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► **To cite this version:**

Björn Lellmann, Francesca Poggiolesi. Nested Sequents or Tree-hypersequents -A survey. R. Padro and Yale Weiss. Saul Kripke on Modal Logic, Springer, In press, Outstanding Contributions to Logic. hal-03590537v2

HAL Id: hal-03590537

<https://hal.science/hal-03590537v2>

Submitted on 13 Oct 2022

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Nested Sequents or Tree-hypersequents – A survey

Björn Lellmann and Francesca Poggiolesi

Abstract This paper presents an overview of the methods of nested sequents or tree-hypersequents that were originally introduced to provide a comprehensive proof theory for modal logic. The paper retraces the history of how these methods have developed. Its aim is also to present, in an unified and harmonious way, the most recent results that have been obtained in this framework. These results encompass several technical achievements, such as the interpolation theorem and the construction of countermodels. Special emphasis is also given to the application to logics other than the standard modal ones as well as to relations to other proof theoretic formalisms.

1 Introduction

The sequent calculus, introduced in Gentzen (1935), was designed to construct derivations for classical (and intuitionistic) logic and to prove that any of these derivations can be *analytic*, where a derivation is analytic when each formula that occurs in it is a subformula of its conclusion. Several successive studies have shown that the classical sequent calculus **Gcl** (see Figure 1) displays different properties that have both a technical and philosophical value. For example, the logical rules of the classical sequent calculus are:¹

- explicit, i.e. the introduction rules for a constant \star are explicit if they exhibit \star in their lower sequent only and they exhibit only one occurrence of \star .

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¹ See Avron (1996); Poggiolesi (2008c, 2010b) and Wansing (2002) for a discussion these properties, as well as for others.

Axioms:

$$\overline{p \Rightarrow p}$$

Structural rules:

$$\frac{M \Rightarrow N}{\alpha, M \Rightarrow N} \text{WL} \quad \frac{M \Rightarrow N}{M \Rightarrow N, \alpha} \text{WR} \quad \frac{\alpha, \alpha, M \Rightarrow N}{\alpha, M \Rightarrow N} \text{CL} \quad \frac{M \Rightarrow N, \alpha, \alpha}{M \Rightarrow N, \alpha} \text{CR}$$

Logical rules:

$$\frac{M \Rightarrow N, \alpha}{\neg \alpha, M \Rightarrow N} \neg L \quad \frac{\alpha, M \Rightarrow N}{M \Rightarrow N, \neg \alpha} \neg R \quad \frac{\alpha_i, M \Rightarrow N}{\alpha_0 \wedge \alpha_1, M \Rightarrow N} \wedge L \quad \frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta} \wedge R$$

Cut rule:

$$\frac{M \Rightarrow N, \alpha \quad \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q} \text{Cut}_\alpha$$

Fig. 1 The classical sequent calculus **Gcl**

- separated, i.e. the introduction rules for a constant \star are separated if they do not exhibit any other connective other than \star .

Moreover, it can be proved that the classical structural rules are admissible, i.e. a rule is admissible when whatever can be proved with the aide of that rule, can also be proved without, e.g. see Troe-lestra and Schwichtenberg (1996). Finally, the classical sequent calculus introduced by Gentzen does not contain any explicit semantic element, i.e. we say that a sequent S does not contain a semantic element if every element that serves to define the sequent itself can be translated in such a way that it forms, together with the translation of the other elements of S , a formula equivalent to S , see Poggiolesi (2010b); Stouppa (2007); Stewart and Stouppa (2004).

Since 1935, an impressive number of different logics have been developed. Amongst them, one family which has proved to be particularly successful, is that of modal logic, that is obtained by adding to classical logic the two operators \Box and \Diamond for necessity and possibility, respectively. The blossoming of modal logic is directly linked to the introduction, in the middle of the 1960, of the Kripke possible worlds semantics (see Kripke, 1963b, 1965, 1963a); by means of such semantics, the modal connectives \Box and \Diamond are interpreted very naturally. In this interpretation, a formula α is necessarily true if, and only if, it is true in every possible world (of a Kripke model), while a formula α is possibly true if, and only if, there exists some world (of a Kripke model) where it is true. Since its introduction, Kripke semantics have been modified, extended and analyzed in a wide plethora of ways and it has given rise to a great numbers of outcomes.

These outstanding results soon led to the investigation of the links between the sequent calculus and modal logic. From the 1950s until the 1980s, the first approaches consisted in adding to the standard sequent calculus **Gcl** for classical logic (see Figure 1) new rules for introducing on the left or on the right of the sequent the modal operators \Box and \Diamond , see, e.g., Curry (1952), Goble (1974), Leivant (1981), Ohnishi and Matsumoto (1957), and Sambin and Valentini (1982). However, it did not take long to realize that these new modal sequent calculi no longer enjoyed most

of the properties satisfied by **Gcl**: in certain cases, it even turned out to be difficult to have a cut-free sequent calculus. Hence, the problem did not seem to be that of choosing suitable rules for modal logic, but that of finding rules, and calculi, that did satisfy certain properties. In other words, it seemed that the move from classical logic to modal logic could not be simply made by adding a new connective with rules governing it, but by *extending one's conception of the objects to be manipulated by such rules*. Generalizing the sequent calculus in order to obtain an adequate proof theory for modal logic was thus seen as a necessary step.

