

Substitutability and the Quest for Stability; Some Reflections on the Methodology of General Equilibrium in Historical Perspective

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Abstract: In this paper, I propose to interpret the history of stability analysis of a Walrasian exchange economy through the lenses of the concept of substitutability. A purely mathematical approach of this story does not seem sufficient to account for the way economists have studied the question of stability and how they have reacted to the results of this research programme. My point is that mathematical constraints kept apart, the concept of substitutability has shaped the path followed by stability analysis since the publication of *Value and Capital* (Hicks, 1939) up to the Sonnenschein-Mantel-Debreu theorems and beyond.

Thus, I uphold that firstly, the concept of substitutability allows to catch the heuristic behind the work on stability, and secondly, it allows replacing the importance of SMD results in a broader context, giving a more subtle view of the ups and down of general equilibrium theory as a research program.

Key words: stability, general equilibrium, gross substitutability, substitutability, complementarity, Hicks (John Richard), law of demand, Sonnenschein-Mantel-Debreu, methodology.

JEL: B21, B23, B41, C62

1 The heuristic value of the substitutability assumption

It is common view that the aims of general equilibrium theory have been seriously disrupted and reoriented after the famous Sonnenschein-Mantel-Debreu theorems. The hopes for finding general sufficient conditions under which a tâtonnement process is stable for a competitive economy have turned into dark pessimism, and even to disinterest.

The story of the stability issue as a specific research program within GET is rather well known. Ingrao and Israel (1990) have identified the steps and boundaries (see also Kirman (2006)). Some scholars have dealt more specifically with the issue of dynamics, establishing connections with the history of general equilibrium theory (Weintraub (1985), Weintraub (1991), Hands (1994)).

In contrast, the methodological appraisal of this story has not been pushed very far. Hands (2010 and 2016) provides insights regarding the notion of stability of consumer's choice and revealed preference theory in relation with the stability of general equilibrium, but his aim is not to provide a comprehensive analysis of the stability issue. Hence, as a shortcut to the history of stability, the most common opinion on the subject (Guerrien, 1989; Rizvi, 1997; Bliss, 1994) credits the famous Sonnenschein-Mantel-Debreu theorems for having discarded any serious reference to the invisible hand mechanism to reach a competitive market equilibrium. However, one can find a slightly different position regarding the stability literature in Ingrao and Israel (1990). According to them, from the very beginning of the 1960s, mathematical knowledge on dynamical systems and some well known instability results (Scarf, 1960; Gale, 1963) made stability researches already a vain task.

The gap between those two positions is not anecdotal. Firstly, according to the stance adopted, the place of the SMD results is not the same, both theoretically and from a symbolic point of view. Secondly, there are methodological consequences at stake on the way we can represent the development of general equilibrium theory, and more specifically, on the kind of methodological principles at work in a field of research characterized first of all by strong mathematical standards.

The aim of this paper is to identify some methodological principles at stake in the history of the stability of a competitive general equilibrium. More precisely, I would like to identify some criteria, other than mere analytical rigor, that were in use to direct research strategies and to evaluate and interpret theorems obtained in this field. This methodological look at the stability literature may lead to a more progressive view of the history, where results are modifying step by step the feelings of mathematical economists on the successes and failures of a research program.

My aim in this article is to provide a first step into the history of stability of a Walrasian exchange economy, taking Hicks's *Value and capital* (1939) as a starting point. To this end, I will put in the foreground the concept of substitutability. Indeed, substitutability has been a structuring concept for thinking about stability. It is my contention here that the concept of substitutability helps to provide some methodological thickness to the history of general equilibrium theory, not captured by purely mathematical considerations. Hence, I uphold that it allows identifying some methodological and heuristic constraints that were framing the interpretation of the successes and failures in this field.

It is well known that a sufficient condition for local and global stability of the Walrasian tâtonnement in a pure exchange economy is the gross-substitutability assumption (GS) (i.e., that all market excess demands increase when the price of other goods increases). By reconstructing the history of stability analysis through the concept of substitutability, I uphold that the representations attached to substitutability constituted a positive heuristic for the research program on stability. Therefore, tracking the ups and downs of this concept within general equilibrium

provides some clues to appraise the methodology of general equilibrium in historical perspective and to account for the rise and fall of stability analysis in general equilibrium theory.

The research program on stability of a competitive general equilibrium is by itself rather specific within GET, and bears on other subfields of GET such as uniqueness and comparative statics. It is also grounded on some views about the meaning of the price adjustment process. Stability theorems are for the most part of them not systematically microfounded: they are formulated *at first* as properties on the aggregate excess demands (such as gross substitutability, diagonal dominance, weak axiom of revealed preference), and their theoretical value is then assessed against their descriptive likelihood and heuristic potential, and not against their compatibility with the most general hypotheses of individual rationality. The paper upholds that the concept of substitutability, as a tool for expressing market interdependencies, was seen as a common language to mathematical economists, rich enough to develop a research program on stability and to appraise its progress and failures.

An ever recurring question behind different narratives on GET revolves around the principles explaining the logic of its development, the fundamental reasons that explain GET was a developing area of research in the 1950s-1960s while it became depreciated in the 1970s. Adopting a Lakatosian perspective, one would say that the research program on GET was progressive in the 1960s and became regressive in the 1970s. Even this question assumes that we (methodologists, theorists, historians) agree upon the idea that GET is functioning as a research program and that it went through two different periods, one during which new knowledge accumulated and one that made new “positive” results hopeless, even devaluating older results in view of new ones.

Recent trends in economic methodology have left behind the search for such normative and comprehensive systems of interpretation of the developments of economic theories. They focus instead on economics as a complex or intricate system of theories, models, fields of research, each (potentially) using a variety of methods as rationalizing and exploratory tools (econometrics and statistical methods applied to various data, simulations, experiments). The first goal of methodological inquiries is then to bring some order into the ways those various tools and methods are applied in practice, how they are connected (or not) through specific discourses, what are the rationales of the practitioners themselves when they apply them.

As far as we are concerned here, the question lends itself how GET can be grasped as an object of inquiry in itself, and more precisely how a field of questionings and research within this field—the stability of a competitive system—can be analyzed both as an autonomous field and in connection with other parts of GET. The present contribution does not claim to provide *the* structuring principle that explains the ups and downs of the researches on the stability of general competitive equilibrium. It is too evident that various aspects of this research are connected with what is taking place in other parts of the field. First, the kind of mathematical object which is likely to serve as a support for discussing about stability is not independent of the choice of the price-adjustment process that is used to describe the dynamics of the system when it is out-of-equilibrium. Hence, the explanatory power of a set of assumptions (about demand properties) is not disconnected from the explanatory

power of another set of assumptions (the price adjustment process), which himself has to be connected with the assumptions about agents behavior, motivations and perception of their institutional environment (e.g. price taking behaviors, utility maximizing and profit maximizing assumptions). In a sense, while it is useful to analyze the proper historical path of researches on the stability of a Walrasian tâtonnement with a methodological questioning in mind, the historian-methodologist should be aware that various rationales are likely to play a role in its valuation as a relevant or anecdotal result. Second, the kind of assumptions made on a system of interdependent markets will have simultaneous consequences on different subfields of GET. An all too obvious example is that Gross Substitutability is both sufficient for uniqueness and global stability of a competitive equilibrium and allows for some comparative statics theorems. Third, the simple fact of identifying an autonomous subfield of research and to claim that it is stable enough through time to be analyzed independently of some internal issues that surface here and there, is something that needs questioning. I have in mind the fact that it is not something quite justified to take the stability of a competitive equilibrium as a historically stable object on which we may apply confidently various methodological hypotheses. There is first the question of delimiting the kind of tools used to describe such a competitive process. Certainly the Walrasian Tâtonnement (WT) has been acclaimed as the main tool for this, but the methodological rationale for it needs be considered in detail to account for the way theorists interpret the theorems of stability. One set of question could be: What about similar theorems when non-tâtonnement processes are considered? Why discard processes with exchanges out of equilibrium? Another set would be: Why not considering that the auctioneer takes into account some interdependencies on the market to calculate new prices? Should we search for stability theorems that are independent of the speed of adjustment on markets? Should stability be independent of the choice of the numéraire?

