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Uncertain Logical Gates in Possibilistic Networks: Theory and application to human geography[☆]

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Abstract

Possibilistic networks offer a qualitative approach for modeling epistemic uncertainty. Their practical implementation requires the specification of conditional possibility tables, as in the case of Bayesian networks for probabilities. The elicitation of probability tables by experts is made much easier by means of noisy logical gates that enable multidimensional tables to be constructed from the knowledge of a few parameters. This paper presents the possibilistic counterparts of usual noisy connectives (and, or, max, min, ...). Their interest and limitations are illustrated on an example taken from a human geography modeling problem. The difference of behavior between probabilistic and possibilistic connectives is discussed in detail. Results in this paper may be useful to bring possibilistic networks closer to applications.

Keywords: possibility theory, belief networks, noisy gates, expert knowledge, human geography

1. Introduction

A belief network is a convenient way of representing the interaction between uncertain variables in the form of a directed graph, each node of which represents a variable. The graphical structure takes advantage of known conditional independence between these variables. Each variable is directly influenced only by its parent variables in the graph. Given such a directed graph between variables and local conditional probability tables, the joint probability distribution of these variables can be retrieved; see [20] for an introduction to Bayesian belief networks. In fact they can be built in two ways: they can be extracted from data

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or made up by a human expert. In the first case, a supposedly large dataset involving a number of variables is available, and the Bayesian network is obtained by some machine learning procedure. The probability tables thus obtained have a frequentist flavor, and the simplest network possible is searched for. On the contrary, when Bayesian networks can be specified using expert knowledge, the structure of a network relating the variables is first given, often relying on causal connections between variables and conditional independence relations the expert is aware of. Then, subjective probability tables must be filled by the expert. They consist, for each variable in the network, of conditional probabilities for that variable, conditioned on each configuration of its parent variables. Note that, even if causal relations as perceived by the expert are instrumental in building a simple and interpretable network, the joint probability distribution obtained by combining the probability tables no longer accounts for causality. Another difficulty arises for subjective expert-based Bayes networks: if variables are not binary and/or the number of parent variables is more than two, the task of eliciting numerical probability tables becomes tedious, if not impossible to fulfill. Indeed, the number of probability values to be supplied increases exponentially with the number of parent variables.

To alleviate the elicitation task, the notion of noisy logical gate (or connective) has been introduced, based on the assumption of independent causal influences that can be combined. As a result, one small conditional probability table is elicited per parent variable, and the probability table of each variable given its parents is obtained by combining these small tables via a so-called noisy connective [10, 19], which may include a so-called leakage factor summarizing the causal effect of variables not explicitly present in the network.

While the notion of noisy connective solves the combinatorial problem of collecting many probability values to a large extent, the issue remains that people cannot always provide precise probability assessments. Let alone the fact that the probability scale is too fine-grained for human perception of belief or frequencies, some conditional probability values may be ill-known or plainly unknown to the experts. The usual Bayesian recommendation in the latter case is to use uniform distributions, but it is well-known (see for instance [15, 16]) that these distributions do not properly model ignorance. Alternatively, one may use imprecise probability networks (called credal networks) [21], qualitative Bayesian networks [23] or possibilistic networks [5]. While the two first options extend probabilistic networks to ill-known parameters (with an interval-based approach for the first extension and an ordinal approach for the second), possibilistic networks represent a more drastic departure from probabilistic networks. In their qualitative version, possibilistic networks can be defined on a finite chain of possibility values and do not refer to numerical values. This feature may make the collection of expert information on conditional tables easier than requiring precise numbers obeying the laws of probability. However, when it comes to filling conditional uncertainty tables, the dimensionality issue present in Bayesian networks remains the same in the possibilistic environment.

This is why in this paper, we propose possibilistic counterparts of noisy connectives of probabilistic networks. As possibilistic uncertainty is merely epistemic and due to a lack of information, we shall speak of *uncertain connectives*. The idea of possibilistic uncertain gates was first considered empirically by Parson and Bigham [22] directly in the setting of possibilistic logic, at a time when possibilistic networks had not yet been introduced. It seems that the question of possibilistic uncertain gates has not been reconsidered ever since, if we except a recent study in the broader setting of imprecise probabilities [1]. The basic ideas pervading this paper first appear in a French conference paper by the authors [8], then more formally in the SUM conference proceedings [11].

This paper elaborates on these preliminary versions. In particular, we explain the construction of possibilistic gates in greater detail. Moreover, we introduce the leaky version of several such gates, as well as variants needed for describing the reinforcement of the possibility of effects due to the presence of multiple causes. A comparison between probabilistic (noisy) gates and possibilistic gates is carried out, emphasizing their difference in terms of expressive power and respective concerns. Lastly, an extensive account of the application to human geography is provided.

The paper is structured as follows. After recalling probabilistic networks with noisy gates in Section 2, we present the corresponding approach for possibilistic networks and present various uncertain gates, especially the AND, OR, MAX, and MIN functions in Section 3. In Section 4, we compare the uncertain OR-gate and the noisy OR-gate in detail, and propose a variant of the uncertain MAX that behaves more in agreement with the noisy MAX. Algorithms needed to implement this approach are discussed in Section 5. Finally, the approach, including algorithmic issues, is illustrated in Section 6 on a belief network dealing with an application to human geography.

2. Probabilistic Networks with Independent Causal Influences

Consider a set of independent variables $X_1, \ldots, X_n$ that influence the value of a variable Y. In the ideal case, there is a deterministic function f such that $Y = f(X_1, X_2, \ldots, X_n)$. In order to account for uncertainty, one may assume the existence of intermediary variables $Z_1, \ldots, Z_n$, such that Z_i expresses the fact that X_i will have a causal influence on Y, and which value of Y it enforces (Z_i has the same range as Y). It is assumed that the relation between X_i and Z_i is probabilistic and that Z_i is independent of other variables given X_i. Besides, we consider the deterministic function as affected by the auxiliary variables Z_i only. In other words, we get a probabilistic network such that

$$P(Y, Z_1, \ldots, Z_n, X_1, \ldots, X_n) = P(Y, Z_1, \ldots, Z_n) \cdot \prod_{i=1}^{n} P(Z_i \mid X_i), \qquad (1)$$

where $P(Y, Z_1, \ldots, Z_n) = 1$ if $Y = f(Z_1, \ldots, Z_n)$ and 0 otherwise. This is called a *noisy function*. In particular, notice that the dependence tables between Y and $X_1, \ldots, X_n$ can now be obtained by combining simple conditional probability distributions pertaining to single factors. For any effect value y of Y, and every n-tuple $(x_1, \ldots, x_n)$ of input values :

$$P(y \mid x_1, \ldots, x_n) = \sum_{z_1,\ldots,z_n : y = f(z_1,\ldots,z_n)} \prod_{i=1}^{n} P(z_i \mid x_i). \tag{2}$$

This is the assumption of *independence of causal influence* (ICI) [10]. In the case of Boolean variables, it is assumed that $P(Z_i = 0 \mid X_i = 0) = 1$ (if no cause, then no effect), while $P(Z_i = 0 \mid X_i = 1)$ could be positive (the effect may or may not appear when the cause is present).

Canonical ICI models are obtained by means of specific choices of functions f. For instance, if all variables are Boolean, f will be a logical connective. In this case, we speak of noisy OR ($f = \vee$), noisy AND ($f = \wedge$); if the range of the Z_i's and Y is a totally ordered set, usual gates are the noisy MAX ($f = \max$), or MIN ($f = \min$).

The approach may be further refined by allowing f to summarize the potential effect of external variables not taken into account: this is the *leaky model*. Then, Y also depends on a leakage variable Z_L not explicitly related to identified causes, i.e., $Y = f(Z_1, Z_2, \ldots, Z_n, Z_L)$. The range of Z_L is supposed to be the range of f, i.e., the range of Y and this variable is independent of the other ones. Hence, the leaky model may be written as:

$$P(Y, Z_1, \ldots, Z_n, Z_L, \mathbf{x}) = P(Y, Z_1, \ldots, Z_n) \cdot P(Z_L) \cdot \prod_{i=1}^{n} P(Z_i \mid X_i),$$

so that for any value y of Y and any configuration $(x_1, \ldots, x_n)$ of parent variables:

$$P(y \mid x_1, \ldots, x_n) = \sum_{z_1,\ldots,z_n,z_L : y = f(z_1,\ldots,z_n,z_L)} P(z_L) \cdot \prod_{i=1}^{n} P(z_i \mid x_i). \tag{3}$$

For instance, in the case of Boolean variables, $P(Y = 1 \mid X_1 = 0, \ldots, X_n = 0)$ may be positive due to such external causes.

We will now turn to the question whether the same kind of ICI approach can be used to elicit possibilistic networks as well.

3. Uncertain logical gates in Canonical Possibilistic Networks

Possibility theory [12, 26] is based on maxitive set functions associated to possibility distributions. Formally, given a universe of discourse U, a possibility distribution $\pi : U \to [0, 1]$ pertains to a variable X ranging on U and represents the available (incomplete) information about the more or less possible values of X, assumed to be single-valued. Thus, $\pi(u) = 0$ means that $X = u$ is impossible. The consistency of information is expressed by the normalization of π : $\exists u \in U, \pi(u) = 1$, namely, at least one value is fully possible for X. Distinct values u and u' may be simultaneously possible at degree 1. A state of complete ignorance is represented by the distribution $\pi_?(u) = 1, \forall u \in U$. The degree of possibility of an event $A \subseteq U$ is defined by the set function

$$\Pi(A) = \sup_{u \in A} \pi(u)$$

called a *possibility measure*. Possibility measures are maxitive, i.e.,

$$\forall A, \forall B, \Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

The underlying assumption is that the agent focuses on the most plausible values compatible with event A, neglecting other ones. A dual *measure of necessity* $N(A) = 1 - \Pi(U \setminus A)$ expresses the degree of certainty of event A as the degree of impossibility of non-A.

