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**Abstract:** In the 1920s, Carnap and Schlick both made an important use of definitions in their main publications: Schlick, in his *Allgemeine Erkenntnislehre* (1918, 2nd ed. 1925) and Carnap in *Der logische Aufbau der Welt* (1928, mostly written by 1925). In this paper, we first provide an analysis of the kinds of definitions that are distinguished in these books and a few other papers, and we then propose a systematic comparison of Schlick's and Carnap's diverging conceptions of definitions in the 1920s, relating them, in both cases, to their respective philosophical projects in the *Allgemeine Erkenntnislehre* and in *Der logische Aufbau der Welt*.

**Key words:** Schlick, Carnap, Gergonne, explicit definition, implicit definition, definition by axioms

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## Carnap and Schlick on definitions in the 1920s

In the 1920s, both Schlick and Carnap made an important use of definitions in their main writings: Schlick, in his *Allgemeine Erkenntnislehre* (*AE* in what follows) first published in 1918, with a second revised edition in 1925, and Carnap in *Der logische Aufbau der Welt* (often referred to as "the *Aufbau*") published in 1928 although a large part of it was already written in 1925. With respect to definitions—how they are conceived and used—a striking difference between the two books is the following: whereas Schlick concentrates on implicit definitions in *AE*, saying very little about explicit definitions, Carnap uses only explicit definitions in the *Aufbau*, mentioning implicit definitions in passing (in § 15) in a short reference to Schlick's *AE*. This is not to say that Carnap ignores or is not interested in implicit definitions—indeed, he discusses them at length in other publications such as Carnap (1927) or Carnap (1937, § 71e)—but they are not part of the philosophical programme of the *Aufbau*; and this is not to say either that Schlick ignores explicit definitions—indeed, they are important for his theory of knowledge—but as a matter of fact, he does not discuss them explicitly in *AE*. In this paper, I will compare the use Carnap and Schlick make of definitions (respectively in the *Aufbau* and in *AE*) and refer to their respective philosophical programmes in order to account for the differences between the two books regarding this issue.

The title of Schlick's book makes it clear that his goal is to provide a general theory of knowledge whereas the purpose of Carnap's book does not appear as obvious from its title, and as a matter of fact, the exact goal of Carnap in the *Aufbau* is a controversial topic. Carnap explains—and this at least is a factual truth—that "the aim of the present investigations is to establish a constitutive system of all concepts" (*Aufbau*, § 1), and we learn in the following paragraphs that the notion of definition is the main operative tool for this constitutive system, so that on the face of it, the aim of the book can be described as the establishment of a system of definitions; indeed, definitions of a kind we would call "explicit", using contemporary terminology, although Carnap's use of this term is different, as he establishes a distinction between "explicit definitions" and "definitions in use". From the title of Schlick's *AE*, it is not immediately clear what the function of definition is in his theory of knowledge, but the content of the book shows that the main focus is on *implicit* definitions, although a more thorough examination makes it clear that explicit definitions are also considered. Before investigating these points in more detail, it will be useful to start with some considerations about implicit and explicit definitions as such in general, so that we can rely on a common background for the following discussion of what Carnap and Schlick say about them. On the one hand, by referring to the historical origin of the expression "implicit definition", we shall be able to distinguish more easily several of its uses. On the other hand, a brief, somewhat more formal account of some elementary aspects of the theory of definitions will help us clarify the matter, although neither Schlick's nor Carnap's approach, in *AE* and in the *Aufbau*, is strictly speaking formal.

**Implicit definitions in Gergonne's sense**

The term "implicit definition" and the distinction between explicit and implicit definitions were introduced in 1818 by the French mathematician J. D. Gergonne, in a paper on the theory of definition. According to Gergonne, defining a word is to explain its meaning using other words, whose meaning has been previously determined (Gergonne 1818–1819, 20). Like other authors before him, however, Gergonne argues that not every word can be defined in this way. A whole range of words including those for sensations, for individuals, and for metaphysical simple ideas, requires other methods for learning their meaning, that go beyond what is usually called a "definition". Such methods include the careful examination of the various circumstances in which a word is used by people having a good command of its meaning, or the statement of a sentence in which only one word has an unknown meaning that, however, can be determined from the meaning of the other words as they are used in that sentence. Gergonne's example is "each of the two diagonals of a quadrilateral divides it into two triangles" (*op. cit.*, 23), where only "diagonal" is supposed to have an unknown meaning. Although this sentence does not have the usual form of a definition, the meaning of "diagonal", when applied to "quadrilateral", can be understood from it if the meaning of the other words in the sentence is previously known. Gergonne proposes to call "implicit definition" this kind of sentences "that gives the meaning of one of the words of which they are composed, by means of the known meaning of the others" (*ibid.*). Note that if the other words are not known by explicit definition, they must be known by a method that is not definition.

Gergonne then generalizes the idea to the case of several words with unknown meaning, combined with other words in several sentences not having the usual form of a definition, and from which the meaning of the words that are not known can nevertheless be learned. In such implicit definitions, the words are not defined one by one but altogether by a set of sentences taken as a whole. Gergonne insists that in such case, the number of unknown words should be exactly the same as the number of sentences. No process is mentioned by which the implicit definitions could be converted into explicit ones, but the comparison with mathematical unknowns (the value of which can be determined through the resolution of a set of equations in which they occur) makes it clear that according to Gergonne the meaning of words implicitly defined by a set of sentences should not be less precisely characterized than in the case of explicit definitions. As a consequence, if *n* words are implicitly defined by a set of *n* sentences in which none of the other words are explicitly defined, these other words must have been learnt by a method that is not definition. Gergonne's idea is not to explicitly define all the words that can be so defined and then to use implicit definition for those that cannot; implicit definition is not definition by axioms.

