The impact of randomness on the distribution of wealth: Some economic aspects of the Wright-Fisher diffusion process

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Sur le caractère aléatoire de la répartition des richesses : Quelques interprétations économiques du processus de Wright-Fisher.

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Résumé en français : Pour étudier certains effets stochastiques en économie on considère la répartition des fortunes dans une population de N agents qui échangent entre eux par des jeux à somme nulle et équilibrés en espérance. Les comportements de ces économies simplifiées, en présence ou non de mécanismes de redistribution, sont très apparents par simulation. Pour leur étude mathématique, la situation limite, lorsque l'on suppose les transactions petites et fréquentes, est intéressante, on rencontre alors des modèles de diffusion introduits en génétique des populations à partir de considérations très différentes.

Mots clés : Répartion des richesses, Diffusion de Wright-Fisher, Inégalitées, Mécanismes de taxation.

The Impact of Randomness on the Distribution of Wealth: Some Economic Aspects of the Wright-Fisher Diffusion Process

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Abstract

In this paper we consider some elementary and fair zero-sum games of chance in order to study the impact of random effects on the wealth distribution of N interacting players. Even if an exhaustive analytical study of such games between many players may be tricky, numerical experiments highlight interesting asymptotic properties. In particular, we emphasize that randomness plays a key role in concentrating wealth in the extreme, in the hands of a single player. From a mathematical perspective, we interestingly adopt some diffusion limits for small and high-frequency transactions which are otherwise extensively used in population genetics. Finally, the impact of small tax rates on the preceding dynamics is discussed for several regulation mechanisms. We show that taxation of income is not sufficient to overcome this extreme concentration process in contrast to the uniform taxation of capital which stabilizes the economy and prevents agents from being ruined.

Keywords: Wealth distribution, Fair zero-sum games, Wright-Fisher diffusions, Inequalities, Impact of modes of taxation.

JEL classification: C32, C63, D31.

1 Introduction

1.1 Motivations

Since the seminal work of the Italian economist and sociologist Vilfredo Pareto [44], researchers have paid a lot of attention to describe, analyze and model wealth accumulation processes. In 1953, Champernowne [13] was the first to propose an exogenous multiplicative Markov chain model of stochastic wealth returns which generates Pareto-shaped wealth distributions. This paved the way for the so-called proportional random growth approach (see for example, [36] [37] or [5]). More recently, equilibrium models were developed to study the characteristics of the wealth accumulation process as the result of agents' optimal consumption-saving decisions (see for example, [4] or [25]).

In this paper, we consider an economy simplified to the extreme and reduced to random games between agents. The games played are assumed to be fair in expectation. This situation may be seen as a basic model of randomness in the physical world or for modeling the effect of volatility on prices, considering that at least for the first order of magnitude the games may be assumed to occur

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with zero expectations. The focus is therefore on social and collective phenomena appearing in this purely speculative framework with or without regulation mechanisms. In some sense, our approach may be related to multiplicative Random Asset Exchange Models. These have recently been introduced in Econophysics (see [32] or [50], Chap. 8) to describe the evolution of wealth distribution in a population interacting economically. In fact, in these models, the interactions between two agents simply result in a well-chosen random redistribution of their assets, without taking into account the possible underlying micro-foundations. We follow this line to emphasize that, in our simple model, social inequalities are driven primarily by chance, rather than by differential abilities.

Our main objective is to study the dynamics of some elementary Markov games of chance in order to highlight the impact of random effects on the wealth distribution of interacting players. Surprisingly, even if zero-sum games of chance (expected to be fair) are played at any round, wealth distribution converges toward the maximum inequality case. This qualitative behavior has already been empirically observed in the literature (see [49] Chap. 15, [3], [22] and [30]). But few papers have provided a theoretical framework to prove that luck alone may generate extreme disparities in wealth dynamics. Notable exceptions are [6] and [7] for the so-called Yard sale model in Econophysics and [37], [25] and [5] where investors are faced with an uninsurable idiosyncratic investment risk. In this paper, similar conclusions on wealth condensation are both supported by numerical and mathematical arguments, at the very least for small and high-frequency transactions in which Wright-Fisher diffusion processes naturally appear as limit models. To our knowledge, this is the first time this so-called family of stochastic processes, widely studied in theoretical population genetics (see [18] or [19]), is used to examine wealth concentration problems, providing interesting, new economic interpretations of these classical dynamics. We also investigate the impact of some regulation mechanisms on the qualitative behavior of the models considered. In particular, our findings provide a very simple agent-based framework for understanding how the Beta distribution, widely used in the literature as descriptive models for the size distribution of income ([46], [9]), arises in wealth distribution problems.

Let us start by giving some elementary definitions and examples.

1.2   Fair elementary zero-sum games of chance (FEG)

We consider two players playing during $n \in \mathbb{N}^*$ consecutive rounds a zero-sum game of chance. If we denote by $(P_k^i)_{k \in \{1,\dots,n\}}$ the payoff process (defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$) of the player $i \in \{1, 2\}$, starting from a constant initial wealth X_0^i, we have $P_k^1 = -P_k^2$ (zero-sum game) and the wealth process is given by

$$X_k^i = X_0^i + \sum_{1 \leqslant j \leqslant k} P_j^i.$$

By a game of chance, we mean a game whose outcome depends on some random experiments. Contrary to what happens in game theory, we do not consider strategic interactions between players. The evolution of wealth only depends on a random redistribution mechanism between the interacting players through the payoff process. At this stage, our approach may be related to the Random Asset Exchange Models recently introduced in Econophysics (see [32] or [50], Chap. 8), to describe the evolution of wealth distribution in an economically interacting population. In fact, the economy is considered here in its simplest form: every time two players interact, wealth is transferred from one to the other, according to some elementary fixed rules, without taking into considerations their neoclassical foundations. Moreover, the zero-sum hypothesis implies that no wealth is imported, exported, generated, or consumed: wealth can only change hands. Working in a closed economic system will allow us to think in terms of proportions of wealth, instead of in absolute values.

Definition 1. *We say that the preceding zero-sum game of chance is fair in expectation on* $(\Omega, \mathcal{A}, \mathbb{P})$ *(and we will write* **FEG** *for fair elementary game) if* $\forall k \in \{1, ..., n\}$, $\mathbb{E}_{\mathbb{P}}[P_k^i] = 0$.

Remark 1: *Let us emphasize that considering games fair in expectation is philosophically the most natural thing to do, if we have no additional reason for the presence of biases, and that it was indeed the first historical approach used in creating a prize-winning purse, adopted by Louis Bachelier in* [2] *who assumed that "the expectation of the speculator is zero".*

In the next two examples, we suppose that $X_0^1 + X_0^2 = 1$ (reasoning in proportion of the total initial wealth $X_0^1 + X_0^2$ instead of using absolute values) and that each player bets a fixed amount a of his/her initial wealth [1].

- **Example 1: The Yard-Sale model.** (see [6], [14] and [30])

In the so-called Yard-Sale model, each player may win or lose with the same probability a fraction a of the wealth of the poorest. In other words,

$$P_1^i = a\left[min(X_0^i, 1 - X_0^i)\mathbf{1}_{U_1 \leqslant 0.5} - min(X_0^i, 1 - X_0^i)\mathbf{1}_{U_1 > 0.5}\right]$$

where U_1 follows a uniform distribution on $[0, 1]$. This model has been introduced by Hayes in [30] [2] to overcome some economic bias of the Theft-and-Fraud model of Ipsolatov et al. [32] where the exchanged amount is a fraction of the losing agent's wealth. In fact, the last model drastically favors the poorest player while the Yard-Sale one is fair in expectation.

As in the preceding example, a game is fair in expectation as soon as the random variables P_k^i are symmetric (in the sense that P_k^i and $-P_k^i$ have the same distribution) but this condition is not necessary (and is questionable from an economic perspective), as we are going to see in the following exchange mechanism:

- **Example 2: Elementary market games with proportional bets.**

Let player i bet a fixed amount a of its initial wealth and win the game with probability X_0^i, we have

$$P_1^i = a(1 - X_0^i)\mathbf{1}_{U_1 \leqslant X_0^i} - aX_0^i\mathbf{1}_{U_1 > X_0^i}$$

where U_1 follows a uniform distribution on $[0, 1]$ and the game is fair in expectation [3]. Contrary to what happen in the Yard-Sale model [4], the structure of the payoff function induces here a third order bias $(E[(P_1^i)^3] = a^3 X_0^i(1 - X_0^i)(1 - 2X_0^i))$ that is none other than a leverage effect in favor of the

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1. For real world transactions, it is reasonable to consider the multiplicative exchange case (instead of the additive one) where the amount of money bet by each economic agent is proportional to their wealth, because wealthy people tend to invest more than the less wealthy. The parameter $1 - a$ may be seen as a measure of the propensity to save, assumed to be constant among all the participants, in order to ensure an identical involvement. For example, this kind of mechanism is considered in [10], [30] or [32] in the framework of Econophysics.

2. As mentioned by Hayes itself in his paper, the transaction mechanism related to the Yard-Sale model has been first introduced by Chakraborti in [12] where the author empirically observed the concentration of wealth after a long time.

