that:

- (1) If X is quasi-compact, then (iii) is automatically satisfied.
- (2) If the family $\{(C_a, K)\}_{a \in X}$ is reduced to one element $\{(C, K)\}$ and the L -convexity is reduced to the usual convexity of a topological vector space, by putting s the identity function, condition (iii) of Definition 2 is reduced to the condition used in Theorem 4 of [F2].

For any map $F : X \rightarrow Y$, let $F^* : Y \rightarrow X$ be the “dual” map of F defined, for all $y \in Y$, by $F^*(y) = X \setminus F^{-1}(y)$, where $F^{-1}(y) = \{x \in X : y \in F(x)\}$.

Remark 4. By the previous remark and if we take s the identity map, we can see that the sens of coercivity of Definition 2 becomes from the fact that condition (iii) is equivalent, for F^* , to the following one:

$$\forall y \in Y \setminus K, \quad F^*(y) \cap C_y \neq \emptyset,$$

which means that outside of the quasi-compact set K , we have some control on the values of F^* .

To see that our coercivity is different from the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], we give the following example:

Example 1. For all $x \in \mathbb{R}$, let $x^+ = \max\{0, x\}$, $x^- = \min\{0, x\}$ and let define the map $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(x) = [-x^+, x^+]$. The family $\{(C_x, K)\}_{x \in \mathbb{R}}$ defined, for all $x \in \mathbb{R}$, by $C_x = [x^-, x^+]$ and K is any compact set containing $\{0\}$ is L -coercing for F with respect to the identity function if we reduce the L -convexity to the usual convexity and we put, for all finite subset A , $D^A = \text{conv}(\{0\} \cup A)$. This family does not satisfy condition (ii) of Definition 2.1 in [BCF] since if we take $x > 0$ and $y < 0$, we cannot find $z \in \mathbb{R}$ such that $[y, 0] \cup [0, x] \subset C_z$.

The following example show that our coercivity is very general:

Example 2. Let X be a convex subset of a Hausdorff topological vector space and $F : X \rightarrow X$ a map. Suppose that F satisfies the following condition given in Theorem III of [L] or in Lemma 1 of [DT] (in term of the “dual” map): There is a nonempty compact

convex subset X_0 of X such that the set $\left\{ y \in X : y \in \bigcap_{x \in \text{conv}(X_0 \cup \{y\})} F(x) \right\}$ is compact.

Then, if we reduce Γ to the usual convexity and s to the identity map, the family $\{(C_y, K)\}_{y \in X}$ defined for all $y \in X$ by:

$$C_y = \text{conv}(X_0 \cup \{y\}) \text{ and } K = \left\{ y \in X : y \in \bigcap_{x \in \text{conv}(X_0 \cup \{y\})} F(x) \right\}$$

is an L -coercing family for F with respect to s in the sense respectively of Definition 2 and Remark 3 by putting for each $A \in \langle X \rangle$, $D^A = \text{conv}(X_0 \cup A)$.

The following result is an extension of the Lemma in [F1] to L -spaces:

Lemma . Let (X, Γ) be an L -space, Z a nonempty subset of X and $F : Z \rightarrow X$ an L -KKM map with strongly compactly closed values. Suppose that for some $z \in Z$, $F(z)$ is quasi-compact, then $\bigcap_{x \in Z} F(x) \neq \emptyset$.

Proof : The map F have strongly compactly closed values then, in order to prove that $\bigcap_{x \in Z} F(x) \neq \emptyset$, by the quasi-compactness condition, it suffices to prove that for each finite subset B of X , $\bigcap_{x \in B} F(x) \cap F(z) \neq \emptyset$. Let $B = \{x_0, \dots, x_{n-1}\}$ be a finite arbitrary subset of X and take $A = \{x_0, \dots, x_n\}$, where $x_n = z$, and $f : \Delta_n \rightarrow \Gamma(A)$ the corresponding continuous function of A satisfying for all $J \subset \{0, \dots, n\}$, $f(\Delta_J) \subset \Gamma(\{x_j \mid j \in J\})$. Then $\{f^{-1}(F(x_i) \cap f(\Delta_n)) \mid i = 0, \dots, n\}$ is a family of closed subsets of Δ_n such that $\Delta_J \subset \bigcup_{j \in J} f^{-1}(F(x_j) \cap f(\Delta_n))$. It follows by Lemma in [KKM] that $\bigcap_{i=0}^n f^{-1}(F(x_i) \cap f(\Delta_n)) \neq \emptyset$, consequently $\bigcap_{x \in X} F(x) \neq \emptyset$. ■

Remark 5. The following example will show that the lemma can not be extended for compactly closed valued correspondence. Let $Z = X = [0, 1]$ and $Y = X$ endowed with the trivial topology $\{\emptyset, X\}$, and

$$F(x) = \begin{cases}]0, x] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

It is easy to check that F is KKM and that $F(0)$ is compact. Moreover, for this topology, any subset of Y is compactly closed. Finally, it is obvious that the intersection is empty.