Indeed, as correctly remarked in Otero (1970), contrary to what has been asserted by several commentators[1], the issue of axiomatization is not mentioned in Gergonne's paper and the idea of characterizing a system of axioms as implicit definitions is completely foreign to him[2]. Neither of course does his paper discuss any presupposed or chosen logical background within which the implicit definition of a set of words takes place; such issue emerges only much later in the history of logic. Implicit definitions are contrasted with explicit ones, to which no specific discussion is devoted as such in Gergonne's essay. Explicit definitions are just characterised as "ordinary definitions" (*op. cit.*, 23), which are in fact the main focus of the paper. These definitions—ordinary ones—are regarded as sentences by which an "identity of meaning" is established between some chosen word and a phrase composed out of several words the meaning of which has already been determined either by use or by some previous convention. The examples Gergonne gives make it clear that the typical form of a definition of some word *W* is a sentence such as "I call *W* …" or "by *W* I

[1] Otero quotes Kneale and Kneale (1962, 385) and Carruccio (1964, 64).

[2] The same mistake of attributing the idea of definition by axioms to Gergonne is made in Quine (1964, 71).

mean …" Each of these sentences comprises only one unknown word *W* (*op. cit.*, 15), it states an identity of meaning between *W* and a more complex expression (*op. cit.*, 13), and it includes all that is necessary for fixing the meaning of *W*, and nothing more (*op. cit.*, 16).

**Explicit definitions vs definitions by axioms**

In Carnap's and Schlick's writings, explicit and implicit definitions are often discussed in connection with the enterprise of axiomatization of a theory. Let us first explain this connection in general and contemporary terms before turning to Carnap's and Schlick's specific views about it.

Let *L* be a formalized language and *T* be a theory defined as the deductive closure of a recursive set *A* of sentences of *L*, *A* being regarded as a set of axioms for *T*. Let *L'* be the language obtained from *L* by adding an *n*-ary relation sign "*R*" and an *m*-ary function sign "*f*"[3]. The explicit definitions of "*R*" and "*f*" on the basis of *L* and *T* have, respectively, the following forms:

Explicit definition of "*R*": $\forall x_1 \ldots \forall x_n\, (Rx_1 \ldots x_n \leftrightarrow \phi(x_1 \ldots x_n))$

where "$\phi(x_1 \ldots x_n)$" (the so-called "*definiens*") is a well-formed expression of *L* with *n* free variables and "$Rx_1 \ldots x_n$" is regarded as the "*definiendum*".

Explicit definition of "*f*": $\forall x_1 \ldots \forall x_m \forall y\, (fx_1 \ldots x_m = y \leftrightarrow \chi(x_1 \ldots x_m, y))$

where "$\chi(x_1 \ldots x_m, y)$" (the *definiens*) is a well-formed expression of *L* with *m*+1 free variables, "$fx_1 \ldots x_n = y$" is the *definiendum*, and a proof of the following formula can be given in *T*

$$\forall x_1 \ldots \forall x_m \exists y \forall z (\chi(x_1 \ldots x_m, z) \leftrightarrow y=z)$$

showing that the definition of "*f*" satisfies the condition for any interpretation of "*f*" to be a function (a proof has to be given that for all $x_1 \ldots x_m$ there exists a unique *y* such that $\chi(x_1 \ldots x_m, y)$).

Definitions of these forms make it possible to satisfy an important requirement in the classical theory of definition: the possibility of eliminating any explicitly defined relation sign, function sign, or constant sign, in any extensional context[4]. Let *L'* be the new enriched language (*L* plus the relation signs "*R*" and "*f*"), *A'* be the union of *A* and the set of explicit definitions, and *T'* be the deductive closure of *A'*; this requirement of eliminability may be

[3] *L'* may be obtained from *L* by adding any number of relation signs and function signs. We limit these numbers to one to simplify the exposition. The explicit definition of a constant sign is the same as the one for "*f*" in the case where *m*=0.

[4] The requirement of eliminativity may not be satisfied if *L* is not a first order language.

formulated in the following way: for each formula ψ' of *L*', there exists some formula ψ of language *L* such that ψ'↔ ψ is provable in *T*'. This holds more generally for any number of new signs of relation and function added to *L*, if explicit definitions of the foregoing forms are provided for them[5].

