

# Bounding Basis-Risk Using s-convex Orders on Beta-unimodal Distributions

$L^2$  Seminar

Claude Lefèvre, Stéphane Loisel, Pierre Montesinos

July 2019

---

$L^2$



## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."

## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."



**Parametric Insurance**

# Motivations

---

## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."



**Parametric Insurance**

**Enterprise Risk Management in a model-free setting**

# Motivations

---

## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."



**Parametric Insurance**

**Enterprise Risk Management in a model-free setting**



**Identification of worst-case scenarios**

# Motivations

## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."



Parametric Insurance

## Enterprise Risk Management in a model-free setting



Identification of worst-case scenarios



Measurement of basis risk

# Motivations

## Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."



Parametric Insurance

## Enterprise Risk Management in a model-free setting



Identification of worst-case scenarios



Measurement of basis risk



Definition of some risk limit for basis risk management

# Model-free approach ?

---

$Z$ : parametric loss index,  
 $L$ : true loss of the insurer



# Model-free approach ?

---

$Z$ : parametric loss index,  
 $L$ : true loss of the insurer



Efficient choice of the Index:

$$Z - c \leq L \leq Z + a,$$

# Model-free approach ?

---

$Z$ : parametric loss index,  
 $L$ : true loss of the insurer



Efficient choice of the Index:

$$Z - c \leq L \leq Z + a,$$



Link between the Loss and the Index:

$$L = Z - c + X,$$

where  $X = SY$ ,  $X \in [0, a + c]$ .

# Beta-unimodal Distributions

## Definition

An  $\mathbb{R}_+$ -valued rv  $X$  has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (1)$$

where  $Y$  is a positive continuous rv, and  $S \sim \mathcal{B}(\alpha, \beta)$  is a random contracting factor independent of  $Y$ .

# Beta-unimodal Distributions

## Definition

An  $\mathbb{R}_+$ -valued rv  $X$  has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (1)$$

where  $Y$  is a positive continuous rv, and  $S \sim \mathcal{B}(\alpha, \beta)$  is a random contracting factor independent of  $Y$ .



**From  $Y$  to  $X$ :** if  $Y \sim F_Y$  and if  $X$  is Beta-unimodal

$$\bar{F}_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} x^\alpha (I_\beta \phi)(x), \quad (2)$$

where:

- $(I_\beta \phi)(x) = \int_x^\infty \frac{1}{\Gamma(\beta)} (t-x)^{\beta-1} \phi(t) dt$  is called the Weyl fractional-order integral operator,
- $\phi(t) = \bar{F}_Y(t) t^{-\alpha-\beta}$ .

# Beta-unimodal Distributions

## Definition

An  $\mathbb{R}_+$ -valued rv  $X$  has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (3)$$

where  $Y$  is a positive continuous rv, and  $S \sim \mathcal{B}(\alpha, \beta)$  is a random contracting factor independent of  $Y$ .

# Beta-unimodal Distributions

## Definition

An  $\mathbb{R}_+$ -valued rv  $X$  has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (3)$$

where  $Y$  is a positive continuous rv, and  $S \sim \mathcal{B}(\alpha, \beta)$  is a random contracting factor independent of  $Y$ .



**From  $X$  to  $Y$ :** if  $X \sim F_X$  and if  $X$  is Beta-unimodal

$$\bar{F}_Y(x) = (-1)^n x^{\alpha+\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (I_\delta D^n \psi)(x), \quad (4)$$

where:

- $\delta \in [0, 1]$  such as  $\beta + \delta = n \in \mathbb{N}$ ,
- $\psi(t) = \bar{F}_X(t)t^{-\alpha}$ ,
- $D^n$  denotes the  $n$ -fold derivative operator.

# Model-free approach ?

$Z$ : parametric loss index,  
 $L$ : true loss of the insurer



Efficient choice of the Index:

$$Z - c \leq L \leq Z + a,$$



Link between the Loss and the Index:

$$L = Z - c + X,$$

where  $X = SY$ ,  $X \in [0, a + c]$ .

**Identify, Measure and Limit the consequences of uncertainty**

## Definition: s-convexity

A function  $\phi$  defined on  $\mathcal{S}$  is said to be **s-convex** if the inequality

$$[x_0, \dots, x_s; \phi] \geq 0,$$

holds for any choice of distinct points  $x_0, \dots, x_s$  in  $\mathcal{S}$ .

$[x_0, \dots, x_s; \phi]$  denotes a divided difference of the function  $\phi$  at the different points  $x_0, \dots, x_s$ .



## Definition: s-convexity

A function  $\phi$  defined on  $\mathcal{S}$  is said to be **s-convex** if the inequality

$$[x_0, \dots, x_s; \phi] \geq 0,$$

holds for any choice of distinct points  $x_0, \dots, x_s$  in  $\mathcal{S}$ .

$[x_0, \dots, x_s; \phi]$  denotes a divided difference of the function  $\phi$  at the different points  $x_0, \dots, x_s$ .



