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CONVEX EXTREMA FOR NONINCREASING DISCRETE DISTRIBUTIONS: EFFECTS OF CONVEXITY CONSTRAINTS

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In risk management, the distribution of underlying random variables is not always known. Sometimes, only the mean value and some shape information (decreasingness, convexity after a certain point,...) of the discrete density are available. The present paper aims at providing convex extrema in some cases that arise in practice in insurance and in other fields. This enables us to obtain for example bounds on variance and on Solvency II related quantities in insurance applications. In this paper, we first consider the class of discrete distributions whose probability mass functions are nonincreasing on a support $\mathcal{D}_n \equiv \{0, 1, \ldots, n\}$. Convex extrema in that class of distributions are well-known. Our purpose is to point out how additional shape constraints of convexity type modify these extrema. Three cases are considered: the p.m.f. is globally convex on $\mathbb{N}$, it is convex only from a given positive point m, or it is convex only up to some positive point m. The corresponding convex extrema are derived by using simple crossing properties between two distributions. The influence of the choice of n and m is discussed numerically, and several illustrations to ruin problems are presented. These results provide a complement to two recent works by Lefèvre and Loisel (2010), (2012).

Keywords: Discrete convex ordering, extrema, nonincreasing p.m.f., convexity type constraints, ruin problems.

1. Introduction

The convex order is a classical stochastic order that compares any two distributions with equal means. More precisely, a random variable (r.v.) X is smaller in the convex sense than a random variable Y, which is denoted by $X \leq_{cx} Y$, if

$$E[f(X)] \leq E[f(Y)] \text{ for all convex functions } f. \quad (1)$$

This implies that $E(X) = E(Y)$ as desired, and $Var(X) \leq Var(Y)$, hence sometimes the convex order is named the variability order. Properties and applications of the convex order can be found e.g. in the books by Ross (1996), Rolski (1976), Müller and Stoyan (2002), Goovaerts *et al.* (1990;1998) and Shaked and Shanthikumar (2007).

The construction of convex extrema is an important theoretical problem with many practical implications. This problem has been investigated for discrete random variables by e.g. Lefèvre and Utev (1996), Denuit and Lefèvre (1997), Denuit *et al.* (1999a;1999c), Courtois *et al.* (2006), Lefèvre and Picard (1993) and for continuous random variables by e.g. Denuit *et al.* (1998;1999b); see also the recent works by Lefèvre and Loisel (2010;2012). In risk management, when one has limited information about the risk distribution (e.g. mean value and some information on the shape of the density), convex extrema provide bounds on the variance of risky variables and on some other risk indicators.

The present paper is concerned with r.v.'s that are valued in a finite set $\mathcal{D}_n \equiv \{0, 1, \ldots, n\}$, for some fixed $n \in \mathbb{N}^*$. Let us recall that for such variables, a condition of convex ordering equivalent to (1) is that

$$E(X) = E(Y) \quad \text{and} \quad \sum_{i=k}^{n} F_X(i) \geq \sum_{i=k}^{n} F_Y(i), \quad (2)$$

where $0 \leq k \leq n$ and F_X denotes the distribution function (d.f.) of X.
More precisely, our starting point is the special class of r.v.'s whose probability mass function (p.m.f.) is nonincreasing on a support $\mathcal{D}_n \equiv \{0, 1, \ldots, n\}$. Distributions of that type are met in various probability models proposed in economics and biosciences. This is

especially true in insurance for which the claim number distributions are often observed to be nonincreasing.

Proposition 1 (Denuit *et al.* (1999c)) *Let N be an arbitrary r.v. with a nonincreasing p.m.f. on $\mathcal{D}_n$. Denote $\nu = E(N)$. In this case $N_{min} \leq_{cx} N \leq_{cx} N_{max}$ where*

$$N_{max} = \begin{cases} 0 & \text{with probab. } 1 - 2\nu/(n+1), \\ 1, \dots, n & \text{with equal probab. } 2\nu/n(n+1), \end{cases} \tag{3}$$

and, defining ξ as the integer on $[0, n-1]$ satisfying $\xi < 2\nu \leq \xi + 1$,

$$N_{min} = \begin{cases} 0, \dots, \xi & \text{with equal probab.} \\ & 2(\xi + 1 - \nu)/((\xi+1)(\xi+2)), \\ \xi + 1 & \text{with probab. } (2\nu - \xi)/(\xi+2). \end{cases} \tag{4}$$

These two extrema convex bounds are illustrated with a numerical example in Figure 1.

Now, for certain situations, it may be possible to obtain further information on the shape of the p.m.f. In this paper, we make the assumption that the p.m.f. is not only nonincreasing on $\mathcal{D}_n$, but it is also convex, globally on $\mathbb{N}$ or at least on some parts of $\mathbb{N}$. Our purpose is to point out how the convex extrema (3), (4) are modified under an additional convexity constraint.

The method of proof will exploit some simple crossing properties between two distributions that are briefly presented in Section 2. Three different cases are then discussed. Section 3 deals with the case where the p.m.f. is assumed to be convex on $\mathbb{N}$. This situation has been studied in Lefèvre and Loisel (2012), but in a more complex framework and using another approach that is based on the concept of multiple monotonicity. In Sections 4 and 5, convexity is assumed to hold only from a fixed point $m = 1$ and $m > 1$, respectively. Section 6 is concerned with r.v.'s with p.m.f. that are decreasing

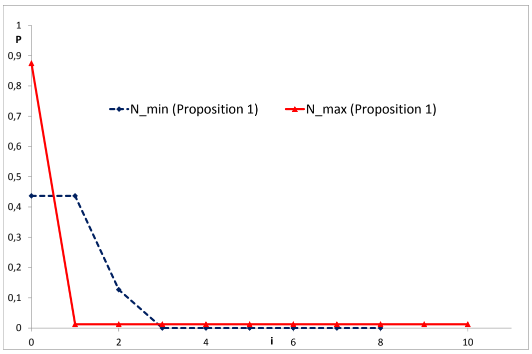

Fig. 1. Convex extrema in the class of discrete r.v.'s with decreasing p.m.f. on a support $\mathcal{D}_n$, $\nu = 0.69$ and $n = 10$.

and convex up to a fixed point $m \geq 2$. Finally, in Section 7, the influence of the choice of n and m on the extrema is discussed numerically, and several applications to some ruin problems are presented for illustration.

To simplify the presentation, positive (resp. negative) a function means below nonnegative (resp. nonpositive), and increasing (resp. decreasing) means nondecreasing (resp. nonincreasing).

2. Crossing properties and convex ordering

This section recalls some simple crossing results that will be used in the sequel. Although known, these results are presented with a short proof for the ease of presentation. Let S^- be the operator which, applied to a function f, counts the number of sign changes of f over its domain, zero terms being discarded. Two functions f and g cross each other k times, $k \geq 0$, if $S^-(f-g) = k$. Let X and Y be two r.v.'s valued on $\mathcal{D}_n$, with (distinct) p.m.f. P_X, P_Y and distribution functions (d.f.) F_X, F_Y.

Proposition 2

$$\textit{If } E(X) = E(Y), \textit{ then } S^-(P_Y - P_X) \geq 2. \tag{5}$$

Proof. As $\sum_{i=0}^{n}[P_X(i) - P_Y(i)] = 0$, the two p.m.f. have at least one crossing point. Suppose that there is a single crossing. Then, one of the two r.v.'s has necessarily a larger mean, which is in contradiction with the assumption made. ⋄

Lemma 1

$$\textit{If } S^-(P_Y - P_X) = 2, \textit{ then } S^-(F_Y - F_X) = 1. \tag{6}$$

Proof. $S^-(P_Y - P_X) = 2$ means that the function $P_Y - P_X$ has opposite signs on three consecutive intervals I_1, I_2, I_3 of $\mathcal{D}_n$. Suppose that the sequence of signs is $+, -, +$, for instance. Then, the function $F_Y - F_X$ is positive increasing on I_1, decreasing on I_2, and negative increasing on I_3 since $F_Y(n) = F_X(n) = 1$. Thus, $F_Y - F_X$ has one sign change on I_2. ⋄

We are now ready to derive a crossing type condition that implies a convex ordering. This result is not new (see e.g. Denuit and Lefèvre (1997)).

Proposition 3

$$\begin{array}{l} \textit{If } E(X) = E(Y) \textit{ and } S^-(P_Y - P_X) = 2 \textit{ with} \\ P_Y \geq P_X \textit{ near } n, \textit{ then } X \leq_{cx} Y. \end{array} \tag{7}$$

Proof. The two means being equal, $P_Y - P_X$ has at least two sign changes by Proposition 2. Thus, the assumption that there are exactly two sign changes is admissible. By Lemma 1, $F_Y - F_X$ has then one sign change. Moreover, as $P_Y \geq P_X$ near n, one has $F_Y \leq F_X$ near n, so that the

consecutive signs of $F_Y - F_X$ are $+, -$.
Now, $E(X) = E(Y)$ is equivalent to

$$\sum_{i=0}^{n} F_Y(i) = \sum_{i=0}^{n} F_X(i).$$

For any integer $k \in [0, n-1]$, this can be rewritten as

$$\sum_{i=k+1}^{n} [F_Y(i) - F_X(i)] = -\sum_{i=0}^{k} [F_Y(i) - F_X(i)]. \quad (8)$$

If k is before the sign change of $F_Y - F_X$, then $F_Y(i) - F_X(i) \geq 0$ for $i = 0, \ldots, k$, so that the left hand side of (8) is negative i.e.

$$\sum_{i=k+1}^{n} F_X(i) \geq \sum_{i=k+1}^{n} F_Y(i). \quad (9)$$

If k is after the sign change of $F_Y - F_X$, then $F_Y(i) - F_X(i) \leq 0$ for $i = k+1, \ldots, n$, so that (9) holds too. In other words, the condition (2) is satisfied, hence $X \leq_{cx} Y$. $\diamond$
Roughly speaking, Proposition 3 states that X is smaller than Y in the convex order if the p.m.f. of Y is heavier on the extremes (near 0 and n).