But how does one generalize the sequent calculus? Consider the main object of the sequent calculus, namely a *sequent*:

$$M \Rightarrow N$$

where M and N are multisets of classical formulae, the *antecedent* and *consequent*, respectively, separated by the *sequent arrow*. Now try to find a more abstract version of it. Then, several possibilities arise:²

- one might want to deal with more than one antecedent or one consequent;
- one might want to deal with more than just one type of sequent arrow;
- one might want to deal with more than just one way (the comma) of separating one formula from another;
- one might want to deal with more than one sequent at a time.

Each of these possibilities has given rise to a different rigorous way for improving the standard sequent calculus; in particular, they have led to higher-arity sequent calculi (e.g. see Blamey and Humberstone, 1991), multiple sequent calculi (e.g. Indrezejczak, 1997), display calculi (e.g. Wansing, 2002), and the framework which we consider here, introduced independently under the names of *nested sequents* in Bull (1992); Kashima (1994), *deep sequents* in Brünnler (2006), and *tree-hypersequents* in Poggiolesi (2008b). Arguably, tree-hypersequents or nested sequents are one of the most successful approaches to the proof theory of modal logic: not only because of the several results and applications that have been obtained with them and which will be described in detail in the next sections (i.e. admissibility of the structural rules, cut-elimination, decidability, interpolation), but also because tree-hypersequents or nested sequents seem to best preserve the original spirit of Gentzen sequent calculus for classical logic: for example, these calculi do not contain any explicit semantic element, and the rules are (often) explicit and separated.

The aim of this paper is to present an overview³ of the tree-hypersequents or nested-sequents calculi in the proof theory for modal logic and beyond. As far as we know, no such overview exists so far, and given the wide variety of results obtained, it seems to be needed. We have tried to be reasonably comprehensive but

² Note that here we do not mention all extensions of the sequent calculus, the list being too long. Moreover, we completely omit the sequent calculi for modal logic obtained not by changing the standard sequent, but by indexing formulae of the language of modal logic, e.g. see Kanger (1957); Simpson (1994); Viganò (2000).

³ We will present only sketches of proofs and give references to the original papers.

due to space constraints we needed to concentrate on some aspects. Of course we might also have unintentionally overlooked some results. The reader is referred in addition, e.g., to the list of publications on nested sequents on the website <http://alessio.guglielmi.name/res/cos/index.html>.

The paper is organized as follows. In Sec. 2 we provide the basic definitions of nested sequent calculi for the standard normal modal logics, we prove the admissibility of structural rules, soundness and completeness plus cut-elimination. In Sec. 3 we show how nested sequents can be applied for countermodel construction (Sec. 3.1) as well as for proving decidability and interpolation results (Sec.s 3.2 and 3.3). Sec. 4 contains an overview of other logics which have been considered in the nested sequent framework, including intuitionistic and first-order logic (Sec. 4.1) as well as other kinds of modal logics such as intuitionistic and constructive modal logic (Sec. 4.2), epistemic modal logics and propositional dynamic logic (Sec. 4.3), and non-normal modal logics and conditional logics (Sec. 4.4). Finally, in Sec. 5 we situate the nested sequent framework in relation to other proof theoretic frameworks including standard sequents (Sec. 5.1), hypersequents (Sec. 5.2), prefixed tableaux (Sec. 5.4), labelled sequents (Sec. 5.5) and display calculi (Sec. 5.7). We also consider the restriction to linear nested sequents (Sec. 5.3) and the extension to indexed nested sequents (Sec. 5.6).

2 The formalism

We use this section to introduce the nested sequents method.⁴ We will do it in the following way: we will first recall some basic notions of modal logic which, although quite well-known, will be important for what follows. Secondly, we will explain the basic ideas behind nested sequents; finally, we will introduce the method formally.

Definition 2.1 Let \mathcal{L} be the *modal propositional language* composed of propositional atoms p, q, r, \dots , the logical symbols \neg, \wedge, \square , and parentheses $(,)$. The connectives \vee, \rightarrow and \leftrightarrow are defined as usual and might be occasionally used as abbreviations. The same holds for the diamond \diamond , which is defined as $\neg\square\neg$. *Formulae* are build up in the usual way and are denoted by greek letters α, β, \dots . We use the capital letters $M, N \dots$ to denote multisets of formulae.

Definition 2.2 A *normal modal system* is a set of well formed formulae of \mathcal{L} such that it contains all the classical tautologies and the distribution axiom :

$$\square(\alpha \rightarrow \beta) \rightarrow (\square\alpha \rightarrow \square\beta)$$

It also contains modus ponens, uniform substitution and the *necessitation rule* namely the rule:

$$\frac{}{\text{if } \vdash \alpha \text{ then } \vdash \square\alpha}$$

⁴ From now till the end of the paper, we adopt the name *nested sequent* instead of *tree-hypersequent*, or *deep sequent* since it is historically the first and also the most used one.