3. This model may be seen as a randomized version of the traditional African Calabash game (see Appendix A) and is similar to the greedy proportional exchange model mentioned in the discussion of [32]

4. In a recent paper [8], Boghosian et al. introduce an extension of the Yard-Sale model, to cope with the substantial evidence that the actual bias in the real world favors the wealthier agent rather than the poorer, using the Wealth-Attained Advantage. Nevertheless this modification is very different in nature than elementary market games with proportional bets because in this case the game is no more fair in expectation.

wealthier agent as usually observed in financial markets. In fact, the wealthier agent has for his gain a very unbalanced law of probability: he loses a lot with a small probability and wins little with a probability that may be close to 1.

While the Yard-Sale model and some of its extensions has been both extensively studied numerically and theoretically in the literature (see [6], [8], [14] and [30]), the objective of the present paper is to provide an analogous understanding of elementary market games with proportional bets. The remainder is organized as follows. In Section 2, we provide a numerical study of elementary market games with proportional bets [5]. We then show, in agreement with the conclusions of [26], that in our model, economic mobility decreases with inequality and increases with the tax rate. In Section 3, we study theoretically our two-player Markov chain game in order to prove high concentration phenomena and, for small transactions, we find its continuous time limit by transforming the time scales and state spaces appropriately. In Section 4, the results are extended in the presence of capital tax rates where the symmetric Beta distribution appears as a steady state probability of the system. Section 5 concludes. Some mathematical proofs are postponed in Appendix where we extend the two player analysis to the N player case.

2 Numerical study of elementary market games with proportional bets

In this section we consider a population of $N = 100$ players [6] with uniformly distributed initial wealth: $\forall i \in \{1, ..., 100\}$, $X_0^i = 1/100$. For each stage k, we select two different players (i_k, j_k) randomly and independently who play an elementary market game with proportional bets (see Example 2) with $a = 10\%$, thus,

$$X_{k+1}^{i_k} = X_k^{i_k} + aX_k^{j_k}\mathbf{1}_{U_k \leqslant \frac{X_k^{i_k}}{X_k^{i_k}+X_k^{j_k}}} - aX_k^{i_k}\mathbf{1}_{U_k > \frac{X_k^{i_k}}{X_k^{i_k}+X_k^{j_k}}},$$

$$X_{k+1}^{j_k} = X_k^{j_k} + aX_k^{i_k}\mathbf{1}_{U_k \leqslant \frac{X_k^{j_k}}{X_k^{i_k}+X_k^{j_k}}} - aX_k^{j_k}\mathbf{1}_{U_k > \frac{X_k^{j_k}}{X_k^{i_k}+X_k^{j_k}}},$$

$$X_{k+1}^{i} = X_k^{i}, \ i \notin \{i_k, j_k\}$$

where the U_k are i.i.d uniform distributions on $[0, 1]$. Even if FEG are played in each round, the randomness clearly induces disparities in wealth between economic agents.

In Figure 1, we represent the distribution of the wealth and of its increasing rearrangement after a large number of steps. We can see that inequalities become greater and greater, while a poor player may become richer and a rich player may become poorer at each stage [7]. In particular the percentage of players that own less than the average wealth increases from around 50% after 100 transactions, to around 90% after 100,000: poverty traps appear, as underlined in Figure 2. To support this intuition, we represent in Figure 3 the evolution of the Gini coefficient [27] and of the Lorenz curve [39] as functions of the number of transactions. We note without ambiguity that the distribution of

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5. A similar analysis has been performed in [14] for the Yard-Sale model with analogous conclusions. Nevertheless, one of the main advantages of elementary market games with proportional bets is their natural extension to the N-agent case (see Appendix B).

6. Analogous results have been numerically obtained for $N = 1000$ players even if the number of simulations increases drastically.

7. The emergence of inequalities is also numerically observed in [49] for other fair games of chance even if convergence towards the maximum inequality situation is not reported.
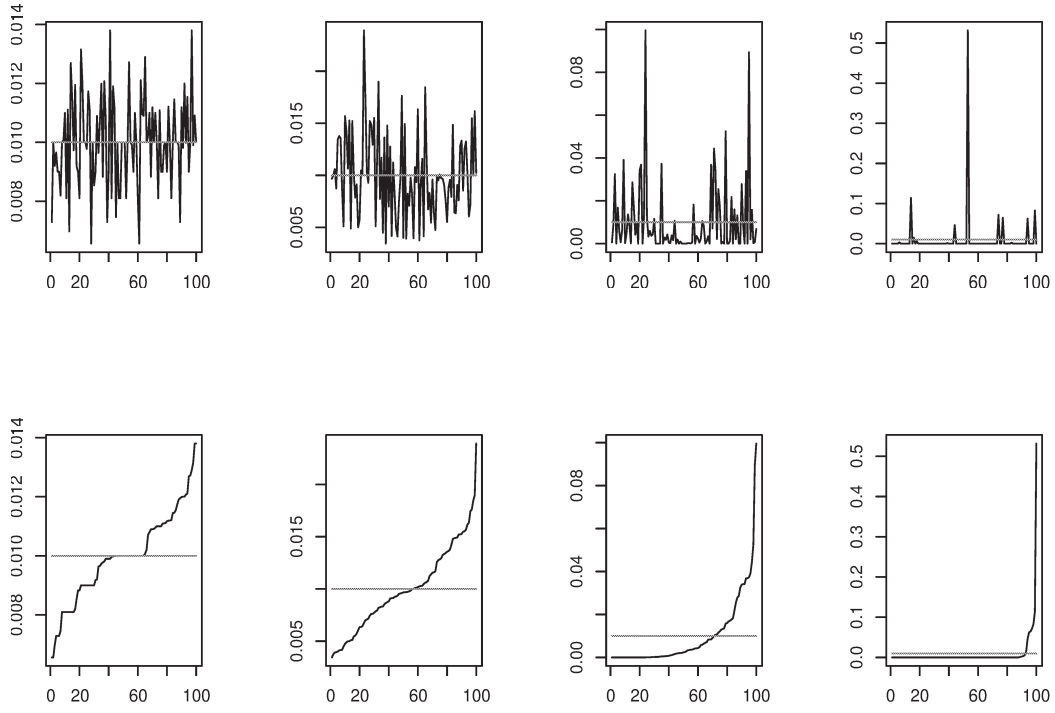
Figure 1: *The first line (resp. the second line) represents the distribution of the wealth (resp. the distribution of the increasing rearrangement of wealth) of the $N = 100$ players after $n = 100$ (first column), $n = 1,000$ (second column), $n = 10,000$ (third column) and $n = 100,000$ (fourth column) FEG starting from uniformly distributed initial wealth.*

wealth converges, according to these classical indicators, towards the case of maximum inequality, even if elementary exchange mechanisms are fair in expectation[8].
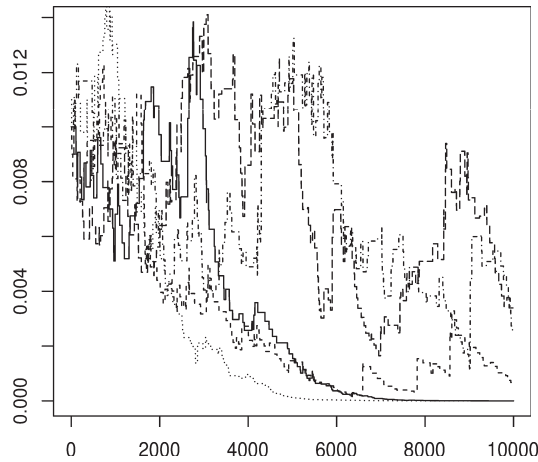

Figure 2: *Five individual wealth trajectories: creation of poverty traps*

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8. In simulations that are not reported here, we can also observe that the Gini coefficient of the economy is an
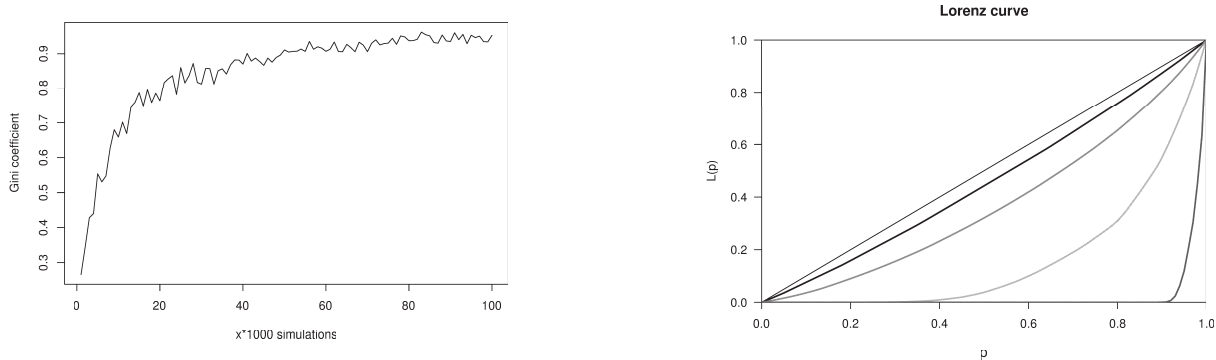
Figure 3: *Gini coefficient (left) and Lorenz curve (right). The value of the Lorenz curve $L(p)$ represents the percentage of the poorer players that possess p percent of the total wealth. For 100, $1,000$, $100,00$ and $100,000$ transactions, the Lorenz curve moves away further and further from the line of perfect equality. The Gini coefficient equals $1 - 2A$ where A is the integral of the Lorenz curve. A Gini coefficient of zero expresses perfect equality. A Gini coefficient of 1 expresses maximum inequality.*

We have exhibited numerical evidence indicating that our model gives rise to a situation of strong concentration in which all the wealth ends up in the hands of a single wealth-holder in spite of a uniform initial distribution of wealth and the hypothesis that fair games are played each round. Here, randomness is the only source of inequalities. Recently, similar results have been numerically obtained in the literature for the Yard-sale model ([6], [14], [30]) and in slightly different frameworks where households, which are identical in terms of their patience and their abilities, are faced with an uninsurable idiosyncratic investment risk ([25], [5]).