A possibilistic network [5] has the same structure as a Bayesian network. The joint possibility for n variables linked by an acyclic directed graph is defined by the *chain rule*:

$$\pi(x_1, \ldots, x_n) = *_{i=1,\ldots,n} \pi(x_i \mid pa(X_i)),$$

where x_i is an instantiation of the variable X_i, and $pa(X_i)$ an instantiation of the parent variables of X_i. The operation $*$ is generally chosen as the minimum (in the qualitative case) [3], or the product (in the numerical case) [6], and this is what we shall assume in the sequel. Note that the behavior of product-based possibilistic nets is very close to the one of Bayes nets, while min-based possibilistic networks have specific properties. For instance, starting from possibilistic conditional tables, and building the joint possibility distribution using the chain rule, one cannot generally retrieve the same conditional tables, due to the drowning effect of the min operation [5].

3.1. Uncertain causal functions in possibilistic networks

Deterministic models $Y = f(X_1, \ldots, X_n)$ are defined like in the probabilistic case:

$$\pi(y \mid x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } y = f(x_1, \ldots x_n); \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Note that if $y = f(x_1, \ldots x_n)$, then $\pi(y \mid x_1, \ldots, x_n) = 1$ indicates the certainty of y because other values of Y are treated as impossible since f is a function.

Let us define possibilistic models with *independent causal influences* (ICI). We use a deterministic function $Y = f(Z_1, \ldots, Z_n)$ with n intermediary causal variables Z_i, as for the probabilistic models, which indicate that the cause X_i has produced its effect. Now, $\pi(y \mid x_1, \ldots, x_n)$ is of the form:

$$\pi(y \mid z_1, \ldots, z_n) * \pi(z_1, \ldots, z_n \mid x_1, \ldots, x_n),$$

where $\pi(y \mid z_1, \ldots, z_n)$ obeys Equation 4. Again, each variable Z_i only depends (in an uncertain way) on the variable X_i. Thus, we have $\pi(z_1, \ldots, z_n \mid x_1, \ldots, x_n) = *_{i=1,\ldots,n} \pi(z_i \mid x_i)$. This leads to the equality

$$\pi(y \mid x_1, \ldots, x_n) = \max_{z_1, \ldots, z_n : y = f(z_1, \ldots, z_n)} *_{i=1,\ldots,n} \pi(z_i \mid x_i), \tag{5}$$

whose similarity with Eq. 2 is striking. Notice that, when $* = \min$, Eq. 5 boils down to applying the extension principle [26] to function f, assuming fuzzy-valued inputs $F_1, \ldots, F_n$, where the membership function of F_i is defined by $\mu_{F_i}(z_i) = \pi(z_i \mid x_i)$.

In case we suppose that y also depends in an uncertain way on other causes summarized by a leakage variable Z_L, giving birth to a leaky ICI model, we then get the counterpart of Eq. 3, which reads:

$$\pi(y \mid x_1, \ldots, x_n) = \max_{z_1,\ldots,z_n,z_L : y = f(z_1,\ldots,z_n,z_L)} *_{i=1,\ldots,n} \pi(z_i \mid x_i) * \pi(z_L). \qquad (6)$$

In the following, we provide a detailed analysis of possibilistic counterparts of noisy gates.

3.2. Uncertain OR

The variables are assumed to be Boolean (i.e., $Y = y$ or $\neg y$, etc.). The uncertain OR (counterpart of the probabilistic “noisy OR”) assumes that $X_i = x_i$ for at least one variable X_i represents a sufficient cause for getting $Y = y$, and $Z_i = z_i$ indicates that $X_i = x_i$ has caused $Y = y$. This gives $f(Z_1, \ldots, Z_n) = \bigvee_{i=1}^{n} Z_i$. The uncertainty indicates that the causes may fail to produce their effects. $Z_i = \neg z_i$ indicates that $X_i = x_i$ did not cause $Y = y$ due to the presence of some inhibitor that prevents the effect from taking place. We assume it is more possible that $X_i = x_i$ causes $Y = y$ than the opposite (otherwise one could not say that $X_i = x_i$ is sufficient for causing $Y = y$). Then we must define $\pi(z_i \mid x_i) = 1$ and $\pi(\neg z_i \mid x_i) = \kappa_i < 1$. Besides, $\pi(z_i \mid \neg x_i) = 0$, since when X_i is absent, it does not cause y. Hence the elementary causal possibility table, where each column should contain 1, to get normal conditional possibility distributions:

$\pi(Z_i \mid X_i)$	x_i	$\neg x_i$
z_i	1	0
$\neg z_i$	κ_i	1

Table 1: Elementary causal possibility table

Note that in the case of a probabilistic network, $\pi(z_i \mid x_i) = 1$ is replaced by $P(z_i \mid x_i) = 1 - \kappa_i$ in Table 1.

Let $\mathbf{x}$ be a configuration of $(X_1, \ldots, X_n)$, where $\mathbf{x}_i$ denotes a literal (x_i or $\neg x_i$) for X_i (and the same convention for Z_i). We can then obtain the table of the conditional possibility distribution $\pi(Y \mid X_1, \ldots, X_n)$ by means of Eq. 5.

$$\begin{aligned}
\pi(y \mid \mathbf{x}) &= \max_{Z_1,\ldots,Z_n : Z_1 \vee \cdots \vee Z_n = y} *_{i=1}^{n} \pi(Z_i \mid \mathbf{x}_i) \\
&= \max_{i=1}^{n} \pi(z_i \mid \mathbf{x}_i) * (*_{j \neq i} \max(\pi(z_j \mid \mathbf{x}_j), \pi(\neg z_j \mid \mathbf{x}_j)) \\
&= \max_{i=1}^{n} \pi(z_i \mid \mathbf{x}_i); \\
\pi(\neg y \mid \mathbf{x}) &= \max_{Z_1,\ldots,Z_n : Z_1 \vee \cdots \vee Z_n = \neg y} *_{i=1}^{n} \pi(Z_i \mid \mathbf{x}_i) \\
&= \pi(\neg z_1 \mid \mathbf{x}_1) * \cdots * \pi(\neg z_n \mid \mathbf{x}_n).
\end{aligned}$$

Note that in the second line of the computation of $\pi(y \mid \mathbf{x})$, one must enforce $Z_i = y$ for one variable Z_i, while other variables take arbitrary values (we have n possible choices of Z_i). Of course $\max(\pi(z_j \mid \mathbf{x}_j), \pi(\neg z_j \mid \mathbf{x}_j)) = 1$ due to normalisation. Besides, in the computation of $\pi(\neg y \mid \mathbf{x})$, the condition $Z_1 \vee \cdots \vee Z_n = \neg y$ can be obtained for sure only if $pa(Y) = (\neg z_1, \ldots, \neg z_n)$.

Let $I_+(\mathbf{x}) = \{i : X_i = x_i\}$ and $I_-(\mathbf{x}) = \{i : X_i = \neg x_i\}$. Then, if the above causal elementary possibility table is adopted, we get:

- $\pi(\neg y \mid \mathbf{x}) = *_{i=1,\ldots,n} \pi(\neg z_i \mid \mathbf{x}_i) = *_{i \in I_+(\mathbf{x})} \kappa_i$;
- $\pi(y \mid \mathbf{x}) = 1$ when $\mathbf{x} \neq (\neg x_1, \ldots, \neg x_n)$ (since the term $\pi(z_i \mid x_i) = 1$ appears for some i in $\max_{i=1}^{n} \pi(z_i \mid \mathbf{x}_i)$);
- $\pi(\neg y \mid \neg x_1, \ldots, \neg x_n) = 1$, $\pi(y \mid \neg x_1, \ldots, \neg x_n) = 0$: $\neg y$ (no effect) can be obtained for sure only if all the causes are absent.

For $n = 2$, this gives the conditional tables:

$\pi(y \mid X_1 X_2)$	x_1	$\neg x_1$
x_2	1	1
$\neg x_2$	1	0

$\pi(\neg y \mid X_1 X_2)$	x_1	$\neg x_1$
x_2	$\kappa_1 * \kappa_2$	κ_2
$\neg x_2$	κ_1	1

Table 2: Uncertain OR for 2 inputs

More generally, if there are n causes, we have to provide the values of n parameters κ_i.

For the uncertain leaky OR, we now assume that the function f takes the form $f(Z_1, \ldots, Z_n) = \bigvee_{i=1}^{n} Z_i \vee Z_L$, where Z_L is an unknown external cause. We assign $\pi(z_L) = \kappa_L < 1$ (hence, $\pi(\neg z_L) = 1$) considering that z_L is not a usual cause. We thus obtain

- $\pi(\neg y \mid \mathbf{x}) = *_{i=1,\ldots,n} \pi(\neg z_i \mid \mathbf{x}_i) * \pi(\neg z_L) = *_{i \in I_+(\mathbf{x})} \kappa_i$;
- $\pi(y \mid \mathbf{x}) = 1$, if $\mathbf{x} \neq (\neg x_1, \ldots, \neg x_n)$;
- $\pi(\neg y \mid \neg x_1, \ldots, \neg x_n) = 1$;

- $\pi(y \mid \neg x_1, \dots, \neg x_n) = \kappa_L$ (even if the causes x_i are absent, there is still a possibility for having $Y = y$, namely if the external cause is present).

Indeed, we get (letting $\neg x = (\neg x_1, \dots, \neg x_n)$),

$$\begin{aligned}\pi(y \mid \neg x_1, \dots, \neg x_n) &= \max(\pi(y \mid \neg x, z_L) * \pi(z_L), \pi(y \mid \neg x, \neg z_L) * \pi(\neg z_L))) \\ &= \max(1 * \kappa_L, 0 * 1) = \kappa_L.\end{aligned}$$

For $n = 2$, the conditional table becomes:

$\pi(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	1	1
$\neg x_2$	1	κ_L

$\pi(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$\kappa_1 * \kappa_2$	κ_2
$\neg x_2$	κ_1	1

Table 3: The leaky uncertain OR for 2 inputs

The only 0 entry has been replaced by the leakage coefficient. For n causes, we have now to provide the values of $n + 1$ parameters κ_i.