It is my contention here that the methodology of economics cannot hope to find out one regulatory principle adapted for describing and rationalizing the evolutions of a field of research when the studied object is by itself under a set various forces from inside and outside that make it rather unstable. If I do not claim for an explanatory principle of the research on stability theorems, what does this historical piece of research pretend to add to existing literature? It provides a principle that is in tune with most recent research on the methodology of GET, as exemplified in Hands (2016). It argues that the mathematical economists involved in the search for stability theorems adopted a strategy that focused on the ability to provide an interpretative content to their theorems, which by itself was necessary to formulate ways of improvement and generalization. In this respect, the concept of substitutability offered a way to connect the properties of individual behaviors with system-wide assumptions (such as GS) and to appraise those assumptions as more or less satisfactory or promising, in consideration of the kind of interpretable modifications that can be elaborated upon, using the language of substitutability. In so doing, using economically interpretable and comparable sets of assumptions is presented as a criteria for valuating theorems, confronting them and fostering new research strategies; while at the same time it does not pretend to exhaust the reasons for interpreting those results with respect to the developments in other fields of GET. Even if the

language of substitutability would lead to some new results (in the 1970s-1990s), their valuation would become too weak in comparison with what was expected as a satisfactory assumption after the critical results obtained by Sonnenschein, Mantel and Debreu. The paper aims at putting some historical perspective on how the concept of substitutability failed to convey enough economically interpretable and fruitful content.

The paper is organised as follows. Section 2 deals with Hicks' *Value and capital* (1939) and its subsequent influence on stability issues until the middle of the 1950s. During this time span, stability is linked intimately with the search for comparative static results. It is a founding time for the heuristic of substitutability, and more precisely for the idea of a relation between substitutability and stability. (2. Stability and Substitutability: A Hicksian Tradition). With the axiomatic turn of GET, there are hopes for finding relevant conditions of stability. On the one hand, substitutability remains a good guiding principle, while on the other hand, the first examples of instability are presented, making findings of reasonable conditions of stability more urgent. (3. From Gross Substitutability to instability examples). The last time period in this story is much more uneasy and agitated. It is characterised with hidden pessimism and with difficulties in making substitutability a fruitful concept for stability theorems. One among other results, the SMD theorems come as a confirmation that the search for stability of the Walrasian tâtonnement is a dead-end. But as we will see, it is not the only result that played a role in the neglect of stability analysis (4. The end of a research program). As a conclusion, I provide an evaluation of the SMD results and of their consequences within the context of many other results (5. Concluding comments).

2 Stability and substitutability: a Hicksian tradition

In *Value and Capital* (1939), Hicks makes a systematic use of substitutes and complements to express stability conditions. He upholds a narrow link between stability and substitutability, giving to the concept of substitutability an explanatory value of the stability of market systems and praising its qualities to describe the main features of market interdependencies. This view would imprint the future of the search for stability conditions. I will first present Hicks' ideas on stability and substitutability (2.1 Stability and substitutability according to Hicks). Then, I show how a Hicksian tradition in GET was established in the 1940s and 1950s (2.2 A Hicksian tradition).

2.1 Stability and substitutability according to Hicks

Let's remind first some technical definitions. In *Value and Capital*, Hicks provides a definition of substitutes and complements on the basis of the Slutsky (1915) fundamental equation of value. The Hicks-Slutsky decomposition of the derivative of the demand for good i , x_i with respect to the price of a good j , p_j is:

$$\frac{\partial x_i(p, r)}{\partial p_j} = \frac{\partial h_i(p, u)}{\partial p_j} - x_j \frac{\partial x_i(p, r)}{\partial r} \quad (1)$$

with r the income of an agent, $x_i(p, r)$ the Marshallian demand for i , $h_i(p, u)$ the compensated demand (or Hicksian demand) for i where u is the (indirect) level of utility attainable with (p, r) , noted $v(p, r)$.

From (1) we say that i and j are net substitutes, independent or net complements if the change in the compensated demand of i due to a change in p_j is positive, null, or negative:

$$\frac{\partial h_i(p, v(p, r))}{\partial p_j} \begin{matrix} \geq \\ \equiv \\ < \end{matrix} 0 \quad (2)$$

From the equation (1) we say that i and j are gross substitutes, independent or gross complements if the change in the Marshallian demand of i due to a change in p_j verifies

$$\frac{\partial x_i(p, r)}{\partial p_j} \begin{matrix} \geq \\ \equiv \\ < \end{matrix} 0 \quad (3)$$

At an aggregated level, definitions (2) and (3) are usable for a general description of substitution between different markets, and equation (1) can serve to discuss the direction and the strength of revenue effects.

As can be inferred from (1), two goods may be locally net substitutes (resp. net complements) and gross complements (resp. gross substitutes) depending on the direction and magnitude of income effects in the Slutsky-Hicks decomposition. What is true at the individual level is also true at the aggregate. Hence, as is well known, the symmetry property of Hicksian demand functions $\frac{\partial h_i(p, r)}{\partial p_j} = \frac{\partial h_j(p, r)}{\partial p_i}$ is not true for Marshallian demands, except of course when income effects can be neglected.

In 1874, Walras had launched the idea of a sequential and iterative process—a tâtonnement—to model the price dynamics on competitive markets and to establish the possibility for such idealized markets to "discover" by groping the equilibrium—whose existence was theoretically assumed by the equality of equations and unknowns in the model. Walras would also connect the tâtonnement with some comparative static results.¹

In *Value and Capital*, Hicks reinstates Walrasian general equilibrium analysis, which had been deemed fruitless by Marshall.² This renewal of interest for general equilibrium, it is worth noting, arises precisely from the availability of new tools to analyze choice and demand, notably the Slutsky equation, hence also the distinction

¹Walras (1874), after discussing the tâtonnement when the price vector is not at equilibrium, applies the same technic to discussing the effects of a simple change in the parameters of the model, e.g. a change in the initial endowment in one good to one agent. The whole set of results—a mix of stability and comparative statics—is *precisely* what Walras calls the "Law of supply and demand"

²Actually, the mathematical analysis of stability was already published in 1937 in the booklet presenting the appendix of *Value and capital, Théorie mathématique de la valeur en régime de libre concurrence* (Hicks, 1937).

between income and substitution effects and the new definition of substitutes and complements build from it.³

In Hicks's view, even more certainly than for Walras, there is no doubt that the law of supply and demand leads the economy to an equilibrium. Hicks will follow Walras's reasoning on stability, with the aim of providing a precise mathematical account for it and to discuss with much more attention the effect of interdependencies between markets. Since the first part of the analysis proceeds in an exchange economy, the Slutsky equation then becomes:

$$\frac{\partial z_i(p, r)}{\partial p_j} = \frac{\partial h_i(p, u)}{\partial p_j} - z_j \frac{\partial x_i(p, r)}{\partial r} \quad (4)$$

This leads to the well-known distinction between perfect and imperfect stability and its mathematical treatment in the Appendix of *Value and capital*. Consider the Jacobian matrix of the normalized system of n goods, that is, the matrix JZ containing all the cross derivatives of excess demand functions relative to all prices $[z_{ij}(p^*)]$, $i, j \in [1, \dots, n]$ (the price of good $n + 1$ being set equal to 1). Stability will be perfect if the principal minors of JZ , calculated at equilibrium p^* , alternate in sign, the first one being negative. The system is imperfectly stable if only the last of these determinants respects the sign condition.