3. For decreasing p.m.f. that are convex on $\mathbb{N}$

Consider the class of r.v.'s which have discrete p.m.f. Our aim is to derive explicit expressions of convex extrema in this class of r.v.'s under constraint of global decreasingness and convexity on $\mathbb{N}$.
We start by introducing the following lemma which will be useful in some following proofs.

Lemma 2 *Consider two discrete r.v.'s X and Y valued on $\mathcal{D}_n$ with globally decreasing and convex p.m.f. on $[s, \infty)$ where $s \in [1, n]$. In particular we consider that the p.m.f. of X in $[s, n+1]$ is linear. If we have $j \in [s, n]$ such that $P(Y = j) \leq P(X = j)$, then by the decreasingness and the convexity we have*

$$P(Y = k) \leq P(X = k)\,,\ \forall\, k > j. \quad (10)$$

Proof. Consider two r.v.'s as defined in Lemma 2 and recall that by global decresingness $P(Y = n+1) = P(X = n+1) = 0$. Let j and k be two positive integers such that $j \in [s, n]$ and $k > j$. In addition assume that $P(Y = j) \leq P(X = j)$ and $P(Y = k) \geq P(X = k)$. It is clear that, on $[j, n]$, the p.m.f. of Y is above the segment of line Z (see Figure 2). Hence we conclude that Y is not convex which is absurd. $\diamond$
Denote by $\mathbb{J}_n$ the set of all r.v.'s with discrete distribution function valued on $\mathcal{D}_n$ (i.e $P(X = j) = 0\ \forall\, j \geq n+1$), with decreasing p.m.f. and fixed mean ν.

3.1. The upper bound.

Let M and $\widetilde{N}_{\max}$ be two arbitrary r.v.'s in $\mathbb{J}_n$ that have globally convex p.m.f. on $\mathbb{N}$ such that $Var(M) \neq Var(\widetilde{N}_{\max})$. Using a reasoning by contradiction, we prove that the convex maximum bound is attained for the r.v. $\widetilde{N}_{\max}$ such that

$$\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } a, \\ i \in (1, \ldots, n) & \text{with probab. } b(n+1-i), \end{cases} \quad (11)$$

where a and b are defined such that (11) is a true p.m.f., with fixed mean ν i.e.

$$\begin{cases} a + \sum_{i=1}^{n} b(n+1-i) = 1, \\ \sum_{i=1}^{n} b(n+1-i)i = \nu, \end{cases} \quad (12)$$

hence,

$$\begin{cases} b = 6\nu/(n(n+1)(n+2)), \\ a = 1 - bn(n+1)/2. \end{cases} \quad (13)$$

In this case it is possible to have explicit expression for a and b in function of ν and n. The p.m.f. of $\widetilde{N}_{\max}$ is then as defined in the following proposition:

Proposition 4 *The convex maximum in the class of r.v.'s with discrete distribution that are with globally decreasing and convex p.m.f. on $\mathbb{N}$ is attained for the r.v. $\widetilde{N}_{\max}$ where for $0 < \nu \leq \frac{n}{3}$*

$$\widetilde{N}_{\max} = \begin{cases} 0 & \textit{with probab. } \frac{1-3\nu}{(n+2)}, \\ i \in (1, \ldots, n) & \textit{with probab. } \frac{6\nu(n+1-i)}{(n(n+1)(n+2))}. \end{cases} \quad (14)$$

To compare graphically bounds (14) and (3) see Figure 3. Note that the condition made on ν comes from the constraints of global decreasingness and convexity on $\mathbb{N}$. The bound (14) is not in line with the one derived in Lefèvre and Loisel (2010) for r.v.'s with decreasing and convex p.m.f. in a support $\{0, \ldots, n\}$. At the same time this bound is in line with bound (3.15) in Lefèvre and Loisel (2012) available for r.v.'s with decreasing and convex p.m.f. on $\mathbb{N}$.

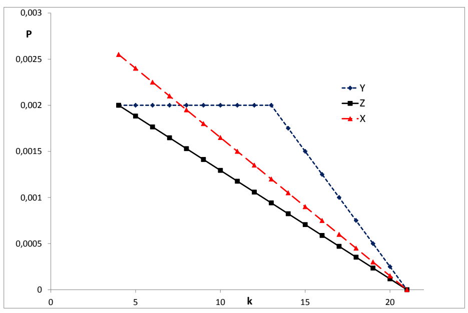

Fig. 2. Crossing situation, $j = s = 4$ and $n = 20$.

Proof. The proof is based on Propositions 2 and 3. In fact, according to Proposition 2, we have at least two crossing points between p.m.f. of r.v.'s M and $\widetilde{N}_{\max}$. In addition, if there exist exactly two crossing points such that $P_{\widetilde{N}_{\max}} \geq P_M$ near n, then under Proposition 3, we conclude that $M \leq_{cx} \widetilde{N}_{\max}$.
We denote by $P(M = 0) = a_1$, $P(\widetilde{N}_{\max} = 0) = a$, $P(M = 1) = c_1$ and $P(\widetilde{N}_{\max} = 1) = c$.

1. $a_1 \geq a$
   Assume that $c_1 \leq c$, in this case we observe, at most, one crossing point in $]0,1]$. Furthermore, by Proposition 2, it is necessary to have, at least, another crossing point in $]1,n]$. However, if this is the case then under Lemma 2 the constraint of convexity is violated. Thus this is absurd. Now consider that $c_1 > c$, then there is at most one crossing point over $\mathcal{D}_n$. This is absurd by Proposition 2.

2. $a_1 < a$
   Assume that $c_1 > c$, then a crossing point is observed in $]0,1]$. By Lemma 2, we cannot get more than one crossing point in $]1,n]$. If not, the convexity constraint is not fulfilled. Hence, we have a situation with at most two crossing points over $\mathcal{D}_n$ such that $P_M \leq P_{\widetilde{N}_{\max}}$ near n. In this case under Proposition 3, $M \leq_{cx} \widetilde{N}_{\max}$. The case where $c_1 \leq c$ is absurd according to Lemma 2 and Proposition 2.$\diamond$

Consequently, the maximum variance denoted by $Var(\widetilde{N}_{\max})$ is obtained for

$$Var(\widetilde{N}_{\max}) = \nu(n+1)/2 - \nu^2. \tag{15}$$

3.2. The lower bound. Consider the class of r.v.'s with discrete p.m.f. that are globally convex and decreasing on $\mathbb{N}$. In this case the convex minimum order is given by the Proposition (5).

Proposition 5 *Let ξ be an integer in $[0, n-1]$ such that $\xi \leq 3\nu \leq \xi + 1$ where $n \geq \xi + 1$. The convex minimum $\widetilde{N}_{\min}$ in the class of r.v.'s which have a globally decreasing and convex distribution is as follows:*
$\forall\, i \in \{0, \ldots, \xi+1\}$

$$P(\widetilde{N}_{\min} = i) = (\xi + 1 - i)\pi_1 + (\xi + 2 - i)\pi_2\,, \tag{16}$$

where $\pi_1 = 2(\xi + 1 - 3\nu)/((\xi+2)(\xi+1))$ and $\pi_2 = 2(3\nu - \xi)/((\xi+2)(\xi+3))$.

For a comparison between the bounds (16) and (4) see Figure 4.
Proof. This bound is obtained as follows: we consider that the distribution of $\widetilde{N}_{\min}$ is defined by

$$P(\widetilde{N}_{\min} = i) = a - bi \quad \forall\, i \in \{0, \ldots, \xi+1\}\,, \tag{17}$$

where $0 < b < a < 1$ thus, we have $P(\widetilde{N}_{\min} = \xi + 2) = 0$. Consider $\pi_2 = P(\widetilde{N}_{\min} = \xi + 1)$ where under the convexity assumption $P(\widetilde{N}_{\min} = \xi) \geq 2\pi_2$. Consider that $P(\widetilde{N}_{\min} = \xi) = \pi_1 + 2\pi_2$, with $\pi_1 > 0$, then immediately we have $b = \pi_1 + \pi_2$ and $a = (1+\xi)\pi_1 + (2+\xi)\pi_2$, so (16) is true, where π_1 and π_2 are obtained under the following constraints

$$\begin{cases} \sum_{i=0}^{\xi+1} (\xi + 1 - i)\pi_1 + (\xi + 2 - i)\pi_2 = 1, \\ \sum_{i=1}^{n} i((\xi + 1 - i)\pi_1 + (\xi + 2 - i)\pi_2) = \nu. \end{cases} \tag{18}$$

Let M and $\widetilde{N}_{\min}$ be two arbitrary r.v.'s in $\mathbb{J}_n$ with p.m.f. globally convex on $\mathbb{N}$ such that $V(M) \neq V(\widetilde{N}_{\min})$.

1. $a_1 > a$
   From Lemma 2 it is easy to see that it is impossible to get more than two crossing points over $\mathcal{D}_n$. This

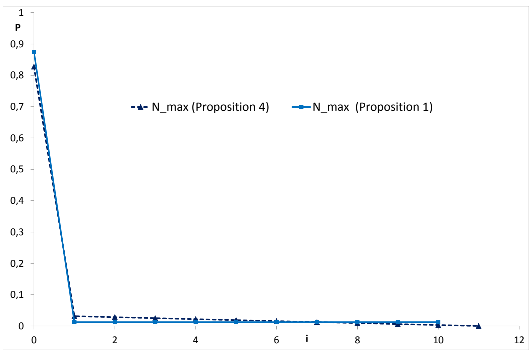

Fig. 3. The upper bound (14) vs the upper bound (3), $\nu = 0.69$ and $n = 10$.