Another way of looking at explicit definitions is to start with some formalized language *L* and some theory *T*, defined as the deductive closure of some recursive set *A* of sentences of *L*. A sign *s* of *L* is said to be explicitly definable on the basis of *L*-{*s*} and *T* if there exists a sentence δ of *L* that is an explicit definition of *s* on the basis of *L*-{*s*} and *T*. Any sentence of *T* is then provably equivalent to a sentence of *L*-{*s*} (in other words, "*s*" may be eliminated). More generally, the signs of a subset $L_e$ of the signs of *L* are said to be explicitly definable on the basis of $L$-$L_e$ and *T* if there exists a set Δ of sentences of *L* that consists of explicit definitions of each sign in $L_e$ on the basis of $L$-$L_e$ and *T*. Any sentence of *T* is then provably equivalent to a sentence of $L$-$L_e$, which means that the signs of $L_e$ may be eliminated. From an epistemological viewpoint, the question may be raised to find some maximal set $L_e$, in such a way that no sign of $L$-$L_e$ is explicitly definable, thus circumscribing a minimal set of signs for a theory logically equivalent to *T*, although such minimal set is generally not unique. A classical question is then: if none of the signs of such minimal set can be explicitly defined on the basis of the others, how is it possible to determine their meaning? How to provide a definition for them? When such minimal set of signs is regarded as a set $L_0$ of primitive signs for *T*, a typical move is to formulate a set of sentences in which no other non-logical sign occurs and regard them as postulates or axioms[6]. A further typical move, which was taken by Schlick in *AE* and by Carnap at the time of the *Aufau*[7], is to call these postulates "implicit definitions" of the primitive signs in $L_0$. It is not clear, however, that such postulates deserve to be called "definitions" as long as no argument has been provided to show that the meaning of the primitive signs in $L_0$ are completely determined by them. In his essay, Gergonne had been cautious to use the term "implicit definition" only for sets of sentences by which the meaning of the unknown terms could be no less precisely characterized than the value of *n* unknowns by a system of *n* equations. Both Schlick and Carnap are aware of this issue but they cope with it in different ways in the 1920s.

Carnap discusses the method of defining concepts by an axiom system (AS) in Carnap (1927). Generally speaking, axioms are often regarded as statements about known objects or concepts and in that case, they consist of sentences made out of words having a definite meaning. But when an AS is regarded as a set of implicit definitions, their purpose is to confer meaning to the unknown words that occur in them although in that case, no proper

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[5] Another important requirement—non-creativity—is often formulated in the classical theory of definition. Keeping the foregoing notations in mind, a sentence δ of language *L*' is called *creative* with respect to theory *T* if there is some sentence ψ of language *L* that is provable in *T*∪{δ} and not provable in *T*. In the classical theory of definition, a sentence δ of *L*' is regarded as a definition of a sign *s* not in *L* (with respect to *T*) only if δ is not creative with respect to *T*. Non-creativity will not be further discussed in this paper.

[6] Taking up the foregoing notations, these axioms could be obtained from set *A* by converting each sentence ψ of *L* that is an element of *A* into a formula ψ' of $L$-$L_e$ provably equivalent to ψ, using the explicit definitions in Δ.

[7] In Carnap (1927). This move is taken up again in Carnap (1937, § 71e).

concept is defined because the AS fails to confer a definite and complete meaning to them: not only are there several models—indeed, an infinite number of them—if the AS is consistent, but these models may also be non-isomorphic. The AS expresses constraints on the unknown words that occurs in the axioms, but not to the point of characterizing a definite meaning. As a consequence, Carnap considers that they are words for “improper concepts”, which he analyses as variables (they may have several values) rather than constants, and he construes consistent AS as *theory schemes* rather than theories. Only when a specific meaning is attributed to the basic unknown words do these schemes become theories properly speaking. Their fruitfulness consists in their possible application to different cases, which are either “formal models” or what Carnap calls “realizations” (empirical models). Implicit definitions construed as definitions by axioms do not satisfy the requirement of eliminability, and clearly, they are not what Gergonne meant by the same term.

**Definitions in the *Aufbau***

Because a characterization (*Kennzeichnung*) of concepts is needed for Carnap’s project of a constitutive system of all concepts, implicit definitions regarded as AS have no place in the *Aufbau*. The characterization of a concept requires that “in the object domain in question, at least one object must exist that answers the description, and at most one such object must exist” (*Aufbau*, § 15). Concepts are introduced in the system by definitions we would call “explicit” although they come in two kinds in Carnap’s terminology, with no common name for both of them[8]. What Carnap calls an explicit definition (strictly speaking) in the *Aufbau* is the explanation of a new sign (*neues Gegenstandszeichen*) as being equivalent (*gleichbedeutend*) to a sign composed of already known signs, these being either fundamental (*Grundzeichen*) or having already been previously defined. What is peculiar to explicit definitions thus construed is that the “old sign” can always take the place of the new one when this one has to be eliminated (*Aufbau*, § 38). A typical example of an explicit definition is “$2 =_{Df} 1+1$” where “2” is the new sign while “1+1” is the old one, “1” and “+” being either primitive or previously defined.

This form of definition, however, does not work in the case of predicates such as “prime” or of relations such as “less than” (in the domain of the natural numbers), because terms of this kind have arguments and the *definiens* usually have a complex form with arguments occurring at several places. As a consequence, the *definienda* have to be “*x* is prime” and “*x* is less than *y*” (where “*x*” and “*y*” are variables for possible arguments), not “prime” and “less than”. Taking up a name from *Principia Mathematica* (Russell and Whitehead 1910, 69), Carnap speaks of a “definition in use” (*Gebrauchsdefinition*) in such cases. What the definition in use explains is not the new sign alone (Carnap also takes up Russell’s idea that the new sign without its argument has no meaning in itself) but “its use in

[8] Although Carnap remarks that the term “explicit definitions (in the wider sense)” is sometimes used as a common name when they need to be distinguished from implicit definitions (*Aufbau*, § 39).

complete sentences" (*Aufbau*, § 39). For example, in a definition in use of "prime number", the *definiendum* "*x* is prime" is explained by a complex expression such as "*x* is a natural number and *x* has exactly two divisors". When the new sign has to be eliminated, the elimination process needs to take the arguments into account. The elimination of "prime" when used in an expression such as "*t* is prime" (where "t" is a name) consists in the replacement of this expression with "*t* is a natural number and *t* has exactly two divisors".