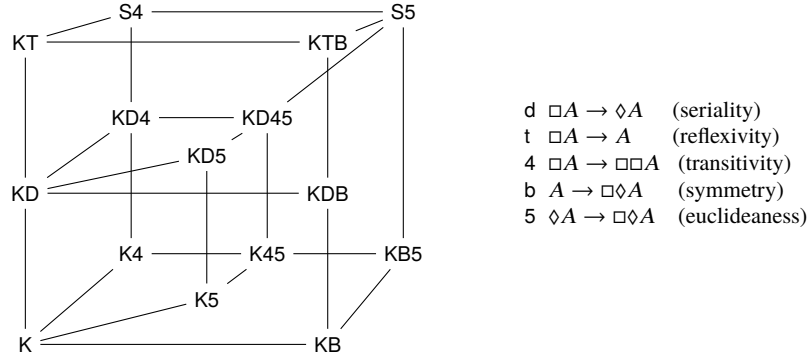


Fig. 2 The modal logic cube together with the axioms and the corresponding frame porperties.

The modal Hilbert system K is the weakest normal modal system: it contains the axioms and rules that we have listed above. The other normal modal systems are obtained by adding to K combinations of the axioms listed in Figure 2: they are named according to the concatenation of the names of their axioms. An exception is provided by systems $S4$ and $S5$: the former is obtained by adding to the system K the axioms T and 4 , the latter the axioms T , 4 and B . These Hilbert normal modal systems are taken to be the main or standard systems of modal logic and they form the so-called *modal logic cube*, see Figure 2.

Let us introduce some basic semantic notions.

Definition 2.3 A *frame* \mathcal{F} is a pair (W, R) such that:

- W is a non empty set (of possible worlds), and
- $R \subseteq W \times W$ is a binary relation on W . R is usually called the *accessibility relation*.

Definition 2.4 A *tree-frame* \mathcal{T} , or simpler a *tree*, is a frame (W, R) that forms a directed, connected and acyclic graph; the nodes of the graph are labelled by the variables i, j, \dots of the set W , and the connection between nodes is established by the relation R .

Definition 2.5 A *model* \mathfrak{M} is a pair (\mathcal{F}, v) , where \mathcal{F} is a frame and v is the following valuation on \mathcal{F} :

$$v := W \otimes PL \rightarrow \{0, 1\}$$

We say that a model $\mathfrak{M} = (\mathcal{F}, v)$ is *based on* the frame \mathcal{F} .

Definition 2.6 Given a model $\mathfrak{M} = \langle (W, R), v \rangle$, $i \in W$, and a formula α , the relation

$$i \models_{\mathfrak{M}} \alpha \quad [\text{or } v(i, \alpha) = 1, \text{ or } \alpha \text{ is true at the world } i \text{ of the model } \mathfrak{M}]$$

usually called the *satisfiability relation*, is inductively defined in the following way:

- $i \models_{\mathfrak{M}} p$ iff $v(i, p) = 1$
- $i \models_{\mathfrak{M}} \neg\beta$ iff $i \not\models_{\mathfrak{M}} \beta$
- $i \models_{\mathfrak{M}} \beta \wedge \gamma$ iff $i \models_{\mathfrak{M}} \beta$ and $i \models_{\mathfrak{M}} \gamma$
- $i \models_{\mathfrak{M}} \Box\beta$ iff $(\forall j \in W) (iRj \rightarrow j \models_{\mathfrak{M}} \beta)$

Definition 2.7 We say that a formula α is *true in a model* \mathfrak{M} , in symbols: $\models_{\mathfrak{M}} \alpha$, if it is true at every world in the model. We say that a formula α is *valid in a frame* \mathcal{F} , in symbols: $\models_{\mathcal{F}} \alpha$, if α is true in every model based on that frame. We say that a formula α is *valid in a class* \mathcal{C} of frames, in symbols: $\models_{\mathcal{C}} \alpha$, if α is valid in every frame which belongs to that class.

Each of the Hilbert normal modal systems forming the modal logic cube is sound and complete with respect to a different class of frames enjoying the property(ies) linked to the corresponding axiom(s), e.g. see (Poggiolesi, 2010b, p. 44).

Let us now introduce the nested sequents method by presenting first of all the basic intuition behind it. We do that by passing through the semantic link between classical and modal logic. One way of looking at this link is the following: both classical and modal logic evaluate (the truth values of) formulae with respect to possible worlds, but while in modal logic we have a set of different possible worlds, in classical logic there is only one possible world (which is therefore often omitted).

Let us look at a sequent $M \Rightarrow N$ as describing one possible world; we will use the expression *world-sequent* to denote this perspective. Since in classical logic there is only one world, only one sequent is required. In modal logic, instead, we deal with n possible worlds; syntactically, we will then need to deal with n sequents at a time. This is the first step toward a generalization of the sequent calculus. However, as clarified by the previous definitions, worlds obey a structure regulated by an accessibility relation. In the case where the accessibility relation is an equivalence relation (this is what happens in S5), i.e. in the case where the n possible worlds can be looked at as just a set of worlds divided by an equivalence relation, syntactically, we can simply work with a set of n sequents, those in a single equivalence class. A set of n sequents is the object standardly called a *hypersequent*.⁵ In the cases where, on the other hand, the accessibility relation amongst worlds enjoys some properties (possibly none), then the n world-sequents need to be arranged somehow. Thanks to the property called the *tree-model property* (which tells us that any satisfiable formula is satisfiable in a model based on a tree-frame, see Blackburn et al., 2001), attention in modal logic can be restricted to tree-frames; consequently, world-sequents can be arranged so as to form a tree. Such a structure is what is called a *nested sequent*, *deep sequent*, or *tree-hypersequent* structure. Hence a nested sequent is simply a bunch of sequents arranged as a tree.