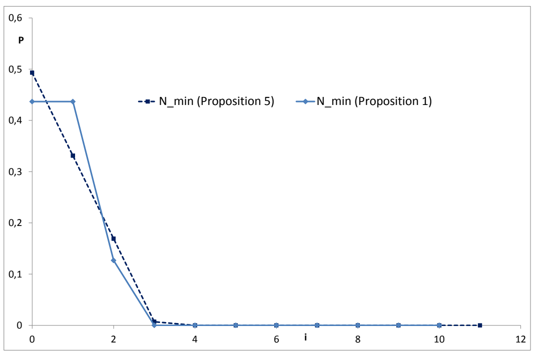

Fig. 4. The lower bound (16) vs the lower bound (4), $\nu = 0.69$ and $n = 10$.

is due to the assumptions of decreasingness and convexity. Also, by the same Lemma, a situation with exactly two crossing points is observed if and only if there is $i \in \{0, \ldots, \xi+1\}$ such that $P(M = i) < P(\widetilde{N}_{\min} = i)$ and $P(M = \xi+2) > 0$. In this case it is clear that $P_{\widetilde{N}_{\min}} < P_M$ near n. Hence, by Proposition 3, $\widetilde{N}_{\min} \leq_{cx} M$.

2. $a_1 \leq a$
   By convexity and decreasingness we have at most one crossing point which is absurd by Proposition 2.$\diamond$

Consequently, we have

$$Var(\widetilde{N}_{\min}) = (\xi+1)(\nu - \xi/6) - \nu^2. \qquad (19)$$

Note that the lower bound (16) is in agreement with Corollary 5.4 in Lefèvre and Loisel (2010) hold for discrete r.v.'s with p.m.f. that are decreasing and convex on $\mathcal{D}_n$. It is also in agreement with bound (3.14) in Lefèvre and Loisel (2012) hold for discrete r.v.'s with p.m.f. that are decreasing and convex globally on $\mathbb{N}$.
The bounds (16) and (14) are illustrated in Figure 5.

4. For decreasing p.m.f. that are convex in $[1, +\infty)$

Note that if a random variable X has a convex p.m.f. in $[1, +\infty)$, then it is either convex (see Section 3) or concave until 2. This section is devoted to extrema convex for r.v.'s which have decreasing p.m.f. that are convex in $[1, +\infty)$ and concave until 2. In application, a typical example is to consider a Poisson distribution with parameter (λ): $(2-\sqrt{2}) < \lambda \leq 1$.

4.1. The upper bound.

Proposition 6 *Consider $\frac{1}{3} < \nu < \frac{n}{3}$. Consider the class of r.v.'s with discrete decreasing p.m.f. that are concave until $[0, 2]$ and convex in $[1, +\infty)$. The convex maximum in this class of r.v.'s is obtained for the r.v. $\widetilde{N}_{\max}$ defined by (20)*

$$\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } \frac{2\,n(n+4)+6-6\,\nu\,(n+1)}{3(n+2)(n+1)}, \\ 1 & \text{with probab. } \frac{n(n+4)-3\,\nu\,(n-2)}{3(n+2)(n+1)}, \\ i \in (2, \ldots, n) & \text{with probab. } \frac{2(3\,\nu-1)(n+1-i)}{(n-1)(n+2)(n+1)}. \end{cases} \qquad (20)$$

The proof of Proposition 6 is given in Appendix A. For a comparison between (20) and (14) see Figure 6.
Consequently, we have

$$Var(\widetilde{N}_{max}) = (n/2+1)\nu - n/6 - \nu^2. \qquad (21)$$

Corollary 1 *Let X be a r.v. in $\mathbb{J}_n$ valued on $\mathcal{D}_n$. In addition, consider that the p.m.f. of X is decreasing on $\mathbb{N}$ and convex in $[1, \infty)$. We know that in the set $[0, 2]$ the distribution of X is either convex or concave (see Section 3). Then the upper bound of X is the supremum of the two bounds (20) and (14). From bounds (21) and (15) the supremum is attained when the random variable is convex on the set $[0, 2]$ i.e for (15).*

4.2. The lower bound.

4.2.1. Case: $\frac{1}{3} \leq \nu \leq \frac{1}{2}$.

Proposition 7 *Consider the class of discrete r.v.'s which have a decreasing p.m.f. that are concave until $[0, 2]$ and*

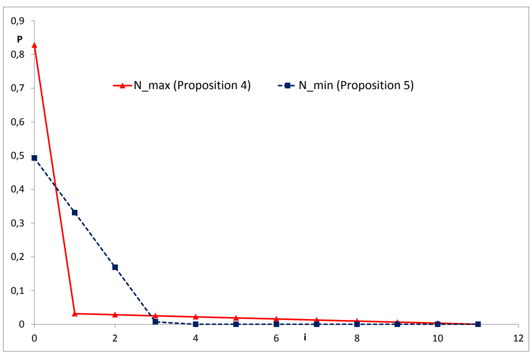

Fig. 5. Convex extrema in the class of decreasing p.m.f. that are convex on $\mathbb{N}$ with $\nu = 0,69$ et $n = 10$

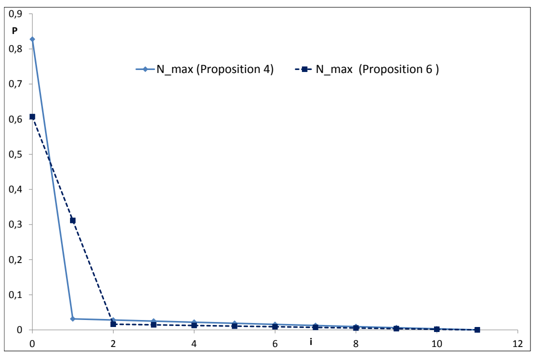

Fig. 6. The upper bound (20) vs the upper bound (14) with $\nu = 0.69$, $n = 10$.

convex in $[1, +\infty)$. *The convex minimum is attained for the r.v.* $\widetilde{N}_{\min}$ *given by (22)*

$$\widetilde{N}_{\min} = \begin{cases} 0 & \text{with probab. } 1-\nu, \\ 1 & \text{with probab. } \nu, \\ 2,\ldots,n & \text{with equal probab. } 0, \end{cases} \tag{22}$$

Proof. Let M be a r.v. in $\mathbb{J}_n$ with a p.m.f. concave in $[0,2]$ and convex in $[1, +\infty)$. It is easy to see that there cannot exist more than two crossing points between p.m.f. of M and $\widetilde{N}_{\min}$.
In addition if a two crossing situations holds, then necessarily we must have $P(M = 2) > P(\widetilde{N}_{\min} = 2)$ hence $\widetilde{N}_{\min} \leq_{cx} M$. $\diamond$
Consequently we have,

$$Var(\widetilde{N}_{\min}) = \nu(1-\nu). \tag{23}$$

4.2.2. Case: $\xi < 3\nu \leq \xi+1+\frac{2}{(\xi+4)}$ where $3\nu-\xi > \frac{2}{\xi+3}$ for all $\xi \geq 1$ and $n \geq \xi+1$.

Proposition 8 *Consider* $\xi < \xi + \frac{2}{(\xi+3)} < 3\nu \leq \xi + 1 + \frac{2}{(\xi+4)}$ *where* $3\nu - \xi > \frac{2}{\xi+3}$. *Taking the class of discrete r.v.'s with decreasing p.m.f. that are concave until* $[0,2]$ *and convex in* $[1, +\infty)$, *the convex minimum is attained for the r.v.* $\widetilde{N}_{\min}$ *given by*

$$\widetilde{N}_{\min} = \begin{cases} 0 & \text{with probab.} \\ & \pi_2\,(1+\xi) + \pi_1\xi, \\ i \in (1,\ldots,\xi+1) & \text{with probab.} \\ & \pi_2\,(2+\xi-i) \\ & +\pi_1\,(1+\xi-i)\,, \\ \xi+2,\ldots,n & \text{with equal probab. } 0, \end{cases} \tag{24}$$

where

$$\pi_1 = \frac{2\,\left(\xi^2 - 3\,\nu\,\xi + 5\,\xi - 12\,\nu + 6\right)}{\left((\xi+2)\,(\xi+5)\,\xi\right)}, \tag{25}$$

and

$$\pi_2 = \frac{2\,\left(-3\,\xi - 2 + 9\,\nu - \xi^2 + 3\,\nu\,\xi\right)}{\left((\xi+5)\,(\xi+1)\,(\xi+2)\right)}, \tag{26}$$

where $\pi_1 \geq 0$ *and* $\pi_2 \geq 0$.

For a comparison between bounds (24) and (16) see Figure 7. The proof of Proposition 8 is given in Appendix B. Consequently, we have

$$Var(\widetilde{N}_{\min}) = (1/12)\,(2+\xi)\,(1+\xi) \\ \left(\pi_1\xi^2 + \pi_2\xi^2 + \pi_1\xi + 5\,\pi_2\xi + 6\,\pi_2\right) - \nu^2. \tag{27}$$

In Figure 8 we present the upper bound (20) and the lower bound (24) for illustration.

5. For decreasing p.m.f. that are convex in $[m, +\infty)$ for fixed $m > 1$

In this section we consider discrete r.v.'s with decreasing p.m.f. that are convex from a fixed point $m > 1$. Our aim is to find the convex extremal bounds in this family. We first consider the upper bound.