Generally speaking, definitions may have several purposes. One may want to characterise a natural kind or the essence of some object or concept that is supposed to exist in some way, and this is typically what is done when someone looks for a definition of gold or silver as exemplified by a given sample. A quite different motivation may be to circumscribe the meaning of a word used by a linguistic community at some specific place and time, and this is typically what linguists working on a dictionary have to do. As a third example, one may want to introduce or make precise some use, old or new, by stipulation, as mathematicians often do through sentences such as "by a group, let's understand so and so", or "let's call so and so a topological space". The goal here is to point to some meaning and conventionally decide that it will be expressed by such and such word. Mere abbreviation is still another use of a definition, without which discourse would often be intolerably prolix and intricate.

In the *Aufbau*, none of the foregoing examples of a goal for a definition is put forward. Generally speaking, whatever the purpose of definition may be, it is usually admitted that it should satisfy the eliminability requirement. Now in the *Aufbau*, this condition of a possible elimination of the *definiendum* is actually exactly what a constituted object (or concept[9]) has to satisfy. Constituting an object requires providing a rule showing how the latter can be eliminated, and this is precisely why a system of constitution is a system of definitions. What is specific of the definitions as they are conceived in the *Aufbau* is that their essence is just the requirement of eliminability: "a rule must be specified that allows the name of the new object to be eliminated from any sentence in which it may occur; in other words: a *definition* of the name of the object must be specified" (*Aufbau*, § 38). As the title of § 38 makes explicit, "constitution happens through definition" (*Konstitution geschieht durch Definition*), and definition, we may add, through elimination rules, i.e. rules that makes the elimination of the defined word possible.

At this point, the difference Carnap points out between what he calls "explicit definitions" and definitions in use becomes crucial. When an explicit definition such as "$2 =_{Df} 1+1$" is formulated, this is almost mere abbreviation; a shorter notation is introduced for an object that belongs to the same sphere of objects. By contrast, when the equivalence of "*x* is prime" and "*x* is a natural number and *x* has exactly two divisors" is put forward as a definition in use, the gain is not only a new notation but also a new concept, which Carnap

[9] The words "object" and "concept" can be used here interchangeably. Carnap explains that "the word 'object' is here always used in its widest sense, namely for anything about which a statement can be made" (*Aufbau*, § 1). "It makes no logical difference whether a given sign denotes the concept or the object or whether a sentence holds for objects or concepts" (*Aufbau*, § 5).

regards as a "pseudo-object" (*Quasigegenstand*) with respect to the objects to which it may apply (in the foregoing example, the natural numbers). Carnap takes here advantage of Russell's idea that "prime" alone (as well as any name for a predicate or relation without its arguments) has no meaning; the *definiendum* is "*x* is prime", not "prime", and only the use of "prime" in a larger context is meaningful. If "*t*" is the name (complex or simple) of an object, the *definiendum* is not "*t* is prime" either, because this sentence has no variable and the elimination rule, or "translation rule" (*Aufbau*, § 39), "would not apply to different sentences, but only to this one" (*ibid*.). In the *Aufbau*, constitution happens through definition *in use*, not through any definition, and only through the definition of propositional sentences, not through the definition of either names or sentences.

For Russell, the definition in use of a predicate such as "prime" amounts to the logical construction of the class of prime numbers, which Russell construes as a fiction in Russell (1919, 46). In the *Aufbau*, Carnap writes that the Russellian idea of classes as fictions corresponds to his own conception of classes as pseudo-objects (§ 33). A crucial difference, however, is that Carnapian pseudo-objects do not carry any ontological commitment as Russellian fictions do. Whereas Russell's construction of logical fictions is a strategy for reducing the ontological cost of our believes, Carnap is extremely careful not to commit his project of a constitutive system of all concepts to any ontological issue (*Aufbau*, § 5, § 27)[10]. As Carnap conceives it, a definition in use does not presuppose the existence of the *definiendum* and it does not bring it to existence either. Nor does it consist in circumscribing the use of a term by a linguistic community. And it does not reduce to a mere abbreviation either. Carnap's very specific notion of a definition in use is a tool for the constitution of a concept and because the constitution of a concept requires the possibility of a reduction, such a definition may be construed as the formulation of an elimination rule.

**Definitions and exact knowledge in Schlick's *Allgemeine Erkenntnislehre***

While Carnap's project of a system of constitution stands out from the traditional epistemological goal of accounting for knowledge, Schlick clearly aims to achieve a "general theory of knowledge". In this context, he outlines aspects of his conception of definition because according to him we only have access to genuine exact knowledge through definitions. The effect of a definition is to make a concept precise through an enumeration of the characters that an object needs to possess for the defined concept to apply to it. Whereas ordinary knowledge is based on intuitive representations that lack in precision, exact knowledge depends on concepts characterized and strictly delimited by their definition:

[10] See Wagner (2022).