Nested sequent calculi were independently introduced by Bull (1992), Kashima (1994), Brünnler (2006), and Poggiolesi (2008b, 2010b). Different notations come

⁵ See Subsection 4.2 (and also Poggiolesi, 2010c) for the links between nested sequents and hypersequents.

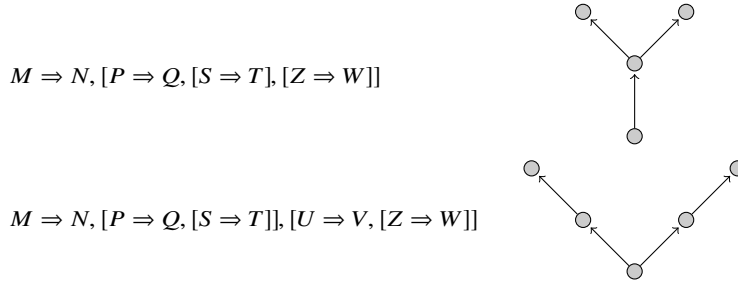


Fig. 3 Two examples of nested sequents with the corresponding tree structure

with the different authors, where the main difference is that between a single-sided and a two-sided formulation. Here we adopt the two-sided formulation in a notation which seems to have become standard and can be found in Bull (1992), Fitting (2014), Ramanayake (2018). See Sec. 2.1 below for a brief comparison with the single-sided formulation.

Definition 2.8 (Nested sequent). A *nested sequent* is a finite object defined recursively as follows:

- if $M \Rightarrow N$ is a sequent, then it is a nested sequent,
- if $M \Rightarrow N$ is a sequent and G_1, \dots, G_n are nested sequents, then $M \Rightarrow N, [G_1, \dots, [G_n]]$ is a nested sequent.

In the following we often use G, H, \dots to stand for nested sequents. Intuitively then, where a nested sequent can be seen as a tree with each node labelled with an ordinary sequent, the *nesting operator* $[.]$ in a nested sequent captures the successor relation in the underlying tree. From a more syntactic perspective, the nesting operator can be seen as a *structural connective* corresponding to the logical connective \Box , in the same way that, e.g., the comma on the right hand side of an ordinary sequent can be seen as the structural connective corresponding to disjunction. This gives rise to the following formula interpretation of nested sequents.

Definition 2.9 (Nested sequent). The *standard interpretation* τ of a nested-sequent is defined recursively as follows:

- $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$,
- $(M \Rightarrow N, [G_1], \dots, [G_n])^\tau := (M \rightarrow N)^\tau \vee \Box(G_1)^\tau \vee \dots \vee \Box(G_n)^\tau$.

Figure 3 shows two examples of nested sequents with the corresponding tree-frames of Kripke semantics. Before considering the rules of the nested calculi for the main systems of modal logic, we need to introduce a notational device that allows us to display these rules. Such a device is called a *context* and it is the occurrence of the symbol $\{ \}$ in a nested sequent (which does not occur inside formulae). Informally, a context allows us to focus on any part of a nested sequent, where the symbol $\{ \}$

acts as a placeholder for a subtree starting at a node of the nested sequent; formally it is defined as follows.

Definition 2.10 The notion of a *nested sequent with a context* is inductively defined by:

- $\{ \}$ is a nested sequent with a context of depth 0,
- if $M \Rightarrow N, [H_1], \dots, [H_n]$ is a nested sequent and $G\{ \}$ is a nested sequent with context of depth m , then $M \Rightarrow N, [H_1], \dots, [H_n], [G\{ \}]$ is a nested sequent with context of depth $m + 1$.

The result of *filling* a nested sequent with a context $G\{ \}$ with another nested sequent H is written as $G\{H\}$ and is obtained by replacing the symbol $\{ \}$ in $G\{ \}$ with H .

Note that the depth of a context in a nested sequent is simply the length of the path from the node labelled with $\{ \}$ to the root. Note also that when showing the rules of the calculi, we might abuse notation and only write N instead of $N, [G_1], \dots, [G_n]$

As stressed by (Poggiolesi, 2010b, p. 8-9) and (Troelstra and Schwichtenberg, 1996, p. 51), Gentzen systems generally come in *variants*. Consider for example classical logic; there are at least two variants of the sequent calculus for classical logic: the *general* variant and the *logical* variant. The general variant for classical logic is a sequent calculus where structural and logical rules are taken as primitive and their roles are kept separate. The logical variant (see Dragalin, 1988) is a sequent calculus that is composed by generalized axioms as well the rules $\neg R$ and $\neg L$ and $\wedge R$; as for the rule introducing the conjunction connective on the left side of the sequent, we consider its multiplicative version, namely the rule

$$\frac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N} \wedge L'$$

The logical variant is called in this way because it contains no structural rule: the force of the structural rules is absorbed by the axioms and the logical rules. Variants of Gentzen systems are different sequent calculi; however, since they are different sequent calculi for the same logic, i.e. different sequent calculi that prove the same sequents, it is profitable to think of them as variants of a same core. In what follows, we chose to mainly work with the logical variant of nested calculi. Indeed, not only is the logical variant the closest to the semantic interpretation of modal logic, but also it is through this variant that most of the results can be more elegantly presented.