Hicks's analysis proceeds from the generalization of the results obtained in a two-good economy. He thinks that, except for particular cases, income effects to buyers and sellers on each market should tend to compensate each others:

Therefore, when dealing with problems of the stability of exchange, it is a reasonable method of approach to begin by assuming that income effects do cancel out, and then to inquire what difference it makes if there is a net income effect in one direction or the other. (Hicks, 1939, 64-65)

Thus, actually, through this thought experiment, the Jacobian of the system is identical to the matrix of substitution effects (the Slutsky matrix) since income effects on each market—i.e. associated to each price variation—are assumed to cancel out. And after a rather clumsy discussion about introducing income effects in the reasoning, Hicks comes to the following conclusion:

To sum up the negative but reassuring conclusions which we have derived from our discussion of stability. There is no doubt that the existence of stable systems of multiple exchange is entirely consistent with the laws of demand. It cannot, indeed, be proved a priori that a system of multiple exchange is necessarily stable. But the conditions of stability are quite easy conditions, so that it is quite reasonable to assume that

³Hicks and Allen (1934) provided the state of the art of the ordinalist theory of choice and demand, obtaining independently of Slutsky (1915) a decomposition of the single price effect on demand into two effects (Chipman and Lenfant, 2002). They also corrected Pareto's insufficiencies (Pareto, 1971) as regards the definition of complements and substitutes, which implied to recognize them as a relationship between two goods as regards a third one (or money). Slutsky (1915) did not provide a new definition, which, he suggested, would have been disconnected from human feelings

they will be satisfied in almost any system with which we are likely to be concerned. The only possible ultimate source of instability is strong asymmetry in the income effects. A moderate degree of substitutability among the bulk of commodities will be sufficient to prevent this cause being effective. (Hicks, 1939, 72-73, emphasis mine)

What kind of substitutability is referred to here? That the goods are net substitutes to one another. Consequently, the argument goes, symmetrical revenue effects at the aggregate level will have only a weak effect compared with the aggregate substitution effect, so that the Jacobian matrix is approximately symmetric. In so doing, Hicks develops a descriptive and explicative point of view on stability, and substitutability is given a prominent role. Substitutability is entrusted to produce a stylised representation of the interdependencies between markets, likely to receive a validation a priori. Thus, the idea that substitutes are dominating over the system is regarded as a natural and virtuous property of the economic system.

In the wake of Samuelson's discarding of Hicks' mathematical treatment of stability, there has been a tendency to evaluate Hicks' analysis of stability exclusively from the standpoint of the mathematical apparatus of *Value and capital*, i.e. from the perfect/imperfect stability distinction, with a view of pinpointing its wrong mathematical conclusions. Instead, to our story, it is worth insisting that Hicks' reasoning in the text provides insights about the importance of substitutability as a structuring device. The heart of Hicks' reasoning, actually, is a discussion of interdependencies in a three-good case.

The gist of the discursive argument about stability of multiple markets in *Value and Capital* is in Chapter V, §§4-5, after Hicks' introduction of the distinction between perfect and imperfect stability. It is to be noted also that the perfect/imperfect stability distinction is aimed at being powerful to consider cases of intertemporal non clearing Keynesian equilibria.

Here, the main question is whether an intrinsically stable market (say of good X) can be made unstable through reactions of price adjustments on other markets (themselves being out of equilibrium following the initial variation of the price p_x and the subsequent reallocations of budgets). The interactions between markets for X and Y (T being the third composite commodity) are first studied under the assumption that net income effects can be neglected. Hicks discusses the case of price elasticities of excess demand for Y on the excess demand for X and T . He ends with the reassuring idea that in the three-good case, and even more when the number of goods widens, cases of strong complementarity are seldom and X will most of the time be "mildly substitutable" with most of the goods constitutive of the composite commodity T .⁴

Hence, the whole discussion of the mathematical apparatus is conducted through the idea of neglecting asymmetric income effects, focusing on complementarity relations to deal with instability and on a reasonable spread of substitutability to ensure stability. This latter argument would be corrected in the second edition of *Value*

⁴If the the market for X is unstable taken by itself, price reactions will tend to increase market disequilibrium, hence it will not be made stable through reactions with other markets (Hicks (1939), 71-72, §5). Again, this result would probably be different if a number of market interactions are taken into account.

and Capital. Our point, here, is that from a heuristic or interpretative standpoint, Hicks' overall discussion is biased not specifically by its mathematical treatment, which is constrained by focusing on symmetrical systems; it is also biased by the strong separation between discussion of net complementarity and substitution on the one side and discussion of income effects on the other side. Hicks' discursive focus on complementarity has two opposite effects. First, it introduces the language of substitutability as the prominent device to discuss stability issues (as will be the case again when discussing intertemporal equilibrium). Second, it isolates the analysis of substitutability from the one on income effects, thus introducing a strong separation between arguments in terms of income effects and arguments in terms of substitutability to deal with stability analysis.

2.2 The establishment of a Hicksian tradition on stability analysis

Hicks' analysis of system stability was first challenged by Samuelson (1941; 1942; 1944)⁵, who rejected his method and results. Samuelson's criticism was the starting point for a series of restatements of Hicks' intuitions by Lange (1944; 1940), Mosak (1944), Smithies (1942) and Metzler (1945). Hicks' views and intuitions were partially saved, pointing out its usefulness to think about stability issues. This led to establishing the language of substitutability as a heuristically fruitful concept to think about stability of general equilibrium. However, Hicks' narrow view which concentrated on net substitutability was abandoned and the analysis would now be conducted in terms of gross-substitutes and complements

According to Samuelson, Hicksian stability is a static approach to stability. It consists mainly "to generalize to any number of markets the stability conditions of a single market" (Samuelson, 1947, 270). Instead, Samuelson (1941) proposes the first true dynamic expression of Walrasian tâtonnement as a simultaneous process of price adjustment on markets. It takes the form of a differential equation system, $\dot{p}_i = H(z_i(p))$, H being a monotonic positive function (actually, Samuelson considers the simple case $\dot{p}_i = z_i(p)$). Such a system is stable if and only if real parts of the eigenvalues associated to the Jacobian $[Z_{ij}]$ of the system are all negative.

More precisely, Samuelson introduces a distinction between local stability and global stability of an equilibrium price vector p^* . The former obtains when applying the differential equation to a system of demand functions starting from the neighborhood of p^* tends to p^* . Global stability obtains if the price path tends to p^* starting from any initial positive price vector. In the local stability case, the Jacobian can be approximated by linearization around p^* and the tâtonnement process will be stable if and only if the real part of all the characteristic roots of the matrix are strictly negative. Samuelson would show that the conditions of Hicksian perfect stability are neither necessary (Samuelson, 1941) nor sufficient (Samuelson, 1944) for local linear stability (the same, a fortiori, for Hicksian imperfect stability). Hence, to Samuelson, Hicksian matrices (ie matrices with principal minors alternating in sign, starting with a negative sign) are not a sound starting point for thinking about stability. The main lesson to be drawn from Samuelson's analysis is that taking

⁵see also Samuelson (1947) which contains in substance the three articles mentioned

the income effect seriously is indispensable to making a serious analysis of stability. However, one weakness of Samuelson's analysis of stability is that he does not strive to provide interpretable conditions of stability. In the time period following immediately Samuelson's reformulation of stability conditions, Metzler, Lange, Mosak and Smithies would rework the Hicksian intuition in line with Samuelson's mathematical apparatus.

Smithies (1942), apparently independently of Samuelson, discussed the case of stability of a monopolistic price competition economy and arrived at different necessary and sufficient properties of the roots of the characteristic polynomial of his system, (which, by the way, followed a sequential process of adjustment): those roots should be less than unity in absolute value. Interestingly, he noted that the advantage of his method over Samuelson's is that his result "leads more readily to general economic interpretation than Mr. Samuelson's method" (Smithies, 1942, 266)

As for Mosak, Lange and Metzler, their investigations on the stability of economic equilibrium circa 1942-1945 can be interpreted as a series of work aiming at developing a Hicksian method in the analysis of Keynesian economic ideas, focussing on the general equilibrium framework and promoting the idea that unemployment can result from intertemporal durable underemployment of some resources. Those various contributions will deal with international trade, imperfect competition, monetary theory and financial behaviors to build on Hicks's intuitions. Passages dealing with the stability of general static equilibrium are occasions to adapt Hick's results to the modern treatment of the price adjustment process provided by Samuelson.