In a second step we study the impact of a simple redistribution mechanism on the preceding dynamics: we consider, at any stage, a proportional capital tax rate b that is collected and uniformly reallocated to the players: If the players (i_k, j_k) are chosen to play the game at stage k:

$$X_{k+1}^{i_k} = (1 - bX_k^{i_k}) + aX_k^{j_k}\mathbf{1}_{U_k \leqslant \frac{X_k^{i_k}}{X_k^{i_k} + X_k^{j_k}}} - aX_k^{i_k}\mathbf{1}_{U_k > \frac{X_k^{i_k}}{X_k^{i_k} + X_k^{j_k}}} + \frac{b}{N},$$

$$X_{k+1}^{j_k} = (1 - bX_k^{j_k}) + aX_k^{i_k}\mathbf{1}_{U_k \leqslant \frac{X_k^{j_k}}{X_k^{i_k} + X_k^{j_k}}} - aX_k^{j_k}\mathbf{1}_{U_k > \frac{X_k^{j_k}}{X_k^{i_k} + X_k^{j_k}}} + \frac{b}{N},$$

$$X_{k+1}^{i} = (1 - bX_k^{i}) + \frac{b}{N}, \ i \notin \{i_k, j_k\}$$

where the U_k are i.i.d uniform distributions on $[0, 1]$.

We perform a numerical study similar to the untaxed case: in a population of $N = 100$ players with uniformly distributed initial wealth: $\forall i \in \{1, ..., 100\}$, $X_0^i = 1/100$, we select randomly and independently two players that play an elementary market game with proportional bets and taxes with $a = 10\%$ and $b = 1\%$. The impact of the small tax parameter b is substantial as observed comparing Figures 1 and 4. The wealth distribution remains mostly uniform each successive stage, the effect of randomness is regulated [9]. In particular the percentage of players that own less than the average wealth stabilizes around 60%. In Figure 5, we also see that the proportion of wealth owned at each time by the richest player goes to 1 in the untaxed case and is around 0.03 when $b = 1\%$. Similarly, we have represented in Figure 6 the impact of b on the Gini coefficient of the population,

increasing function of a: inequalities are greater in more speculative economies.

9. A similar mechanism of taxation is studied numerically in [3] with analogous conclusions.

after $n = 100,000$ transactions. Even for very small values of b the Gini coefficient reduces drastically. Nevertheless, even if we consider very high tax rates, the Gini coefficient of the economy seems to be stable around at 0.1, and never converges toward the egalitarian situation.

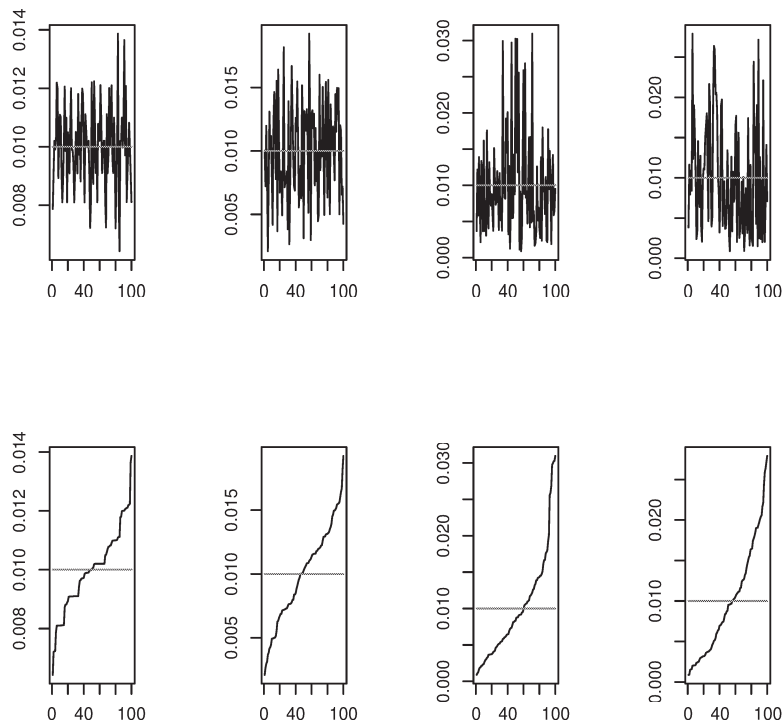**Figure 4**: *The first line (resp. the second line) represents the distribution of the wealth (resp. the distribution of the increasing rearrangement of wealth) of the $N = 100$ players after $n = 100$ (first column), $n = 1,000$ (second column), $n = 10,000$ (third column) and $n = 100,000$ (fourth column) FEG starting from uniformly distributed initial wealth with $a = 0.1$ and $b = 0.01$.*
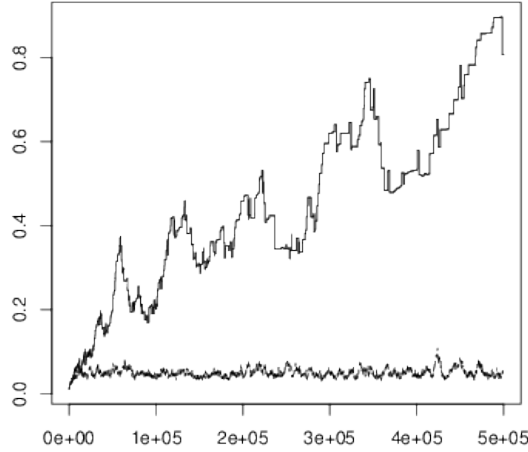
Figure 5: *Proportion of wealth owned at each stage by the richest player in the $N = 100$ players and $n = 100,000$ transactions game. The top curve corresponds to the untaxed case and lower curve to a tax rate of 1%.*
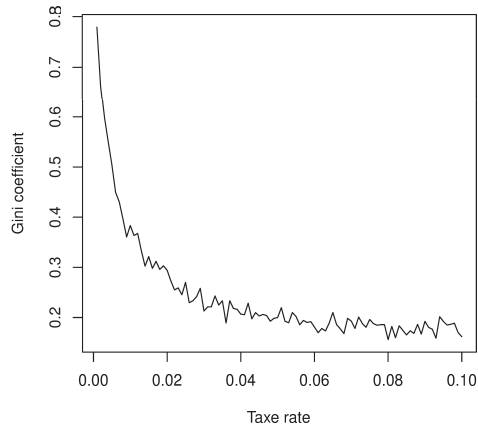
Figure 6: *Dependence of the Gini coefficient on the tax rate b in a population of $N = 100$ players after $n = 100000$ transactions.*

To conclude this section, we examine the economic mobility implied by the model with or without taxes to capture the capacity of individuals to move across the wealth distribution through time. Accordingly, we consider two mobility indicators. First we represent in Figure 7 the rank of the wealthiest agent in the population, then, we measure in Figure 8 the Pearson correlation function $C(t)$ of the rank of the players: $\forall t > 5000$ [10]

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10. Since our model starts from a situation where all the players own the same proportion of the wealth, we first run 5000 transactions and then compute the Pearson correlation function $C(t)$ of the rank of the players at time $t > 5000$ with respect to the ranks observed at time $t = 5000$.

$$C(t) = \frac{\sum\limits_{i=1}^{N} (R_i(t) - \frac{N}{2})(R_i(5000) - \frac{N}{2})}{\sqrt{\sum\limits_{i=1}^{N} (R_i(t) - \frac{N}{2})^2 \sum\limits_{i=1}^{N} (R_i(5000) - \frac{N}{2})^2}}$$

where $R_i(t)$ is the rank of the i-th player at time t.
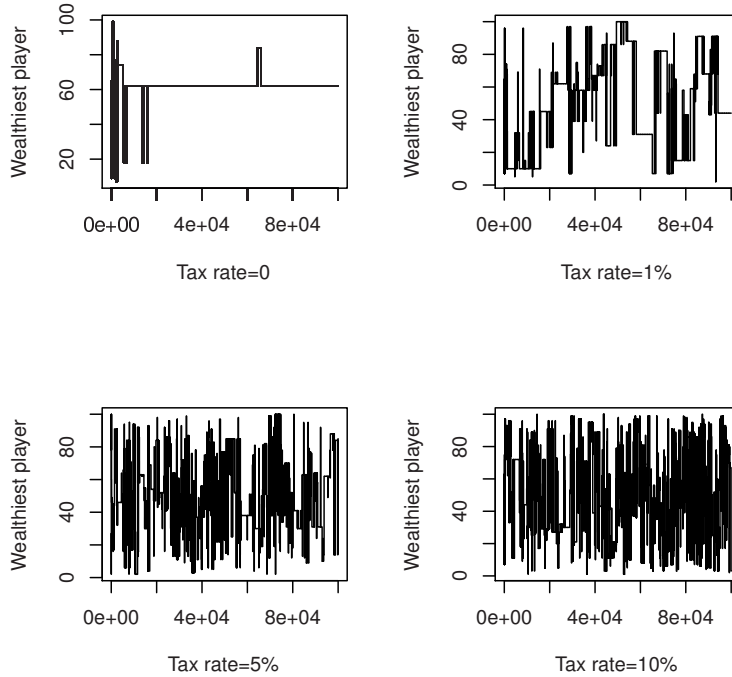
Figure 7: *Identification of the wealthiest player in the $N = 100$ players and $n = 100,000$ transactions game with and without capital taxes.*

When the capital tax rate increases, we observe that richest individuals tend not to remain the richest ones over time. Concentration of wealth and mobility are intrinsically linked in our approach. As already numerically observed in [26], we can conclude that economic mobility is decreasing with inequality and increasing with the tax rate.

