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# Epistemological and methodological foundations of creating a learning trajectory of children's mathematics

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*This paper argues for creating learning trajectories of children's mathematics by integrating evidence of shifts in the mathematics of students, theory of goal-directed instructional design, and evidence of instructional supports. We networked two theories in support of this stance: a radical constructivist theory of learning, and Duality, Necessity, Repeated Reasoning (DNR)-based instruction. We exemplify how our networking of theories guided methodological choices by drawing on a program of research aimed at understanding and supporting students' ways of understanding quadratic growth as a representation of a constantly changing rate of change. We close by discussing challenges for creating and sharing learning trajectories.*

*Keywords: Learning trajectory, constructivism, cognition, learning theories, epistemology.*

## Learning trajectories in mathematics education

Attention to learning trajectories and progressions remains a prominent strand of research in mathematics education (e.g., Clements & Sarama, 2004). The influence of this domain of research is evident in the development of mathematics standards (e.g., National Governor's Association Center for Best Practices, 2010; UK Department of Education, 2009), funding priorities, topics conferences (e.g., the Third International Realistic Mathematics Education Conference), and special reports (e.g., Daro, Mosher, & Corcoran, 2011). Given the relatively wide-spread promulgation of trajectories and progressions in our discourse in research and practice, there is a need to articulate theoretical and methodological foundations of learning and teaching (Simon, 2013). This paper argues for the importance of creating learning trajectories as research-based models of teaching and learning that take seriously a commitment to understanding and supporting students' mathematics.

The notion of a learning trajectory has different meanings among mathematics education researchers. Simon's (1995) original discussion offered a description of a *hypothetical learning trajectory (HLT)* consisting of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). An HLT constitutes a starting point for task design, and is then modified into a learning trajectory based on empirical data, often in the form of a teaching experiment. Clements and Sarama (2004) described a learning trajectory as an elaboration of children's thinking and learning in a specific mathematical domain, connected to a conjectured route through a set of tasks. These definitions emphasize the construct as a tool for *hypothesizing* what students might understand about a particular mathematical topic and how that understanding may change over time in interaction with carefully designed tasks and teaching actions.

Building on this body of work, we argue that the main purpose of a learning trajectory is to effectively convey the relationships between teaching, task-design, and shifts in student

conceptions. We advocate for a stance that articulates an integrated system of ways of bringing about conceptual change in the mathematics of students in relation to theory-driven instructional support. Thus, in order to create learning trajectories, we need mutually informing theories of learning and teaching. This paper elaborates two such theories, the radical constructivist theory of learning and Duality, Necessity, and Repeated Reasoning-based instruction, and identifies how these two related theories influence our methodological approach to building learning trajectories. We provide an example situated in a rate of change approach to quadratic function.

## **Theoretical and epistemological framing**

### **A theory of learning mathematics**

In discussing the theory that guides our approach to crafting learning trajectories, we address three major issues: (a) distinguishing students' mathematics from the mathematics of students, (b) leveraging the epistemic student, and (c) leveraging a model of learning. Theory of instructional design is treated in the next section.

Learning trajectories articulate students' evolving conceptions within a particular instructional context. In order to do this, we distinguish from our own mathematics as researchers and teachers, our students' mathematics, and the models we create of our students' mathematics. The need to distinguish our own mathematics from students' mathematics is borne out of our epistemological stance. We consider mathematical knowledge to develop as part of a process in which children gradually construct and then experience a reality as external to themselves (von Glasersfeld, 1995; Piaget, 2001). From this perspective, knowledge is considered viable if it stands up to experience, enables one to make predictions, and allows for the enactment of desired objectives.

The term *students' mathematics* refers to the models students construct to organize, comprehend, and control their experiences – i.e., students' knowledge. The *mathematics of students* is the set of models we construct of our students' knowledge (Steffe & Olive, 2010). Often there is little distinction between these two notions in curricula or in standards documents. We believe, however, that the mathematical knowledge we attribute to students in the creation of a learning trajectory must be viewed as different from our own knowledge. Our goal is to determine how to engender and explain students' productive thinking. By distinguishing our mathematics from the students' mathematics, we recognize that students bring significant knowledge to bear when engaging in school mathematics, and we position students as logical, coherent thinkers and doers of mathematics. The job of establishing a learning trajectory then becomes one of explaining students' thinking in a way that portrays it as coherent and internally consistent (Steffe, 2004).

Individual students differ in their personal backgrounds, knowledge, and dispositions. A learning trajectory should depict not one particular student or group of students, but rather the *epistemic student*. An epistemic student is an organization of schemes that researchers build to explain students' characteristic mathematical activity and how that activity changes in the context of teaching (Steffe & Norton, 2014). Researchers construct epistemic students through teaching interactions with specific students, but epistemic students are not specific to those particular interactions. Instead we conceive them to be useful models of students' schemes that one can leverage to describe, explain, and predict the mathematical actions of similar students who may be

operating at the same level.