Lange's (1944) *Price Flexibility and Employment* is probably the most representative account of those various attempts. He had already identified that the theory of complementarity would be a debated topic (Lange, 1940). The main point here is that the interplay of markets depends partly on individual's behaviors towards money. Indeed, Lange (1940) provided a systematic analysis of complementarity relationships at the market level. He refrains from Hicks's tendency to identify complementarity as a possible cause of instability. Lange also introduces a notion of partial stability of order m , expressing the fact that a system can be stable for a given subset of m prices that are adjusted ($m < n$). He discusses Hick's dynamic stability conditions and notes that since Samuelson leaves out the derivative of the function H in the characteristic determinant, it is tantamount to assuming that the flexibility of all prices is the same. He highlights that Hicksian stability makes sense in case when the Jacobian (the characteristic determinant) is symmetric: then, all roots are real, and the Hicksian conditions are necessary and sufficient for perfect stability.⁶ The meaning of symmetry, he goes on, is that "the marginal effect of a change in the price p_s upon the speed of adjustment of the price p_r equals the marginal effect of a change in the price p_r upon the speed of adjustment of the price p_s " (Lange, 1944, 98). Thus, stability analysis does not require an equal speed of adjustment on each market, but that the effects of a price change (dp_r) upon the speed of adjustment on another market $\frac{dp_s}{dt}$ are symmetric.

Mosak's exposition of the theory of stability of the equilibrium in *General-*

⁶Symmetry of the characteristic determinant of order m implies (and requires) symmetry of all its principal minors

Equilibrium Theory in International Trade also discusses the flaws of Hick's stability analysis. Its main merit, in this respect, is to operate a shift in the interpretation of stability. Instead of focusing on the symmetry properties of the Jacobian, the analysis of stability now revolves around the properties of excess demands, which can be conducted either in terms of gross-substitutability vs gross-complementarity or in terms of asymmetrical vs symmetrical income effects.⁷ ... If the rate of change of consumption with respect to income is the same for all individuals then this net income effect will be zero. In order that the net income effect should be at all large, $\frac{\partial x_s}{\partial r}$ must be considerably different for buyers of x_s from what it is for sellers. It is not too unreasonable to assume therefore that ordinarily the income effects will not be so large as to render the system unstable" (Mosak, 1944, p.42).

Mosak would also mention the assumption that usually, goods that are net substitutes are also gross-substitutes (Mosak, 1944, p.45).

Metzler (1945) established that under gross substitutability (GS), the conditions of Hicksian stability are the same as the conditions for true dynamic stability. Metzler insists that Hicks's analysis of stability aims at giving some ground to comparative statics results by providing a theory of price dynamics when a system is out of equilibrium. The conclusion that lends itself from Samuelson's results is that "Hicksian stability is only remotely connected with true dynamic stability" (Metzler, 1945, 279). However, Metzler's argues, "the Hicks conditions are highly useful despite their lack of generality"(Metzler, 1945, 279):

In the first place ... Hicks conditions of perfect stability are necessary if stability is to be independent of ... price responsiveness. Second, and more important, in a certain class of market systems Hicksian perfect stability is both necessary and sufficient for true dynamic stability. In particular, if all commodities are gross substitutes, the conditions of true dynamic stability are identical with the Hicks condition of perfect stability. (Metzler, 1945, 279-280).

The idea to take into account speeds of adjustment on each market is congenial to Samuelson's dynamic stability conditions and was further identified as a defect of Hicks' analysis by Lange (1944). This point is quite interesting since it illustrates how some properties of a mathematical tool can be entrusted with important descriptive qualities. Clearly, imposing that a price-adjustment should not be independent of the speed of adjustment may be taken as a gain in generality in some sense, but to some clearly it was not and it would appear as an unnecessary constraint.⁸

Given that the knowledge of speeds of adjustment is likely to be dependent upon specific institutional properties of an economic system, it is desirable to for-

⁷Stability can be destroyed only if the market income effects are sufficiently large to overcome the relationships which prevail between the substitution terms. It cannot be destroyed by any possible degree of complementarity.

⁸Smithies (1942) analyses the stability of a monopolistic competitive framework. Starting from profit maximization conditions of n producers with their own market demand expectations. Each producer will change its price according to a continuous adjustment process proposed by Lange (1935) taking into account the difference between the last period price expectation and the price of the last period.

ulate stability conditions in terms that are independent of such speeds. For all that, the fact that Hicks conditions of perfect stability are necessary in his case does not make them sufficient for stability. At least, when the assumption of gross substitutability is made, Hicksian stability is necessary and sufficient for true dynamic stability. Metzler agrees that this property may not be useful since "almost all markets have some degree of complementarity" (Metzler, 1945, 291). Hence ignoring gross complementarity in the system can lead to "serious errors" (Metzler, 1945, 284). However, Metzler's feeling is in tune with other researchers interested in stability analysis, and it is upholding the interest of Hicks fundamental intuitions:

It is natural to speculate about the usefulness of these conditions for other classes of markets as well. The analysis presented above does not preclude the possibility that the Hicks conditions may be identical with the true dynamic conditions for certain classes of markets in which some goods are complementary. Indeed, Samuelson has previously demonstrated one such case [Samuelson, 1941, 111]; ... Further investigation may reveal other cases of a similar nature. In any event, an investigation which relates the true stability conditions to the minors of the static system will be highly useful, whether or not the final results are in accord with the Hicks conditions. (Metzler, 1945, 292)

In a few words, the main contribution of the Metzler analysis (together with those mentioned above) is to introduce once more the concept of substitutability into the analysis of stability, and to focus on the interpretative content of the analysis. The Hicksian tradition is reformulated around the gross substitutability hypothesis. And this hypothesis is taken as a fruitful point of departure.

To make a conclusion on this first group of works on stability, one can say that the concept of substitutability has been worked out in the 1930s and 1940s so that it can be used to describe and to interpret the main stability properties of an economic system. So, beyond the mathematics of stability, there is an interpretative content and a "positive heuristic" attached to substitutability. It is enhanced first by the idea that substitutes are good for stability (Hicks), and then, following Samuelson's criticism, that income effects should not be so distributed as to disturb the stabilizing properties of symmetric systems, which implies in turn to consider that net substitution will dominate over income effects. Hence, even if a theory of aggregate income effects is needed, most of the arguments and the dynamic of research will take gross-substitutability as the starting point for further results.

It was so much shared by the researcher involved in GET at the time that Newman could write more than a decade later:

A good deal of the work on the analysis of stability has been directed towards establishing intuitively reasonable—or at least readily comprehensible—conditions on the elements of A , that will ensure stability. (Newman, 1959, 3)

Research in this direction was on the tracks since the beginning of the 1950s. It was notably explored by Morishima (1952) by formulating a stability theorem

when some complementarity is introduced into the model. A Morishima is characterized by a complementarity-substitutability chain hypothesis, (CS): Substitutes of substitutes (and complements of complements) are substitutes, and substitutes of complements (and complements of substitutes) are complements.

Morishima derives a number of theorems from such a system, notably that dynamic stability conditions are equivalent to the Hicksian conditions for perfect stability. This result, in turn, was important for establishing the interest of Hicksian stability conditions despite Samuelson's criticism. However, Morishima's analysis introduced new unexpected constraints regarding the choice of a numéraire and would lead to further comments in the next decades.

Now, by the end of the 1950s, the idea that stability should be analyzed through the properties of the matrix representing the derivatives of excess demand functions relatively to prices was well established. Moreover, research focused on systems implying gross substitutability or on systems whose content could be described as properties of the $[z_{ij}]$ involving substitutability. The existence and optimality theorems of the 1950s put stability issues in the background for a few years, only to surface in the end of the 1950s in a more serious form. Now, the question is: What will be left of this analysis of stability after the axiomatic turn in general equilibrium theory, once the issue of global stability will become central?

3 From “gross-substitutability” to instability examples (1958-1963)

The turn of the 1960s represents the heyday of the research on the stability of a system of interdependent markets connected through a simple dynamics of price adjustments, the Walrasian tâtonnement. At this moment in time, the use of the language of substitutability is structuring research towards theorems of stability. Morishima even proclaimed that "Professor Hicks is the pioneer who prepared the way to a new economic territory—a system in which all goods are gross substitutes for each other" (Morishima, 1960, 195). Indeed, work on stability of general equilibrium was rather limited in the 1950s, researchers being more focussed on existence and welfare theorems, and there was a sudden boom by the very end of the 1950s. The time span between 1958 and 1963 is fundamental both for the structuring of the research on stability and the importance of the language of substitutability as the main interpretative device to think about stability.