In the next part we start the mathematical study of elementary market games with proportional bets [11]. This is done to recover theoretically the extreme concentration phenomena observed empirically.

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11. The same elementary study has been provided for the Yard-Sale model of Example 1 in a companion paper [14] that is interesting in itself because it allows some classical results of Econophysics to be recovered using a purely probabilistic approach.

Figure 8: *Pearson correlation functions as a function of t corresponding (from the top to the bottom) to a tax rate b of 0, 1, 5 and 10%. A strong correlation coefficient over time corresponds to a lack of economic mobility.*
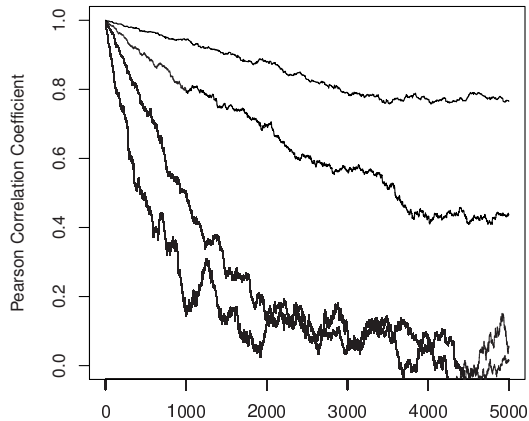
3 Mathematical study of elementary market games with proportional bets: the two players case

3.1 Elementary market games with proportional bets

Let us consider the repeated version of the elementary market game with proportional bets described in Example 2. If we denote by X_n^i the wealth of player i after n transactions, we have $X_n^1 + X_n^2 = 1$ (zero-sum game) and

$$X_{n+1}^i = X_n^i + a(1 - X_n^i)\mathbf{1}_{U_{n+1} \leqslant X_n^i} - aX_n^i\mathbf{1}_{U_{n+1} > X_n^i}$$

where $(U_k)_{k \in \mathbb{N}^*}$ is a sample of the uniform distribution on $[0, 1]$. The sequence $(X_n^i)_{n \in \mathbb{N}}$ is a Markov chain with

$$\mathbb{E}[X_{n+1}^i \mid X_n^i] = (X_n^i + a(1 - X_n^i))X_n^i + (X_n^i - aX_n^i)(1 - X_n^i) = X_n^i.$$

Thus $(X_n^i)_{n \in \mathbb{N}}$ is a non-negative and bounded martingale that converges almost-surely and in L^p $(1 \leqslant p < \infty)$ toward a random variable X_∞^i that is invariant with respect to the transition probability of the chain given by

$$P(x, dy) = x\varepsilon_{x+a(1-x)}(dy) + (1 - x)\varepsilon_{x-ax}(dy)$$

where $\varepsilon_\alpha(dy)$ is the Dirac mass at the point $\alpha \in \mathbb{R}$. Passing to the limit in the relation

$$\mathbb{E}[(X_{n+1}^i - X_n^i)^2 \mid X_n^i] = X_n^i a^2(1 - X_n^i)^2 + (1 - X_n^i)a^2(X_n^i)^2 = a^2 X_n^i(1 - X_n^i)$$

we deduce that $X_\infty^i \in \{0, 1\}$ and from $\mathbb{E}[X_\infty^i \mid X_n^i] = X_n^i$ that X_∞^i follows a Bernoulli distribution of parameter X_0^i.

After an infinite number of transactions, one player concentrates all the wealth as empirically observed in the numerical exercise of Section 2. Nevertheless it is easy to see that it is not possible for one player to be ruined after a finite number of rounds because the random variable X_n^i ranges across the interval $[(1-a)^n X_0^i, 1+(1-a)^n(X_0^i-1)]$. If we want to obtain theoretical approximations of times of near-bankruptcy, analytic computations quickly become prohibitive. In the next part, we give a precise answer to this question, at the very least, in the case of small and high-frequency transactions.

Remark 2: *In the repeated version of the elementary Calabash game described in Appendix A, the situation is different, due to the fact that bets are discrete. Let X_n^i be the wealth of player i after n transactions and N_n^i the number of seeds bet by player i at stage n with $0 < N_n^i \leqslant X_n^i$ if $X_n^i \neq 0$ and $N_n^i = 0$ otherwise (the game is finished when one of the player is ruined). We have $X_n^1 + X_n^2 = N$ (zero-sum game) and*

$$X_n^1 = X_0^1 + \sum_{1 \leqslant n \leqslant k} P_j^1$$

where

$$P_n^1 = N_n^2 \mathbf{1}_{U_n \leqslant \frac{N_n^1}{N_n^1 + N_n^2}} - N_n^1 \mathbf{1}_{U_n > \frac{N_n^1}{N_n^1 + N_n^2}}$$

and where $(U_n)_{n \in \mathbb{N}^*}$ is a sample of the uniform distribution on $[0,1]$. If we denote by $(\mathcal{F}_n)_{n \in \mathbb{N}^*}$ the filtration generated by the $(U_n)_{n \in \mathbb{N}^*}$, supposing that the processes $(N_n^i)_{n \in \mathbb{N}^*}$ and $(N_n^2)_{n \in \mathbb{N}^*}$ are predictable, the process $(X_n^1)_{n \in \mathbb{N}}$ is a bounded martingale fulfilling

$$\mathbb{E}[(X_{n+1}^1 - X_n^1)^2 \mid \mathcal{F}_n] = N_{n+1}^1 N_{n+1}^2.$$

Thus, in this case, one of the players is almost-surely ruined in finite time.

3.2 Elementary market games with proportional bets: Continuous time case

In this section we prove that the sequence of stochastic processes obtained from the preceding Markov chains by transforming the time scales and state spaces appropriately weakly converges to a diffusion process, the latter being more amenable to analysis [12].

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded mapping. For all $a \in \mathbb{R}_+$ and $x \in]0,1[$ we define the generator A_a of the elementary market game with parameter a:

$$A_a[f](x) = \mathbb{E}[f(X_1^1) - f(X_0^1) \mid X_0^1 = x] = xf(x + a(1-x)) + (1-x)f(x - ax) - f(x).$$

In particular, when f is of class C^∞ with a compact support in the interval $]0,1[$, we obtain from Taylor expansion that $\frac{1}{a^2} A_a f$ uniformly converges toward $\frac{1}{2}x(1-x)f''(x)$ when a goes to 0.

Considering the process $(Z_t^a)_{t \in \mathbb{R}_+}$ that is the rescaled (at frequency a^2) continuous time linear interpolation of the sequence $(X_n^1)_{n \in \mathbb{N}}$ with $X_0^1 = x$:

$$\begin{aligned} Z_{na^2}^a &= X_n^1 \quad \forall n \geqslant 0 \\ Z_{(n+\theta)a^2}^a &= Z_{na^2}^a + \theta(Z_{(n+1)a^2}^a - Z_{na^2}^a) \quad \theta \in [0,1] \quad \forall n \geqslant 0, \end{aligned}$$

we obtain from classical arguments (see for example [51] Chap. 11) the uniform weak convergence of $(Z_t^a)_{t \in \mathbb{R}_+}$ when a goes to 0 toward the diffusion process $(X_t)_{t \in \mathbb{R}_+}$, associated to the infinitesimal generator

$$A[f](x) = \frac{1}{2}x(1-x)f''(x), \tag{1}$$

that is the unique strong solution of the Stochastic differential equation

$$dX_t = \sqrt{X_t(1-X_t)}dB_t \qquad 0 < X_0 < 1 \tag{2}$$

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12. The convergence result we obtain in this section is in some sense an answer to the following remark of Sheng in [49] p. 493: *"Then there arise some difficulties in the conversion problem, which place some restrictions on the choice of the size and number of bets."*

where B_t is a standard Brownian motion [13].