3.3. Uncertain AND

Let us consider Boolean variables ($Y = y$ or $\neg y$, etc.). The uncertain AND (counterpart of the probabilistic "noisy AND") uses the same local conditional tables but it assumes that $X_i = x_i$ represents a *necessary* cause for $Y = y$. We again build the conditional possibility tables $\pi(Y \mid X_1, \dots, X_n)$ by means of Eq. 5 using $f(Z_1, \dots, Z_n) = \bigwedge_{i=1}^n Z_i$ instead. This is the De Morgan dual to the uncertain OR gate:

$$\begin{aligned}\pi(y \mid \mathbf{x}) &= \max_{Z_1,\dots,Z_n : Z_1 \wedge \dots \wedge Z_n = y} *_{i=1}^n \pi(Z_i \mid \mathbf{x}_i) \\ &= \pi(z_1 \mid \mathbf{x}_1) * \dots * \pi(z_n \mid \mathbf{x}_n); \\ \pi(\neg y \mid \mathbf{x}) &= \max_{Z_1,\dots,Z_n : Z_1 \wedge \dots \wedge Z_n = \neg y} *_{i=1}^n \pi(z_i \mid \mathbf{x}_i) \\ &= \max_{i=1}^n \pi(\neg z_i \mid \mathbf{x}_i) * (*_{j \neq i} \max(\pi(z_j \mid \mathbf{x}_j), \pi(\neg z_j \mid \mathbf{x}_j)) \\ &= \max_{i=1}^n \pi(\neg z_i \mid \mathbf{x}_i).\end{aligned}$$

We notice that $\bigwedge_{i=1}^n Z_i = y$ can be obtained only if $pa(Y) = (z_1, \dots, z_n)$. Thus, we find

- $\pi(\neg y \mid x_1, \dots, x_n) = \max_{i=1}^n \pi(\neg z_i \mid x_i) = \max_{i=1}^n \kappa_i$;
- $\pi(y \mid x_1, \dots, x_n) = 1$;
- $\pi(\neg y \mid \mathbf{x}) = 1, \pi(y \mid \mathbf{x}) = 0$ if $\mathbf{x} \neq (x_1, \dots, x_n)$ (if at least one of the causes is absent, the effect is necessarily absent).

$\pi(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	1	0
$\neg x_2$	0	0

$\pi(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$\max(\kappa_1, \kappa_2)$	1
$\neg x_2$	1	1

Table 4: Uncertain AND for 2 inputs

For $n = 2$, Eq. 5 yields the conditional tables for the uncertain AND:

More generally, if there are n causes, we have to assess n values for the parameters κ_i.

The case of the uncertain AND with leak corresponds to the possibility $\pi(z_L) = \kappa_L < 1$ that an external factor $Z_L = z_L$ causes $Y = y$ independently of the values of the X_i. Namely $f(Z_1, \ldots, Z_n, Z_L) = (\bigwedge_{i=1}^n Z_i) \vee Z_L$. For $n = 2$, Eq. 5 then gives the combined conditional possibility in Table 5, similar to Table 3. The difference is that the leakage coefficient appears in three entries of the matrix for y, as the effect is then given a chance to appear when the two causes are not simultaneously present.

$\pi(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	1	κ_L
$\neg x_2$	κ_L	κ_L

$\pi(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$\max(\kappa_1, \kappa_2)$	1
$\neg x_2$	1	1

Table 5: Leaky uncertain AND for 2 inputs

3.4. Uncertain MAX

The uncertain MAX is a multiple-valued extension of the uncertain OR, where the output variable Y (hence the variables Z_i) is valued on a finite, totally ordered, severity or intensity scale $L = \{0 < 1 < \cdots < m\}$. We assume that $Y = \max(Z_1, \ldots, Z_n)$. The statement $Z_i = z_i \in L$ represents the fact that X_i alone has increased the value of Y at level z_i. In this subsection, y, z_i denote any values in L, and x_i any value in the range of X_i. The elementary conditional possibility distributions $\pi(y \mid x_i)$ are supposed to be given. We can then compute the conditional tables $\pi(y \mid \mathbf{x})$ where $\mathbf{x} = (x_1, \ldots, x_n)$, as:

$$\begin{aligned}
\pi(y \mid \mathbf{x}) &= \max_{(z_1,\ldots,z_n) \in L^n : y = \max(z_1,\ldots,z_n)} *_{i=1}^n \pi(z_i \mid x_i) \\
&= \max_{i=1}^n \pi(Z_i = y \mid x_i) * (*_{j \neq i} \Pi(Z_j \leq y \mid x_j)) \,.
\end{aligned}$$

In a causal setting, we assume that $y = 0$ is a normal state (no effect), and $y > 0$ is more or less abnormal, $y = m$ being fully abnormal (strong effect). Suppose that the range of X_i is L as well. It is natural to assume that:

- if $X_i = j$ then $Z_i = j$ is completely possible, which means $\Pi(Z_i = j \mid X_i = j) = 1$;

- if $X_i = 0$ then $Z_i = 0$, which means $\Pi(Z_i \neq 0 \mid X_i = 0) = 0$ (no cause, no effect);
- $0 < \Pi(Z_i < j \mid X_i = j) < 1$ (a cause having strong intensity possibly induces an effect with weak severity, or may even have no effect at all, but this is abnormal);
- an effect with severity weaker than the intensity of a cause is all the less plausible as the effect is weaker. This leads to suppose the following inequalities:

$$\pi(Z_i = 0 \mid X_i = j) \leq \pi(Z_i = 1 \mid X_i = j) \leq \cdots \leq \pi(Z_i = j \mid X_i = j) = 1.$$

- an effect with severity higher than the intensity of a cause is all the less plausible as the effect is stronger. This leads to suppose the following inequalities:

$$\pi(Z_i = m \mid X_i = j) \leq \pi(Z_i = m{-}1 \mid X_i = j) \leq \cdots \leq \pi(Z_i = j \mid X_i = j) = 1.$$

This leads to state the elementary conditional table on the left-hand side of Table 6 (for 3 levels of strength 0, 1, 2).

$\pi(Z_i \mid X_i)$	$X_i = 2$	$X_i = 1$	$X_i = 0$
$Z_i = 2$	1	κ_i^{21}	0
$Z_i = 1$	κ_i^{12}	1	0
$Z_i = 0$	κ_i^{02}	κ_i^{01}	1

$\pi(Z_i \mid X_i)$	$X_i = 2$	$X_i = 0$
$Z_i = 2$	1	0
$Z_i = 1$	κ_i^{12}	0
$Z_i = 0$	κ_i^{02}	1

Table 6: Elementary conditional tables in the many-valued case

where $\kappa_i^{02} \leq \kappa_i^{12}$. In case we have m levels of strength, we have to assess $\frac{m(m+1)}{2} + \frac{m(m-1)}{2} = m^2$ coefficients. There are two interesting special cases:

- $\kappa_i^{21} = \Pi(Z_i > j \mid X_i = j) = 0$: if we assume that a cause having a weak intensity cannot induce an effect with strong severity;
- $\kappa_i^{21} = \kappa_i^{01} = 1$: if we remain in total ignorance of what a cause having a weak intensity can produce.

On the right-hand side is the corresponding table when the variables X_i are Boolean (then the middle column is dropped).

The global conditional possibility tables are then obtained by applying Eq. 5, using the values of $\pi(Z_i \mid X_i)$, as given in the above Table 6.

$$\pi(y|\mathbf{x}) = \max_{i=1}^{n} \pi(Z_i = y|x_i) * (*_{j \neq i} \Pi(Z_j \leq j|x_j)).$$

As above, in the case of the leaky uncertain max, we consider the output Y is of the form $\max(Z_1, \ldots, Z_n, Z_L)$ where Z_L is an unknown cause that may affect Y. The expression $\pi(y \mid \mathbf{x})$ is now expressed as

$$\begin{aligned}\pi(y \mid \mathbf{x}) &= \max_{(z_1,\ldots,z_n,z_L)\in L^{n+1}: y=\max(z_1,\ldots,z_n,z_L)} *_{i=1}^{n} \pi(z_i \mid x_i) * \pi(z_L) \\ &= \max\begin{cases}\max_{i=1}^{n} \pi(Z_i = y \mid x_i) * \Pi(Z_L \le y) * (*_{j\neq i}\Pi(Z_j \le y \mid x_j)), \\ \pi(Z_L = y) * (*_{i=1}^{n}\Pi(Z_i \le y \mid x_i))\end{cases}\end{aligned}$$

The possibility distribution for the leak variable is given by $m+1$ values $\pi_L(i) = \kappa_L^i$, where $\kappa_L^0 = 1$ (it is completely possible that the external cause has no effect on Y, and $\kappa_L^i \ge \kappa_L^{i+1}$ (it is all the more unlikely that the external cause is present as the observed effect is strong). Under these assumptions the above expressions simplify since $\Pi(Z_L \le y) = 1$.