In this section, I would like to put to the foreground the mode of development of general equilibrium analysis after Arrow-Debreu-McKenzie theorems of existence. I will begin by enhancing the intuitive privilege that was attributed to the stability hypothesis, and as a consequence, the interest for the gross-substitutability assumption (2.1. “Gross substitutability” as a reference assumption). Then, I will focus on instability examples and the way the results have been received by theoreticians (2.2 Scarf and Gale's counter-examples). The discussion of alternative sufficient conditions of stability are then discussed (2.3 Gross Substitutability, Diagonal Dominance and WARP)

3.1 “Gross substitutability” as a reference assumption

Most of the work on stability in the fifties and sixties is centered on the hypothesis of gross substitutability. It is a sufficient hypothesis for unicity of equilibrium (Arrow and Hurwicz, 1958). It is also the hypothesis with which Arrow, Block and Hurwicz established the global stability of the tâtonnement in 1959. This result was presented as a confirmation of the importance for stability of the substitution among goods. Let’s make a short digression on the status of concept and hypothesis within the axiomatic phase of general equilibrium theory. Axiomatisation is usually at odds with the interpretative content of the concepts and hypothesis (Debreu, 1986; Ingrao and Israel, 1990). The question is thus whether the heuristic properties of substitutability should remain relevant in this context. The answer is yes. Leaving aside the relevance of the tâtonnement as a descriptive tool, the fact is that most of the theoreticians, I mean those who were interested in the work on stability, tended to consider that the concepts and assumptions used should have some heuristic properties and descriptive qualities. This aspect of the work on stability, compared with other fields of general equilibrium theory, is hardly underlined (Hands, 2016). In any case, it is certainly a key to study the development of stability analysis and to understand the reactions of the main protagonists. Otherwise stated, everything happens as if the descriptive content of general equilibrium was not only at the level of the dynamic process but also at the level of the properties of the excess demand functions giving stability. In this sense, substitutability plays a heuristic role in the stability analysis, in conformity with Hicks’s ideas. It is also to be mentioned that some theoreticians have always privileged a use of axiomatics bounded by the constraints of providing interpretable theorems. This is exemplified in Arrow and Hahn (1971) and it can be traced back to Abraham Wald (1936). The assumption of gross-substitutability, as such, could appear to any economist with a solid background in mathematics as the most natural assumption to obtain global or local stability of the price adjustment process. Indeed, GS appears in many studies on dynamic stability in the late 1950s. Let’s mention Hahn (1958), Negishi (1958), McKenzie (1960), Arrow and Hurwicz (1960), and the now classical Arrow and Hurwicz (1958) and Arrow, Block and Hurwicz (1959) articles.⁹ Arrow and Hurwicz (1958) testifies for the optimist flavor of the time. Under GS, homogeneity of demand functions and the Walras Law, they show that the tâtonnement process is globally stable. The proof makes use of Lyapunov’s second method to study dynamical systems. In this article, they show that certain kinds of complementarity relations are logically impossible within the framework of a Walrasian economy. This is taken as reducing the conditions of instability:

[The] theorem ... suggests the possibility that complementarity situations which might upset stability may be incompatible with the basic assumptions of the competitive process. (Arrow and Hurwicz, 1958, 550)

In the same time, the gross substitutability assumption is seen as not realistic. But gross substitutability is after all nothing more than a sufficient condition for

⁹We know from Negishi (1958, 445, fn) that his contribution and those of Hahn (1958) and Arrow and Hurwicz (1958) were prepared independently and submitted to *Econometrica* in between April and July 1957.

stability, and the field of investigation seems to be open for less stringent hypotheses, introducing complementarity. So, during the axiomatic turn, there is a slight epistemological shift in stability analysis. On the one hand, there is still the Hicksian idea that substitutes are good for stability, but it is quite clear that substitutes, as opposed to complementary goods, will not do all the work, and that the task will not be so easy to achieve. The fact is that generalization of the gross substitutability assumption (the weak gross substitutability) was not that easy to obtain. On the other hand, it is clear also that substitutability is still regarded as the most important concept in order to express stability conditions and to describe the structural properties of an economy. Neither the diagonal dominance, nor the weak axiom of revealed preferences in the aggregate caught that much interest (see below).

Through this theorem, Arrow, Block and Hurwicz were confirming the importance of GS for global stability after other results of local stability (Hahn, 1958; Negishi, 1958). Arrow and Hurwicz (1958) had already provided a proof for this theorem in a three-good economy. Even if GS could appear as an *ad hoc* assumption, given its strong mathematical properties on the price path, it was regarded however as a central assumption and a relevant and promising starting point for further inquiries. Other aspects of the Arrow, Block and Hurwicz contribution were regarded as strong results. Notably, the fact that the global stability was obtained both in an economy with or without numéraire (the normalized and non-normalized cases). Moreover, global stability implied that the price adjustment process was not necessarily linear: any sign-preserving adjustment was accepted.

On this occasion, we see that various constraints are likely to operate on the judgment about the quality or the importance of a result. It turns out that the superiority of having a theorem independent of the choice of a numéraire is something that could appear as justified by the search for the greatest generality, while its genuine significance from an interpretative content was not discussed or debated.

As a consequence of this optimism and of the heuristic content of substitutability, one can understand the situation of the work on stability at the end of the 1950s. The idea that it would be necessary to find out stable systems with complementarity was clearly identified. Enthoven and Arrow (1956) show that if A is stable, the DA is stable iff the diagonal elements in D are all positive. They address the limits of such a model:

In any actual economy, however, we must be prepared to find substantial, asymmetrical income effects and a goodly sprinkling of gross complementarity. It is desirable, therefore, to try to find other classes of matrices about which useful statements about stability can be made. (Enthoven and Arrow, 1956, 453)

For all that, due to its structuring role, there is a kind of benevolence towards the gross substitutability assumption. It is the task for unsatisfied people to prove that gross substitutability is not appealing, and that the concept of substitutability may not be enough to study stability. What makes the interest of this story is that counter-examples of unstable economies will arrive a few months later.

3.2 Scarf and Gale's counter-examples

The two important contributions of Scarf (1960) and Gale (1963) will shift the debate on stability. I will not enter precisely into their construction here. Just to go to the point of my analysis, they construct a general equilibrium model with three goods, based on individual rational agents, so that the tâtonnement process of the economy does not converge to the unique equilibrium. Scarf's example implies complementarities between two goods, and asymmetrical income effects. Scarf comments on his results, underlying that instability comes from pathological excess demand functions. Scarf's attitude towards this result is ambiguous. On the one hand, he asserts that "Though it is difficult to characterise precisely those markets which are unstable, it seems clear that instability is a relatively common phenomenon" (Scarf, 1960, 160). On the other hand, he gives some possible objections to the empirical relevance of his model:

As a final interpretation, it might be argued that the types and diversities of complementarities exhibited in this paper do not appear in reality, and that only relatively similar utility functions should be postulated, and also that some restrictions should be placed on the distribution of initial holdings. This view may be substantiated by the known fact that if all the individuals are identical in both their utility functions and initial holdings, then global stability obtains. (Scarf, 1960, 160–161)

Scarf's comment shows negatively how the language of substitutability makes sense to interpret results, be there positive or negative ones. The presence of complementarity in the system is a guarantee of descriptive relevance of the model. And Scarf goes even farther in suggesting that complementarity may be a cause for instability while a sufficient degree of substitutability may ensure stability.

As for Gale (1963, 8), he would insist on Giffen goods to explain the instability examples obtained. In line with this tendency to entrust substitutability with an explanatory power, the same kind of interpretation can be found in Negishi (1962) and also in Quirk and Saposnik (1968, 191), who are of opinion that the stability of a tâtonnement "is closely tied up with the absence of strongly inferior goods".