The nested calculus \mathbf{NS}_k for the basic modal system K is composed of the axioms and rules given in Figure 4. The rules that introduce negation and conjunction on the left and on the right of the sequent are the standard rules adapted to the nested framework: they state that the classical propositional rules can be applied in any world-sequent. As for the modal rules, they reflect at the syntactic level the satisfaction of a formula of the form $\Box\alpha$ in a world x of a Kripke-model. Indeed, the rule $\Box R$ states, if read bottom up, that if $\Box\alpha$ is false in a world-sequent x , then there exists a world-sequent y accessible from x where α is false. The rule $\Box L$, in contrast, states that if $\Box\alpha$ is true in a world-sequent x , then in any world-sequent y accessible from x , α is true.

Initial Nested Sequents

$$\overline{G\{p, M \Rightarrow N, p\}}$$

Logical Rules

$$\frac{G\{M \Rightarrow N, \alpha\}}{G\{\neg\alpha, M \Rightarrow N\}} \neg L \qquad \frac{G\{\alpha, M \Rightarrow N\}}{G\{M \Rightarrow N, \neg\alpha\}} \neg R$$

$$\frac{G\{\alpha_0, \alpha_1, M \Rightarrow N\}}{G\{\alpha_0 \wedge \alpha_1, M \Rightarrow N\}} \wedge L \qquad \frac{G\{M \Rightarrow N, \alpha_0\} \quad G\{M \Rightarrow N, \alpha_1\}}{G\{M \Rightarrow N, \alpha_0 \wedge \alpha_1\}} \wedge R$$

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L \qquad \frac{G\{M \Rightarrow N, [\Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box R$$

Fig. 4 The basic nested sequent calculus \mathbf{NS}_K for modal logic K.

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha \Rightarrow]\}}{G\{\Box\alpha, M \Rightarrow N\}} d \qquad \frac{G\{\Box\alpha, \alpha, M \Rightarrow N\}}{G\{\Box\alpha, M \Rightarrow N\}} t$$

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} 4 \qquad \frac{G\{\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}} b$$

Fig. 5 The special logical rules for extensions of modal logic K.

In order to obtain the calculi for the remaining modal normal systems, we add combinations of the special logical rules given in Figure 5 to the calculus \mathbf{NS}_K . The resulting calculus is written $\mathbf{NS}_K \cup X$ where X is the set of additional rules. Indeed, each rule corresponds to one of the axioms listed in Figure 2. In the calculi containing the rules 4 and b , we also need to add the following rule:

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}} 5$$

Contrary to all the other rules, 5 does not reflect the strength and the power of the corresponding axiom: indeed the calculus \mathbf{NS}_K plus 5 is not cut-free, while the calculus \mathbf{NS}_K plus the rules 5 and t is not complete. Therefore, the rule 5 just serves to complete the calculi obtained by adding the rules 4 and b to the calculus \mathbf{NS}_K . We call this proviso the *5-closure* of a calculus:⁶

Definition 2.11 A set $X \subseteq \{d, t, b, 4\}$ satisfies the *5-closure proviso* if whenever $\{b, 4\} \subseteq X$, then $5 \in X$.

⁶ See Brünnler (2009); Poggiolesi (2008d).

2.1 Single-sided nested sequents

As in the case of standard sequents, nested sequents have also been considered in a single-sided formulation, most prominently in the series of works starting from Kashima (1994) and Brünnler (2006). Similarly to the case of classical propositional logic, for normal modal logics based on classical logics the two formulations can be translated into each other. The benefit of the single-sided formulation is that the presentation of nested sequents and nested sequent rules is much more economic. The downside of this is that formulae are often assumed to be in *negation normal form*, i.e., negation is allowed only immediately in front of propositional variables, and that because of this we need to assume that we have the dual operators of the modal box, i.e., the modal diamond, as well as of conjunction, i.e., disjunction, in the language.

Definition 2.12 A *single-sided nested sequent* is a structure

$$M, [G_1], \dots, [G_n]$$

where M is a multiset of formulae in negation normal form, $n \geq 0$ and G_1, \dots, G_n are single-sided nested sequents. The *standard interpretation* τ of single-sided nested sequents is given by $(M, [G_1], \dots, [G_n])^\tau := \bigvee M \vee \Box G_1^\tau \vee \dots \vee \Box G_n^\tau$.

Thus, in a single-sided nested sequent the nodes of the underlying tree are labelled with single-sided sequents, i.e., multisets of formulae in negation normal form, instead of two-sided sequents. The notions of *context*, *depth*, etc., are defined analogously to the two-sided formulation. The modal rules of the single-sided nested sequent system for modal logic K then take the following form:

$$\frac{G\{M, [\alpha]\}}{G\{M, \Box\alpha\}} \Box_R \quad \frac{G\{M, \Diamond\alpha, [H, \alpha]\}}{G\{M, \Diamond\alpha, [H]\}} \Box_L$$

The rules for extensions are formulated analogously. We refer the reader to, e.g., Kashima (1994), Brünnler (2006), and Marin and Straßburger (2014) for the detailed description of the systems.

2.2 Admissibility and invertibility results

We use this section for showing that all structural rules are admissible in the nested calculi for the main normal systems of modal logic. We will also show that the logical rules are invertible.