The diffusion process (2) is known in mathematical genetics as the Wright-Fisher process. It is often encountered as diffusion approximation of classical discrete stochastic models used in population genetics (see [18] Chap. 7 or [19] Chap. 3) [14]. To our knowledge, only few applications of this process (also known as Jacobi diffusion) exist in the economic or financial literature. Notable exceptions are: [15] to model interest rates; [35] in the framework of exchange rate modeling, [28] that use the Wright-Fisher diffusion' multi-dimensional extension (see Appendix B) to capture the dynamics of discrete transition probabilities; and [42] who proves that the distribution of market weights is related to Wright-Fisher diffusions in the volatility-stabilized market models of [23] (see also [24] for a survey concerning stochastic portfolio theory). One of the interesting aspects of the present paper, is to provide a new and simple Markov chain approximation of (2) in the framework of wealth distribution problems.

The points 0 and 1 are absorbing since the constant processes 0 and 1 are solutions of (2). The process $(X_t)_{t\in\mathbb{R}_+}$ is then a continuous and uniformly integrable martingale that converges almost-surely toward 1 with probability X_0 and toward 0 with probability $(1 - X_0)$. The mapping

$$u(x) = -2[(1 - x)\log(1 - x) + x \log x]$$

being null at the boundary of $[0, 1]$ and fulfilling $Au = -1$ on $]0, 1[$, we obtain from the Dynkin's formula (see [17] Chap. 13) that

$$u(x) = \mathbb{E}[T \mid X_0 = x]$$

where T is the hitting time of the boundary $\{0, 1\}$. Thus, T is in L^1 and so almost-surely finite. In particular starting from $X_0 = \frac{1}{2}$, the mean hitting time is $2 \log 2$. [15]

Remark 3: *Using the same approach, we can even show that e^T is in L^1. In fact, the mapping $h(x) = x(1 - x)$ fulfills $Ah = -h$, thus defining $\xi = h + u + 1$ we have $A\xi + h + 1 = 0$ and $\lim \xi = 1$ at the boundary of $[0, 1]$. From the proof of Th. 13.17 in [17] we deduce that*

$$\xi(x) = \mathbb{E}_x[\exp\{\int_0^T \frac{1 + h(X_s)}{\xi(X_s)}\}ds].$$

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13. This result of convergence requires the existence and the unicity of the martingale problem associated to the generator A that is equivalent to the weak existence and the unicity in distribution of the solution of the associated stochastic differential equation. Here, (2) having Hölderian coefficients of order $\frac{1}{2}$, from the Yamada-Watanabe theorem (see [47] p. 360) we even deduce the strong existence and unicity of the solution of (2).

14. If we consider a fixed population of size N (representing for example genes) with individuals that can be of two different types (two alleles), the simplest neutral Wright-Fisher model of evolution assumes that generation $(k + 1)$ is formed from generation k by choosing N genes at random with replacement. If we denote by Y_n^N the number of individuals of type 1 in generation n, we have

$$\mathbb{P}(Y_{n+1}^N = i | Y_n^N = j) = C_N^i \left(\frac{j}{N}\right)^i \left(1 - \frac{j}{N}\right)^{N-i}$$

and the process $X_t^{(N)} = \frac{1}{N} Y_{[tN]}^{(N)}$ weakly converges toward the Wright-Fisher diffusion (see [29]). Moreover, in this case, the fixation time corresponding to the disappearance of type 2 individuals is finite almost surely (contrary to what happens for the ruin time in elementary games with proportional bets).

15. For the Yard-Sale model of Example 1, a similar study is performed in [14]. Nevertheless, in this case, the mean hitting time of the associated diffusion approximation is almost surely infinite: the concentration phenomena are slower in this model. This difference between the two models may be explained by the fact that, in its simplest version, the Yard-Sale model has a symmetric payoff contrary to elementary market games with proportional bets where a leverage effect (third order moment of the payoff) tends to favor the richer player.

The result follows because the continuous mapping $\frac{1+h(x)}{1+u(x)+h(x)}$ is bounded below (its minimum is reached at $\frac{1}{2}$ and is equal to $\frac{5}{5+8log(2)}$).

In the Markov chain case, it is not possible for one player to be ruined after a finite number of rounds. When we pass to the limit, the stepsize of the chain is shortened proportionally to a^2 that goes to zero. If for small $\varepsilon > 0$, T_ε denotes the Markov chain exit time from $[\varepsilon, 1-\varepsilon]$, when a is small enough, T_ε should [16] be of order $u(x)/a^2$ that is a decreasing function of a (the ruin time is smaller for more speculative games). This intuition is confirmed by the simulations presented in Table 1.

a	"Theoritical" ruin time	Empirical ruin time
0.1	139	144
0.08	216	218
0.05	552	529
0.03	1533	1464
0.01	13900	13294

Table 1: *Theoretical and empirical ruin times for elementary market games with proportional bets: the theoretical ruin time corresponds to the Wright-Fisher approximate $\frac{2log(2)}{a^2}$ while the empirical one is obtained as the average first exit time from $[\varepsilon, 1-\varepsilon]$ obtained for $1,000$ independent runs of the Markov chain game with $X_0^1 = X_0^2 = 0.5$.*

4   The buffering effect of a tax rate in the economy

In this section we study the impact of a small tax rate on the dynamics of the preceding Markov chain. One of the simplest hypotheses is to consider a proportional capital tax rate that is collected at any stage and uniformly reallocated to the players. With a tax rate b (b fulfilling $0 < a + b < 1$), the transition of the Markov chain becomes

$$X_{n+1}^i = (1-b)X_n^i + a(1-X_n^i)\mathbf{1}_{U_{n+1} \leqslant X_n^i} - aX_n^i\mathbf{1}_{U_{n+1}>X_n^i} + \frac{b}{2}$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sample of the uniform distribution on the unit interval [17]. This game remains a zero-sum game that is not a FEG in general. In fact, we deduce easily from the preceding dynamics that

$$\mathbb{E}[X_{n+1}^1 - X_n^1] = -b(1-b)^n(X_0^1 - \frac{1}{2}).$$

Thus the game is FEG if and only if $X_0^1 = \frac{1}{2}$ (uniform initial wealth) when $b > 0$ and favors the poorest player at any step [18] for different initial endowments. The state space of this Markov chain

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16. This intuition is reinforced from the following theoretical result: if T_ε^a denotes the exit time of the process Z^a from $[\varepsilon, 1-\varepsilon]$, T_ε^a converges in distribution when a goes to zero toward the corresponding diffusion exit time ([21] Problem 3, Chap. 10).

17. Another natural choice for the redistribution mechanism is to impose the following dynamics

$$X_{n+1}^i = (1-ba)X_n^i + a(1-X_n^i)\mathbf{1}_{U_{n+1} \leqslant X_n^i} - aX_n^i\mathbf{1}_{U_{n+1}>X_n^i} + \frac{ba}{2},$$

in this case the parameter b may be interpreted as a proportional transaction cost. All the results of this section remain valid taking $b = \lambda a$ instead of $b = \lambda a^2$.

18. Nevertheless, it may be seen as asymptotically FEG because $\mathbb{E}[X_{n+1}^1 - X_n^1] \to 0$.

being compact there exists at least one invariant distribution that is not a priori unique. [19]

In spite of its simplicity, it is a priori difficult to obtain explicitly one invariant measure of such a Markov chain when $b \neq 0$ [20]. Therefore, we study the diffusion limit of the model for small and high-frequency transactions. Following a similar approach as in Section 3.2, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable and bounded mapping, $\forall a \in \mathbb{R}_+$ and $x \in]0, 1[$ the generator A_a of the elementary taxed market game with parameters a and b becomes

$$A_a[f](x) = \mathbb{E}[f(X_1^1) - f(X_0^1) \mid X_0^1 = x] = xf((1-b)x + a(1-x) + \frac{b}{2}) + (1-x)f((1-b)x - ax + \frac{b}{2}) - f(x).$$

In particular, when f is of class C^∞ with a compact support in the interval $]0, 1[$, we obtain from Taylor expansion that $\frac{1}{a^2} A_a f$ uniformly converges toward

$$A[f](x) = \frac{1}{2}x(1 - x)f'' + \frac{\lambda}{2}(1 - 2x)f'$$

when a goes to 0 and $b = \lambda a^2$. The infinitesimal generator A is associated to the diffusion

$$dX_t = \sqrt{X_t(1 - X_t)}dB_t + \frac{\lambda}{2}(1 - 2X_t)dt \qquad 0 < X_0 < 1 \tag{3}$$

where B_t is a standard Brownian motion. This diffusion process is classically known (see [18] Chap. 7.2) as the one dimensional Wright-Fisher diffusion with mutations, the mutation rates being identical (equal to $\frac{\lambda}{2}$) for the two alleles [21]. Here, the mutation rate is simply interpreted as a proportional capital tax rate that will counter the natural tendency of wealth to concentrate at the top and will ensure economic diversity in the same way that the mutation rate shapes genetic diversity. [22] In fact, if we are interested in the players' ruin problem (equivalent to know if the points 0 or 1 are accessible for the Wright-Fisher diffusion with mutations) remind that, for an infinitesimal generator of the form

$$A[f](x) = \frac{1}{2}x(1 - x)f'' + b(x)f',$$

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19. The proof of the unicity of such a stationary distribution is not crucial for our purpose because this distribution won't be explicitly known. Moreover, any of these possible invariant distributions will be well approximated, for small and high-frequency transactions, by the unique invariant stationary distribution of the Wright-Fisher diffusion with mutations (see the proof of the Th. 2.2 of [21] p. 418).