The different reasons invoked to comment the instability examples should not be overlapped. They stem also from the tendency to disconnect analytical cases. The appearance of a Giffen effect is linked to situations when substitution is difficult and is not independent with the specific situation of the some agents in terms of initial endowments compared with other agents in the economy.

Nevertheless, the Scarf and Gale examples are received with a kind of perplexity. Everything happens as if their models where singular models, and thus as if they were not affecting the general idea that systems including enough substitutability may be stable. At the same time, it is now felt urgent to find less stringent conditions including complementarity, guaranteeing stability. At this moment in time, the interpretative content of substitutability is at stake. With Scarf's and Gale's examples, the situation is reversed. The suspicion is now clearly on stable systems, and it is the task for all those who have a positive a priori in favor of stability to produce examples of stable systems including complementarity relations. In fact, Scarf's results make it possible to question the heuristic content of substitutability.

Actually, by identifying many possible sources of instability, relating to the spread of initial endowments and to the variety of preferences, and to their implications on demand, the interpretative and descriptive content of substitutability loses some ground. It does not seem possible to express only with substitutes and complements the characteristics of an economy, and its properties for stability. Nevertheless, substitutability remains the main concept with which it is thinkable to search for stability conditions. As a proof for this, it is remarkable that neither the diagonal dominance hypothesis, nor the weak axiom of revealed preference would be serious candidates for serving as a starting point to think about stability, at least in those years.

3.3 Discussing Diagonal Dominance and WARP in the aggregate

The general idea that we uphold here is of a methodological nature. Whereas some authors would tend to apply an external set of criteria for success and failure of a research program, we would like to focus on some complementary criteria to appraise the history of the research done on stability. Research on stability was structured on a set of soft constraints in terms of methods and tools to be used, which are regarded as more fundamentally in tune with the spirit of the general equilibrium research program. To name just a few, such soft constraints concern the choice of the price adjustment process, the interpretative and descriptive potential of conditions for stability, the relative importance of global vs local stability results, the search for results that are independent of the choice of a numéraire, a tendency to prefer models implying unicity of equilibrium. All those constraints are structuring the expectations and valuations of the results obtained. So far, we have seen that until the beginning of the 1970s, substitutability was able to meet a number of constraints and to offer a good starting point for a descriptive interpretation of GET. We have to discuss more in depth why substitutability was privileged compared with the assumptions of Diagonal Dominance and the Weak Axiom of Revealed Preferences in the aggregate.

The research programme on the stability of general equilibrium is very specific within GET, and it has consequences also on uniqueness and comparative statics because it is disconnected from direct microfoundations of the statements. Sets of conditions proposed as sufficient conditions for stability are related to market properties and the study of the microeconomic foundations of those properties is postponed. Meanwhile, stability conditions are valued according with their heuristic content or likelihood. Actually, no alternative condition on the properties of excess demands would appear as a promising alternative before the end of the 1950s. One such alternative is the Diagonal Dominance condition. It states that the terms of the matrix JZ are such that

(DD)

$$z_{ii} < 0 \quad \text{and} \quad |z_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^n |z_{ij}| \quad j = (1, 2, \dots, n) \quad (5)$$

This condition appeared at first in Newman (1959) but was independently explored by Arrow and Hahn¹⁰. This condition states that the effect of a price change of good i on the excess demand for good i must be negative and greater in absolute value than the sum of absolute values of the indirect effects of the same variation in price of good i on the excess demands of all other goods.¹¹

In some sense, DD has much to be recommended. It is less stringent than gross substitutability, because (GS) implies (DD). But practically, no utility function has been found that may imply diagonal dominance without implying also gross substitutability. Moreover, only certain forms of diagonal dominance do guarantee stability. Then, it seems easier to provide less stringent conditions by taking gross substitutability as a starting point that can be amended and weakened than by taking diagonal dominance as a starting point. Other reasons can also explain why DD was not taken as an interesting basis for research on stability in those years. Actually, one can figure out the economic content of DD, expressing that the own price effect on a market dominates the whole set of indirect effects from other prices. On second thought, it turns out that this idea of domination involves some quantitative property that are better avoided. At least, it is in those terms that general equilibrium theorists conceived of the search for general theorems. In this respect, as long as one could hope to find satisfactory results only with qualitative assumptions, quantitative constraints, interpretable as they may, were not favored. Moreover, it does not seem easy to use DD as a starting point to search for less stringent assumptions. For instance, Arrow and Hahn would point out that it has a “Marshallian flavor” (Arrow and Hahn, 1971, 242) and that it does not carry with it enough heuristic power. Such views on DD would change later on, as the set of constraints would weaken.

What about WARP and stability? It was already known since Wald that WARP in the aggregate was a sufficient condition for uniqueness of equilibrium. Arrow et al. (1959) showed that WARP is a sufficient condition for local stability. Actually, WARP is a necessary and sufficient condition for the uniqueness of equilibrium in certain cases. GS thus implies WARP, with the advantage that the GS property is preserved through aggregation while it is not the case for WARP (in a more than three-good case). Actually, those relationships would not be discussed in the 1960s, hence DD appeared as the only alternative starting point for discussing about stability, even though research on stability was somehow disconnected from the immediate search for microfoundations of the assumptions made on the properties of excess demand functions.

¹⁰"This condition appears to be new in the literature on general equilibrium, although Dr. Frank Hahn has informed me that he and Kenneth Arrow have used it in some as yet unpublished work. It is common in the mathematical literature." (Newman, 1959, 4)

¹¹An alternative statement for (DD), (DD') is that the own price effect z_{ii} be greater in absolute value to the sum of all cross price effects from the variation of the prices of other goods $|z_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^n |z_{ij}|$ $i = 1, 2, \dots, n$. A JZ satisfying both DD and DD' is strongly dominant diagonal. Newman also mentions another set of stability conditions based on (DD)—quasi-dominant main diagonal—that was proposed by Hahn and Solow.

4 The end of a research programme?

So far, I have indicated how a general framework of interpretation of the work on stability of an exchange economy was constructed. As was seen in the first section, the idea to search for sufficient conditions introducing complementarity pre-existed to the Arrow Block Hurwicz result and to the Scarf and Gale counter examples. In this section, I want to focus on two different kinds of obstacles that were put on the road. Firstly, from an internal point of view. All the attempts that were made to generalise the gross substitutability assumption did not give many results. What is clear from Scarf and Gale is that it was no more possible to introduce complementarity arbitrarily (4.1 The impossible generalisation). From an external point of view, then, some work in the seventies and eighties questioned radically the research programme, as it had been formulated by Walras (4.2 Through the looking-glass, and what Sonnenschein, Mantel, Debreu, Smale and others found there).

4.1 The impossible generalization

This is an important point for my thesis. The time period following immediately the Scarf results shows that researchers have not much hopes to obtain much better than the Arrow et al. (1959) Arrow, Block and Hurwicz (1959) theorem. It is the true moment when the heuristic of substitutability failed and faded away within one decade. It is striking for instance how this set of conditions is treated in Arrow and Hahn (1971) General competitive analysis, a book which was (and still is in many respects) the state of the art of GET.