Lemma 2.13 (Generalized initial sequents) *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, the nested sequents of the form $G\{\alpha, M \Rightarrow N, \alpha\}$ are derivable in $\text{NS}_k \cup X$.*

$$\begin{array}{ccc}
\frac{G\{M \Rightarrow N\}}{G\{\alpha, M \Rightarrow N\}} \text{WL} & \frac{G\{M \Rightarrow N\}}{G\{M \Rightarrow N, \alpha\}} \text{WR} & \frac{G\{M \Rightarrow N\}}{G\{M \Rightarrow N, [P \Rightarrow Q]\}} \text{EW} \\
\frac{G\{\alpha, \alpha, M \Rightarrow N\}}{G\{\alpha, M \Rightarrow N\}} \text{CL} & \frac{G\{M \Rightarrow N, \alpha, \alpha\}}{G\{M \Rightarrow N, \alpha\}} \text{CR} & \frac{G\{M \Rightarrow N, [P \Rightarrow Q], [S \Rightarrow T]\}}{G\{M \Rightarrow N, [P, S \Rightarrow Q, T]\}} \text{merge} \\
\frac{G}{\Rightarrow [G]} \text{NR} & &
\end{array}$$

Fig. 6 Structural Rules

Proof By induction on α . □

The structural rules of standard sequent calculi are adapted to the nested structure: they can be applied to any world-sequent (see rules WL, WR and CL, CR). In addition to them we also have rules manipulating the nested sequent structure, namely the external structural rules of *external weakening* (EW), *merge* (merge) and *necessitation* (NR), see Figure 6. It can be shown that all these rules are *height-preserving admissible*, i.e., that whenever their premisses are derivable with derivations of height at most n , then so is their conclusion.

Lemma 2.14 (Admissibility of the structural rules) *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, the rules WL, WR, EW, CL, CR, NR and merge (see Figure 6) are height-preserving admissible in $\mathbf{NS}_k \cup X$.*

Proof By induction on the derivation of the premiss. □

Using the previous lemma we can then show that the logical and special logical rules are *height-preserving invertible*, i.e., whenever their conclusion is derivable with a derivation of height at most n , then so are their premisses.

Lemma 2.15 (Invertibility) *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, the logical, modal and special logical rules are height-preserving invertible in $\mathbf{NS}_k \cup X$.*

Proof By induction on the derivation of the premiss, using Lem. 2.14. □

As we have seen above, each of the axioms listed in Figure 2 corresponds to a special logical rule (see Figure 3); however, it also corresponds to a *special structural rule*, see Figure 8. Each of the special structural rules mirrors the frame property corresponding to the axiom to which it is linked. E.g., rule \tilde{d} ensures that every node in a nested sequent can be equipped with a successor, corresponding to the seriality property linked to the axiom d. Given the special logical rules, we then have admissibility of the corresponding special structural rules.

Lemma 2.16 (Admissibility of the special structural rules) *For any $X \subseteq \{d, t, b, 4, 5\}$, let \tilde{X} be the corresponding subset in $\{\tilde{d}, \tilde{t}, \tilde{b}, \tilde{4}, \tilde{5}\}$ (see Figure 8). For any $X \subseteq \{d, t, b, 4, 5\}$, the rules in \tilde{X} are height-preserving admissible in $\mathbf{NS}_k \cup X$.*

Proof By induction on the derivation of the premiss. □

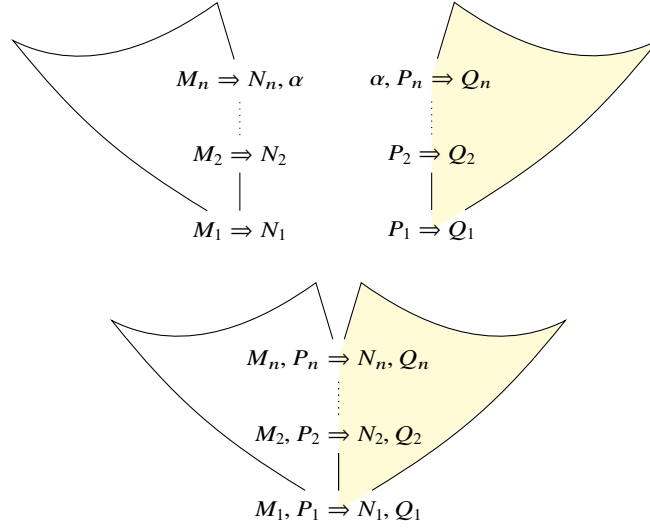


Fig. 7 A visual representation of the premisses and conclusion of the (context-splitting) cut rule for nested sequents.

2.3 Cut-elimination, soundness and completeness

One of the technical benefits of the nested sequent framework is that it can be used in purely syntactic *cut elimination* proofs. However, defining the cut rule in this framework is not entirely trivial, since we need to generalise it to the tree structure of nested sequents. As usual we have the choice between the *context-sharing* and the *context-splitting* variant of the rule. The context-sharing variant is given by:

$$\frac{G\{M \Rightarrow N, \alpha\} \quad G\{\alpha, M \Rightarrow N\}}{G\{M \Rightarrow N\}} \text{ cut}_{c,s}$$

While cut elimination can indeed be shown syntactically for the context-sharing variant of the cut rule (see, e.g., Brünnler, 2009), this entails a certain overhead in terms of the structural rules to ensure that the context of the premisses is always the same. Hence in the following we consider the context-splitting variant. The intuition is that considering the rule bottom-up from conclusion to premisses, the nested sequent is split along the branch from the root to the node where the cut formula is introduced. Dually, considering the rule from premisses to conclusion, the nested sequents corresponding to the premisses are *merged* or *zipped together* along the branch from their roots to the node containing the cut formula. The graphical intuition for this is shown in Figure 7. Formally, this is captured as follows (recall that we abuse notation and write, e.g., $M_1 \Rightarrow N_1$ for $M_1 \Rightarrow N_1, [G_1], \dots, [G_m]$):