20. The same holds for the Wright-Fisher Markov chain with mutations.

21. In order to obtain at the limit the Wright-Fisher diffusion with different mutation rates

$$dX_t = \sqrt{X_t(1 - X_t)}dB_t + \frac{\lambda_2}{2}(1 - X_t)dt - \frac{\lambda_1}{2}X_t dt$$

we simply have to consider two different tax rates $b_i = \lambda_i a^2$ for the players.

22. In [25], the authors study the characteristics of the wealth accumulation process as the result of agents' optimal consumption-savings decisions when investors are faced with an uninsurable idiosyncratic investment risk represented by independent geometric Brownian motions with the same coefficients. In particular, for the two players case, the proportion X_t^1 of the total wealth in the economy held by the first player at time t is given by the following diffusion process

$$dX_t^1 = \Gamma^2 X_t^1(1 - X_t^1)(1 - 2X_t^1)dt + \sqrt{2}\Gamma X_t^1(1 - X_t^1)dB_t$$

where Γ is the instantaneous standard deviation of his return. Interestingly, this diffusion is once again a classical model of population genetics known as the Karlin model (see for example [31] Eq. (37)) with a stabilizing drift toward $\frac{1}{2}$ sufficiently strong to make $\{0, 1\}$ inaccessible. In this case, X^1 oscillates back and forth between the boundaries but it takes a long time to move from the neighborhood of one boundary to the other. With respect to our study, we are in a mixed situation where wealth concentration occurs during relatively long periods but is tempered by mobility effects. Here, contrary to our untaxed case, the economy never collapses even if the social mobility induced by the drift of the diffusion process remains smaller than the one observed in the presence of the capital tax rate.

the scale function is given by

$$s(x) = \int_{1/2}^{x} \exp[-\int_{1/2}^{y} \frac{2b(z)}{z(1-z)} dz] dy.$$

Thus, the process being recurrent (0 or 1 are inaccessible) if and only if $s(0+) = -\infty$ and $s(1-) = +\infty$ (see [47] Ex. 3.21 p. 298), for the Wright-Fisher diffusion with mutations no players are ruined in finite time as long as $\lambda > 1$. When it is strictly greater than 1, the tax rate λ induces a diversity in the economy similar to the mutation rate impact in the genetic mixing. For the Markov chain game, this condition implies $b > a^2$. For example, when $a = 10\%$ the only condition to stabilize the economy is to impose a tax rate strictly greater than the realistic value of 1%.

Remark 4: *Interestingly, it is easy to see in our economy simplified to the extreme that a tax on the income yields a completely different asymptotic result from a tax on the capital owned. In fact, in the Markov chain game, the rate b (and so λ) may be seen as a capital tax rate because a fixed proportion of the wealth is collected. If we consider instead an income tax uniformly redistributed among players, we obtain the following FEG*

$$X_{n+1}^i = X_n^i + a(1 - \frac{b}{2})(1 - X_n^i)\mathbf{1}_{U_{n+1} \leqslant X_n^i} - a(1 - \frac{b}{2})X_n^i \mathbf{1}_{U_{n+1} > X_n^i}.$$

Here, we simply recover the untaxed dynamics with changed parameters, thus, when b is a constant, the diffusion limit is related to the Wright-Fisher diffusion without mutations up to the factor $(1 - \frac{b}{2})^2$:

$$dX_t = (1 - \frac{b}{2})^2 \sqrt{X_t(1 - X_t)} dB_t.$$

In particular, the parameter b is not sufficient to prevent the convergence toward the maximal inequality case and its only effect is to slow down the ruin. In fact, when T_b is the hitting time of the boundary $\{0, 1\}$ for the associated diffusion, we have by analogy with the untaxed case

$$\mathbb{E}[T_b \mid X_0 = x] = -\frac{2}{(1 - \frac{b}{2})^2}[(1 - x)\log(1 - x) + x\log x].$$

In [45] where players follow optimal consumption-bequest plans the same distinction has been pointed out between capital and income taxes with similar conclusions.

In the presence of tax $(\lambda > 0)$[23], looking for an invariant measure m on $]0, 1[$ fulfilling

$$\int_0^1 A[f](x)g(x)dm(x) = \int_0^1 A[g](x)f(x)dm(x)$$

for f and g of class C^∞ with a compact support in the interval $]0, 1[$, we find a Beta probability distribution

$$m(dx) = \frac{[x(1-x)]^{(\lambda-1)}}{\beta(\lambda, \lambda)} dx$$

that is symmetric with respect to the uniform initial wealth case $x = \frac{1}{2}$.

If $\lambda > 1$, there is a strong restoring force in the direction of $x = \frac{1}{2}$ because the tax rate is high, if $\lambda = 1$, the Lebesgue measure on $[0, 1]$ is invariant and if $0 < \lambda < 1$, the restoring force is

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23. When $\lambda < 0$ (meaning that a fixed tax is collected and redistributed proportionally to the wealth of the players), we can prove following [43] that the limit of the continuous time wealth process almost surely belongs to the vertices of the simplex.

partially offset, the density of the invariant measure approaches infinity near the boundaries 0 and 1 remaining of finite mass. In all these cases, the Wright-Fisher diffusion is ergodic and converges in distribution toward the invariant probability measure. In Figure 9, we have represented the invariant density functions of the Wright-Fisher diffusion with mutations for different values of the tax rate λ to compare them with the empirical distribution of $X_{100,000}^1$ in the Markov chain market game with $b = \lambda a^2$. In both cases we start from a maximal inequality case ($\lambda = 0$) to be concentrated more and more around $x = \frac{1}{2}$ when λ increases.
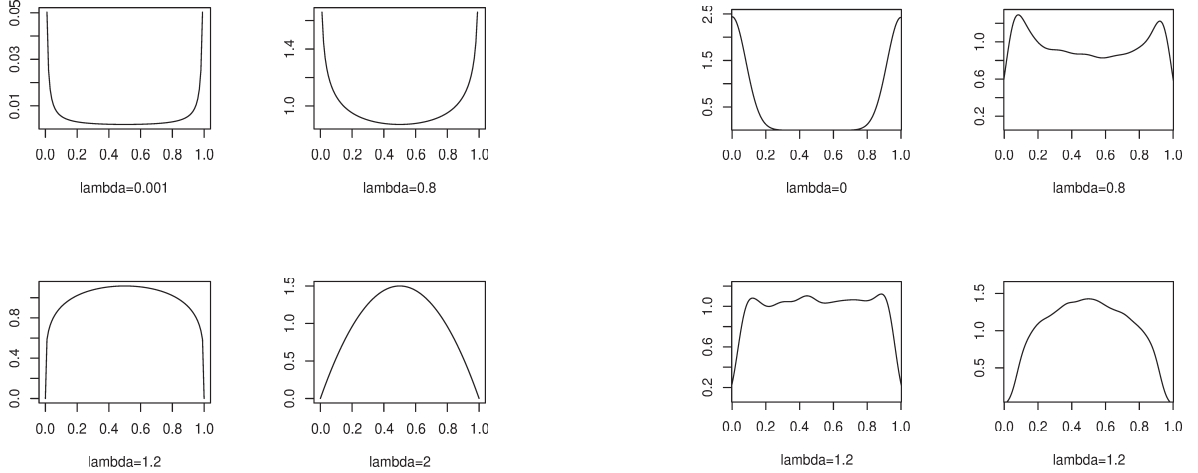

Figure 9: *Invariant density function of the Wright-Fisher diffusion with mutations for different values of λ (left part) and empirical distribution (using 1000 independent Monte Carlo simulations) of X_{100000}^1 in the Markov chain market game with parameters $a = 0.1$ and $b = \lambda a^2$ (right part).*

To complete the analogy, between the Markov chain market game and the limiting Beta distribution, we compare their associated Gini coefficients and small wealth probabilities (that is the probability that at least one player owns less than a fixed small ε). For this purpose, remind that (see [46]) the Gini coefficient associated with the Beta(λ, λ) distribution is given by

$$\text{Gini}(\lambda) = \frac{2\beta(2\lambda, 2\lambda)}{\lambda\beta(\lambda, \lambda)^2}.$$

Moreover, at stationarity, the small wealth probability fulfills

$$m\left(\{X_t^1 \leqslant \varepsilon\} \cup \{X_t^2 \leqslant \varepsilon\}\right) \underset{0^+}{\sim} \frac{2\varepsilon^\lambda}{\beta(\lambda, \lambda)\lambda} = p(\lambda).$$

In particular, when $\varepsilon < \frac{1}{4}$, from the properties of the Beta function (see [1] Chap. 6), we can prove [24] that p is naturally a decreasing function of λ (the small wealth probability decreases when the tax

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24. If we denote by ψ_n the polygamma function of order n (see [1] Chap. 6), we have $log(p(\lambda))' = log(\varepsilon) + g(\lambda)$ where

$$g(\lambda) = -2(\psi_0(\lambda) - \psi_0(2\lambda)) - \frac{1}{\lambda} = \psi_0(\lambda + \frac{1}{2}) - \psi_0(\lambda + 1) + 2log(2).$$

Since $g'(\lambda) = \psi_1(\lambda + \frac{1}{2}) - \psi_1(\lambda + 1)$ we can see that g is an increasing function (ψ_1 being a decreasing one) bounded by $2log(2)$. Thus, if $\varepsilon < \frac{1}{4}$, p is strictly decreasing. From $\beta(\lambda, \lambda) \underset{0^+}{\sim} \frac{2}{\lambda}$ and $\beta(\lambda, \lambda) \underset{+\infty}{\sim} \frac{2\sqrt{\pi}}{\sqrt{\lambda}}$ we deduce easily that

$$\lim_{\lambda \to 0} p(\lambda) = 1 \text{ and } \lim_{\lambda \to \infty} p(\lambda) = 0.$$

rate increases) and that, in accordance with intuition,

$$\lim_{\lambda \to 0} p(\lambda) = 1 \text{ and } \lim_{\lambda \to \infty} p(\lambda) = 0,$$

when λ is small, one player concentrates, with a probability of one, an arbitrary large proportion of the wealth while sufficiently strong capital taxation almost surely prevents this concentration from occuring. In Table 2 (resp. Table 3) we compare, for different values of λ, the Gini coefficient (resp. small wealth probability) obtained from the Beta(λ, λ) distribution and the empirical Gini coefficient (resp. empirical small wealth probability) obtained from $1,000$ independent realizations of $X^1_{100,000}$ in the Markov chain market game with $b = \lambda a^2$: the results are close together.