Actually, as we have mentioned before, the hopes to find stability results with complementarity relations was on the agenda from 1945 on (Metzler), and it had just been confirmed by the results on global stability. From the beginning of the fifties on, Morishima was working on this agenda (Morishima, 1952, 1954, 1960, 1970). Morishima's idea was to introduce complementarity relations between certain goods. He thus proposed an economy whose excess demands would verify that goods were grouped together so that all the substitutes of substitutes are substitutes to each other, and so that all complementary goods of complementary goods were complementary to each other. In the same spirit, McKenzie (1960) established the dynamic stability of an exchange economy in which certain sums of the partial derivatives of the excess demand with respect to the prices are positive, which allows introducing a certain amount of complementarity into the system.¹²

The above discussion on the relative merits of GS, DD and WARP is thus conditional on the kind of constraints that the theoretician takes as granted. This would certainly have an impact on further research (section 4). It was already apparent here and there in the 1950s and 1960s. For instance, McKenzie (1958) offers one of the first study in which GS implies global stability of the unique equilibrium. when there is no numéraire in the model. He also provides a model with a numéraire which makes it possible to consider the stability of a system in which certain weighted sums of the partial derivatives of excess demands with respect to prices are positive, a

¹²For any partition of the set of goods $J = (1, \dots, n)$ into two subsets J_1 and J_2 , we have $\sum_{i \in I_1} z_{ij_2} + \sum_{i \in I_2} z_{ij_1} > 0$ for all $j_1 \in J_1, j_2 \in J_2$

"natural generalization". McKenzie's comment shows that the descriptive potential of an assumption is linked with the choice of constraints. Indeed, the case in which some complementarity is introduced in the model leads only to a local stability result and acknowledges the multiplicity of equilibria, a situation which he finds descriptively adequate, i.e. in accordance with his own ideas about the stylized facts: "In this case, one must be content with a local stability theorem, but one hardly need apologize for that. Global stability is not to be expected in general" (McKenzie, 1958, 606)

Finally, Nikaido (1964) proposed the generalised gross substitutability assumption, i.e. that the sum of the symmetrical terms relative to the diagonal of the Jacobian be positive. In such a system, if tea is a gross complement to sugar, then sugar must be a gross substitute to tea. Some remarks on all these developments are in order: Firstly, all the results obtained have strong limitations relatively to the programme of general equilibrium theory. They are not independent of the choice of the *numéraire* good and they are valid only locally. For example, the Morishima case was showed to be incompatible with a Walrasian economy, because of the properties of the numéraire commodity. That stability should be invariant under a change of numéraire seemed "reasonable" (Newman, 1959, 4). Ohyama (1972) added a condition on the properties of substitution of the numéraire with other goods to ensure stability. Secondly, most of the results I have mentioned suppose that they refer to quantitative constraints on excess demand functions, in the sense that they suppose comparing the relative strength of partial derivatives. From this point of view, the diagonal dominance hypothesis goes in the same direction.

All these limitations illustrate the doubts that arose regarding the hopes for finding a true generalisation of the gross substitutability hypothesis. Indeed, this kind of quantitative constraint is something general equilibrium theorists would prefer to dispense with. At least this general view was not debated. Meanwhile, the heuristic of substitutability is shrinking. The question now is to discuss the relevance of quantitative and structural restrictions on excess demand functions. The change in the spirit and in the state of mind of the theoreticians can be clearly felt. Just to give a quotation by Quirk (1970) Quirk (1970), who focuses specifically on the limits of a purely qualitative approach to GET: "In contrast to the Arrow-Hurwicz results, here we do not prove instability but instead show that stability cannot be proved from the qualitative properties of the competitive model alone, ... except in the gross substitute case" (Quirk, 1970, 358). It is a very clear way to renounce establishing general properties compatible with stability. This reduces quite naturally the analytical appeal of substitutability as a single comprehensive tool to deal with stability issues (see also Ohyama (1972, 202). Finally, at this moment on, theoreticians have realised to what degree the gross substitutability assumption was specific, as the only *qualitative* hypothesis on excess demand functions guaranteeing stability of a tâtonnement process. Of course, it may be that some mathematical economists were perfectly aware that GS is too much an *had hoc* mathematical assumption to obtain stability, but still its specificity as a qualitative assumption was much better acknowledged by the end of the 1960s and beginning of the 1970s.

A word is in order regarding the presentation of the sufficient conditions for stability in Arrow and Hahn (1971)'s *General competitive analysis*. The presentation

of stability results in surveys such as Newman (1959) and Negishi (1962) were clearly transmitting the view of a progressive program of research, with a need to understand the links between different sets of conditions. Yet, Negishi introduced some more temperate view, both enhancing the GS assumption as concentrating the essence of the knowledge on stability and pointing out that due to unstability examples, theoreticians would better concentrate on alternate adjustment processes (such as non-tâtonnement processes).¹³ To sum up, in the beginning of the seventies, the work on stability gives a very pessimistic, and even negative, answer to the agenda originally formulated by Metzler and then by Arrow, Block and Hurwicz. Two kinds of results will come and evacuate a bit more any interest with this kind of work: the well-known Sonnenschein-Mantel-Debreu theorem on the one hand, and the Smale-Saari-Simon results on the other hand.

4.2 Through the looking glass and what Sonnenschein, Mantel, Debreu, Smale and others found there.

Already in the thirties it was known that some properties of individual demand behavior would not be preserved at the aggregated level in general (see for example Schultz et al. (1938) and Hicks (1939)). Clearly, there was a gap between weak restrictions on the demand side and stringent sufficient conditions for stability. So, while the work on stability was progressing only very slowly, and not with the results that were expected, a group of theoreticians was engaged in taking the problem from the other side, that is, from the hypothesis of individual maximising behaviour:

“Beyond Walras’ Identity and Continuity, that literature makes no use of the fact that community demand is derived by summing the maximizing actions of agents” (Sonnenschein, 1973, 353)

If it is not possible to demonstrate that an economic system with complementarity relations among markets is stable, is it not possible to show that any general equilibrium system based on rational agents exhibits some properties regarding the excess demand functions. This would be at least a way to "measure" the gap between what the logic of GET gives us and what we expect from it in order to arrive at stability theorems. The answer to this question is well known. It is a series of negative results known as Sonnenschein-Mantel-Debreu theorems or results¹⁴. Market excess demand generated by an arbitrary spread of preferences and initial endowments will exhibit no other properties than Walras Law and the Homogeneity of

¹³Negishi’s reaction to Scarf examples is interesting in its way to put the emphasis on the choice of the "computing device" and not on the interpretative content of stability conditions. This is another instance of the fact that the proper balance between different attitudes regarding the research program was not discussed and can only be grasped here and there from passing remarks: "We must admit that the tâtonnement process is not perfectly reliable as a computing device to solve the system of equations for general economic equilibrium. It is possible to interpret these instability examples as showing that the difficulty is essentially due to the assumption of tâtonnement (no trade out of equilibrium) and to conclude that the tâtonnement process does not provide a correct representation of the dynamics of markets." (Negishi (1962, 658-9))

¹⁴SMD theorems are named after a series of articles published in 1972–1974.

degree zero of excess demands relative to prices. Otherwise stated, given an arbitrary set of excess demands, one can always construct an economy that will produce those excess demands. The question raised by Sonnenschein, Mantel, Debreu and others goes against the usual stream of investigation concerning stability. But it is the most natural stream in terms of the individualistic methodological foundations of the general equilibrium program. Nevertheless, this result raised some perplexity from the field of econometrics. After all, the distribution of endowments and of preferences allowing for such arbitrary excess demand in the aggregate may well be as much (or even more) unrealistic as the ones generating a representative agent (Deaton, 1975). Kirman and Koch (1986) showed that the class of excess demands would not be restrained even if the agents had the same preference relations and co-linear endowments. To improve further on the constraints would mean to construct a representative agent. So, the SMD result would imply that Giffen goods are quite “normal” goods in a general equilibrium framework, and following Scarf and Gale conclusions, “instability” would be a common feature of economic systems. Then, the S-M-D theorem reduces still a bit more the relevance of quantitative restrictions that would yield stability. The change in the spirit of the economists has been portrayed by Mantel:

“Another field in which new answers are obtained is that of stability of multimarket equilibrium. It is not so long ago that the optimistic view that the usual price adjustment process for competitive economies is, as a rule, stable, could be found—an outstanding representative is that of Arrow, Block, and Hurwicz (1959). Counterexamples with economies with a single unstable equilibrium by Scarf (1960) and Gale (1963) had a sobering effect, without destroying the impression that the competitive pricing processes show some kind of inherent stability. Here the question arises whether such counterexamples are likely, or whether they are just unlikely exceptions” (Mantel et al., 1977, 112)