> By using defined concepts, scientific knowledge raises itself far above ordinary knowledge. Whenever we have at our disposal suitably defined concepts, knowledge becomes possible in a form practically free from doubt. (*AE*, § 6, 27)[11]

Concepts have to be distinguished from images and intuitions, which inevitably involve imprecise elements:

> A concept is to be distinguished from an intuitive image above all by the fact that it is completely determined and has nothing uncertain about it. (*AE*, § 5, 20)

In scientific knowledge, concepts made precise through definitions take the place of images and intuition that are used in ordinary knowledge.

> It is through definitions that we seek to obtain what we never find in the world of images but must have for scientific knowledge: absolute constancy and determinateness. (*ibid*.)

Elements of Schlick's conception of definition can be found in § 5 (on "knowing by means of concepts") and in § 6 (on "the limits of definitions"), although no systematic exposition of "ordinary definitions" (*die gewöhnliche Art des Definierens*) as he calls them in § 7 is offered. Nothing is said, in particular, about the general form of a definition or about the logical background it presupposes. What Schlick writes about definitions, however, allows us to recognize what he has in mind here as what are usually called "explicit definitions", especially because he later contrasts them with another kind of definition, introduced in § 7 under the name of "implicit definitions" [12]. What we do learn about definitions as Schlick conceives them in these paragraphs—this is in fact the main reason why ordinary explicit definitions are of key importance for his theory of knowledge—is that they make exact knowledge possible through the denomination of an object by its correct name: "the definition specifies the common name we are to apply to all objects that possess the characteristics set forth in the definition" (*AE*, § 5, 20). Schlick then adds: "Or, to use the traditional language of logic, every definition is a nominal definition" (*ibid*.). The term "nominal definition" (which is not used elsewhere in *AE*) does indeed belong to the traditional language of logic, where it has been given several meanings and is usually contrasted with "real definition".

This traditional qualification[13] is used here to underline two different points. First, what the *definiendum*[14] designates is nothing real, precisely because it is a concept, as opposed to an image or a representation; indeed, a concept is nothing but a sign: "a concept

[11] The page number refers to the English translation although this translation is sometimes slightly modified.

[12] Although the term "*explizite Definition*" does not occur in the German text of *AE*, the English and French translators read "*explizite Definition*" instead of "*implizite Definition*" in the second sentence of the last subparagraph of § 7, a reading that is justified by the context. Schlick clearly opposes implicit definitions to explicit ones in this passage. The German editor of *AE* in the *Gesamtausgabe* published by Springer agrees that Schlick means "explicit", not "implicit" in this occurrence, as is remarked in a footnote of Schlick (2008, 60).

[13] Schlick mentions Aristotelian real definitions in § 7, p. 30.

[14] Note that the term "*definiendum*" that we use here does not occur in *AE*.

plays the role of a sign or symbol" (*ibid.*). "Concepts are not real. […] Strictly speaking, concepts do not exist at all. What does exist is a *conceptual function*" (*AE*, § 5, 22). In exact thinking, representations are replaced by concepts construed as signs. The first reason why ordinary definition is "nominal" is that the *definiendum* designates a sign. Second, a definition does not presuppose anything existing that the concept defined would itself designate and that the definition would aim to characterize. A definition can perfectly well be formulated that does not apply to anything existing, and this gives a second justification for considering ordinary definitions as being "nominal". Whereas so called "real definitions" are supposed to characterize the essence of something existing, the function of nominal definitions is to create concepts:

> In science generally, the purpose of definitions is to create concepts as clearly determined signs, by means of which the work of knowledge can go forward with full confidence. (*AE*, § 7, 33)

Although concepts are signs, they are not to be confused with words, because the concept that is represented by a word may change, in case the use of words itself changes: "In speech, concepts are designated by words or names" (*AE*, § 5, 21) the meaning of which may vary. The function of a definition is precisely to fix that meaning. For this reason, the lack of definition is the source of error and inexactness:

> The use of images as proxies for concepts has probably been the most prolific source of error in philosophic thinking in general. Thought takes flight without testing the load capacity of its wings, without determining whether the images that carry it correctly fulfil their conceptual function. Now this can be established only by going back, again and again, to the definitions. (*AE*, § 5, 21)

After showing in § 5 the importance of definitions for exact or scientific knowledge based on concepts, § 6 is devoted to an examination of the limits of ordinary (i.e. explicit, nominal) definitions. Schlick takes up the well-known argument according to which defining all concepts by ordinary definitions is not possible because this would lead to infinite regress. This is because Schlick assumes that ordinary definitions define concepts by resolving them into simpler ones (*AE*, § 7, 32) so that the undefinable concepts are also the simplest ones (*AE*, § 5, 30). To the classical issue that results (how to define the undefinable?), the classical answer is that the meaning of some basic concepts need to be given by another method such as intuition or direct experience: "We cannot learn what 'blue' or what 'pleasure' is by definition but only by intuiting something blue or experiencing pleasure" (*AE*, § 6, 29), what is sometimes called "definition by ostension", or "concrete" definition (*AE*, § 6, 30). Although this kind of answer is usually sufficient in practice, it will not satisfy the epistemologist's goal of accounting for exact knowledge because such knowledge excludes any reliance on intuition or direct experience. A different answer, which is supposed to meet the requirement of the *Erkenntnistheoretiker*, is explained in § 7, devoted to implicit definitions, and in § 11 on definitions, conventions and empirical judgments.