5.1. The upper bound.

Proposition 9 *Let* n *and* m *be integers and let* ν *be a positive real number such that* $0 < 3\nu \leq (m^2 + nm + m + n^2 + 2n)/(m+n+2)$. *The convex maximum order in the class of r.v.'s with p.m.f. that are globally decreasing on* $\mathbb{N}$ *and convex from a fixed point* $m > 1$ *is attained for the*

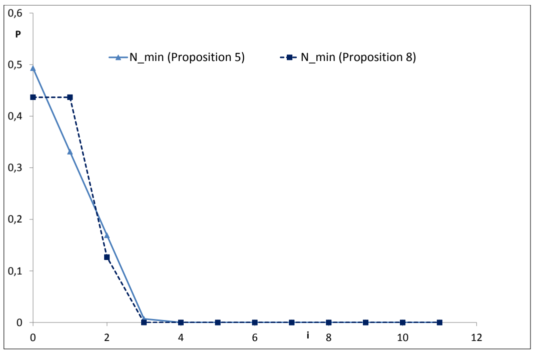

Fig. 7. The lower bound (24) vs the lower bound (16), $\nu = 0.69$, $n = 10$.

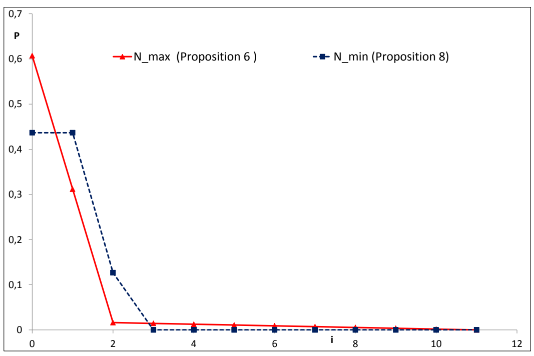

Fig. 8. The lower bound (24) vs the upper bound (20), $\nu = 0.69$, $n = 10$.

r.v. $\widetilde{N}_{\max}$ *given by*

$$\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } 1 - \frac{3\nu(n+m)}{m^2+nm+m+n^2+2n}, \\ 1,\ldots,m-1 & \text{with equal probab. } \frac{6\nu(n+1-m)}{m+n^3+3n^2+2n-m^3}, \\ i \in (m,\ldots,n) & \text{with probab. } \frac{6\nu(n+1-i)}{m+n^3+3n^2+2n-m^3}. \end{cases} \tag{28}$$

See Figure 9 for an illustration. The proof of Proposition 9 is given in Appendix C. Consequently, the supremum of the variance is attained for the predefined r.v. $\widetilde{N}_{\max}$ where for all $1 < m \leq n$, we have

$$Var\left(\widetilde{N}_{\max}\right) = \frac{\nu\left(m^2\left(1-m^2\right)+n\left(2+n^3+4n^2+5n\right)\right)}{2\left(m+n^3+3n^2+2n-m^3\right)} - \nu^2. \tag{29}$$

Remark 1 *If $m = 1$ or $m = 0$, then this bound corresponds to the bound (15) obtained for discrete r.v.'s with globally decreasing and convex p.m.f.*

Remark 2 *Let $\mathcal{F}_m$ be a set of decreasing r.v.'s valued on $\mathcal{D}_n$ with equal mean ν and with convex p.m.f. for a fixed point m: $1 < m \leq n$. Let us denote by $\widetilde{N}_{\max,m}$ the convex maximum bound in the set $\mathcal{F}_m$ and define the more general set $\mathcal{G} = (\mathcal{F}_m)_{(m\in\{1,\ldots,n-1\})}$. As $\mathcal{F}_p \subset \mathcal{F}_m$ for all $p \leq m$, we deduce that the convex maximum bound in the set $\mathcal{G}$ is $\widetilde{N}_{\max,n}$.*

5.2. The lower bound.

5.2.1. Case: $0 \leq \xi < 2\nu \leq (\xi+1) \leq m$.

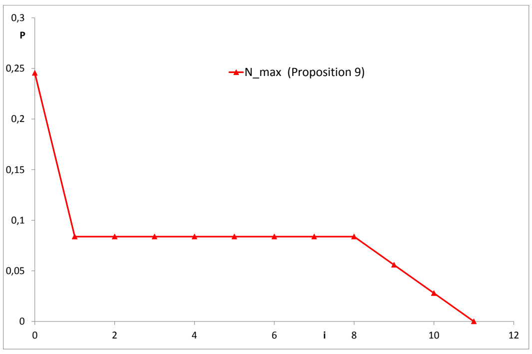

Fig. 9. The upper bound (28), $m = 8$, $\nu = 3.8$, $n = 10$.

Proposition 10 *Let n and m be two integers and ν be a non-negative real number such that $0 \leq \xi < 2\nu \leq \xi + 1 \leq m$. The convex minimum in the class of r.v.'s with p.m.f. that are globally decreasing on $\mathbb{N}$ and convex from a fixed point $m > 1$ is attained for the r.v. $\widetilde{N}_{\min}$ where*

$$\widetilde{N}_{\min} = \begin{cases} 0,\ldots,\xi & \text{with equal probab. } \frac{2(\xi+1-\nu)}{(\xi+1)(\xi+2)}, \\ \xi+1 & \text{with probab. } \frac{(2\nu-\xi)}{(\xi+2)}, \\ \xi+2,\ldots,n & \text{with equal probab. } 0 \end{cases} \tag{30}$$

Note that this bound is equal to the bound (3.12) in Lefèvre and Loisel (2012) obtained for discrete r.v.'s with decreasing p.m.f. globally on $\mathbb{N}$.
It is also equal to the bound (5.3) derived in Lefèvre and Loisel (2010) for discrete r.v.'s with decreasing p.m.f. on $\mathcal{D}_n$. Consequently,

$$Var\left(\widetilde{N}_{\min}\right) = -\frac{\xi}{3}\left(\xi - 4\nu + 1\right) + \nu - \nu^2. \tag{31}$$

The proof of Proposition 10 is given in Appendix D.

5.2.2. Case: $m < \xi \leq 3\nu \leq \xi + 1$, $\frac{m(m+1)}{m+\xi+2} \leq 3\nu - \xi$ and $n \geq \xi + 1$.

Proposition 11 *Let m, ξ and n be three integers where $1 < m < n$ and ν be nonnegative real number such that $m < \xi \leq \xi + \frac{m(m+1)}{(m+2+\xi)} \leq 3\nu \leq \xi + 1 + \frac{m(m+1)}{(m+3+\xi)}$ and $\frac{m(m+1)}{m+\xi+2} \leq 3\nu - \xi$. The convex minimum in the class of r.v.'s with p.m.f. that are globally decreasing on $\mathbb{N}$ and convex from a fixed point $m > 1$ is attained for the r.v. $\widetilde{N}_{\min}$ such that*

$$\widetilde{N}_{\min} = \begin{cases} 0,\ldots,m & \text{with equal probab. } \pi_2\left(2+\xi-m\right) + \pi_1\left(1+\xi-m\right), \\ i \in (m+1,\ldots,\xi+1) & \text{with probab. } \pi_2\left(2+\xi-i\right) + \pi_1\left(1+\xi-i\right), \\ \xi+2,\ldots,n & \text{with equal probab. } 0, \end{cases} \tag{32}$$

where

$$\pi_1 = \frac{m(m-3\nu+\xi+2)+3-9\nu+\xi(\xi+4-3\nu)}{2^{-1}(\xi+2)(2m+\xi+3)(\xi+1-m)}, \tag{33}$$

and

$$\pi_2 = \frac{6\nu - m\left(m+\xi+1-3\nu\right) - \xi(\xi+2-3\nu)}{2^{-1}(2m+\xi+3)(\xi+2-m)(\xi+2)}, \tag{34}$$

where $\pi_1 \geq 0$ and $\pi_2 \geq 0$.

See Figure 10 for an illustration of the lower bounds (32) and (30). Consequently we have,

$$
\begin{aligned}
&Var\left(\widetilde{N}_{\min}\right) = \\
&(1+\xi-m)\left(\frac{1}{6}\pi_1 m\,(m+1)\,(2m+1)\right. \\
&+\frac{1}{6}\,(\pi_1\,(1+\xi)+\pi_2\,(2+\xi)) \\
&\left(2m^2+5\,m+2\xi m+6+7\xi+2\xi^2\right) \\
&\left.-(\pi_1+\pi_2)(\frac{1}{4}m+\frac{1}{2}+\frac{1}{4}\xi)(m^2+m+2+3\,\xi+\xi^2)\right) \\
&+\frac{1}{6}\,\pi_2\,(2+\xi-m)\,m\,(m+1)\,(2\,m+1)-\nu^2.
\end{aligned}
\tag{35}
$$

Note that if we take $m = 1$ and $\xi = 1$, then in this case this bound coincides with the bound obtained in Proposition 7. The proof of Proposition 10 is given in Appendix E.

6. For decreasing p.m.f. that are convex up to a point $m \geq 2$

6.1. The upper bound.

Proposition 12 *Let ν be a positive real number such that $2\nu < n+1$. The convex maximum order in the class of r.v.'s with p.m.f. that are globally decreasing on $\mathbb{N}$ and convex up to a fixed point m where $m \geq 1$ is attained for the r.v. $\widetilde{N}_{max}$ given by*

$$
\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } 1-2\nu/(n+1), \\ 1,\ldots,n & \text{with equal probab.} \\ & 2\nu/(n(n+1)). \end{cases}
\tag{36}
$$

See Figure 11 for illustration. Note that this bound is equal to the bound obtained in Lefèvre and Loisel (2010) in Corollary 5.2 valid for r.v.'s with p.m.f. that are decreasing and convex on $\mathcal{D}_n$.

Proof. Denote by $P(M=0) = a_1$, $P(\widetilde{N}_{max}=0) = a$, $P(M=1) = b_1$ and $P(\widetilde{N}_{max}=1) = b$.

1. $a_1 > a$
   As the sum of probabilities must be equal we have a crossing point on $[1,\infty)$. As $\widetilde{N}_{max}$ is decreasing, we cannot have a second crossing point between M and $\widetilde{N}_{max}$.