Definition 2.17 For two nested sequent contexts

$$\begin{aligned} G\{\} &= M_1 \Rightarrow N_1, [\dots [M_n \Rightarrow N_n, [\{\}]] \dots] \\ H\{\} &= P_1 \Rightarrow Q_1, [\dots [P_n \Rightarrow Q_n, [\{\}]] \dots] \end{aligned}$$

the *merge* is the following nested sequent context:

$$G \oplus H\{\} := M_1, P_1 \Rightarrow N_1, Q_1, [\dots [M_n, P_n \Rightarrow N_n, Q_n, [\{\}]] \dots].$$

Intuitively, in $G \oplus H\{\}$ the two nested sequent contexts are “zipped together” along the path from root to the context symbol $\{\}$. In particular, this means that the context symbol $\{\}$ occurs at the same depth both in G and H .

The *context-splitting cut rule* in the nested sequent framework then is the following rule, where the context has the same depth in $G\{\}$ and $H\{\}$:

$$\frac{G\{M \Rightarrow N, \alpha\} \quad H\{\alpha, P \Rightarrow Q\}}{G \oplus H\{M, P \Rightarrow N, Q\}} \text{ cut}$$

The cut elimination result is then shown as follows.

Lemma 2.18 (Cut elimination) *Let $G\{M \Rightarrow N, \alpha\}$ and $H\{\alpha, P \Rightarrow Q\}$ be two nested sequents where the context symbol $\{\}$ occurs at the same depth both in G and H . If*

$$\frac{\begin{array}{c} \vdots d_1 \\ G\{M \Rightarrow N, \alpha\} \end{array} \quad \begin{array}{c} \vdots d_2 \\ H\{\alpha, P \Rightarrow Q\} \end{array}}{G \oplus H\{M, P \Rightarrow N, Q\}} \text{ cut}$$

and d_1 and d_2 do not contain any other application of the cut-rule, then we can construct a derivation of $G \oplus H\{M, P \Rightarrow N, Q\}$ with no application of the cut-rule.

Proof The proof is by induction on the complexity of the cut-formula α , with subinduction on the sum of the heights of the derivations of the premisses of the cut-rule. The details of the proof can be found in (Poggiolesi, 2008b, Section 7); here we just illustrate some significant case. Suppose the cut-formula α is of the form $\Box\beta$ and it has been introduced in the left premiss by the rule $\Box R$ and in the right premiss by the rule $\Box L$. Hence we have the following situation:

$$\frac{\frac{G\{M \Rightarrow N, [\Rightarrow \beta]\}}{G\{M \Rightarrow N, \Box\beta\}} \Box R \quad \frac{H\{\Box\beta, P \Rightarrow Q, [\beta, S \Rightarrow T]\}}{H\{\Box\beta, P \Rightarrow Q, [S \Rightarrow T]\}} \Box L}{G \oplus H\{M, P \Rightarrow N, Q, [S \Rightarrow T]\}} \text{ cut}$$

We reduce to

$$\frac{G\{M \Rightarrow N, [\Rightarrow \beta]\} \quad \frac{G\{M \Rightarrow N, \Box\beta\} \quad H\{\Box\beta, P \Rightarrow Q, [\beta, S \Rightarrow T]\}}{G \oplus H\{M, P \Rightarrow N, Q, [\beta, S \Rightarrow T]\}} \text{ cut}}{\frac{G \oplus G \oplus H\{M, M, P \Rightarrow N, N, Q, [S \Rightarrow T]\}}{G \oplus H\{M, P \Rightarrow N, Q, [S \Rightarrow T]\}} X} \text{ cut}$$

where the first cut is eliminable by induction on the height of the proof, whilst the second cut is eliminable by induction on the complexity of the cut-formula. The X denotes several applications of internal contraction rules and merge.

Several other interesting cases of cut emerge whenever the cut-formula α is still of the form $\Box\beta$, the left premiss has been introduced by the rule $\Box R$ and the right premiss has been introduced by one of the special logical rules Y . In the cases where Y is either t , d or b , the strategy is the same: in order to reduce, we need to use the corresponding admissible special structural rule \tilde{Y} , see Lemma 2.16, on the nested sequent above the left premiss of the cut. As an example, we show the case of the rule t .

$$\frac{\frac{G\{M \Rightarrow N, [\Rightarrow \beta]\}}{G\{M \Rightarrow N, \Box\beta\}} \Box R \quad \frac{H\{\Box\beta, \beta, P \Rightarrow Q\}}{H\{\Box\beta, P \Rightarrow Q\}} t}{G \oplus H\{M, P \Rightarrow N, Q\}} \text{cut}$$

We reduce to

$$\frac{\frac{G\{M \Rightarrow N, [\Rightarrow \beta]\}}{G\{M \Rightarrow N, \beta\}} \tilde{t} \quad \frac{G\{M \Rightarrow N, \Box\beta\} \quad H\{\Box\beta, \beta, P \Rightarrow Q\}}{G \oplus H\{\beta, M, P \Rightarrow N, Q\}} \text{cut}}{G \oplus G \oplus H\{M, M, P \Rightarrow N, N, Q\}} \text{cut}}{G \oplus H\{M, P \Rightarrow N, Q\}} X$$

where the first cut is eliminable by induction on the height of the proof, whilst the second cut is eliminable by induction on the complexity of the cut-formula. The X denotes several applications of internal contraction rules and merge.