Formulating an explanation of changes in students' concepts and operations is not merely an empirical matter. We also bring to bear a set of conceptual tools in order to interpret students' activity and problem solving. These tools have their origins in the radical constructivist (RC) model of knowing (von Glasersfeld, 1995; Piaget, 2001). For the purposes of learning trajectory construction, we rely particularly on the constructs of mental operation, scheme, assimilation and accommodation, and abstraction. A *mental operation* is an internalized, reversible mental action that is an element of a larger structure, such as a scheme, constituted by the coordination of operations. A *scheme* is an organization of actions or operations which enables anticipation of results without having to engage in mental activity. As an example, Piaget (2001) described the mental operation of combining objects (such as addition). Several successive additions are the equivalent of a single addition (so one can compose additions), and they can be inverted into the operation of taking away, or subtraction.

Treating new material as something already known is an act of *assimilation*. When assimilating, one encounters an experience and incorporates it into a scheme. When the enactment of a scheme results in an unexpected outcome, a learner may experience perturbation or disequilibrium. One response can be a change in the learner's recognition, in effect spurring a reorganization of one's scheme. This reorganization is *accommodation*, which many consider to be the source of conceptual change.

### **DNR-based instructional design**

The theory of DNR-based instruction (*Duality, Necessity, and Repeated Reasoning*; Harel, 2008a; 2008b) informs our instructional design principles. Drawing on the RC theory of knowing, Harel (2008b) noted that "any observations humans claim to have made is due to what their mental structure attributes to their environment" (p. 894). He emphasized that researchers' observations are merely models of students' conceptions; using our language, these are models of the mathematics of students (Harel, 2013). Drawing on the mechanisms of assimilation and accommodation, Harel (2013) characterized knowing as a developmental process that proceeds through a back-and-forth between the two in order to reach equilibrium.

The duality principle addresses two forms of knowledge, *Ways of Understanding* (WoU) and *Ways of Thinking* (WoT). WoU can be thought of as subject matter, consisting of students' definitions, theorems, proofs, problems, and their solutions (Harel, 2008a). WoT are students' conceptual tools, such as deductive reasoning, heuristics, and beliefs about mathematics (Harel, 2013). The Duality Principle states that students develop WoT through the production of WoU, and, conversely, the WoU they produce are afforded and constrained by their WoT (Harel, 2008a). We contend that the mathematical content of a learning trajectory must be formulated in terms of both WoU and WoT.

The *necessity principle* states that in order for students to learn the mathematics we intend to teach them, they must have an intellectual need for it (Harel, 2008b). We can engender intellectual need through problematic situations that necessitate the creation of new knowledge in order to be resolved. Finally, the repeated reasoning principle addresses the need for teachers to ensure that their students internalize, retain, and organize knowledge (Harel, 2008a). Repeated reasoning should not be confused with drill and practice of routine problems. Rather, it is an instructional

principle that advocates providing students with sequences of problems that require thinking through puzzling situations and solutions; the problems must respond to students' intellectual need.

## Methodological approach and rationale

Our methodological approach to establishing learning trajectories is a direct consequence of our theoretical and epistemological framing. We elaborate how our networked theories (RC, DNR-based instruction) informed our methodologies. We describe three aspects: (a) leveraging theory to create an HLT, (b) ongoing refinement of an HLT into an LT through enacting a teaching experiment, and (c) finalization of an LT through retrospective analysis.

### Creation of an HLT

We enact a form of design-based research to simultaneously engender and study innovative forms of learning (Cobb & Gravemeijer, 2008). The planning phase involves creating an HLT (Simon, 1995) informed by the networking of the RC theory of learning and the theory of DNR-based instruction. Our HLT was a tentative progression of student concepts and associated tasks that we hypothesized would necessitate a WoU that quadratic functions represent a constantly-changing rate of change between two covarying quantities. Simultaneously, our aim was to support a WoT that functions can be representations of covariation and can be explored and understood through a covariational lens (Thompson & Carlson, 2017). Consequently, we devised a dynamic representation of proportionally-growing rectangles in which students could investigate situations that, to us, entailed the three continuously covarying quantities, height, length, and area (Figure 1).