2. $a_1 < a$ and $b_1 \leq b$
   It is clear that we do not have any crossing point between M and $\widetilde{N}_{max}$.

3. $a_1 < a$ and $b_1 > b$
   It is clear that we can have a situation with two crossing points between M and $\widetilde{N}_{\max}$. Near n, as $b_1 > b$, we have $P_M < P_{\widetilde{N}_{\max}}$. Hence we have $M \leq_{cx} \widetilde{N}_{\max}$. ⋄

Consequently we have

$$
\widetilde{Var}_{\max} = (1/3)\,(2n+1)\,\nu - \nu^2. \tag{37}
$$

6.2. The lower bound.

6.2.1. Case: $m \leq \xi \leq 2\nu < \xi+1 \leq n$.

Proposition 13 *The convex minimum in the class of r.v.'s with p.m.f. that are globally decreasing on $\mathbb{N}$ and convex up to a fixed point m where $m > 2$ and such that $m \leq \xi \leq 2\nu < \xi+1 \leq n$, is attained for the r.v. $\widetilde{N}_{\min}$ where*

$$
\widetilde{N}_{\min} = \begin{cases} 0,\ldots,\xi & \text{with equal probab.} \\ & \frac{2(\xi+1-\nu)}{(\xi+1)(\xi+2)}, \\ \xi+1 & \text{with probab. } \frac{2\nu-\xi}{(\xi+2)}, \\ \xi+3,\ldots,n & \text{with equal probab. } 0. \end{cases}
\tag{38}
$$

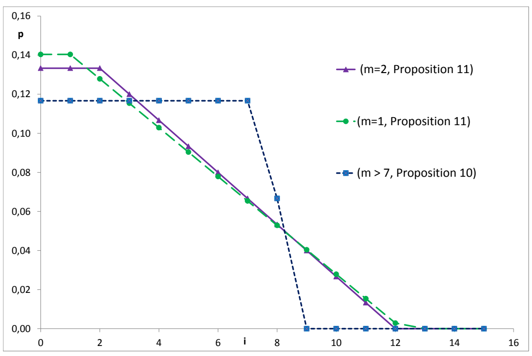

Fig. 10. The lower bounds for decreasing p.m.f. that are convex on $[m, +\infty)$, $\nu = 3.8$, $n = 15$ for $m = 2$, $m = 1$ and $m > 7$.

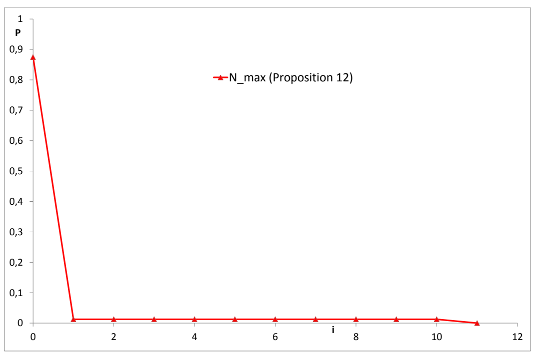

Fig. 11. The upper bound (36) for r.v.'s with p.m.f. that are decreasing and convex up to a point $m > 1$, $\nu = 0.69$, $n = 10$.

Note that this bound has the same shape as the one obtained in Proposition 10.

Proof.

Denote by $P(M = 0) = a_1$ and $P(\widetilde{N}_{min} = 0) = a$.

1. $a_1 > a$
   Three cases arise :
   - If $P(M = \xi) < a$, then it is easy to see that we can have at most one crossing point in $]\xi, \xi + 2[$. This is the case where $P(M = \xi + 1) > P(\widetilde{N}_{\min} = \xi + 1)$.
   - If $P(M = \xi) > a$, then we have at most two crossing points. This is the case when $P(M = \xi + 1) < P(\widetilde{N}_{\min} = \xi + 1)$ and $P(M = \xi + 2) > 0$. Hence $\widetilde{N}_{\min} \leq_{cx} M$.
   - If $P(M = \xi) = a$ then at most we can have one crossing point which is absurd.
2. $a_1 \leq a$
   In this case we can easily see that we have at least one crossing point. This is absurd.$\diamond$

6.2.2. Case: $\xi \leq 3\nu \leq \xi + 1$ and $m \geq [2\nu] + 1$.

Proposition 14 *The convex minimum in the class of r.v.'s with p.m.f. that are globally decreasing on $\mathbb{N}$ and convex up to a fixed point m where $m \geq 2$ such that $\xi \leq 3\nu \leq \xi+1$ and $m \geq [2\nu]+1$, is attained for the r.v. $\widetilde{N}_{\min}$ where $\forall\ i \in \{0, \ldots, \xi + 1\}$*

$$P\left(\widetilde{N}_{\min} = i\right) = (\xi + 1 - i)\,\pi_1 + (\xi + 2 - i)\,\pi_2, \quad (39)$$

where $\pi_1 = 2(\xi + 1 - 3\nu) / ((\xi + 2)(\xi + 1))$ and $\pi_2 = 2(3\nu - \xi) / ((\xi + 2)(\xi + 3))$.

Note that this bound coincides with the one obtained in Proposition 5 for r.v.'s with p.m.f. that are globally decreasing and convex on $\mathbb{N}$.

Proof.

Denote by $P(M = 0) = a_1$ and $P(\widetilde{N}_{min} = 0) = a$.

1. $a_1 \leq a$
   By assumption of decreasingness and convexity we cannot have more than one crossing point between p.m.f. of r.v.'s M and $\widetilde{N}_{\min}$ which is absurd.
2. $a_1 > a$
   In this case we have a two-crossing situation if and only if $P(M = \xi + 2) > P(\widetilde{N}_{\min} = \xi + 2)$. In this case it is clear that $P_{\widetilde{N}_{\min}} < P_M$ near n, hence by Proposition 3, $\widetilde{N}_{\min} \leq_{cx} M$.$\diamond$

7. Some numerical illustration

Since 2001, the European Commission has begun to establish the new regulation 'Solvency II' common to all countries members of the European Union. This regulation will be applied from October 2014. In particular, it sets the harmonization of methods of valuation of liabilities with a risk margin and the implementation of new rules to estimate the solvency capital requirement (SCR). This last is the main monitoring tool of the control authorities and is calculated so that all risks to which the entity is exposed are taken into consideration. The Basic SCR is estimated firstly for each business line, then all individual SCR are aggregated with respect to a correlation matrix. Insurance companies have the choice between two options: they can either adopt a standard approach or an internal model.

In the standard model of 'Solvency II' we distinguish two approaches: a scenario based approach and a factor based approach. In the scenario based approach, we measure the sensitivity to a shock of each individual risk and in the factor based approach we apply fixed factors to approximate the risk. In this last case the SCR is defined by $SCR = q\sigma$ where σ is the standard error of the random loss and $q > 0$ is called a quantile factor. So, $q = 3$ is usually chosen for claim amount with a moderate tail distribution. For heavy tailed risks, a more relevant value is $q = 5$ or 6. In the sequel we use a factor based approach to approximate the SCR.

Traditionally, computing or approximating the distribution function of the aggregate claim amount has been one of the central points in insurance mathematics. In order to investigate the distribution of the aggregate claim amount we consider individual model or collective model. Note that a collective risk model is often adopted to describe the occurrence of large claim. The total number N of claim occurring in a given period is random, typically it has a Poisson, binomial or negative binomial distribution. Further r.v.'s claim size W are strictly positive and are assumed to be independent and identically distributed and independent of N.

Consider the r.v. $S = \sum_{i=1}^{N} W_i$ describing the aggregate claim amount. In this case we say that S has a compound distribution where $E(S) = E(N)E(W)$ and $Var(S) = E(N)Var(W) + [E(W)]^2 Var(N)$. In practice, we have limited information about the distribution of N. In fact, in general only the mean $E(N)$ is available and the distribution of N is assumed to be Poisson with parameter λ. For example with respect to some French data , one could use Poisson distribution with parameter $\lambda(C_{27}) = 0.37$ for business line C_{27} (drought and earthquake) and $\lambda(C_{35}) = 0.69$ for business line C_{35} (construction-damages to building). Recall that we are in a situation where a large amount of claim occurs without being in a catastrophic situation. In reality practitioners consider that observing more than a certain number n of claims correspond to a catastrophe. For this reason we consider that n is fixed.

Instead of Poisson distribution we consider distributions that are decreasing on $\mathbb{N}$ which are globally or partially convex. Recall that the probability mass function of a Poisson distribution is decreasing convex if $\lambda \leq 2-\sqrt{(2)}$ and decreasing but not convex if $2-\sqrt{(2)} < \lambda \leq 1$. The bounds for $SCR = q\sigma$ are simply given by

$$SCR(\widetilde{N}_{min}) = q\sqrt{Var\,(W)\,E\,(N) + E^2\,(W)\,Var(\widetilde{N}_{min})}, \tag{40}$$

and

$$SCR\widetilde{N}_{max}) = q\sqrt{Var\,(W)\,E\,(N) + E^2\,(W)\,Var(\widetilde{N}_{max})}. \tag{41}$$

In the numerical illustrations we choose, $E[W(C_{27})] = 1000$, $Var[W(C_{27})] = 2500^2$ and $E[W(C_{35})] = 2000$, $Var[W(C_{35})] = 7000^2$(in thousand of euros).

7.1. Convex extrema for decreasing convex distribution globally on $\mathbb{N}$.

This gives $Var(\widetilde{N}_{max}) = 1.898$ and $SCR(\widetilde{N}_{max}) = 12311.85$ for business line C_{27} where $n(C_{27}) = 10$. These bounds are of course sharper than those obtained for distribution that are decreasing and convex not globally on $\mathbb{N}$ studied by Lefèvre and Loisel (2010) which are recalled here: $Var(\widetilde{N}_{max}) = 2.453$ and $SCR(\widetilde{N}_{max}) = 13098.2$.
(For maximum bounds see Tables 1 and 2 and for minimum bounds see Table 3).