The main result of this paper is the following extension of the well known matching theorem of Fan (Theorem 4 in [F2]):

Theorem 1. *Let Z be an arbitrary set in the L-space (X, Γ) , Y an arbitrary topological space and $F : Z \rightarrow Y$ a strongly compactly closed maps. Suppose that there is a function $s \in \zeta(X, Y)$ such that:*

- (1) *The map $R : Z \rightarrow X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM;*
- (2) *There exists an L-coercing family $\{(C_x, K)\}_{x \in X}$ for F with respect to s ;*

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$, more precisely $K \cap (\bigcap_{x \in Z} F(x)) \neq \emptyset$.

Proof: The map F has compactly closed values, then in order to prove that:

$$K \cap \left(\bigcap_{x \in Z} F(x) \right) \neq \emptyset,$$

it suffices to prove that for each finite subset A of Z , $\bigcap_{x \in A} F(x) \cap K \neq \emptyset$.

Let $A \in \langle Z \rangle$, by condition (ii) of Definition 2, there exists a quasi-compact L-convex set D^A containing A such that for all $y \in D^A$, $C_y \cap Z \subset D^A \cap Z$. Consider now the map $R^A : D^A \cap Z \rightarrow D^A$ defined by $R^A(x) = s_0^{-1}(F(x) \cap s(D^A))$ where $s_0 : D^A \rightarrow s(D^A)$ be the continuous function given by $s_0(x) = s(x)$ for $x \in D^A$. By hypothesis (1) and the L-convexity of D^A , it is immediate that $R^A = R(x) \cap D^A$ is an L-KKM map. Next, by the continuity of s , $F(x) \cap s(D^A)$ is closed in $s(D^A)$ then $R^A(x)$ is closed in D^A . Since $(D^A, \Gamma|_{\langle D^A \rangle})$ is also an L-space, we deduce by the Lemma that $\bigcap_{x \in D^A \cap Z} R^A(x) \neq \emptyset$.

for all $x \in D^A \cap Z$, $s(R^A(x)) \subset F(x) \cap s(D^A)$, we have: $\bigcap_{x \in D^A \cap Z} \{F(x) \cap s(D^A)\} \neq \emptyset$. To

finish the proof, we will show that: $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x) \cap K$.

Indeed, it is clear that $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x)$, then it only remains to show that: $\bigcap_{x \in D^A \cap Z} \{F(x) \cap s(D^A)\} \subset K$. Let $y \in \bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A))$, then $y \in s(D^A)$ which implies that there exists $z \in s^{-1}(y) \cap D^A$. By condition (ii) of Definition 2, $C_z \cap Z \subset D^A \cap Z$, it follows that $y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x)$. Hence, by hypothesis (2), $y \in K$ and the theorem is proved. ■

Remark 6. *Taking a continuous function s in the previous theorem is inspired by Aubin ([A], p. 400), and has been used by [BC1], [BC2] and [L].*

Now we can use Theorem 1 to prove the following result on fixed points :

Theorem 2. *Let (X, Γ) be an L-space, Y an arbitrary topological space, $s \in \zeta(X, Y)$ and $S : X \rightarrow Y$ a map such that:*

- (i) *For each $x \in X$, $S(x)$ is strongly compactly open in Y ;*
- (ii) *For each $y \in Y$, $S^{-1}(y)$ is nonempty and L-convex;*
- (iii) *There exists an L-coercing family $\{(C_x, K)\}_{x \in X}$ for the map $Q(x) = Y \setminus S(x)$ with respect to s .*

Then, there exists $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, if s equal to identity map, S has a fixed point.

Proof : It follows by (i) that Q has strongly compactly closed values and by (iii) that $\{(C_x, K)\}_{x \in X}$ is an L-coercing family for Q . Since for each $y \in Y$, $S^{-1}(y)$ is nonempty, then $\bigcap_{x \in X} Q(x) = \emptyset$. Now, let $R : X \rightarrow X$ be the map defined by $R(x) = s^{-1}(Q(x))$. We deduce by Theorem 1 that R is not L-KKM. Let A be a finite subset of X and $x_0 \in \Gamma(A)$ such that $x_0 \notin \bigcup_{x \in A} R(x)$, it follows that $s(x_0) \in \bigcap_{x \in A} S(x)$. This means that $A \subset S^{-1}(s(x_0))$, then by (ii), $\Gamma(A) \subset S^{-1}(s(x_0))$. Hence $s(x_0) \in S(x_0)$. ■

Though [L] and [BC1], [BC2] use the notion of compactly-closed (without precisising the compactness notion), in view of Remark 5, a cautious look at their papers shows that their assumption of compactly-closed corresponds (in our terminology) to strongly compactly closed.

Note that Theorem 1 extends Theorem 1 in [BC1] and Theorem I in [L]. Theorem 2 generalizes Theorem 1 in [BC2] and Theorem 1.1 in [L]. If the L-convexity is reduced to the usual convexity, then by Example 2, Theorem 1 and Theorem 2 extends respectively Theorem III in [L] and Lemma 1 in [DT]. Obviously, if X is compact, the result of Theorem 1 follows immediately.

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