For $n = 2$, $m = 2$, the conditional Table 7 is obtained when the X_i's are three-valued. Let us justify some expressions appearing in this table.[1]

- $$\pi(2 \mid 11) = \max\begin{cases}\pi(Z_1 = 2 \mid X_1 = 1) * \pi(Z_2 \le 2 \mid X_2 = 1), \\ \pi(Z_1 \le 2 \mid X_1 = 1) * \pi(Z_2 = 2 \mid X_2 = 1), \\ \kappa_L^2 * \pi(Z_1 \le 2 \mid X_1 = 1) * \pi(Z_2 \le 2 \mid X_2 = 1)\end{cases}$$
  $$= \max(\kappa_1^{21} * 1, 1 * \kappa_2^{21}, \kappa_L^2 * 1 * 1) = \max(\kappa_1^{21}, \kappa_2^{21}, \kappa_L^2)$$

- $$\pi(1 \mid 22) = \max\begin{cases}\pi(Z_1 = 1 \mid X_1 = 2) * \pi(Z_2 \le 1 \mid X_2 = 2), \\ \pi(Z_1 \le 1 \mid X_1 = 2) * \pi(Z_2 = 1 \mid X_2 = 2), \\ \kappa_L^1 * \pi(Z_1 \le 1 \mid X_1 = 2) * \pi(Z_2 \le 1 \mid X_2 = 2)\end{cases}$$
  $$= \max(\kappa_1^{12} * \kappa_2^{12}, \kappa_2^{12} * \kappa_1^{12}, \kappa_L^1 * \kappa_1^{12} * \kappa_2^{12}) = \kappa_1^{12} * \kappa_2^{12}$$

- $$\pi(1 \mid 21) = \max\begin{cases}\pi(Z_1 = 1 \mid X_1 = 2) * \pi(Z_2 \le 1 \mid X_2 = 1), \\ \pi(Z_1 \le 1 \mid X_1 = 2) * \pi(Z_2 = 1 \mid X_2 = 1), \\ \kappa_L^1 * \pi(Z_1 \le 1 \mid X_1 = 2) * \pi(Z_2 \le 1 \mid X_2 = 1)\end{cases}$$
  $$= \max(\kappa_1^{12} * 1, \kappa_1^{12} * 1, \kappa_L^1 * \kappa_1^{12} * 1) = \kappa_1^{12}$$

- $$\pi(y \mid 00) = \max\begin{cases}\pi(Z_1 = y \mid X_1 = 0) * \pi(Z_2 \le y \mid X_2 = 0), \\ \pi(Z_1 \le y \mid X_1 = 0) * \pi(Z_2 = y \mid X_2 = 0), \\ \kappa_L^y * \pi(Z_1 \le y \mid X_1 = 0) * \pi(Z_2 \le y \mid X_2 = 0)\end{cases}$$
  $$= \max(0 * 1, 1 * 0, \kappa_L^y * 1 * 1) = \kappa_L^y \text{ if } y > 0 \text{ and } 1 \text{ otherwise.}$$

Note that in general, we can expect the fact that the external cause is less likely to produce a strong effect than a regular cause, so that in column $\pi(2 \mid \mathbf{x})$, we may assume $\kappa_L^2 \le \min(\kappa_1^{21}, \kappa_2^{21})$ so that the leakage coefficient should only appear in the last line of Table 7.

[1] The expressions of $\pi(1 \mid 21)$ and $\pi(1 \mid 22)$ were erroneous in [8, 11] for the uncertain MAX, and are corrected here.

$\mathbf{x}$	$\pi(2 \mid \mathbf{x})$	$\pi(1 \mid \mathbf{x})$	$\pi(0 \mid \mathbf{x})$
$(2,2)$	1	$\kappa_1^{12} * \kappa_2^{12}$	$\kappa_1^{02} * \kappa_2^{02}$
$(2,1)$	1	κ_1^{12}	$\kappa_1^{02} * \kappa_2^{01}$
$(2,0)$	1	κ_1^{12}	κ_1^{02}
$(1,2)$	1	κ_2^{12}	$\kappa_1^{01} * \kappa_2^{02}$
$(1,1)$	$\max(\kappa_1^{21}, \kappa_2^{21}, \kappa_L^2)$	1	$\kappa_1^{01} * \kappa_2^{01}$
$(1,0)$	$\max(\kappa_1^{21}, \kappa_L^2)$	1	κ_1^{01}
$(0,2)$	1	κ_2^{12}	κ_2^{02}
$(0,1)$	$\max(\kappa_2^{21}, \kappa_L^2)$	1	κ_2^{01}
$(0,0)$	κ_L^2	κ_L^1	1

Table 7: Uncertain leaky MAX

$\mathbf{x}$	$\pi(2 \mid \mathbf{x})$	$\pi(1 \mid \mathbf{x})$	$\pi(0 \mid \mathbf{x})$
$(2,2)$	1	$\kappa_1^{12} * \kappa_2^{12}$	$\kappa_1^{02} * \kappa_2^{02}$
$(2,0)$	1	κ_1^{12}	κ_1^{02}
$(0,2)$	1	κ_2^{12}	κ_2^{02}
$(0,0)$	κ_L^2	κ_L^1	1

Table 8: Uncertain MAX with Boolean inputs

When the X_i's are Boolean, we get Table 8, where only 4 lines remain:

More generally, if we have m levels of strength, and n causal variables, we need nm^2 coefficients for defining the uncertain MAX. If we take into account the leak, we have to add $\frac{m(m+1)}{2}$ coefficients per variable, in order to replace the 0 by a leak coefficient in the conditional tables $\pi(Z_i \mid X_i)$ (assuming that an effect of strong severity may take place even if the causes present have a weak intensity).

3.5. Uncertain MIN

As for the uncertain MAX wrt uncertain OR, the uncertain MIN is a multiple-valued extension of the uncertain AND, where variables are valued on the intensity scale $L = \{0 < 1 < \cdots < m\}$. We assume that $Y = \max(\min(Z_1, \ldots, Z_n), Z_L)$, taking into account a leak variable. We can then compute the conditional tables, under the same assumptions as before, as

$$\pi(y \mid \mathbf{x}_1) = \max_{(z_1,\ldots,z_n,z_L) \in L^{n+1} : y = \max(\min(z_1,\ldots,z_n), z_L)} (*_{i=1}^n \pi(z_i \mid x_i)) * \pi(z_L)$$
$$= \max \begin{cases} \max_{i=1}^n \pi(Z_i = y \mid x_i) * \Pi(Z_L \leq y) * (*_{j \neq i} \Pi(Z_j \geq y \mid x_j)), \\ \pi(Z_L = y) * (\max_{i=1}^n \Pi(Z_i \leq y \mid x_i)) \end{cases}$$

The conditional possibility tables are thus obtained by applying Eq. 5, using the same values of $\pi(Z_i \mid X_i)$, and κ_L^y as in the case of the uncertain leaky

MAX. For $n = 2$, $m = 2$, this gives the following conditional Table 9 for ternary inputs: Note that the leakage coefficients are more present in the leaky MIN than in the leaky MAX, even if the leakage coefficients are small. This is not surprizing as it is enough to miss one of the two causes to fail the regular effect, and the external cause may then emerge as the reason for some unexpected effect in these more numerous situations. For binary inputs, it reduces to the conditional Table 10.

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12}, \kappa_2^{12})$	$\max(\kappa_1^{02}, \kappa_2^{02})$
$(2,1)$	$\max(\kappa_2^{21}, \kappa_L^2)$	1	$\max(\kappa_1^{02}, \kappa_2^{01})$
$(2,0)$	κ_L^2	$\kappa_1^{12} * \kappa_L^1$	1
$(1,2)$	$\max(\kappa_1^{21}, \kappa_L^2)$	1	$\max(\kappa_1^{01}, \kappa_2^{02})$
$(1,1)$	$\max(\kappa_1^{21} * \kappa_2^{21}, \kappa_L^2)$	1	$\max(\kappa_1^{01}, \kappa_2^{01})$
$(1,0)$	κ_L^2	κ_L^1	1
$(0,2)$	κ_L^2	$\kappa_2^{12} * \kappa_L^1$	1
$(0,1)$	κ_L^2	κ_L^1	1
$(0,0)$	κ_L^2	κ_L^1	1

Table 9: Uncertain leaky MIN

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12}, \kappa_2^{12})$	$\max(\kappa_1^{02}, \kappa_2^{02})$
$(2,0)$	0	κ_1^{12}	1
$(0,2)$	0	κ_2^{12}	1
$(0,0)$	0	0	1

Table 10: Uncertain MIN with Boolean inputs

4. Comparison with Probabilistic Gates

It is interesting to compare the possibilistic and probabilistic tables as they do not behave in the same way. The elementary probabilistic causal table takes the following form, where $\kappa_i = P(\neg z_i \mid x_i)$:

$P(Z_i\|X_i)$	x_i	$\neg x_i$
z_i	$1 - \kappa_i$	0
$\neg z_i$	κ_i	1

Table 11: Elementary causal probability table

Consider the conditional table of the noisy OR [10] (Table 12), to be compared with the conditional table of the uncertain OR (Table 2):

$P(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$1-\kappa_1\kappa_2$	$1-\kappa_2$
$\neg x_2$	$1-\kappa_1$	0

$P(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$\kappa_1\kappa_2$	κ_2
$\neg x_2$	κ_1	1

Table 12: Noisy OR

We shall distinguish between min-based and product based possibilistic networks.

4.1. The min-based case

There is an important difference between the behavior of conditioning in the probabilistic and the possibilistic cases. In the qualitative possibility setting, the conditional possibility $\Pi(Y \mid X)$ is the largest value λ such that $\min(\lambda, \Pi(x)) = \Pi(Y \wedge X)$, that is

$$\Pi(Y \mid X) = \begin{cases} 1 \text{ if } \Pi(Y \wedge X) = \Pi(X); \\ \Pi(Y \wedge X) \text{ if } \Pi(Y \wedge X) < \Pi(X). \end{cases}$$

and the conditional necessity is $N(Y \mid X) = 1 - \Pi(\neg Y \mid X)$. As a consequence it is impossible to have that $\Pi(Y) < \Pi(Y \mid X) < 1$, which dually reads the impossibility that the certainty of an event can decrease while remaining somewhat certain (one cannot have the strict inequality $N(Y) > N(Y \mid X) > 0$) [13]. The framework thus does not capture the idea of graceful degradation of belief in the min-based case.