7.2. Convex extrema for decreasing distributions that are concave until 2 and convex on $[1,\infty)$.

For business line C_{27}, $Var(\widetilde{N}_{max}) = 0.416$ and $SCR(\widetilde{N}_{max}) = 9911.68$ where $n(C_{27}) = 10$. For business line C_{35} (construction-damages to building), $Var(\widetilde{N}_{max}) = 3.781$ and $SCR(\widetilde{N}_{max}) = 41971.70$ for $n(C_{35}) = 20$. It is clear that these bounds are sharper than those derived in Section 7.1. (For maximum bounds see Tables 4 and 5 and for minimum bounds see Table 6.)

Table 1. Maximum bounds for decreasing convex distribution globally on $\mathbb{N}$ for business line C_{27}.

n	$Var(\widetilde{N}_{max})$	$SCR(\widetilde{N}_{max})$
5	0.973	10875.73
10	1.898	12311.85
20	3.748	14770.97
30	5.598	16875.47
40	7.448	18745.18

7.3. Decreasing distribution globally on $\mathbb{N}$ and convex on $[m,+\infty)$ for fixed $m \geq 1$.

For different fixed values of n and m we evaluate the variance and the SCR. (For maximum bounds see Tables 7, 8, 9 and 10, and for minimum bounds see Table 11.)
In Table 11 we note that for business line C_{27} we use Proposition 10 for any fixed $m \geq 1$ in fact, for this business line $0 < 2 \times 0.37 \leq 1$. For business line C_{35} we use Proposition 11 for $m = 1$ and Proposition 10 for all $m > 1$ as $1 < 2 \times 0.69 \leq 2$.

Table 2. Maximum bounds for decreasing convex distribution globally on $\mathbb{N}$ for business line C_{35}.

n	$Var(\widetilde{N}_{max})$	$SCR(\widetilde{N}_{max})$
5	1.594	11858.77
10	3.319	41171.37
20	6.769	46817.53
30	10.219	51852.50
40	13.669	56440.07

Table 3. Minimum bounds for decreasing convex distribution globally on $\mathbb{N}$ for business lines C_{27} and C_{35}.

	C_{27}	C_{35}
$Var(\widetilde{N}_{min})$	0.27	0.594
$SCR(\widetilde{N}_{min})$	9641.7	36092.7

Table 4. Maximum bounds for decreasing distributions that are concave until 2 and convex on $[1,\infty)$ for business line C_{27}.

n	$Var(\widetilde{N}_{max})$	$SCR(\widetilde{N}_{max})$
5	0.324	9742.01
10	0.416	9911.68
20	0.600	10239.22
30	0.783	10556.59
40	0.966	10864.67

Table 5. Maximum bounds for decreasing distributions that are concave until 2 and convex on $[1,\infty)$ for business line C_{35}.

n	$Var(\widetilde{N}_{max})$	$SCR(\widetilde{N}_{max})$
5	1.105	37099.22
10	1.997	38790.82
20	3.781	41971.70
30	5.564	44926.18
40	7.347	47698.30

Table 6. Minimum bounds for decreasing distributions that are concave until 2 and convex on $[1,\infty)$ for business lines C_{27} and C_{35}.

	C_{27}	C_{35}
$Var(\widetilde{N}_{min})$	0.233	0.467
$SCR(\widetilde{N}_{min})$	9572.96	35838.24

From Tables 7, 8, 9 and 10 we remark that for fixed n the choice of the maximal value m influences considerably the value of the variance and of the SCR. Also, for fixed m we observe that the choice of the maximal value n influences considerably the value of the variance and of the SCR.

7.4. Convex extrema for decreasing distribution globally on $\mathbb{N}$ and convex up to a point $m \geq 2$. Consider r.v. with p.m.f. that are decreasing globally on $\mathbb{N}$ and convex up to a point $m \geq 2$. In Table 12 we have maximum bounds and in Table 13 we have minimum bounds. We note that for minimum bounds we use Proposition 14.

Table 7. Maximum variances for decreasing distribution globally on $\mathbb{N}$ and convex on $[m, +\infty)$ for fixed $m \geq 1$ (business line C_{27}).

$n \setminus m$	1	2	18
5	0.973	0.995	–
10	1.898	1.906	–
20	3.748	3.750	4.69
30	5.598	5.599	6.182
40	7, 448	7.449	7.840

Table 8. Maximum SCR for decreasing distribution globally on $\mathbb{N}$ and convex on $[m, +\infty)$ for fixed $m \geq 1$ (line of business C_{27}).

$n \setminus m$	1	2	18
5	10875.73	10911.75	–
10	12311.85	12322.95	–
20	14770.98	14773.65	15877.23
30	16875.49	16876.65	17487.19
40	18745.18	18745.75	19118.17

Table 9. Maximum variances for decreasing distribution globally on $\mathbb{N}$ and convex on $[m, +\infty)$ for fixed $m \geq 1$ (line of business C_{35}).

$n \setminus m$	1	2	18
20	6.769	6.773	8.525
30	10.219	10.221	11.307
40	13.669	13.670	14.400

Table 10. Maximum SCR for decreasing distribution globally on $\mathbb{N}$ and convex on $[m, +\infty)$ for fixed $m \geq 1$(line of business C_{35}).

$n \setminus m$	1	2	18
20	46817.53	46823.99	49444.95
30	51852.50	51855.28	53343.07
40	56440.07	56441.60	57365.65

Table 11. Minumum bounds for decreasing distribution globally on $\mathbb{N}$ and convex on $[m, +\infty)$ where $m \geq 1$.

	C_{27}	C_{35}
$Var(\widetilde{N}_{min})$	0.233	0.467
$SCR(\widetilde{N}_{min})$	9572.96	35838.24

7.5. The influence of the choice of n and m. Let us prove algebraically that a is an increasing function of n and that b is a decreasing function of n. From (28) for all fixed $m \geq 1$ we denote by

$$\begin{aligned} a(n) &= 1 - \frac{3\nu(n+m)}{m^2 + nm + m + n^2 + 2n} \\ &= 1 - \frac{3\nu}{(m+1) + \frac{n(n+1)}{m+n}}. \end{aligned}$$

Hence the behavior of $a(n)$ follows the behavior of $\beta(n) = \frac{n(n+1)}{m+n}$, where

$$\frac{\partial \beta(n)}{\partial n} = \frac{(n+m)(2n+1) - n(n+1)}{(n+m)^2}.$$

Note that the sign of this derivative always follows the sign of the numerator. We note that $(n+m)(2n+1) - n(n+1) = 2n^2 + 2nm + m = 0$ has a general form of a second degree equation where the value of the discriminant denoted by Δ is equal to $4m(m-1)$. Two cases arise: the first one is when m is equal to 1. In this case $\Delta = 0$ and we have a double solution equal to -1. As n is a positive integer, the sign of the partial derivative is always positive in the first case. The second case is when $m > 1$. In this case $\Delta > 0$ and the equation has two distinct solutions denoted by $n_1 = -m - \sqrt{m(m-1)}$ and by $n_2 = -m + \sqrt{m(m-1)}$ which are both negative. As $n > 0$ the sign of this polynomial is always positive on $[1, \infty)$. Hence we deduce that $a(n)$ is an increasing function of n for all fixed $m \geq 1$. Similarly, denote by

$$b(n) = \frac{6\nu}{m - m^3 + n(n^2 + 3n + 2)}.$$

We note that the behavior of $b(n)$ when n varies follows the inverse behavior of $\alpha(n) = (n(n^2 + 3n + 2))^{-1}$. Hence for all fixed $m \geq 1$, it is very easy to see that $b(n)$ is a decreasing function of n. By the same argument we can prove that for fixed n

$$a(m) = 1 - \frac{3\nu(n+m)}{m(m+n+1) + (n^2 + 2n)}$$

is a decreasing function of m and

$$b(m) = \frac{6\nu}{m(1-m^2) + n(n^2 + 3n + 2)}$$

Table 12. Maximum bounds for decreasing distribution and convex up to $m \geq 2$.

n	$Var(\widetilde{N}_{max} C_{27})$	$Var(\widetilde{N}_{max} C_{35})$
10	2.453	4.354
20	4.920	8.954
40	9.853	18.154

Table 13. Minumum bounds for decreasing distribution and convex up to $m \geq 2$ where $n = 10$.

	C_{27}	C_{35}
$Var(\widetilde{N}_{\min})$	0.27	0.594
$SCR(\widetilde{N}_{\min})$	9641.7	36092.7

is an increasing function of m.

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Appendix A
Proof of Proposition 6

This bound is obtained as follows: we assume that

$$\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } a, \\ 1 & \text{with probab. } c, \\ i \in (2, \dots, n) & \text{with probab. } b(n+1-i), \end{cases} \tag{A1}$$

where a, b and c are such that (A1) is a true p.m.f. with fixed mean ν. Moreover, note that if N is a r.v. with p.m.f. that is concave in $[0, 2]$ and convex in $[1, +\infty)$, then by the constraint of convexity we must have

$$P(N=2)-P(N=1) \leq (P(N=3)-P(N=1))/2 \tag{A2}$$

and by concavity constraint we have

$$P(N=0) - P(N=1) \leq (P(N=0) - P(N=2))/2. \tag{A3}$$

These constraints lead to

$$\begin{cases} bn \leq c, \\ a - c \leq c - b(n-1). \end{cases} \tag{A4}$$

Following Proposition 3 we have $X \leq_{cx} Y$ if the p.m.f. of the r.v. Y has heavier values on extremes than those taken by the r.v. X. For this we shall consider that

$$P(N=0) - P(N=1) = (P(N=0) - P(N=2))/2.$$

Hence a, b and c are determined by solving the following system

$$\begin{cases} a - c = c - b(n-1), \\ a + c + \sum_{i=2}^{n} b(n+1-i) = 1, \\ c + \sum_{i=2}^{n} b(n+1-i)i = \nu, \\ where\ 0 < bn < c < a < 1. \end{cases} \tag{A5}$$

Consider two r.v's M and $\widetilde{N}_{\max}$ in $\mathbb{J}_n$ with p.m.f. concave in $[0, 2]$ and convex in $[1, +\infty)$. Denote by $P(M = 0) = a_1$, $P(M = 1) = c_1$, $P(M = 2) = d_1$ and $P(\widetilde{N}_{\max} = 2) = d$.