**Schlick on implicit definitions**

Assuming some concepts are undefinable by explicit definitions and assuming absolutely exact knowledge is possible, how to define undefinable concepts? Schlick presents his own answer to this question as inspired by the history of modern geometry and by the strivings of mathematicians for exactness. Before considering implicit definitions in "real science", he examines their use in what he calls "conceptual science" (*AE*, § 11, 69). The key idea, which is to be found in geometry, is the following: "to stipulate that the basic or primitive concepts are to be defined just by the fact that they satisfy the axioms" (*AE*, § 7, 33). Instead of stating axioms whose validity would be based on the meaning of the primitive terms, the axioms of geometry are considered as fixing it. This procedure is called either "implicit definition" or "definition by axioms" (*ibid*.). In the case of geometry, Schlick attributes this move to Hilbert; his own idea, on which his theory of knowledge is based, consists in generalizing it to science as such, including empirical science.

Hilbert himself does not use the term "implicit definition" although he defends the idea that axioms may be considered as definitions for the primitive terms of a science. In Hilbert (1899), he asserts that axioms of group II "define the concept 'between' " (§ 3) while those of group IV "define the concept of congruence or displacement" (§ 6), and in his 1900 conference on mathematical problems, he generalizes the idea of definition by axioms to any science:

> When we are engaged in investigating the foundations of a science, we must set up a system of axioms that contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas. Hilbert (1900, 264)

In the early 1900s, the use of the term "implicit definition" for "definitions by axioms" (or by postulates) expanded rapidly, often with a confusing reference to Gergonne. Gabriel (1978) is a short but precise and very informative historical analysis of the confusion between several meanings of the term "implicit definition" and in particular of the erroneous association of Gergonne's notion of implicit definition with the idea of definition by axioms, an error that Gabriel spots for example in Enriques (1907, 11)[15]. Schlick does not refer to Gergonne and what he means by "implicit definition" is nothing but definition by axioms.

In his correspondence with Hilbert, Frege criticized the Hilbertian use of axioms as definitions (see Blanchette 2018) and he subsequently expanded his criticism in a series of papers on the foundations of geometry (Frege 1903, 1906). Frege argues that axioms cannot be construed as definitions because the very idea of an axiom presupposes that the meaning of any term occurring in it has already been fixed. Hilbert's conception of axiom, however, is

[15] Enriques remarks that the reference to Gergonne in the context of a discussion of definition by axioms is already made in Vacca (1899).

different from the traditional notion Frege has in mind: a system of axioms in Hilbert's sense is supposed to fix the meaning of some elementary terms that occur in them, as can be seen in the foregoing quotations from Hilbert. But in this exchange, Frege's point is not terminological: indeed, he insists that a system of axioms cannot be meant as fixing the meaning of elementary terms that occur in it. Carnap was Frege's student, and (although he does not mention Frege's criticism of Hilbert) he completely agrees that an axiom system cannot be regarded as a definition of the primitive terms, if only because a consistent AS admits an infinite number of formal (i.e. logical or mathematical) interpretations:

> Strictly speaking, it is not a definite object (concept) that is implicitly defined through the axioms, but a class of them or, what amounts to the same, an 'indefinite object' or 'improper concept'. (*Aufbau*, § 15)

But instead of rejecting Hilbert's idea of definitions by axioms altogether, he proposes an interpretation of this idea that makes it meaningful: in Carnap (1927), he argues that what is explicitly defined by an AS (assuming it satisfies some conditions of consistency and of completeness) is a second order relation or what is called today a structure. For example, Peano's axiom system for arithmetic surely does not define the primitive terms "zero", "number", "successor" although it defines the structure of progressions[16]. In his correspondence with Hilbert, Frege had already remarked that the axioms of geometry in Hilbert's sense could not be read as being first order:

> The characteristic marks you give in your axioms are apparently all higher than first-level; i.e. they do not answer to the question "What properties must an object have in order to be a point (a line, plane, etc.)?", but they contain, e.g., second-level relations, e.g. between the concept *point* and the concept *line*. It seems to me that you really want to define second-level concepts but do not clearly distinguish them from first-level ones. Frege, (1900, 46)

While Frege wants to maintain a strict distinction between definitions and axioms and rejects any confusion between the two notions, Carnap suggests regarding axiom systems in Hilbert sense as defining second order relations.

Returning now to Schlick, what about his notion of implicit definition? Does Frege's and Carnap's criticism of definition by axioms apply to the doctrine defended in *AE*? Carnap's reservations toward AS as definitions of primitive terms rely on the fact that an AS does not have a unique model; indeed, on the fact that an infinite number of interpretations satisfy any consistent AS, which clearly shows that no unique object (or concept, or relation, etc.) corresponding to each primitive term has been determined by the AS. The reason why this is no objection for Schlick is that for him, the primitive terms of an AS as he conceives them do

[16] On implicit definitions and the definition of structures, see Giovannini and Schiemer (2019). On Carnap's requirement of completeness, see Awodey and Carus (2001).

not designate anything real anyway; what they designate are concepts, i.e. signs, which are nothing real: at least in the case of what he calls “conceptual science”,