This is a striking difference with conditional probability where this limitation of expressive power does not occur. While this property is sometimes viewed as a major impediment to considering qualitative possibility as a reasonable representation of belief ([25] p. 265), this pessimistic view can be challenged. Note that one may simultaneously have $N(y) > 0$ and $N(y \mid x) = 0$ (and even $N(\neg y \mid x) > 0$), so that the qualitative framework allows for severe belief change. Moreover, this limitation just indicates that the qualitative setting is rougher than the quantitative one, and that qualitative necessity degrees are not proportional to intensity of belief. In some situations this rough model is sufficient for the purpose at hand, as being more expressive than classical logic (since it allows for non-monotonic reasoning [4]).

A similar lack of expressiveness occurs when comparing the conditional possibility and probability tables in case of the OR connective (Tables 2 and 12). In the possibilistic table, we see (using the associated necessity measure N) that

$$N(y \mid x_1x_2) = \max(N(y \mid x_1\neg x_2), N(y \mid \neg x_1x_2)) = 1 - \min(\kappa_1, \kappa_2)$$

while

$$P(y \mid x_1x_2) = 1-\kappa_1\cdot\kappa_2 > \max(P(y \mid x_1\neg x_2), P(y \mid \neg x_1 x_2)) = \max(1-\kappa_1, 1-\kappa_2),$$

so that in qualitative possibility networks, the uncertain OR gate does not allow the reinforcement the certainty of the effect in the presence of two causes, because connectives are idempotent.

4.2. The product-based case

If possibility degrees are numerical and $*$ = product, the conditional possibility is just defined by the usual division ($\Pi(Y \mid X) = \frac{\Pi(X\wedge Y)}{\Pi(X)}$), so that we can model the graceful degradation of beliefs ($\Pi(Y) < \Pi(Y \mid X) < 1$ may occur). The possibilistic network then behaves like a probabilistic network because $N(y \mid x_1x_2) = 1 - \kappa_1 \cdot \kappa_2 > \max(N(y \mid x_1\neg x_2), N(y \mid \neg x_1x_2))$ is also retrieved.

However, another major difference in behavior between uncertain and noisy OR-gates will occur in case the effects of causes are not frequent (weak causes), namely when $P(\neg z_i \mid x_i) = \kappa_i > 0.5, i = 1, 2$. Then it may happen that $P(y \mid x_1x_2) = 1 - \kappa_1\kappa_2 > 0.5$, that is the simultaneous presence of two causes that individually do not frequently produce an effect may make this effect more frequent than not. Then a possibilistic rendering of this case must be such that $\pi(\neg z_i \mid x_i) = 1 > \pi(z_i \mid x_i) = \lambda_i$. Then the uncertain OR-gate with two weak causes behaves as follows:

$\pi(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	$\max(\lambda_1, \lambda_2)$	λ_2
$\neg x_2$	λ_1	0

$\pi(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	1	1
$\neg x_2$	1	1

Table 13: Uncertain OR for 2 weak causes

However, there is no way of observing a reversal effect, since $\pi(y \mid x_1x_2) = \max(\lambda_1 * \lambda_2, \lambda_1, \lambda_2) = \max(\lambda_1, \lambda_2) < 1$. Hence $\pi(\neg y \mid x_1x_2) = 1$ and $N(y \mid x_1x_2) = 0$. In other words, using the uncertain OR, two causes that are individually insufficient to make an effect plausible are still insufficient to make it plausible if joined together, because on the one hand there is no reinforcement effect in this case, and there is no way of producing 1 from operands that are less than 1. Note that this fact reminds of the property of closure under conjunction for necessity measures in possibility theory ($N(y_1) > N(\neg y_1)$ and $N(y_2) > N(\neg y_2)$ imply $N(y_1 \wedge y_2) > N(\neg(y_1 \wedge y_2))$ which fail to hold in probability theory, where a reversal effect is possible in this case.

The case with one weak cause and one strong one is also worth studying, say cause 1 is weak ($\pi(\neg z_1 \mid x_1) = 1 > \pi(z_1 \mid x_1) = \lambda_2$) and the other is strong ($\pi(z_2 \mid x_2) = 1 > \pi(\neg z_2 \mid x_i) = \kappa_2$).

Then, one observes that the strong cause alone makes the effect somewhat certain to the same degree as in the elementary causal table, independently of the presence or not of the weak one. When the strong cause is absent, the effect is absent with a weak certainty as per the presence or not of the weak

$\pi(y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	1	1
$\neg x_2$	λ_1	0

$\pi(\neg y \mid X_1X_2)$	x_1	$\neg x_1$
x_2	κ_2	κ_2
$\neg x_2$	1	1

Table 14: Uncertain OR for strong and weak causes

cause. Note that in the possibilistic case, we need the three tables 2, 13, 14 that represent a distinct behavior each case, while the probability table 12 is valid in the three cases, while the believed effects depend on the numerical values given in the table.

4.3. Should possibilistic logical gates be mended ?

Note that insofar as the behavior of the uncertain possibilistic gates is judged counterintuitive in a given context, it would be possible to change the combination of the elementary conditional tables. For instance one may define the global conditional possibility tables $\pi(Y \mid X_1, X_2)$ enforcing $\pi(y \mid x_1x_2) > \pi(\neg y \mid x_1x_2)$ even if $\pi(y \mid x_1) < \pi(\neg y \mid x_1)$ and $\pi(y \mid x_2) < \pi(\neg y \mid x_2)$, which is perfectly compatible with possibility theory. However, one may also claim that the possibilistic OR gate behaves as expected and that the behavior of the noisy OR is questionable, depending on what we intend to model. Consider the case when $P(z_i \mid x_i) = P(\neg z_i \mid x_i) = \kappa_i = 0.5$ for $i = 1, 2$. Note that then $P(y \mid x_1x_2) = 0.75$.

- Interpreting κ_i as a frequency: Then this result can be easily explained. As when cause x_i is present irrespective of the other cause, the effect y is present 50% of the results, and causes are independent of each other, this effect is produced 25% of the time when x_1 is present and x_2 is absent, 25% of the time when x_2 is present and x_1 is absent, and 25% of the time when x_1 and x_2 are present. So no surprize that the reinforcement effect in favor of y can be observed. However, the possibilistic model has no way of representing equiprobability, hence cannot model this situation.

- Interpreting κ_i as a degree of belief: then $\kappa_i = 0.5$ represent the agent's ignorance whether x_i causes y or not. Under this view, the probabilistic approach produces a counterintuitive result. Indeed, it is very hard to make sense of the reasoning line whereby given that the agent ignores whether x_i causes y or not, for $i = 1, 2$, this agent should believe that the presence of both causes makes the effect y more likely than its negation. It is one more example of production of knowledge out of sheer ignorance, which is usual when uniform probability is interpreted as lack of information.

Actually, the uncertain OR gate behaves consistently with the situation of ignorance: if it is believed that each x_i causes $\neg y$, rather than y, then there is no way of starting to believe y when observing two reasons not to believe it. And in the case of ignorance, the uncertain OR-gate just produces ignorance.

These results extend to other gates like the uncertain MAX, for instance. Again, the simultaneous presence of a number of causes, which, taken in isolation, do not normally produce an effect, may lead to a plausible effect under a noisy MAX, which can never be the case with an uncertain MAX.

However, in the following, we are interested in representing the same dataset by probabilistic and possibilistic networks for the sake of comparing both models on an application. Then we try to modify the construction of the conditional possibilistic table from elementary ones in order to get closer to the probabilistic model. Note that we can imagine various ways of completing the possibilistic conditional tables from the knowledge of local conditional tables $\Pi(Z_i \mid X_i)$ For instance we may use an aggregation operation $\otimes$ [2] on the possibility scale and define $\Pi(Y \mid X_1X_2)$ as $\Pi(Z_1 \mid X_1) \otimes \Pi(Z_2 \mid X_2)$ with the constraint $Y = Z_1 = Z_2$. However, since aggregation operations are order-preserving and such that $1 \otimes 1 = 1$ and $0 \otimes 0 = 0$, we cannot address the case when weak causes join to make a strong one in the possibilistic setting using this approach.

4.4. Uncertain MAX with Thresholds

As observed in Section 4 when comparing the uncertain OR to the noisy OR, the simultaneous presence of a number of causes, which, taken in isolation, do not normally produce an effect, may lead to a plausible effect under a noisy MAX, which can never be the case with an uncertain MAX. Yet, situations of this kind do arise in applications and are fully compatible with the expression of a conditional table in possibility theory.

In order to make the construction of possibility tables in agreement with the mutual reinforcement of weak causes an appropriate uncertain gate has to be designed, by means of a suitable uncertain function f which can produce this effect. One idea we have tested in order to approximate such behavior is the proposal of *uncertain MAX with thresholds*. In addition to the usual parameters of an uncertain MAX, this uncertain gate requires that a threshold θ_j be specified for each value y_j of the effect variable Y. Such threshold is an integer expressing the minimum number of causes that have to simultaneously occur in order for effect y_j to become possible. To this end, a cause X_i may be considered to "occur" if the value of its corresponding intermediary causal variable Z_i differs from the zero level, i.e., $Z_i > 0$. Note that threshold gates also exist in the probabilistic setting [10]

More precisely, as in the case of the uncertain MAX, we assume that the output variable Y and the variables Z_i are valued on a finite, totally ordered, severity or intensity scale $L = \{0 < 1 < \cdots < m\}$, but the function f describing this gate is based upon may be written as

$$Y = \max(Z_1, \ldots, Z_n, \max_{i=1}^{m}\{i \cdot 1_{[\|\{j : Z_j > 0\}\| \geq \theta_i]}\}),$$

using the inverson bracket notation, whereby

$$1_{[condition]} = \begin{cases} 1, & \text{if } condition \text{ is true;} \\ 0, & \text{otherwise.} \end{cases}$$

For instance, suppose $n = 4, m = 3$, $\theta_1 = 1, \theta_2 = 2, \theta_3 = 3, \theta_4 = 4$. Then $f(1,1,0,0) = 2$, $f(1,1,1,0) = 3$, $f(1,1,1,1) = 4$.