The relationship between height,  $h$ , and area,  $A$ , can be expressed as  $A = ah^2$  where  $a$  is the ratio of length to height. We wanted the students to develop the following specific WoUs: (a) the rate of change of a rectangle's area grows at a constantly-changing rate for each same-unit increase in height (or length); (b) the rate of the rate of change of the rectangle's area is constant for same-unit height (or length) increases; (c) given a height  $h$ , the rectangle's area could be determined by  $ah^2$ ; and (d) the rate of the rate of change of area is dependent on the change in height. In order to engender these WoUs, we devised tasks in which students had to predict the nature of growth, determine areas for specific height values and vice versa, and decide whether given tables of values represented rectangles that grew in proportion to one another or not. See Ellis (2011) for an elaboration of the mathematics.

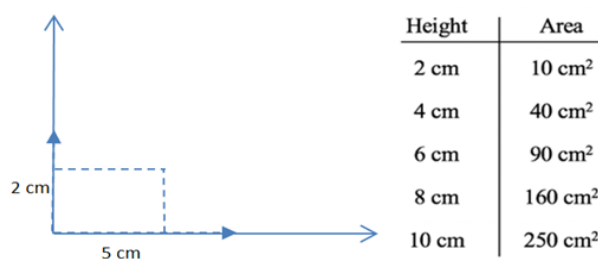


Figure 1: A growing rectangle and associated table of height-area values

### Teaching experiment: Ongoing refinement of an HLT into a tentative LT

Drawing from the theories we networked, we engaged in an in-depth, 15-day teaching experiment following the method of Steffe and Thompson (2000). We taught 15 lessons to a group of 6 middle-

grades students (ages 13–14) who were enrolled in pre-algebra (3 students), algebra (2 students), and geometry (1 student). The second author was the teacher-researcher (TR). One purpose of a teaching experiment is to gain direct experience with students’ mathematical reasoning, which affords the creation and testing of hypotheses about the mathematics of students in real time. This means that our mathematical tasks were not wholly predetermined, but instead were created and revised on a daily basis in response to hypothesized models of students’ mathematics. Because our problem context relied on area models, it was important to first identify the students’ existing schemes and operations for area. After conducting pre-interviews and developing an initial model of the mathematics of students, we created new tasks to necessitate more robust constructions of area as not dependent on whole-unit iterations. During and between each session, we engaged in an iterative cycle of (a) teaching actions, (b) assessment and model building of students’ thinking, and (c) task revision and creation on an ongoing basis. In this manner, during each session, we continually revised our HLT into a tentative, empirically based LT.

### Retrospective analyses: Finalizing an LT

In addition to our ongoing analysis, we relied on retrospective analysis to inform the development of a learning trajectory as a model of the mathematics of students (Steffe & Thompson, 2000). One purpose of retrospective analysis is to build a model of the epistemic student and to characterize students’ changing WoU and WoT throughout the course of the teaching experiment. A secondary purpose, for us, was to contextualize and explain changes in students’ schemes and operations with respect to the tasks and teaching actions they encountered. We aimed to elaborate features of tasks, teacher moves, questioning, and socio-mathematical norms that supported the students’ scheme accommodation. Our inclusion of instructional supports into a learning trajectory relied both on our analyses of empirical data and on our understandings of the local instructional theories grounding our design and enactment of the teaching experiment. Figure 2 identifies the goals, tools and constructs we leveraged in order to create a learning trajectory. Notice how the primary and secondary aims are linked to a coordination of RC and DNR-based instructional design theories.

Analytic Aim	Goals, Tools and Constructs
Primary Aim: Identify the mathematics of students	<i>Identify</i> WoU and WoT: Describe (a) students’ mathematical definitions, theorems, and concepts; (b) heuristics, reasoning, strategies, and beliefs about mathematics
	<i>Characterize</i> WoU and WoT: Identify operations and schemes
	<i>Exemplify</i> WoU and WoT: Provide data evidence of specific WoU and WoT
Secondary Aim: Contextualize and explain changes in the mathematics of students	Identify <i>shifts</i> in WoU and WoT: Articulate instances of scheme accommodation and abstraction
	Identify <i>task features</i> fostering shifts: Identify the WoU necessitated by each task; clarify relationships between tasks elements and students’ existing WoU
	Identify <i>instructional supports</i> fostering shifts: Determine teacher moves promoting accommodation and abstraction; characterize student engagement and discourse; identify socio-mathematical norms

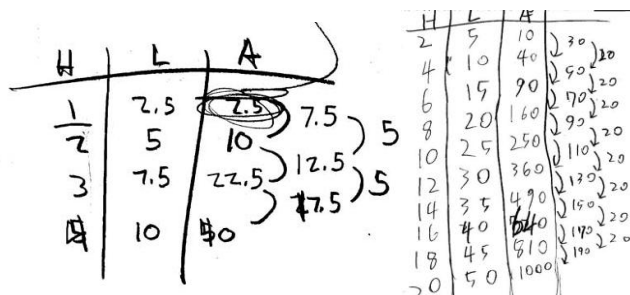
Figure 2: Analytic aims, goals, tools, and constructs for creating a learning trajectory