> implicit definitions have no association or connection with reality at all; specifically and in principle they reject such association; they remain in the domain of concepts. A system of truths created with the aid of implicit definitions does not at any point rest on the ground of reality. On the contrary, it floats freely, so to speak, and like the solar system bears within itself the guarantee of its own stability. […] The construction of a strict deductive science has only the significance of a game with symbols. (*AE*, § 7, 37)

Schlick mentions an important reservation about the possibility of defining concepts by an AS: the axioms that implicitly define a series of concepts “must not involve any contradiction. If the postulates put forward are not compatible, then no concept will satisfy them all” (*AE*, § 7, 38-39). Schlick does not explain what he means by a series of concepts satisfying a set of postulates and he does not clarify the notion of a contradiction either. Whatever he has in mind on these points, it is remarkable that he does not mention completeness (in any sense of the word) among the conditions that an AS must satisfy in order to implicitly define a series of concepts; by contrast, completeness is a major issue in Carnap’s discussion of AS in Carnap (1927). For him, no concept is defined by an AS Δ if there is a sentence ϕ such that ϕ and ¬ϕ are both compatible with Δ, i.e. if both Δ,ϕ and Δ,¬ϕ have a model. When Schlick writes that “in implicit definition we have found a means that allows for perfect determinateness of concepts and thus for strict precision in thinking” (*AE*, § 7, 38), by “perfect determinateness” (*vollkommene Bestimmtheit*) he apparently does not mean this kind of completeness but only the fact that an AS does not depend on the uncertainty of its applicability to real cases. Absolute precision is attainable by implicit definition (as Schlick conceives it) because in it, concept and intuition are separated, as are thought and reality: “the bridges between them are down” (*ibid.*). What is not clear, at this point, is how implicit definitions so construed can be used as an important part of a general theory of knowledge, that includes a theory of empirical real knowledge, not just “conceptual science”. In the *AE*, his point is clarified in the following few paragraphs devoted to judgment, knowledge, and conventions.

### Definitions in the system of science

Schlick analyses a judgment (from a logical, not a psychological viewpoint) as a sign for the existence of a relation between concepts (*AE*, § 8, 41), and thus as a sign for a fact (*AE*, § 8, 42), so that “concepts are linked together by means of judgments” (*AE*, § 8, 45) while it is no less true that “judgments are linked to one another by concepts” (*ibid.*) because each concept occurs in several judgments. When we think about all judgements and concepts in conceptual science (i.e. mathematical science, as opposed to “real science”, i.e. science of reality), the resulting image is that of a network: “Our scientific systems form a network in which concepts represent the nodes and judgements the threads that connect them." (*AE*, § 8, 46). Now because "the definitions of a concept are those judgments that, so to speak, put it in touch with the concepts nearest it" (*ibid.*), it follows that "we must count definitions as genuine judgements" (*ibid.*). Indeed, in purely conceptual systems, "the distinction between definitions and theorems is a relative one" (*ibid*.). Definitions (either explicit or implicit) do not have any "special place" (*AE*, § 8, 47). Schlick then concludes: "Thus we unify the picture we must make of the great connected structure of judgements and concepts that constitute science" (*ibid.*).

This analysis of science as a network of judgments highlights two important points about Schlick's understanding of definitions, including implicit ones. First, definitions do not have a special place among the judgments of conceptual science: the choice of such and such characters for the definition of a concept is a question of pure convenience and once a definition has been adopted, it establishes a link between concepts like any other judgment. Second, understanding the connection between implicit definitions and other judgments is a prerequisite for the discussion of the application of implicit definitions to reality. In other words, the issue of application is not discussed just for implicit definition but for the whole network of judgments that constitutes science. This is in sharp contrast to Carnap who asks, for example, what it is exactly that the AS of Peano's arithmetic defines. Schlick has a different agenda: his analysis focuses on exact science in general and implicit definitions are discussed as parts of the network of science, not as forming isolated AS that would have by themselves a definite connection to reality. The issue of application to reality is not about definitions by axioms but about the network of concepts and judgments that constitutes a self-contained scientific system, including theorems and both explicit and implicit definitions. In a traditional analysis of definitions, explicit definitions establish conceptual relations in the form of a reduction of concepts to simpler ones, until the simplest concepts are reached, which require ostensive, concrete definitions, by which a connection with reality is made. For Schlick, this approach does not account for the nature of exact knowledge. What he proposes is firstly to replace concrete definitions with implicit ones but also, secondly, to approach the issue of application not for single AS but for science regarded as a whole network of judgments:

> Now the remarkable thing is that for a suitable choice of objects (singled out by means of concrete definitions), we can find implicit definitions such that the concepts defined by them may be used to designate uniquely those same real objects. (*AE*, § 11, 70)

Implicit definitions manage to explain the possibility of exact science, provided that science is considered as a conceptual system applied to reality as a whole.