We can then express the conditional table of the uncertain MAX with thresholds by applying the extension principle, as in the case of the uncertain MAX, although the resulting analytical expression is much less legible:

$$\pi_f(y \mid x_1, \ldots, x_n) = \max_{Z_1,\ldots,Z_n : y = \max(Z_1,\ldots,Z_n, \max_{i=1}^{m}\{i \cdot 1_{[\|\{j : Z_j > 0\}\| \geq \theta_i]}\})} *_{i=1}^{n} \pi(Z_i \mid x_i)$$
$$= \max\left(1_{[\|\{i \mid x_i > 0\}\| \geq \theta_y]}, \max_{j=1}^{n} \pi(Z_j = y \mid x_j) * (*_{j \neq i} \Pi(Z_j \leq y \mid x_j))\right).$$

More intuitively, by default $\pi(y \mid x_1, \ldots, x_n)$ has the same value as with the uncertain MAX, except when the number of occurring causes exceeds the threshold for value y, in which case $\pi(y \mid x_1, \ldots, x_n) = 1$.

The global conditional tables are then obtained by applying Eq. 5, using the same values of $\pi(Z_i \mid X_i)$ as in the case of the uncertain MAX. For $n = 2$, $m = 2$ (i.e., Y and the X_i's three-valued), $\theta_2 = 2$, and $\theta_1 = m + 1$ (i.e., no threshold set for $Y = 1$), the following conditional Table 15 is obtained. The only cell where possibility is raised to 1 due to the thresholds is marked by a dagger (†): the fact that two causes weakly present ($(1,1)$) as input cause the strongest effect with full possibility.

It is easy to check that π_f is less specific than π_{MAX}, which means that the imitation of the noisy MAX is imperfect. For instance, one may wish to decrease the value $\pi(1 \mid (1,1))$ in Table 15, since the value $\pi(2 \mid (1,1))$ has been set to 1; this would occur in a probabilistic approach as the sum of probabilities in each line is 1. However, this is not possible using the definition of f. But we are allowed to decrease the value $\pi(1 \mid (1,1))$ manually, as long as the maximum of values in each line of Table 15 is 1.

$\mathbf{x}$	$\pi(2 \mid \mathbf{x})$	$\pi(1 \mid \mathbf{x})$	$\pi(0 \mid \mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12}, \kappa_2^{12})$	$\kappa_1^{02} * \kappa_2^{02}$
$(2,1)$	1	1	$\kappa_1^{02} * \kappa_2^{01}$
$(2,0)$	1	κ_1^{12}	κ_1^{02}
$(1,2)$	1	1	$\kappa_1^{01} * \kappa_2^{02}$
$(1,1)$	$1^\dagger$	1	$\kappa_1^{01} * \kappa_2^{01}$
$(1,0)$	κ_1^{21}	1	κ_1^{01}
$(0,2)$	1	κ_2^{12}	κ_2^{02}
$(0,1)$	κ_2^{21}	1	κ_2^{01}
$(0,0)$	0	0	1

Table 15: Uncertain MAX with thresholds

Algorithm 1 UNCERTAIN-MAX(Y, prm).
Generate a conditional possibility table for variable Y given its causes $X_1, \ldots, X_n$ using the uncertain MAX with its given parameters prm.

Input: Y: the effect variable;
$prm = \{\langle cond_i, \mathbf{k}_i \rangle\}$: a set of normalized possibility distributions $\mathbf{k}_i = (\kappa_{i1}, \ldots, \kappa_{i\|Y\|})$, $\max_{j=1,\ldots,\|Y\|}\{\kappa_{ij}\} = 1$, which apply when condition $cond_i$ holds; $cond_i = (\langle X_i, x_i \rangle)$, a (possibly empty) pair of a cause variable X_i and one of its values x_i; $cond_i$ holds if $X_i = x_i$ holds; an empty condition always holds.
Output: $\pi(Y \mid X_1, \ldots, X_n)$: a conditional possibility distribution of Y given its causes $X_1, \ldots, X_n$.

```
π(Y | X_1, ..., X_n) ← 0
for all x ∈ X_1 × ... × X_n do
   K ← {k : ⟨cond, k⟩ ∈ prm, x ⊨ cond} {Select the elementary possibility
   distributions that apply to x}
   for all y = (y_1, ..., y_‖K‖) ∈ Y^‖K‖ do
      β ← min_{i=1,...,‖K‖}{κ_{i y_i}}
      ȳ ← max_{i=1,...,‖K‖}{y_i}
      π(ȳ | x) ← max{β, π(ȳ | x)}
   end for
end for
return π(Y | X_1, ..., X_n)
```

---

5. Implementation

A prototype involving the uncertain connectives defined above, allowing to execute possibilistic models such as the one described in Section 6 has been implemented in R. Here, we give some details about the practical implementation of the uncertain connectives defined in the paper. We focus in particular on the uncertain MAX (and its variant with thresholds), whose implementation is non-trivial.

The way the uncertain MAX is implemented is shown in Algorithm 1. The parameter prm taken as input by this algorithm may be thought of as representing a set of rules of the form

$$(X_i = x_i) \Rightarrow Y \sim (\kappa_{i,y_0}, \ldots, \kappa_{i,y_m}), \tag{7}$$

where X_i on the left-hand side is a parent variable of Y in the possibilistic graphical model, x_i are one of their values, and $(\kappa_{i,y_0}, \ldots, \kappa_{i,y_m})$ is a normalized possibility distribution over the values of variable Y, i.e., for all $y \in Y$, $\kappa_{i,y} \in [0, 1]$, and $\max_{y \in Y} \kappa_{i,y} = 1$. Note that the X_i's in the above rules can in fact represent vectors of more elementary interacting variables, and allow to encode multi-condition tables not representable by combining simple conditional tables involving such variables in isolation.

The left-hand side of a rule may be empty : in that case, the rule is interpreted as if it were statements of the form

$$Y \sim (\kappa_{L,y_0}, \ldots, \kappa_{L,y_m}). \tag{8}$$

Such rules may be used to represent leakage coefficients, which apply to all possible combinations of causes.

On the one hand, this choice of representation of the parameters generalizes the uncertain gates to the case of multivalued variables; on the other hand, it allows the expert to express its knowledge of the phenomenon in more intuitive terms, in the form of rules, which is entirely in the spirit of making expert knowledge elicitation easier.

Albeit such representation is more intuitive, it requires some additional care: the antecedents of the rules fed into the uncertain MAX must cover all possible combinations $\mathbf{x} \in X_1 \times \ldots \times X_n$ of the values of the parent variables of Y in order to ensure that the resulting conditional possibility distribution $\pi(Y \mid X_1, \ldots, X_n)$ be normalized. However, we may notice that, if a leak rule of the form of Eq. 8 is given, that rule alone already covers all combinations of parent variable values and is thus a sufficient condition for the normalization of $\pi(Y \mid X_1, \ldots, X_n)$; in that case, the parameters of the uncertain MAX may be underspecified.

The algorithm constructs the table of conditional possibility in an incremental way, starting with a table filled with zeros (Line 1), and then considering all combinations of values for the cause variables (Line 2). For a given combination $\mathbf{x}$, which corresponds to a row of the table, a subset K of normalized possibility distributions that apply to $\mathbf{x}$ is extracted from the parameters (Line 3). Lines 4–8 compute one min expression of Eq. 5, by considering all the combinations of parameters in the possibility distributions of K and update the corresponding cell (the one in the column of the maximum y of the combination) if the result of the min exceeds its current value, so that, once this inner loop completed, the max in Eq. 5 will have been computed for all the cells of the row corresponding to $\mathbf{x}$.

The implementation of the uncertain MAX with thresholds follows the same pattern as the previous algorithm.

6. Application

Probabilistic and possibilistic networks using noisy/uncertain logical gates have been used to model the social specialization of municipalities in a metropolitan area, under a human geography perspective (alternative models have been proposed in the economic literature like rent-gap theory models for urban gentrification and hedonic price models for environmental amenities for rural and suburban developments). We will first present the models and their logical gates. We will then compare the uncertainty content of the trend scenarios produced by the two models and we will finally evaluate the sensitivity of model outcomes to probabilistic and possibilistic elicited parameters.

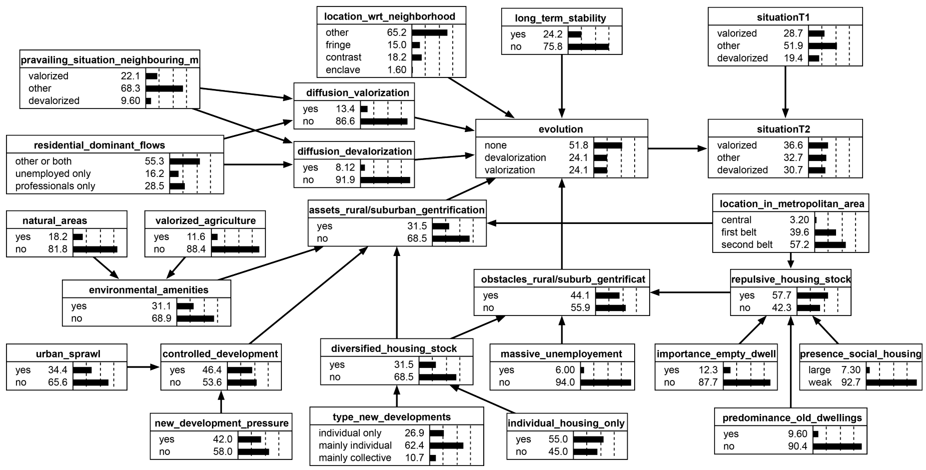


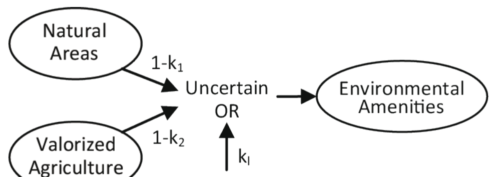
Figure 1: The BN model for the valorization/devalorization of municipalities in the study area (adapted from [24]).