λ	Theoretical Gini coefficient	Empirical Gini coefficient
1	0.33	0.32
2	0.26	0.25
5	0.17	0.18
8	0.14	0.14
10	0.12	0.12

Table 2: *Comparison between the Gini coefficient obtained theoretically from the Beta(λ, λ) distribution and the empirical Gini coefficient obtained from 1000 independent realizations of X^1_{100000} in the Markov chain market game with $a = 0.1$ and $b = \lambda a^2$.*

λ	Theoretical small wealth probability	Empirical small wealth probability
0.1	0.81	0.85
0.5	0.4	0.44
1	0.2	0.23
2	0.06	0.05
3	0.02	0

Table 3: *Comparison between the small wealth probability ($\varepsilon = 0.1$) obtained theoretically from the Beta(λ, λ) distribution and the empirical small wealth probability obtained from $1,000$ independent realizations of $X^1_{100,000}$ in the Markov chain market game with $a = 0.1$ and $b = \lambda a^2$.*

To conclude this section, let us remark that the Beta distribution ([46]) and some of its generalizations (see [40] or [41]) have been widely used in the literature as descriptive models for the size distribution of income and are interesting alternatives to Pareto-like distributions. Here we find a very simple agent-based model for understanding how it can naturally appear in wealth distributions problems, through fair elementary games, and we see that the parameter of the obtained symmetric (about $\frac{1}{2}$) Beta distribution is related to the underlying capital tax rate.

In the technical Appendix B, we extend the results of Sections 3 and 4 to the case of N players with similar conclusions concerning the extreme concentration process without redistribution and concerning the impact of proportional capital taxes leading to an invariant Dirichlet distribution [25].

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25. These results are postponed to the end of the document to make it easier to read and because the proofs for N players follow along the same lines as before.

5 Conclusion

We have shown that a simplified economy of N agents, who exchange by zero-sum games fair in expectation, converges in the almost surely on the situation where a single agent concentrates all the wealth confirming that social inequalities may be driven primarily by chance, rather than by differential abilities. The mathematical study can be pushed more accurately by considering the limit diffusion process obtained for small and frequent transactions, and we recover at the limit some classical models used in population genetics: the Wright-Fisher diffusions.

We also prove, in our framework, that the presence of a tax on owned capital prevents the convergence to extreme inequality even for a low tax level. The economy converges in this case to a random situation which mixes the respective fortunes of the agents. Surprisingly, when income taxes are considered, the dynamics is drastically different: a tax on the income only slows down the dynamics towards the extreme inequality.

Let us mention finally that this study can be extended thanks to other classical mathematical tools from population genetics which provide an interesting insight, namely the passage to an infinite population represented by a measure [26]. This gives a new economic interpretation of the so-called Fleming-Viot process ([19]) that achieves a kind of zoom on the situation where only a few agents are not yet ruined.

Acknowledgments

Appendix A: Elementary market games with proportional bets may be seen as randomized versions of the Calabash game (see [11], p. 57)

In its simplest form, this traditional African game also known as the gourd game takes place between two players. Each player uses seeds of a certain color and all the seeds have an identical form. At each turn, each player puts into the gourd as many seeds as he/she wants. The gourd is a sort of large hollowed-out melon, with care being taken to leave a stem inside on which a single seed can sit. The gourd is shaken until one of the seeds comes to rest on this stem and the player of the corresponding color collects all the seeds in the gourd. After this, players exchange the seeds so as to keep the same color. For a one stage game between two players, if we denote by N_1^i as the number of seeds bet by player i $(0 < N_1^i \leqslant X_0^i)$, we have

$$P_1^1 = N_1^2 \mathbf{1}_{U_1 \leqslant \frac{N_1^1}{N_1^1 + N_1^2}} - N_1^1 \mathbf{1}_{U_1 > \frac{N_1^1}{N_1^1 + N_1^2}}$$

where U_1 follows a uniform distribution on $[0, 1]$. The game is a FEG. Even if strategic interactions are fundamental in this framework, we can prove easily (see Remark 2) that one of the players is almost-surely ruined in finite time. In the following, we prove that elementary market games with proportional bets naturally appear as limits in the Calabash game when particular random strategies are used by each player.

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26. Such a representation has been used as soon as in 1982 by G. Debreu for another problem in [16] p. 125 et seq.

Let us take in the preceding framework, $X_0^1(N) = [p_N N]$, $X_0^2(N) = N - X_0^1(N)$ (where $[x]$ denotes the integer part of x and where $(p_N)_{N\in\mathbb{N}^*}$ is a sequence in $[0,1]$ that converges toward $p \in]0,1[$) and

$$N_1^i(N) = 1 + \sum_{k=1}^{X_0^i(N)-1} H_k^i$$

where the $(H_k^i)_{(i,k)\in\{1,2\}\times\mathbb{N}^*}$ are independent random variables such that H_k^i follows a Bernoulli distribution of parameter $\frac{aX_0^i(N)-1}{X_0^i(N)-1}$. Thus, we obtain a one stage calabash game where the players randomly select the number of seeds with bets that are proportional in expectation to their initial wealth because $\mathbb{E}[N_1^i(N)] = aX_0^i(N)$. From

$$X_1^1(N) = X_0^1(N) + N_1^2(N)\mathbf{1}_{U_1 \leqslant \frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)}} - N_1^1(N)\mathbf{1}_{U_1 > \frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)}},$$

if we suppose that the sequence $(H_k^i)_{(i,k)\in\{1,2\}\times\mathbb{N}^*}$ is independent of U_1 we have $\forall t \in \mathbb{R}$

$$\mathbb{E}\left[e^{it\frac{X_1^1(N)}{N}}\right] = e^{it\frac{X_0^1(N)}{N}}\mathbb{E}\left[e^{it\frac{N_1^2(N)}{N}}\frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)} + e^{-it\frac{N_1^1(N)}{N}}\frac{N_1^2(N)}{N_1^1(N)+N_1^2(N)}\right].$$

Since

$$\sum_{k=1}^{\infty}\frac{Var[H_k^i]}{k} < \infty \text{ and } \frac{1}{N}\mathbb{E}\left[\sum_{k=1}^{N}H_k^i\right] \to a,$$

we deduce from the Kolmogorov's strong law of large numbers (see [48], Th. 2.3.10) that $\frac{1}{N}\sum_{k=1}^{N}H_k^i$ converges almost surely toward a and that $\frac{N_1^1(N)}{N}$ (resp. $\frac{N_1^2(N)}{N}$) converges almost surely toward ap (resp. $a(1-p)$). Thus, by the dominated convergence theorem

$$\mathbb{E}\left[e^{it\frac{X_1^1(N)}{N}}\right] \to e^{itp(1-a)}(1-p) + e^{it(a(1-p)+p)}p$$

and $(\frac{X_0^1(N)}{N}, \frac{X_1^1(N)}{N})$ converges in distribution toward the elementary market game with proportional bets of Example 2.

Appendix B: N players games

The aim of this mathematical appendix is to extend the study of the preceding dynamics (with or without tax) to the N players case. This can be achieved via different transaction mechanisms but we assume here that one player plays against all his/her opponents at each stage and we consider a proportional tax on capital uniformly distributed among all the players. In other words, if we denote by $X_n = (X_n^1, ..., X_n^N)$ the vector of wealth after n rounds we have $\forall n \in \mathbb{N}$, $\forall i \in \{1,...,N\}$,

$$X_{n+1} = (1 - (a+b))X_n + ae_i + \frac{b}{N} \text{ with probability } X_n^i$$

with $a + b < 1$ and where e_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere. It is easy to see by induction that starting from a point X_0 in the simplex

$$\left\{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid \sum_{i=1}^{N} x_i = 1, x_i \geqslant 0\right\},$$

$(X_n)_{n\in\mathbb{N}}$ stays in the simplex.

Now we study the asymptotic behavior of the vectorial Markov chain $(X_n)_{n\in\mathbb{N}}$ with or without tax and we present only the main lines that follow the 2 players case.