1. $a_1 < a$ and $c_1 > c$
   In this case we observe a crossing point in $]0, 1[$. By Proposition 2 there must exist, at least, another crossing point in $]1, n]$. Two cases arise:
   1.1 If $d_1 \leq d$, then we have a crossing point in $]1, 2[$. In addition, by global convexity in $[1, \infty)$ there cannot exist another crossing point in $]2, n]$ (see Figure A1 for illustration).
   1.2 If $d_1 > d$, then we have a single crossing point in $]2, n]$ by Lemma 2 (see Figure A2 for illustration).

   Note that in the previous two cases the p.m.f. of M and $\widetilde{N}_{\max}$ have exactly two crossing points in the set $\mathcal{D}_n$. In addition, near n, $P_M \leq P_{\widetilde{N}_{\max}}$. Thus, by Proposition 3 we have $M \leq_{cx} \widetilde{N}_{\max}$.
2. $a_1 < a$ and $c_1 = c$
   Two cases arise:
   2.1 if $d_1 > d$ then in this case a first crossing point is observed in $]0, 2]$. In addition, by Proposition 2 we have exactly one crossing point in $]2, n]$ where near n we have $P_M \leq P_{\widetilde{N}_{\max}}$ (see Figure A3 for illustration).
   2.2 if $d_1 \leq d$, then the two crossing points must be in $]2, n]$ which is not possible by Lemma 2. Hence, this case is absurd.
3. $a_1 < a$ and $c_1 < c$
   Two cases arise:

3.1 If $d_1 > d$, then the constraint of concavity is not fulfilled.

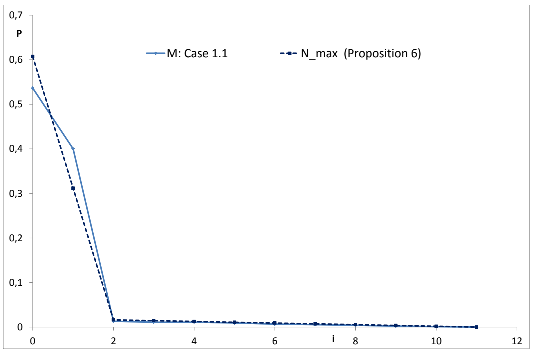

Fig. A1. The upper bound (20) vs the case 1.1. $\nu = 0.69$, $n = 10$.

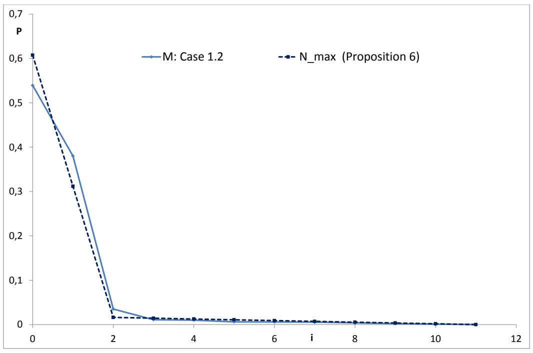

Fig. A2. The upper bound (20) vs the case 1.2. $\nu = 0.69$, $n = 10$.

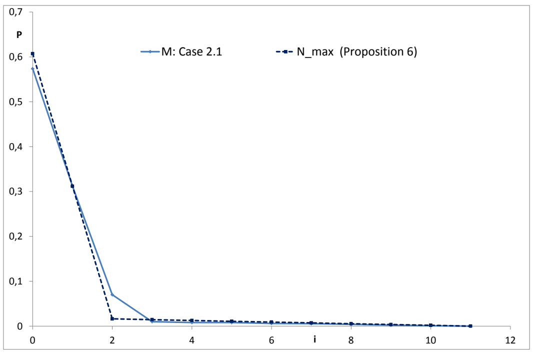

Fig. A3. The upper bound (20) vs the case 2.1. $\nu = 0.69$, $n = 10$.

3.2 If $d_1 \leq d$, then it is clear that there is no crossing point until the point 2. In addition by Lemma 2, in order to ensure the constraint of convexity, there is no crossing between p.m.f. of M and $\widetilde{N}_{\max}$ in $]2, n]$. So, the constraint of equal mean is violated.

4. $a_1 \geq a$ and $c_1 \geq c$
In this case by Lemma 2 we have at most one crossing point over $\mathcal{D}_n$.

5. $a_1 \geq a$ and $c_1 \leq c$
In this case, to ensure the assumption of concavity, we must have $d_1 < d$. Hence we have a crossing point in $]0, 2[$. So, if a second crossing point exists, then it is necessarily in $]2, n]$. If it is the case, then by Lemma 2, the constraint of convexity is violated. So, this case is absurd.

Appendix B
Proof of Proposition 8

The lower bound (24) is derived such that the distribution of $\widetilde{N}_{\min}$ has the following p.m.f. where a and b are two positive integers :

$$\widetilde{N}_{\min} = \begin{cases} 0 & \text{with probab. } a-b, \\ i \in (1, \ldots, \xi+1) & \text{with probab. } a-bi, \\ \xi+2, \ldots, n & \text{with equal probab. } 0, \\ where\ 0 < b < a < 1. & \end{cases} \tag{B1}$$

Therefore, a and b are derived such that (B1) is a true p.m.f. and such that the mean of the r.v. $\widetilde{N}_{\min}$ is equal to ν.

$$\begin{cases} a - b + \sum_{i=1}^{\xi+1}(a-bi) = 1, \\ a - b + \sum_{i=2}^{\xi+1}(a-bi)i = \nu. \end{cases} \tag{B2}$$

Hence it follows that

$$a = \frac{2\left((2\,\xi+3)(\xi+2)(\xi+1) - 3\,\nu\,(4+\xi^2+3\,\xi)\right)}{\xi(\xi+5)(\xi+2)(\xi+1)}, \tag{B3}$$

and

$$b = \frac{6\,(\xi+1-2\,\nu)}{\xi(\xi+5)(\xi+1)}, \tag{B4}$$

where $0 < b < a < 1$ and $\xi < 2\nu \leq \xi + 1$.
Note that from (B1) we have $P(\widetilde{N}_{\min} = 2) = a - 2b$ and $P(\widetilde{N}_{\min} = 0) = P(\widetilde{N}_{\min} = 1) = a - b$. So, the assumption of concavity in $[0, 2]$ is valid. Assume that $P(\widetilde{N}_{\min} = \xi+1) = \pi_2$. Moreover, we have $P(\widetilde{N}_{\min} = \xi + 2) = 0$, then by constraint of convexity we must have $P(\widetilde{N}_{\min} = \xi) = \pi_1 + 2\pi_2$. Hence, it follows that $b = (\pi_1 + \pi_2)$ and $a = \pi_2(2+\xi) + \pi_1(\xi+1)$.
Denote by $P(\widetilde{N}_{\min} = 0) = P(\widetilde{N}_{\min} = 1) = a^*$, $P(M = 0) = a_1^*$, $P(M = 1) = c_1$, $P(M = 2) = d_1$, and $P(\widetilde{N}_{\min} = 2) = d^*$.

1. $a_1^* > a^*$ and $c_1 \geq a^*$
In this case we remark that in $]0, 1[$ we have no crossing point. Now, assume that a first crossing point is observed in $[1, \infty[$. This means that there is $j \in [1, \xi+1[$ such that $P(M = j) < P(\widetilde{N}_{\min} = j)$. Further, consider that a second crossing point exists then, this means that there is $k \in [j+1, \xi+1[$ where $j < k \leq \xi$ such that $P(M =$

$k) > P(\widetilde{N}_{\min} = k)$. If a third crossing point exists, then the constraint of convexity is violated. Hence at most, we have two crossing points where near n, $P_M > P_{\widetilde{N}_{\min}}$. Then $\widetilde{N}_{\min} \leq_{cx} M$ (see Figure B1 for illustration).

2. $a_1^* > a^*$ and $c_1 < a^*$
In this case we have a first crossing point in $]0,1[$. By the constraint of convexity we have at most one crossing point in $]1, \xi+2[$. Hence we are in a situation with exactly two crossing points. As $a_1^* > a^*$ it follows immediately that near n, $P_M > P_{\widetilde{N}_{\min}}$. So, by Proposition 3, $\widetilde{N}_{\min} \leq_{cx} M$ (see Figure B2 for illustration).

3. $a_1^* \leq a^*$
In this case by the assumption of decreasingness we must have $c_1 \leq a^*$. Three cases arise:

   3.1 If $P(M = \xi+1) > P(\widetilde{N}_{\min} = \xi+1)$, then, by the assumption of convexity we have at most one crossing point in the set $\mathcal{D}_n$. This is absurd under Proposition 2.

   3.2 If $P(M = \xi+1) \leq P(\widetilde{N}_{\min} = \xi+1)$ and $P(M = \xi+2) > 0$, then we observe a crossing point in $]\xi+1, \xi+2[$. In addition, by the assumption of convexity and according to Lemma 2 we have no crossing point in $[0, \xi+1]$. So, this case is absurd because of the assumption of equal means which requires at least two crossing points.