How is it possible for Schlick to assert that a network of implicitly defined concepts and judgments can actually be found that will be as suitably coordinated to the system of facts as a system of concretely defined concepts would be ("*we can find* implicit definitions such that…" Schlick writes)? The answer is that we do not *know* that it is actually possible. Such a claim—that there exists a conceptual system of judgments that is perfectly coordinated to the system of facts—"cannot itself be proved to be a true judgment. Rather, it is a hypothesis" (*AE*, § 11, 71). We do not know that exact knowledge is possible:

> We are thus never certain whether a complete conceptual system really is in a position to furnish an unambiguous designation of the facts. (*AE*, § 11, 71)

All we can do is to suppose it is, and on the basis of this hypothesis, to look for a conceptual system that is coordinated to reality:

> Obviously, to suppose that the wold is intelligible is to assume the existence of a system of implicit definitions that corresponds exactly to the system of empirical judgments. (*AE*, § 11, 70)

In the *Aufbau*, Carnap makes a similar hypothesis when he discusses the possibility of a rational science. Taking for granted that "a scientific statement makes sense only if the meaning of the object names that it contains can be indicated" (*Aufbau*, § 13), he distinguishes two possibilities for providing this meaning: either "displaying" (*Aufweisung*), which corresponds to ostensive definition, or "characterizing" (*Kennzeichnung*), which corresponds to explicit definition (in the larger sense). Carnap's main claim in the *Aufbau* is that we can make do without ostensive definition using only explicit definition. Indeed, although the possibility of dispensing with "*Aufweisung*" cannot be established a priori, "any intersubjective, rational science presupposes this possibility" (*ibid*.). A purely structural characterization of all objects (using explicit definitions) is possible "to the extend in which scientific discrimination is possible at all" (*Aufbau*, § 15). This amounts to the formulation of a hypothesis about the possibility of science that is similar to the one we find in *AE*.

Carnap remarks that his notion of a structural characterization of all objects through explicit definitions is related to Schlick's use of implicit definitions in *AE* but he also objects to Schlick's approach that implicit definition does not allow for a structural characterization of each object. Schlick's theory of exact knowledge is based on a larger view of factual sciences, which he sees as "a network of judgments the individual meshes of which are coordinated with individual facts" (*AE*, § 11, 69). While Carnap assumes that a system of constitution—i.e. of explicit definitions—allows for the structural characterization of each object (or concept), Schlick assumes that the use of implicit definitions allows for a coordination of judgements with facts, provided this use is extended to the whole of science. A comparison of the two approaches must take into consideration the fact that the kind of characterization of each object or concept Carnap has in mind is purely extensional so that it does not aim to give the essence of the object and not even the sense of its name.

> Of course, such a definition by distinctive signs or 'constitution' of a concept by no means exhausts the concept. It only specifies its place in the system of concepts, just as, by comparison, a place on the surface of the Earth is specified by its latitude and longitude. (Carnap 1927, 358)

The metaphor is repeated in the *Aufbau*:

> In analogy, the construction of an object corresponds to the indication of the geographical coordinates for a place on the surface of the earth. The place is uniquely determined through these coordinates; any question about the nature of this place (perhaps about the climate, nature of the soil, etc.) has now a definite meaning. (*Aufbau*, § 179)

This comparison is strikingly similar to the one Schlick himself uses in *AE*:

> Concepts are simply imaginary things, intended to make possible an exact designation of objects for the purpose of cognition. Concepts may be likened to the lines of latitude and longitude that span the earth and permit us to designate unambiguously any position on its surface. (*AE*, § 5, 27)

What we see here is that in both Carnap's and Schlick's projects, definitions are used as a means to build a general network in which any concept has its place. In the case of Carnap's project, however, the foregoing comparison with geographical coordinates does not take into account the hierarchical order established by a system of constitution. What explicit definitions provide above all is the possibility of reducing all concepts to a minimal basis. However, the reduction of concepts to fundamental ones—which requires explicit definitions—also has its place in Schlick's theory of knowledge. Explicit definitions are used to account for the difference between ordinary and exact knowledge in § 5. Schlick then explains the limits of explicit definitions in § 6, before introducing the idea of implicit definitions in § 7. In the following paragraphs, explicit definitions (conceived as judgments that resolve given concepts into simpler ones) are seldom mentioned but Schlick by no means ignores the idea of reducing the concepts of science to a minimal basis. In the following quote, "definition" clearly refers to "explicit definitions":

> Those judgments will be taken as definitions that resolve a concept into the characteristics from which one can construct the greatest possible number (possibly all) of the concepts of the given science in the simplest possible manner. (*AE*, § 9, 50)

Unlike Carnap, Schlick is not so much interested in the hierarchy of concepts and the levels it induces, corresponding to spheres of objects, as in the possibility of reducing them *in principle*. Once the possibility of a reduction is admitted, it can be assumed as realized and all that remains is the abstract view of science in which only implicitly defined fundamental concepts are taken into consideration. Explicit definitions no longer need to be mentioned.

Carnap and Schlick are sometimes criticized for having a one-sided approach to definitions. Carnap is supposed to have been interested only in explicit definitions and Schlick only on implicit ones. But the texts of the 1920s say otherwise. Both Schlick and Carnap take explicit and implicit definitions into consideration in their respective projects. In

*AE*, Schlick mentions explicit definitions for his analysis of exact, as opposed to ordinary, knowledge. But after showing the limits of this kind of definitions, he highlights the possible use of implicit definitions to account for factual science construed as a network of judgments coordinated with the system of facts. Carnap uses only explicit definitions in the *Aufbau* because they are the indispensable tool for building a constitutive system; in other publications, however, implicit definitions are carefully studied, preserved from Frege's criticism of definitions by axioms, and used as an important means for understanding the system of science.

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