The case without tax, b = 0

The vector $(X_n)_{n\in\mathbb{N}}$ is once again a bounded martingale that converges almost surely and in L^p $1 \leqslant p < +\infty$, toward X_∞ that is invariant with respect to the transition of the chain. From

$$\mathbb{E}[|\ X_{n+1} - X_n\ |^2|\ X_n] = \sum_{i=1}^{N} X_n^i [\sum_{j \neq i} a^2 X_n^j + a^2(1 - X_n^i)^2] = a^2(1 - \sum_{i=1}^{N}(X_n^i)^2)$$

we deduce that X_∞ almost surely belongs to the vertices of the simplex and that $X_\infty = e_i$ with probability X_0^i.

To study the ruin times of the players, we consider the game with small and high frequency transactions: If F is a function from $\mathbb{R}^N$ into $\mathbb{R}$ of class C^∞ with compact support we have

$$A_a[F](x) = \mathbb{E}[F(X_1) - F(X_0) \mid X_0 = (x_1, ..., x_N)] = \sum_{i=1}^{N} x^i F((1 - a)X_0 + ae_i) - F(X_0).$$

Using the Taylor formula we prove that $\frac{1}{a^2}A_a[F](x)$ uniformly converges, when a goes to 0, toward

$$A[F](x) = \frac{1}{2}\sum_{i=1}^{N} x^i(1 - x^i)F''_{i^2}(x) - \sum_{i<j} x^i x^j F''_{ij}(x) = \frac{1}{2}\sum_{i=1}^{N} x^i(1 - x^i)F''_{i^2}(x) - \frac{1}{2}\sum_{i\neq j} x^i x^j F''_{ij}(x).$$

Letting $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, the infinitesimal generator A becomes

$$A[F](x) = \frac{1}{2}\sum_{i,j=1}^{N} x^i(\delta_{ij} - x^j)F''_{ij}(x).$$

This generator is classically associated to the N-allele Wright-Fisher diffusion $(X_t)_{t\in\mathbb{R}_+}$ (see [18] Chap. 8) and the rescaled continuous time linear interpolation at frequency a^2 of the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution toward $(X_t = (X_t^1, ..., X_t^N))_{t\in\mathbb{R}_+}$ fulfilling $\forall i \in \{1, ..., N\}$

$$dX_t^i = X_t^i\sqrt{X_t^1}dB_t^1 + X_t^i\sqrt{X_t^2}dB_t^2 + \cdots + (X_t^i - 1)\sqrt{X_t^i}dB_t^i + \cdots + X_t^i\sqrt{X_t^N}dB_t^N$$

where $(B^1, ..., B^N)$ are independent standard Brownian motions. [27]

Following [38], we remark that the mapping u_N defined on the simplex by

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27. Classically this convergence holds when the limit diffusion has a unique weak solution. For $N = 2$, we have seen that the result is a simple consequence of the Yamada-Watanabe theorem (see [47] p. 360) for one dimensional diffusions. For the general case, this stochastic differential equation with bounded coefficients has a weak solution (see [34], Th. 2.2, Chap. 4). For the unicity, we can do an induction reasoning on the dimension N because in the interior of the simplex the coefficients are C^1 and Lipschitz (classical conditions for strong existence and unicity) and because the faces of the simplex are absorbing sets for the diffusion with almost-surely finite hitting times (see also [20]).

$$u_N(x) = -2\left[\sum_{i=1}^{N}\varphi(x^i) - \sum_{i<j}\varphi(x^i + x^j) + \sum_{i<j<k}\varphi(x^i + x^j + x^k) - \cdots \right.$$

$$\left. \cdots + (-1)^N \sum_{j_1<j_2<\cdots<j_{N-1}}\varphi(x^{j_1} + x^{j_2} + \cdots + x^{j_{N-1}})\right]$$

where $\varphi(x) = x\log x$, vanishes on the faces of the simplex and fulfills $Au_N = -1$ on its interior [28]. Thus, if T_N denotes the first hitting time of the faces of the simplex (that is the time when a first player is ruined),

$$\mathbb{E}[T_N \mid X_0 = (x_1, ..., x_N)] = u_N(x).$$

In particular, starting from the uniform situation $X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})$ we have

$$\mathbb{E}[T_N \mid X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})] = u_N(X_0) = 2\sum_{k=1}^{N-1}(-1)^k C_N^k \varphi(\frac{k}{N})$$

and this quantity converges toward 0 when N goes to infinity [29]. In the same way, the mapping $w_N(x) = -2\sum_{i=1}^{N}(1 - x^i)\log(1 - x^i)$ also fulfills $Aw_N = -1$ on the simplex except for the vertices and is zero on the vertices. Thus if S_N denotes the first hitting time of the vertices (that is the time when all the players except one are ruined), S_N is almost-surely finite and we have

$$\mathbb{E}[S_N \mid X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})] = -2N(1 - \frac{1}{N})\log(1 - \frac{1}{N}) \to 2.$$

Remark 5: *Using the mapping $h_N(x) = \sum_{i=1}^{N} x^i(1 - x^i)$, we can prove in the spirit of Section 3.2 that S_N has a finite exponential moment.*

The case b $\neq$ 0

Supposing that $b = \lambda a^2 > 0$, we can prove that the generator A_a associated to the N players game with proportional bet and tax converges toward

$$A_\lambda[F](x) = \frac{1}{2}\sum_{i,j=1}^{N} x^i(\delta_{ij} - x^j)F_{ij}''(x) + \sum_{i=1}^{N}\frac{\lambda}{N}(1 - Nx_i)F_i'(x)$$

that is the infinitesimal generator associated to the N-allele Wright-Fisher diffusion with a uniform mutation rate of $\frac{\lambda}{N}$ (see [18] p. 314). The unique invariant probability measure of the associated diffusion is given by the following Dirichlet distribution:

$$m_N(dx) = \frac{\Gamma(2\lambda)}{\Gamma(\frac{2\lambda}{N})^N}\prod_{i=1}^{N}x_i^{(\frac{2\lambda}{N}-1)}dx,$$

---

28. If $N = 2$, $u_2(x) = -2[x^1\log(x^1) + x^2\log(x^2)]$ and we recover the function u of Section 3.2 because $x_1 + x_2 = 1$.

29. If $\varphi = \sum_{p\in\mathbb{Z}}a_p e^{2i\pi px}$ is the Fourier series representation of a the function φ we have $u_N(X_0) = 2\sum_{p\in\mathbb{Z}}a_p[(1 - e^{2i\pi\frac{p}{N}})^N - 1] \to -\sum_p a_p$. But φ being of bounded variation, using the Dirichlet-Jordan test ([33]), we obtain $\sum_p a_p = \varphi(0) = 0$.

in particular, the marginal distributions are $\text{Beta}\left(\frac{2\lambda}{N}, \frac{2(N-1)\lambda}{N}\right)$ distributions and the small wealth probability of any agent fulfills

$$m_N\left(\{X_t^1 \leqslant \varepsilon\}\right) \underset{0^+}{\sim} \frac{N\varepsilon^{\frac{2\lambda}{N}}}{2\beta\left(\frac{2\lambda}{N}, \frac{2(N-1)\lambda}{N}\right)\lambda} = p_N(\lambda).$$

Thus, we can prove that (see [1] Chap. 6 and Footnote 21)

$$p_N(\lambda) \underset{N\to\infty}{\to} 1, \quad \lim_{\lambda\to 0} p(\lambda) = \frac{N-1}{N} \quad \text{and} \quad \lim_{\lambda\to\infty} p(\lambda) = 0.$$

Remark 6: *It will not have escaped the attentive reader that the N-players game studied in Appendix B is slight different than the one empirically analyzed in Section 2 where the players have pairwise interactions. In its taxed version, this Markov chain game may be written in the following way:*

$$X_{n+1} = (1-b)X_n + \frac{b}{N} + \sum_{k\in S_N} Y_{n+1}^{k_1,k_2}\, \mathbf{1}_{V_{n+1}=(k_1,k_2)}$$

with

$$S_N = \left\{k = (k_1, k_2) \in \mathbb{N}^2 \mid N \geqslant k_1 > k_2 \geqslant 1\right\}$$

and

$$Y_{n+1}^{k_1,k_2} = a\, X_n^{k_2}\mathbf{1}_{U_{n+1}^{(k_1,k_2)} \leqslant \frac{X_n^{k_1}}{X_n^{k_1}+X_n^{k_2}}}\, (e_{k_1} - e_{k_2}) + a\, X_n^{k_1}\mathbf{1}_{U_{n+1}^{(k_1,k_2)} > \frac{X_n^{k_1}}{X_n^{k_1}+X_n^{k_2}}}\, (e_{k_2} - e_{k_1})$$

where $(V_n, U_n^k)_{(n,k)\in\mathbb{N}^*\times S_N}$ are independent random variables such that V_n (resp. U_n^k) follows a uniform distribution on S_N (resp. $[0,1]$). Supposing once again that $b = \lambda a^2 > 0$, we can prove that the associated generator converges toward

$$\tilde{A}_\lambda[F](x) = \frac{2}{N(N-1)}A_{\frac{\lambda N(N-1)}{2}}[F](x)$$

that is once again of the Wright-Fisher type. Thus contrary to what happens in the Yard-Sale model (see [14]), in elementary market games with proportional bets, global or pairwise interactions give rise to the same type of dynamics.

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