   3.3 If $P(M = \xi+1) \leq P(\widetilde{N}_{\min} = \xi+1)$ and $P(M = \xi+2) = 0$, then under Lemma 2 we have no crossing point over $\mathcal{D}_n$. This is absurd.

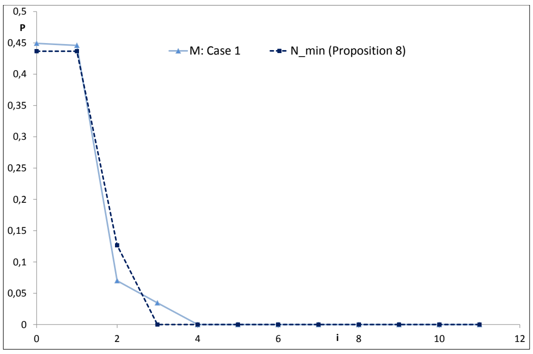

Fig. B1. The lower bound (24) vs the case 1. $\nu = 0.69, n = 10$.

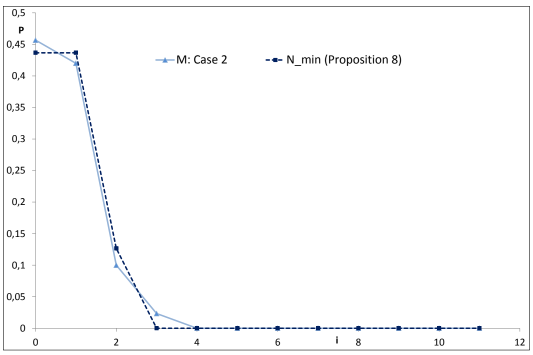

Fig. B2. The lower bound (24) vs the case 2. $\nu = 0.69, n = 10$.

Appendix C
Proof of Proposition 9

This bound is obtained as follows:

$$\widetilde{N}_{\max} = \begin{cases} 0 & \text{with probab. } a, \\ 1, \ldots, m-1 & \text{with equal probab.} \\ & b(n+1-m), \\ i \in (m, \ldots, n) & \text{with probab.} \\ & b(n+1-i), \end{cases} \tag{C1}$$

where $0 < b(n+1-m) < a < 1$ and such that

$$\begin{cases} a + (m-1)b(n+1-m) + \sum_{i=m}^{n} b(n+1-i) = 1, \\ b(n+1-m)\sum_{i=1}^{m-1} i + \sum_{i=m}^{n} b(n+1-i)i = \nu, \end{cases} \tag{C2}$$

Next we present a reasoning by contradiction to prove Proposition 9. Let M be a random variable in $\mathbb{J}_n$ with a p.m.f. that is decreasing globally on $\mathbb{N}$ and convex from a fixed point $m > 1$, where $0 < 3\nu \leq (m^2+nm+m+n^2+2n)/(m+n+2)$. Denote by $P(M = 0) = a_1$, $P(M = 1) = c_1$, $P(\widetilde{N}_{\max} = 0) = a$ and $P(\widetilde{N}_{\max} = 1) = c$. For this, we enumerate all possible p.m.f. that may be taken by the r.v. M and we prove that in all cases $M \leq_{cx} \widetilde{N}_{\max}$.

1. $a_1 < a$ and $c_1 \leq c$
In this case by Lemma 2 there cannot be a crossing point on $\mathcal{D}_n$ which is absurd.

2. If $a_1 < a$ and $c_1 > c$, then we have a first crossing point in $]0,1[$.

   2.1 In addition, if $P(M = m-1) \leq P(\widetilde{N}_{\max} = m-1)$, then by nonincreasingness we have necessarily, a crossing point in $]1, m-1[$. In addition, under Lemma 2 there is not any crossing point in $]m, \infty)$.

   2.2 If $P(M = m-1) > P(\widetilde{N}_{\max} = m-1)$, then by Lemma 2 there is at most one crossing point on $]m, n[$. Hence in both cases we have exactly two crossing points over $\mathcal{D}_n$ where near n we have $M \leq_{cx} \widetilde{N}_{\max}$.

3. $a_1 \geq a$ and $c_1 > c$
In this case, by the same reasoning as in the previous case, there cannot exist more than one crossing point over $\mathcal{D}_n$ which is absurd. In fact, if $P(M = m-1) < P(\widetilde{N}_{\max} = m-1)$, then we observe only one crossing point in $]0, m-1[$ and by Lemma 2 there is not another crossing point on $]m-1, n[$. If $P(M = m-1) \geq P(\widetilde{N}_{\max} = m-1)$, then by Lemma 2 there cannot exist more than one crossing over $\mathcal{D}_n$.

4. $a_1 \geq a$ and $c_1 \leq c$
In this case we observe a first crossing point in $]0,1[$ but, by the assumption of decreasingness and according to lemma 2 there is no other crossing point over $\mathcal{D}_n$ which is absurd.

Appendix D
Proof of Proposition 10

This bound is obtained as follows: we assume that

$$\widetilde{N}_{\min} = \begin{cases} 0,\dots,\xi & \text{with equal probab. } a, \\ \xi+1 & \text{with probab. } b, \\ \xi+2,\dots,n & \text{with equal probab. } 0, \end{cases} \tag{D1}$$

where a and b are such that this is a true p.m.f., with fixed mean ν i.e.

$$\begin{cases} (\xi+1)*a+b=1, \\ \xi(\xi+1)*a/2+(\xi+1)*b=\nu, \end{cases} \tag{D2}$$

Note that we do not have any constraint related with convexity below point $\xi+1$ as $m \geq \xi+1 > 1$. Denote by $P(M=0)=a_1$ and by $P(\widetilde{N}_{\min}=0)=a$.

1. $a_1 > a$ and $P(M=\xi) < P(\widetilde{N}_{\min}=\xi)$
   In this case, by assumption of nonincreasingness, we have exactly one crossing point on $]0,\dots,\xi[$. Two cases arise:

   1.1 $P(M=\xi+1) > P(\widetilde{N}_{\min}=\xi+1)$.

   1.2 $P(M=\xi+1) < P(\widetilde{N}_{\min}=\xi+1)$.

   Note that in these two cases we have at most two crossing points. In addition in the presence of two crossings, we have $P_{\widetilde{N}_{\min}} \leq P_M$ near n. Hence, $\widetilde{N}_{\min} \leq_{cx} M$.

2. $a_1 > a$ and $P(M=\xi) > P(\widetilde{N}_{\min}=\xi)$
   In this case we have at most two crossing points over $\mathcal{D}_n$. More precisely this case arises if and only if we have $P(M=\xi+1) < P(\widetilde{N}_{\min}=\xi+1)$ and $P(M)=\xi+2) > 0$, (see Figure D1 for illustration). Thus in this case $P_{\widetilde{N}_{\min}} \leq P_M$ near n.
   So, by Proposition 3 we conclude that $\widetilde{N}_{\min} \leq_{cx} M$.
   It is clear that if $P(M=\xi) = (\widetilde{N}_{\min}=\xi)$, then we have at most one crossing which is absurd under Proposition 2.

3. $a_1 \leq a$
   We note that we have at most one crossing point over $\mathcal{D}_n$ which is absurd following Proposition 2.

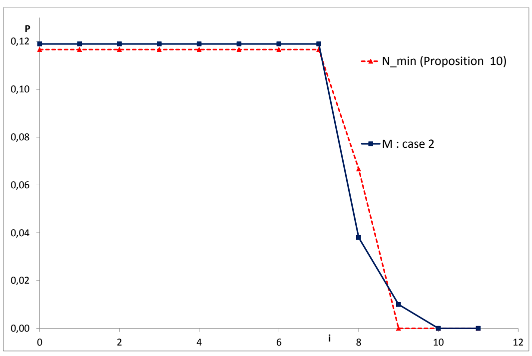

Fig. D1. The lower bound (30) vs the case 2., $m=8$, $\nu=3.8$, $n=10$.

Appendix E
Proof of Proposition 11

The distribution of $\widetilde{N}_{\min}$ is obtained as explained in the following optimization program:

$$\widetilde{N}_{\min} = \begin{cases} 0,\dots,m & \text{with equal probab. } a-bm, \\ i \in (m+1,\dots,\xi+1) & \text{with probab. } a-bi, \\ \xi+2,\dots,n & \text{with equal probab. } 0, \end{cases} \tag{E1}$$

where $0<b<a<1$ and

$$\begin{cases} (m+1)(a-bm)+\sum_{i=m+1}^{\xi+1}(a-bi)=1, \\ m(m+1)(a-bm)/2+\sum_{i=m+1}^{\xi+1}(a-bi)i=\nu. \end{cases} \tag{E2}$$

From (E1), we have $P(\widetilde{N}_{\min}=\xi+2)=0$ and $P(\widetilde{N}_{\min}=\xi+1)=\pi_2$. In addition, to ensure the constraint of convexity, we consider that $P(\widetilde{N}_{\min}=\xi)=\pi_1+2\pi_2$ where $\pi_1 \geq 0$. Hence, we can deduce that $b=(\pi_1+\pi_2)$ and $a=\pi_2(2+\xi)+\pi_1(\xi+1)$. Denote by $P(M=m)=a_1$ and by $P(\widetilde{N}_{\min}=m)=a$.

1. $a_1 > a$
   In this case there are, at most, two crossing points near n $P_{\widetilde{N}_{\min}} \leq P_M$. Thus, by Proposition 3, $\widetilde{N}_{\min} \leq_{cx} M$.
2. $a_1 < a$
   According to the assumptions of nonincreasingness and convexity, there exist at most two crossings over $\mathcal{D}_n$ such that near n $P_{\widetilde{N}_{\min}} \leq P_M$. Thus, by Proposition 3, $\widetilde{N}_{\min} \leq_{cx} M$.
3. If $a_1 = a$, then by convexity and nonincreasingness it is clear that there exists at most one crossing, which is absurd under Proposition 2.