After the SMD results, Scarf and Gale counterexamples could no longer be regarded as improbable, if the excess demand should have arbitrary properties. But from a historical point of view, one must keep in mind that there was a twelve years gap between the reception of the SMD results and the strengthening of these results by Kirman and Koch 1986. What happened during that time span is also very fruitful for our inquiry. For all those who were discouraged by the turn of events, for those who had only a poor faith in the possibility to find satisfactory theorems, the Scarf counterexamples were a starting point for something else. We have seen that Scarf himself felt uncomfortable with the instability result, and that he felt that some disturbing cause of instability may have been arbitrarily introduced in the model. This was the starting point for an inquiry into dynamic systems and algorithmic computation of equilibrium (Scarf and Hansen, 1973). In this field of research, Steve Smale endeavored to cope with the question of stability. His purely mathematical look at the subject kept the interpretative content outside, and he readily understood that in general equilibrium “complexity keeps us from analysing very far ahead” (Smale, 1976c, 290). Rather than concentrating his reproaches on the descriptive content of the tâtonnement process and on the stability conditions that

were found, Smale tackles another question, quite different from that of Sonnenschein, Mantel and Debreu. If equilibrium exists, how is this equilibrium reached? After Scarf and Hansen (1973), Smale (1976) will found a dynamic process much more complex than Walrasian tâtonnement which allows finding the equilibrium, for any arbitrary structure of the excess demand functions. This process, the Global Newton method, is a generalisation of a classical algorithm of computation of equilibrium. In this process, the variations in the prices on each market $\frac{dp_i}{dt}$ will not depend solely on the sign of the excess demand $z_i(p)$ on this market, but also on the excess demands on other markets. This dynamic process is $Dz(\mathbf{p})\frac{dp}{dt} = -\lambda z(\mathbf{p})$

with λ having the same sign as the determinant of the Jacobian.

Smale's shift in the way to attack the issue of stability is of interest when confronted with the constraints that general equilibrium theorists had put on the research program, focusing on Walrasian tâtonnement as a neutral dynamics, whose advantage came essentially from the mathematical simplicity in handling it.

Hahn's (1982) reaction to this kind of process is embarrassed. Indeed, Smale's process is shifting the general equilibrium program. What can be the meaning of a dynamic process in which the behaviour of the prices on each market depends of the situation on every other market? Hahn does not have any answer to give. The fundamental problem is that this process is very demanding in terms of information. While the Walrasian auctioneer does not have anything else to know than excess demands at a given price vector, the fictional auctioneer of the Global Newton method will have to know about the qualitative properties of each excess demands. Saari and Simon (1978) established that this amount of information was the price to be paid to find a computational method independent of the sign of the excess demand. This is precisely this kind of information that the use of a Walrasian tâtonnement dynamics aimed at ignoring. With the Sonnenschein-Mantel-Debreu and Kirman and Koch's theorems on the one hand and with Smale, Saari and Simon's results on the other, the stability research program, in its original form, has collapsed.

It is not the purpose of the present study to tell the details of all the escapes from SMD (Lenfant, 2010). As far a substitutability is concerned, it is worth noting that it appears here and there.

A first consequence of the Scarf-Smale escape from stability issues is that the concept of substitutability becomes at best too weak to serve as a descriptive basis of the properties of stable systems, at least as long as the search for global stability theorems is at stake. Slowly, the condensed structure of constraints pertaining to the research program on stability has disaggregated. For instance, the SMD results have destroyed the perspective to search for reasonable conditions on uniqueness and global stability. Hence, research has been reoriented towards different perspectives, either to concentrate on the algorithm that permits to calculate equilibria (this is the Scarf perspective) or to concentrate on local stability results in various frameworks. It is not the purpose of this article to discuss various ways to react to SMD. Work on the stability of a Walrasian type price adjustment process has led to some new results regarding WARP and DD in relation to GS. To Hahn (1982) even though GS implies DD, there is practically no example of utility functions satisfying DD but not GS. Keenan and Kim (2013) show that DD is a sufficient condition for stability

in an unnormalized tâtonnement process.

The effect of SMD results on the research program on stability cannot be examined independently of their broader impact on the theory of general equilibrium. Once the perspective of obtaining uniqueness vanishes, the interest for local stability comes back to the forefront. Once the idea of treating money as a specific input into GET which cannot be dealt with endogeneously is well accepted, the idea of appraising results that are not independent of the choice of the numéraire may attract more attention. Whatever the global effect of SMD, it is still to be found that some research focus on substitutability as a stabilizing phenomenon. For instance, Keenan (2000) has established that the standard conditions for global stability of WT (either GS, DD or that the Jacobian is Negative semidefinite) "can be translated into ones that need be imposed only on the aggregate substitution matrix" (Keenan, 2001, 317) do depend exclusively on substitution effects:

"Thus for each condition on the matrix of total price effects implying global stability, there is a corresponding one on only the matrix of compensated price effects which also implies global stability" (Keenan, 2001, 317)

Keenan's agenda may seem dubious, in its way to treat substitutability has a concept which is sufficient to support all the relevant information for understanding stability. At least, it could be taken as a remnant of the heyday of stability theory.¹⁵ To us, it is revealing of the still lively importance of the concept of substitutability as a heuristic device for discussing stability. Following a quite different agenda, Grandmont (1992) has established conditions on the interdependence of preferences within the economic system (increasing heterogeneity) with the result that sufficient heterogeneity leads to GS for a growing set of initial values of the price vector and to WARP in the limit, thus guaranteeing stability of the WT. But a different view could be held on the basis of a more tractable and applied approach to GET, in line with Scarf's agenda, such as the one upheld by Kehoe (1992). Kehoe argues that in production economies, the GS assumption loses much of its interest because there are cases of multiple equilibria. He focuses on the number of equilibria rather than on stability issues. In this framework, it is possible to construct economies with Cobb-Douglas consumers (hence well-behaved GS behaviors) and yet a production technology that generates several equilibria. In contrast, WARP (in the aggregate) implies uniqueness even in production economies (Wald, 1936). Hence, in production economies, GS does not imply WARP.

5 Final remarks

In this paper, my aim was to put to the foreground the uses of the concept of substitutability in general equilibrium theory. Substitutability, as the main concept used to describe the qualitative properties of an economic system, was expected to provide also good interpretative properties i.e. it was hoped that substitutability

¹⁵Note that on this occasion, Keenan favors the discussion of conditions on the Jacobian to the use of a Lyapunov function

would be a sufficient way to express general conditions under which the stability of the tâtonnement would be guaranteed. I have interpreted this very general idea as a guiding principle for the researches on stability. It was thought that substitutes and complements should represent enough information to formulate “reasonable” or “hardly credible” stability conditions. The point was then to see how this guiding principle, this positive heuristic, has been affected by the mathematical results that were found, and how it came to be deprived of its interpretative content. Of course, I do not pretend that substitutability was the only concept implied in the elaboration of the research programme. It is quite clear from my presentation that the formalisation of the Walrasian tâtonnement, the reflection on quantitative constraints, have also played a role in this story. They formed a complex system of rules to be followed and were themselves embedded in different representations of the purpose of GET. The present article does not pretend to have identified the one single way of interpreting the history of this research program and its connections with other aspects of GET. It has highlighted that within the development of GET, it is possible to identify descriptive heuristics that seem to have played a role in structuring the research agenda and the interpretation of the results. But in the final analysis, the concept of substitutability has served as a criterion in order to evaluate the relevance of most of the results and to appraise the theoretical consequences of those results on the research programme. It has been a tool for rationalising the path followed by stability analysis. From a methodological point of view, a conclusion that can be drawn from this study of stability is that the weakening of a research program and its reformulation within the framework of a purely mathematical theory, do not depend on a unique result, be it a negative result. The matter depends more pragmatically on the accumulation of many negative or weak results that come to be interpreted as a bundle of results indicating that something else must be done and that the programme must be amended. And it might be that the Sonnenschein-Mantel-Debreu result was not the most important result with regard to this amendment. In this respect, there was not any more important result, because it was more questioning, and it makes sense to us when connected to other results and to the general principles that dominated research on GET in the 1960s and 1970s. This overview, it is hoped, opens more fundamentally to a new representation of the development of GET based on simulations. Thus, a number of disruptive shifts from the original research program have changed the whole understanding of the tenets of GET and of the role played by different assumptions. The evolution of the theoretical involvements of theoreticians into the concept of substitutability offers, we think, a fruitful perspective on the transformations of a complex and intricate research topic such as GET.

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