6.1. Model Specification

The metropolitan area of Aix-Marseille in southern France has experienced ongoing social polarization since the 1980s. The geography of unemployment, on the one hand, and the concentration of high-skilled professionals, on the other, both considerably contribute to the structuring of a contrasted metropolitan social morphology [9, 17]. The knowledge of factors inducing social polarization of the municipalities in the metropolitan area is nevertheless uncertain. Social polarization is analyzed as the opposition of valorized municipalities, hosting wealthier resident populations and namely high-skilled professionals, and devalorized municipalities, hosting lower-income populations and, more particularly, the unemployed. Several factors contribute to the valorization or to the devalorization of the municipal residential space. But these factors have “soft”, uncertain impacts on the phenomena under investigation: the same causes can sometimes produce different effects and observed effects can have multiple possible causes.

A probabilistic model of these socio-spatial mechanisms has already been proposed [24] (cf. Fig. 1) in the form of a Bayesian network (BN). The BN was built using expert knowledge elicited through noisy logical gates (OR, AND, and MAX) with leak parameters (taking into account the impact of factors omitted in the model). We then developed a min-based possibilistic network (PN) using uncertain logical gates (OR, AND and MAX-threshold) with leak parameters in order to link the same 26 variables of the BN. The numerical parameters of the PN were made compatible with the BN parameters using a least committing probability-to-possibility preference preserving transformation [14] in order to transform probability degrees into numerical possibility degrees.

This transformation was used by lack of expert data in the form of possi-



Where (1-k_i) = N (z_i|x_i) can be interpreted as « possibilistic force» of cause x_i
Π (*Environmental_Amenities* | *Natural_Areas*, *Valorized_Agriculture*) =
Uncertain OR (*leak* 0.1, *Natural_Areas* 0.8,*Valorized_Agriculture* 0.9)

Conditional Possibility Table for variable *Environmental_amenities*

Natural_Areas	Val_Agriculture	yes	no
yes	yes	1	min(k_1,k_2)=0.1
yes	no	1	K_1=0.1
no	yes	1	K_2=0.2
no	no	K_l=0.1	1

Figure 2: Generation of a conditional probability table through an Uncertain OR logical gate.

bility distributions. We started with a Bayesian network with already existing probabilistic data. It was not possible to start the data collection again and train experts into forwarding possibility degrees instead of probability degrees. And our intention was to compare the results of possibilistic and probabilistic networks on the same data, which means keeping the ordinal information contained in the probabilistic data. Using a least committed probability-to-possibility preference preserving transformation at the local level was a natural way of generating such possibilistic counterparts of subjective probabilistic data, even if we are aware that making local probability-to-possibility transforms is for instance not equivalent to making probability-to-possibility transforms of the joint probability, as studied in [7]. Note that the same issues occur when trying to learn possibilistic networks from data [18].

In Fig. 2 we show how an Uncertain OR logical gate can be used to generate a conditional probability table. Only three parameters must be elicited: the possible influence of the two parent variables on the child variable (necessity of the consequence given that the parents are sufficient causes) and the leak parameter, which takes into account the activation of the consequence from secondary causes not included in the model. This table allows possibilistic prediction from uncertain knowledge. If, for example, in a given municipality of the study area, we are relatively certain of the presence of natural areas ($\Pi = 1$, $N = 0.5$) and if it is only partially possible that agricultural areas are considered attractive and valorizing for residential use ($\Pi = 0.5$, this is for example the case for vineyards but not for industrial crops), we can infer that it is relatively certain ($N = 0.5$) that the municipality in question has environmental amenities.

Another difference between the min-based possibilistic model and the probabilistic one is the possibility, for the former, of keeping track of the κ_i parameters in the reasoning process, in order to figure out the sensitivity of results to the parameters of uncertain causation. The advantage of uncertain logical gates can be better appreciated in the whole model (Fig. 1).

Evolution is, for example, a ternary variable (having three values: *no evolution*, *valorization*, and *devalorization*) depending on 5 binary variables and one 4-value variable. The conditional probability table is thus made of $3 \times 2^5 \times 4 = 384$ parameters, whereas the uncertain MAX-threshold gate used in our PN model requires at most 27 parameters (indeed only 10 κ_i and θ_j parameters different from 0 and 1 are used in our model).

Evolution is typically a multi-valued variable with a hierarchical order of values. Urban geographers [24] consider that *valorization* is the value with highest priority: when social groups of higher purchasing power decide to live in a given municipality, real estate prices go up and other social groups are crowded out. It corresponds to the highest severity effect in section 3.4. The second priority effect is *devalorization*: when a given set of causes operates in order to specialize the municipality in retaining inhabitants of lower social status, this effect has greater priority (severity) than no effect at all. Finally, the absence of change in the social mix of the municipality is the default outcome (no effect), in the absence of particular triggers for valorization and/or devalorization.

The diffusion of valorization (i.e., the spatial diffusion of suburban and rural gentrification within the metropolitan area through residential flows of high-skilled professionals) and the presence of assets for rural and suburban gentrification are triggers of valorization for a given municipality.

The attraction of residential flows of unemployed people (*diffusion_devalorization* variable in the model) and the presence of obstacles to rural and suburban gentrification are triggers of devalorization. The long-term instability of the social mix in the municipality over the last 20 years and its particular geographic location with respect to the social mix of neighboring municipalities can be triggers of either valorization or devalorization. Valorization and devalorization are nevertheless uncommon outcomes in the presence of only one of these triggering factors, as these are normally relatively weak: in the probabilistic model, several triggers have to be simultaneously present in order to cumulate probability values and make the absence of change less probable. Several specifications of the uncertain MAX connective were considered in order to replicate as much as possible the probabilistic behavior of the BN model. A MAX-threshold connective saturating possibility values of uncommon outcomes when three concurrent causes are present was finally selected.

Again, as discussed in Section 4 we have two ways of understanding the above situation. Viewed in terms of frequencies, the reinforcement effect of the probabilities of residential moves due to several triggers, using a noisy MAX, is in line with the actual phenomenon of people changing their dwelling places, while viewed in terms of subjective probabilities, this reinforcement effect is more difficult to justify, as in this case, equal probabilities of opposite events just represent ignorance, and not equal proportions of moves in one direction and in another. Then the probabilistic approach surprizingly transforms ignorance into the prediction of a trend, while the possibilistic approach using the uncertain MAX with threshold just increases the range of possibilities, and therefore more cautiously increases the imprecision of conclusions, in case of several weak triggering factors.

The use of the uncertain MAX with threshold gives the best results on the empirical values of the 439 municipalities of the study area in terms of proximity to the probabilistic predictions (which were in turn transformed in possibilistic values to allow comparison). Alternative variants of the uncertain max with upper and lower thresholds and even with two different thresholds (a first threshold for making the uncommon effects completely possible and a second threshold for making the absence of change not plausible) were also considered. The two-threshold specifications are in theory better able to replicate the behavior of the probabilistic model, but prove not better than the simpler one-threshold model for the empirical results, although requiring additional elicited parameters. The bias introduced by the simpler one-threshold connective is its inability to foresee the necessity of change, a phenomenon that, although rare, is sometimes found among the 439 municipalities of the study area.

Whatever the specification of the MAX connective, the *Evolution* variable determines, together with the *Situation T1* variable, the value of the *Situation T2* variable in a ten years simulation period. *Situation T1* (the reference year is here 2009) and *Situation T2* (2019) have three possible values: *Valorized* (V), *Devalorized* (D) or *Other* (O). Values of *Situation T1* are considered to be known without uncertainty (Fig. 3-a). Values of *Situation T2* are not observable and have to be inferred by the model.

6.2. Comparative Results

Both the BN and the PN model were thus used to produce trend scenarios for social polarization in the 439 municipalities of the Aix-Marseille metropolitan area through the prediction of *Situation T2* values.

Both scenarios are based on uncertain knowledge of relationships among variables and produce an uncertain evaluation of the future state of the metropolitan area in terms of social polarization. Most plausible values (in terms of probability/possibility) inferred by the two models can be projected in space (Fig. 3). With the probabilistic model, we display a most probable value of *Situation T2* for each municipality (Fig. 3-b). This often gives a fallacious impression of certainty: differences between most probable values can be relatively small and are not taken into account on the figure. The possibilistic model, using a min-max logic, produces in many cases sets of completely possible values ($\Pi = 1$, Fig. 3-c). In Fig. 3, these are represented as circles with multiple colors (two different outcomes equally possible) or as grey circles (three different outcomes equally possible, which is the most uncertain model result). We thus decided to test the significance of the probability differences in the BN model: only probability differences exceeding a given threshold were considered significant. For a given probability threshold, we could thus exhibit small sets of "equally" most probable values for some municipalities, even with the BN. Fig. 3-d, e and f, show results for threshold probability differences of 0.1, 0.2 and 0.3, respectively.

If no threshold is considered, the most probable values inferred by the BN and the completely possible values inferred by the PN coincide only in 54.7% of cases. In the remaining cases, possibilistic results are more uncertain and always include probabilistic results (most probable values are always completely possible for the PN).

The best agreement between the two models is obtained with thresholds 0.20 and 0.25 (lower and higher values give worse results). Respectively 72.4% and 77.2% of the inferred values are then identical. Most probable values are almost always compatible with PN solutions: they are included in the completely possible values as, for example, when $\{V, O\}$ are the most probable values and $\{V, O, D\}$ is the set of completely possible values. The inverse is not always the case: depending on the threshold value, 24% and 18% of possibilistic solutions are not included in the most probable values.

Keeping in mind that possibilistic results tend to be slightly less restrictive (and hence more uncertain) than the probabilistic counterparts, both models show that the outcome for several municipalities is not completely uncertain, but is made of two equally probable/possible states. Complete uncertainty is still the case for 130 spatial units for the PN and 85 for the BN; these are often neither too peripheral nor too central within the metropolitan area. In this respect, these uncertainty-based models could be used to better highlight areas of insufficient knowledge.