**s differentiability condition:** if  $\phi^{(s)}$  exists in  $\mathcal{S}$ , then  $\phi$  is s-convex if and only if  $\phi^{(s)} \geq 0$ .

## Definition: s-convexity

## Definition: s-increasing convexity

A function  $\phi$  is said to be **s-increasing convex** on its domain  $\mathcal{S}$  if and only if for all choices of  $k + 1$  distinct points  $x_0 < x_1 < \dots < x_k$  in  $\mathcal{S}$ , we have

$$[x_0, x_1, \dots, x_k; \phi] \geq 0, \quad k = 2, \dots, s.$$

We denote by  $\mathcal{U}_{s-icx}^{\mathcal{S}}$  the class of the s-increasing convex functions on  $\mathcal{S}$ .

**Definition: s-convexity**

**Definition: s-increasing convexity**

**Definition: s-convex order**

Let  $X_1$  and  $X_2$  be two random variables that take on values in  $\mathcal{S}$ . Then  $X_1$  is said to be smaller than  $X_2$  in the s-convex order, denoted by  $X_1 \leq_{s-cx}^{\mathcal{S}} X_2$  if

$$\mathbb{E}[\phi(X_1)] \leq \mathbb{E}[\phi(X_2)] \text{ for all } \phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}, \quad (5)$$

where  $\mathcal{U}_{s-cx}^{\mathcal{S}}$  is the class of all the s-convex functions  $\phi : \mathcal{S} \rightarrow \mathbb{R}$ .

## Definition: moment space

We denote by  $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$  the moment space of all the random variables valued in  $[a, b]$  and with known  $s - 1$  moments  $\mu_1, \dots, \mu_{s-1}$ .

# Identification: s-convex extrema

## Definition: moment space

We denote by  $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$  the moment space of all the random variables valued in  $[a, b]$  and with known  $s - 1$  moments  $\mu_1, \dots, \mu_{s-1}$ .

## Theorem: s-convex extrema

Let  $Y \in \mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$ .

Within  $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$ , there exist two unique random variables  $Y_{\min}^{(s)}$  and  $Y_{\max}^{(s)}$  such that


$$Y_{\min}^{(s)} \leq_{s\text{-cx}} Y \leq_{s\text{-cx}} Y_{\max}^{(s)}.$$

*Proof.* See Denuit et al. (1999).


**s-convex extrema are actually extremal distributions built from the s-1 first moments of  $Y$ .**

## 3-convex extremal distributions

Let  $Y \in \mathcal{B}_3([a, b], \mu_1, \mu_2)$ .


$$Y_{\min}^{(3)} = \begin{cases} a & \text{with proba } \frac{\mu_2 - \mu_1^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{(a - \mu_1)^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \end{cases}$$

and


$$Y_{\max}^{(3)} = \begin{cases} \mu_1 - \frac{\mu_2 - \mu_1^2}{b - \mu_1} & \text{with proba } \frac{(b - \mu_1)^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{\mu_2 - \mu_1^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}. \end{cases}$$

# Generalization to Beta-unimodal distributions

## Proposition (Beta-unimodal $s$ -convex extrema)

Let  $Y \in \mathcal{B}_s(\mathcal{S}, \mu_1, \mu_2, \dots, \mu_{s-1})$ , and let  $\phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}$ . If  $X$  is Beta-unimodal, then  $X \in \mathcal{B}_s(\mathcal{S}, \nu_1, \nu_2, \dots, \nu_{s-1})$  and the  $s$ -convex extrema in this set are

$$X_{min}^{(s)} = SY_{min}^{(s)}, \text{ and } X_{max}^{(s)} = SY_{max}^{(s)}.$$

Besides, if  $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$ , then

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(SY_{max}^{(s)})] \leq \mathbb{E}[\phi(SY_{max}^{(s-1)})] \leq \dots \leq \mathbb{E}[\phi(SY_{max}^{(2)})],$$

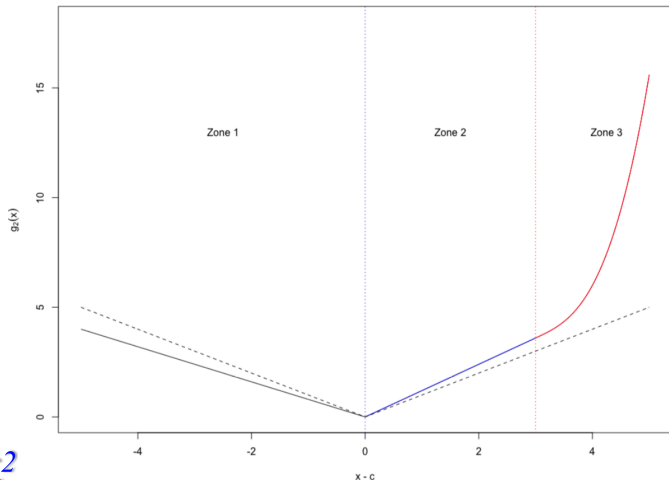
which can be written as,

$$\forall k \in \llbracket 2, s \rrbracket, X \leq_{k-cx} X_{max}^{(s)} \leq_{k-cx} X_{max}^{(s-1)} \leq_{k-cx} \dots \leq_{k-cx} X_{max}^{(2)}.$$

When  $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$ , the more moments, the sharper the bounds !