### An excerpt of a learning trajectory for quadratic function

In our previous work, we identified WoT and WoU in students’ learning of quadratic function from a rates of change perspective (Fonger, Dogan, & Ellis, 2017). Below, we provide an example of a

link between goal-directed instructional supports and a shift in student thinking (i.e., an excerpt of a LT). We focus on one shift from a student's WoU that (a) the rate of the rate of change of the rectangle's area is constant with  $\Delta x$  implicit, to (b) the rate of the rate of change of area is dependent on  $\Delta x$ , which is explicit. In the excerpt, the students had already created well-ordered tables for the growing rectangles (Figure 3). They had also attended to the constantly changing rate of change of area as determined by finding area increases for same-unit increases in, but there was a lack of coordination of change in area with change in height. Prompted to explain the growth in area of a 2 cm by 3 cm rectangle (Task A), one student said, "It goes 4.5 and then 7.5 and then 10.5 and then...just keeps going." In this manner, the students attended to the area's growth, but did not coordinate it with growth in height or length. In response, the TR prompted the students to draw diagrams of the growing area, predicting that the act of drawing would necessitate a coordination of height and area. The students' drawing activity did necessitate a coordination, but many students kept the change in height implicit, as evidenced by their language "every time." For instance, Jim drew a picture of a growing 2 cm by 3 cm rectangle and explained the rate of change of area as "how many new squares it's gaining *every time* it grows."

In an attempt to further encourage an explicit coordination of the rate of area with a quantified change in height, the TR asked the students to create a table for a 2 cm x 5 cm growing rectangle (Task B), anticipating that the students would make tables with different height increments. This did occur: For instance, Jim created a table in which  $\Delta x$  was 1 cm, and Daeshim created a table with  $\Delta x$  as 2 cm (Figure 3). The different tables resulted in a conflict about whether the constantly-changing rate of change of area should be  $5 \text{ cm}^2$  or  $20 \text{ cm}^2$  until one student, Jim, realized that the rate depended on  $\Delta x$ : "I'm going up by 1's and they're going up by evens." After a class discussion in which the students agreed that the rate could legitimately be either  $5 \text{ cm}^2$  or  $20 \text{ cm}^2$ , depending on  $\Delta x$ , Jim exclaimed, "Your rate of growth can change no matter what!" In subsequent days the TR encouraged the students to think about other proportionally-growing rectangles, and, ultimately, to draw diagrams relating changes in area to the rectangles' dimensions. These diagrams further emphasized explicit attention to all three quantities, height, length, and area, and enabled the students to explicitly link the change in area to the change in height. For instance, Daeshim determined that the rate of the rate of change of area would be twice the area of the original rectangle; Jim found that it would be equivalent to twice the length for any 1 cm by  $L$  cm rectangle.



**Figure 3: Jim and Daeshim's well-ordered tables for a 2x5 growing rectangle**

We propose that a learning trajectory should include not only an articulation of particular WoUs and the shifts between them, but also a hypothesized connection between these shifts and specific instructional supports. Although space constraints limit an in-depth discussion of all of the

instructional supports in play, we see that the instructional move to prompt diagram drawing necessitated a functional accommodation: students began to attend to and then coordinate increases in height and length, not just area. In addition, the open structure of the task to determine rates of change of area for a 2 cm by 5 cm rectangle further encouraged explicit attention to  $\Delta x$ .

## Discussion

In the domain of research on learning trajectories, attention to a theory of instructional design is lacking. Moreover, methodological approaches to creating learning trajectories as a retrospective practice are scant in the literature. In this research we networked RC and DNR-based theories to inform our methods of task development, pedagogical actions, and retrospective analyses. This approach and the resulting product (i.e., an empirical LT for quadratic growth) are novel and not elaborated in the literature thus far.

This paper contributes to an understanding of how a networking of theoretical assumptions can guide methodological choices in establishing learning trajectories. Specifically, we argue for learning trajectories research to be guided by theoretical lenses on the mathematics of students as well as on a theory of instructional design. In this domain of research, there is a need for researchers to move beyond a focus on creating hypothetical learning trajectories, attending to their methods of creating learning trajectories.

This paper makes explicit the theoretical perspectives undergirding our approach to learning trajectories research. One challenge we see is leveraging learning trajectories as a way to not only frame a study (e.g., in creating hypothetical learning trajectories), but to also retrospectively create and share learning trajectories in ways that are consistent with the theories undergirding their creation. This research illustrates one approach for addressing the challenge of creating learning trajectories as empirically based models of an interweaving of shifts in students' mathematical understandings and goal-directed, theoretically grounded instructional practices.

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