6.3. An Empirical Sensitivity Analysis to Elicited Parameters

A sensitivity analysis of the two models has been performed with respect to the elicited parameters of the MAX connective governing the *Evolution* variable. This is a particularly crucial variable for the models, directly influencing the *Situation T2* variable and depending, directly or indirectly, from almost all causal relations contained in the models. The sensitivity analysis was conducted using the empirical data: by varying the values of the elicited parameters, we did not assess how the conditional probability/possibility tables were modified. Instead, we assessed how the parameter changes would affect the predictions on the most plausible outcomes of the *Situation T2* variable for the 439 municipalities of the study area. This empirical sensitivity analysis is much more interesting for the practical use of the model in decision making on the study area.

In the reference BN, the Noisy MAX connective for *Evolution* is parametrized as follows. Triggers of valorization or devalorization have probabilistic strength κ_i equal to 0.3 only. Triggers of devalorization and devalorization have probabilistic strength of 0.12 and 0.18, respectively (a slightly higher probabilistic strength is given to the production of the devalorization effect in order to counter the priority rule of the connective and produce an unbiased model with equal marginal probabilities for the two effects). The absence of effect has thus always a conditional probability of 0.7. Under the hypothesis that these elicited values correspond to reality, we will use the most probable values of the *Situation T2* variable for the 439 municipalities as the benchmark of the analysis.

Table 16: Empirical sensitivity analysis for the BN model

	Model with overestimated parameters				Model with underestimated parameters			
Reference model	O	D	V	Total	O	D	V	Total
O	117	30	38	185	185			185
D		100	2	102	17	85		102
V			152	152	26		126	152
Total	117	130	192	439	228	85	126	439
Overall agreement				**84.1%**				**90.2%**
Incompatibilities				**15.9%**				**9.8%**

We first consider a model where experts overestimate the probabilistic strength of triggering factors, by reducing the conditional probability of absence of effect to 0.6 and by increasing to 0.4, 0.25, and 0.15 the κ_i parameters of the Noisy MAX connective. Most probable values of *Situation T2* are in 84.1% of cases the same as in the reference model (Table 16, left). In the remaining 15.9% of cases, they are different and incompatible (for these municipalities, a valorized or a devalorized state is now inferred, whereas the reference model would infer the value *other* as most probable). We subsequently consider a model underestimating the probabilistic strength of factors, by increasing the conditional probability of absence of effect to 0.8 and by decreasing to 0.2, 0.12, and 0.08 the κ_i parameters. The most probable values of *Situation T2* are now in 90.2% of cases the same as in the reference model (Table 16, right). In the remaining cases, the new model predicts the state *other*, whereas the reference model would infer a valorized or a devalorized state as the most probable.

In the reference PN, the Uncertain MAX-threshold connective is parametrized in accordance to a least committing transformation of the corresponding probabilistic parameters. κ_i for triggers of valorization or devalorization only is 0.6, κ_i for triggers of both effects is 0.5 (devalorization) and 0.4 (valorization). The behavior of this connective makes the absence of effect always completely possible, whereas a given effect is completely possible only in the presence of three triggers. Over- and underestimation of possibilistic strength of triggers concern both the κ_i parameters and the θ_j thresholds. Over- and underestimated parameters of the PN are thus obtained through least specific transformations of the probabilistic parameters. θ_j thresholds are set to 2 and 4, respectively, reflecting the number of triggering factors whose presence is sufficient to make absence of effect non-plausible in the modified probabilistic models.

Table 17 (left) shows how the most plausible predictions of the overestimated model coincide only in 80.9% of cases with those of the reference PN. Trend scenario predictions for several municipalities become completely uncertain while previously being D or $\{D, O\}$. The under-estimated model is even more deviant from the reference one (Table 17, right). Only 68.1% of predictions agree

Table 17: Empirical sensitivity analysis for the PN model

	Model with overestimated parameters						Model with underestimated parameters					
Ref. model	O	D	V	VO	VOD	Total	O	D	V	VO	VOD	Total
O	82			3		85	85					85
D		41			29	70		70				70
D,O					52	52	52					52
V			85			85			85			85
V,O				17		17				17		17
V,O,D					130	130			40	48	42	130
Total	82	41	85	20	211	439	137	70	125	65	42	439
Overall agreement						**80.9%**						**68.1%**
Incompatibilities						**0.0%**						**0.0%**

with those of the reference model: many O predictions replace $\{D, O\}$ predictions and V or $\{V, O\}$ predictions replace completely uncertain ones of the form $\{V, O, D\}$. Nevertheless, in the first case the new predictions include the predictions of the reference model by introducing more uncertainty, and in the second case the new predictions are included in the reference predictions by reducing their uncertainty.

At first glance, the BN model seems less sensitive to small disturbances in the probabilistic parameters of the Noisy MAX connective, compared to its possibilistic counterpart. However, the advantage of the PN model lies elsewhere. The 15.9% and 9.8% predictions of the disturbed BN that differ from those of the reference model are incompatible with the latter. By allowing more uncertainty, the 19.1% and 31.9% of predictions of the disturbed PN models that differ from those of the reference model are restrictions or generalizations of them and the predictions of the three models are mutually compatible (in the sense that they are nested sets of possible results).

6.4. Lessons drawn

In conclusion, uncertain logical gates made the construction of the PN model possible. The use of most probable solutions of the BN model often gives a false impression of certainty. In order to compare results from the BN and the PN models, we need to enlarge the notion of most probable values: solutions whose probabilities differ less than 0.20/0.25 must be considered as equally probable. In this case, the solutions of the two models are identical for about three fourths of the municipalities of the study area. Despite this consistency between the two approaches, the possibilistic model integrates a larger amount of uncertainty in the solutions inferred. Indeed, in the remaining fourth of municipalities, completely possible values inferred by the PN are normally larger sets than most probable values inferred by the BN. The BN model also tends to overestimate the valorization of municipalities in the study area: the PN model often predicts complete uncertainty ($\{V, O, D\}$ all equally possible) whereas the most plausible values are just V or $\{V, O\}$ in the probabilistic model. A further analysis of the parametrization of the two models is nevertheless necessary in order to understand the origin of such a bias.

The results of the sensitivity analysis of the model predictions to the parameters governing their key variable (through a MAX connective) are apparently counterintuitive. The probabilistic model seems to be less affected by parameter disturbances than its possibilistic counterpart, but, at a closer look, BN predictions, when restricted to the most probable values, are more unstable than PN predictions: the former can be incompatible under slightly different parameter choices, whereas the latter, by allowing more uncertainty, are always nested sets of solutions generalizing or restricting the results obtained by different parameter choices.

7. Conclusion

This is the first detailed study of the counterpart of the main probabilistic noisy gates for possibilistic networks, together with an illustrative implementation on a human geography application. Uncertain possibilistic gates are of primary interest for the practical use of possibilistic networks, when uncertainty has an epistemic flavor. The study has revealed some noticeable differences of behavior between noisy gates and uncertain possibilistic gates, in particular when the cumulation of causes having a rare effect may increase the plausibility of the effect. Generally speaking, possibilistic modeling appears to be more cautious. A detailed comparative study of the expressive power of Bayesian nets and possibilistic networks is a topic for further investigation, as well as the development of a complete panoply of uncertain possibilistic gates.

Acknowledgments

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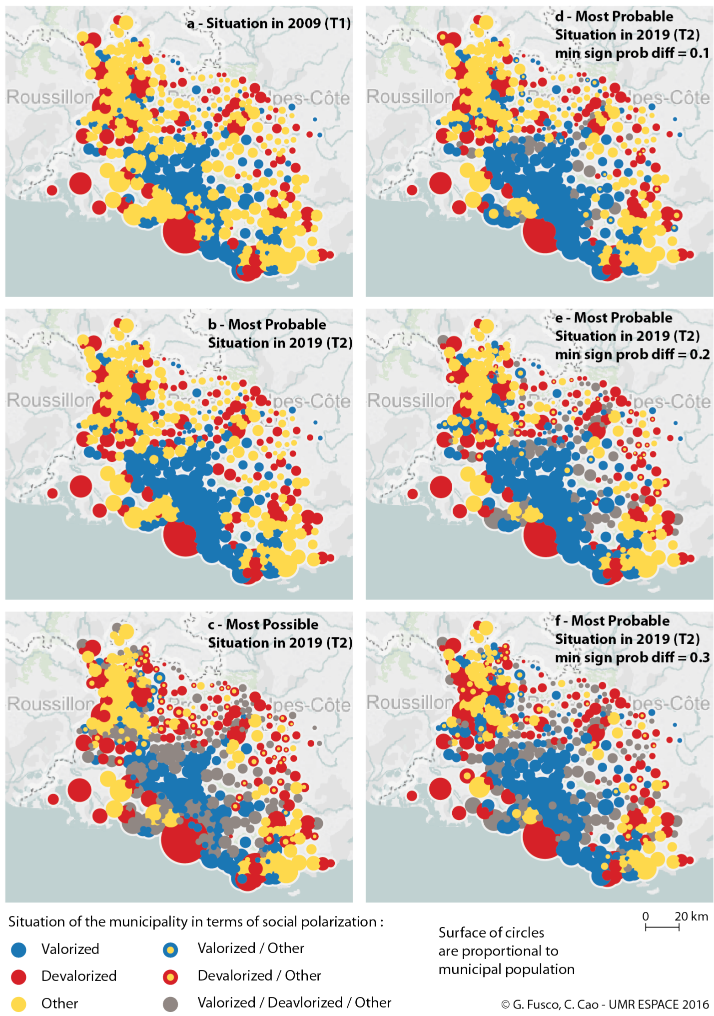

Figure 3: Probabilistic and possibilistic results projected in space.