## Penalty functions

A penalty function  $g$  allows us to represent the consequences of a positive or negative difference between the Loss and the Index.





## Two examples

$$g(L-Z) = \eta(c-X)_x + \gamma(X-c)_+ + \begin{cases} \psi_n(x), \\ -\gamma(x-(c+d))_+ + \varphi_\kappa(x), \end{cases}$$

where

$$\psi_n(x) = \gamma(x-(c+d))_+^n,$$

and

$$\varphi_\kappa(x) = \gamma\kappa \left( \exp\left(\frac{(x-(c+d))_+}{\kappa}\right) - 1 \right)$$

Note:  $\psi_n$  is  $n$  increasing convex,  $\varphi_\kappa$  is  $\infty$  increasing convex.

# From basis risk budget to risk limits

## Framework

First, we consider a symmetric range of possible values for uncertainty:  $a = c$ , i.e.  $L = Z - a + X$ , with  $X \in [0, 2a]$ .

Let  $BR^{Sym}$  be the two-sided uncertainty budget. The way we limit the consequences of a difference is finding the window of uncertainty (the value of  $a$ ) such that the budget is not overpassed.

We look for  $\hat{a}_f^{(s)}$  defined as,

$$\hat{a}_f^{(s)} = \operatorname{argmin} \left| BR^{Sym} - \mathbb{E}[g(X_{max}^{(s)} - c)] \right|, \text{ with } f = \psi_n, \varphi_\kappa.$$

# Numerical illustrations

---

## Parameters

$$a = c = 5, \quad d = 2.5, \quad \eta = \gamma = 1$$
$$Y \sim \mathcal{U}[0, 10], \quad \alpha = 4, \beta = 2, \quad BR^{Sym} = 3.5417.$$

# Numerical illustrations

## Parameters

$$a = c = 5, \quad d = 2.5, \quad \eta = \gamma = 1$$
$$Y \sim \mathcal{U}[0, 10], \quad \alpha = 4, \beta = 2, \quad BR^{Sym} = 3.5417.$$

Convex order used	$\hat{a}_{\psi_2}^{(s)}$	$\hat{a}_{\psi_3}^{(s)}$	$\hat{a}_{\psi_4}^{(s)}$	$\hat{a}_{\psi_6}^{(s)}$
2	4.5268	4.4686	4.4015	4.2746
3	-	4.6292	4.5521	4.4002
4	-	-	4.6335	4.4730

Table: Values of  $\hat{a}_{\psi_n}^{(s)}$  for  $s = 2, 3$  and  $4$ , for  $n = 2, 3, 4$  and  $6$

# Numerical illustrations

## Parameters

$$a = c = 5, \quad d = 2.5, \quad \eta = \gamma = 1$$
$$Y \sim \mathcal{U}[0, 10], \quad \alpha = 4, \beta = 2, \quad BR^{\text{Sym}} = 3.5417.$$

Convex order used	$\hat{a}_{\psi_2}^{(s)}$	$\hat{a}_{\psi_3}^{(s)}$	$\hat{a}_{\psi_4}^{(s)}$	$\hat{a}_{\psi_6}^{(s)}$
2	4.5268	4.4686	4.4015	4.2746
3	-	4.6292	4.5521	4.4002
4	-	-	4.6335	4.4730

Table: Values of  $\hat{a}_{\psi_n}^{(s)}$  for  $s = 2, 3$  and  $4$ , for  $n = 2, 3, 4$  and  $6$

Convex order used	$\hat{a}_{\varphi_{\kappa_2}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_3}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_4}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_6}}^{(s)}$
2	4.6324	4.4783	4.3351	4.1020
3	5.0058	4.8033	4.6172	4.3206
4	5.2321	5.0000	4.7870	4.4510

Table: Values of  $\hat{a}_{\varphi_{\kappa}}^{(s)}$  for  $s = 2, 3$  and  $4$ , for  $\kappa_2, \kappa_3, \kappa_4$  and  $\kappa_6$

# Conclusion

---

Promise kept ?



**Identification** of s-convex extrema for Beta-Unimodal random variables,

# Conclusion

---

## Promise kept ?



**Identification** of s-convex extrema for Beta-Unimodal random variables,



**Measurement** of basis risk in "model-free" context, and show the impact of information (in terms of moments) on basis risk assessment,

# Conclusion

---

## Promise kept ?

- ➔ **Identification** of s-convex extrema for Beta-Unimodal random variables,
- ➔ **Measurement** of basis risk in "model-free" context, and show the impact of information (in terms of moments) on basis risk assessment,
- ➔ Propose a way to set up **risk limit** for basis risk, relative to extremal s-convex bounds.

**Is basis risk always bounded ?**