In the cases where the special logical rule used to introduce the right premiss of the cut-rule is either 4 or 5, then the strategy is analogous to the one above except for the fact that one should further distinguish cases on what may have introduced the nested sequent above the right premiss of the cut-rule. \square

Theorem 2.19 *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, every derivation in $\mathbf{NS}_k \cup X \cup \{\text{cut}\}$ can be effectively transformed into a derivation d' in $\mathbf{NS}_k \cup X$, i.e., where there is no application of the cut-rule.*

Proof It follows from the previous Lemma by induction on the number of cuts. \square

We can now close the section by establishing soundness and completeness for the nested calculi.

Theorem 2.20 *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, the nested calculus $\mathbf{NS}_k \cup X$ is sound and complete with respect to the corresponding Hilbert system.*

Proof The soundness proof is by induction on the height of derivations in Hilbert systems and can be found in (Poggiolesi, 2010b, §6.3). The completeness proof consists in showing that we can construct in nested sequent calculi derivations of the axioms and the inference rules that compose the corresponding Hilbert system. Here we show the derivation of the distribution axiom; the others derivations can be found in (Poggiolesi, 2010b, §6.3).

$$\begin{array}{c}
\frac{G\{M \Rightarrow N, [\Rightarrow]\}}{G\{M \Rightarrow N\}} \bar{d} \quad \frac{G\{M \Rightarrow N, [P \Rightarrow Q]\}}{G\{M, P \Rightarrow N, Q\}} \bar{t} \\
\frac{G\{M \Rightarrow N, [P \Rightarrow Q]\}}{G\{M \Rightarrow N, [\Rightarrow [P \Rightarrow Q]]\}} \bar{4} \quad \frac{G\{M \Rightarrow N, [S \Rightarrow T, [P \Rightarrow Q]]\}}{G\{M, P \Rightarrow N, Q, [S \Rightarrow T]\}} \bar{b} \\
\frac{G\{M \Rightarrow N, [S \Rightarrow T, [P \Rightarrow Q]]\}}{G\{M \Rightarrow N, [S \Rightarrow T, P \Rightarrow Q]\}} \bar{5}
\end{array}$$

Fig. 8 The special structural rules for extensions of modal logic K.

$$\begin{array}{c}
\frac{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha \Rightarrow \alpha, \beta] \quad \Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha, \beta \Rightarrow \beta]}{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha \rightarrow \beta, \alpha \Rightarrow \beta]} \rightarrow_L \\
\frac{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha \rightarrow \beta, \alpha \Rightarrow \beta]}{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha \Rightarrow \beta]} \Box_L \\
\frac{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\alpha \Rightarrow \beta]}{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\Rightarrow \beta]} \Box_L \\
\frac{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow [\Rightarrow \beta]}{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow \Box\beta} \Box_R \\
\frac{\Box(\alpha \rightarrow \beta), \Box\alpha \Rightarrow \Box\beta}{\Box(\alpha \rightarrow \beta) \Rightarrow \Box\alpha \rightarrow \Box\beta} \rightarrow_R \\
\frac{\Box(\alpha \rightarrow \beta) \Rightarrow \Box\alpha \rightarrow \Box\beta}{\Rightarrow \Box(\alpha \rightarrow \beta) \rightarrow \Box\alpha \rightarrow \Box\beta} \rightarrow_R
\end{array}$$

Note that in the completeness proof, use of a nested form of the *cut rule* to simulate applications of modus ponens in the Hilbert-system is required. There also exists a completeness proof based on the construction of counter-models that can be found in Brünnler (2006). We will look at this proof in detail in Sec. 3.1. \square

The nested calculi so far introduced enjoy several interesting properties that we hope to have underlined; however they also display a defect which is the lack of *modularity*.⁷ The nested calculi presented are not modular because the axiom 5 is not systematically reflected by the rule 5. As shown by Marin and Straßburger (2014), this lack of modularity can be repaired by giving up the logical variant of the nested calculi. Indeed, if we consider nested sequent calculi where: (i) the contraction rules CL and CR are added (see Figure 6), (ii) in the rule \Box_L as well as in the special logical rules, the formula $\Box\alpha$ which is not the main formula of the rule is erased, (iii) for each modal axiom, the corresponding special logical and structural rule are taken as primitive rules, then we have a more general version of the rules 5 and $\bar{5}$ which allow the calculi to be modular.

3 Applications

We use this section to illustrate three applications of the nested sequent method; in particular, we will dwell on countermodels, decidability and interpolation results.

⁷ Consider a Hilbert system S' which is obtained by adding an axiom A to the Hilbert system H . The modularity property demands the Gentzen system G' for the system S' to be obtained from the system G for the Hilbert system S by adding n rules that systematically reflect at the proof-theoretical level the axiom A .

3.1 Countermodel construction

In this section we will focus on the completeness of nested calculi via the construction of countermodels. More precisely, we show the strategy of this proof in the case of the nested calculus \mathbf{NS}_k . Note that the completeness proof via countermodels can be proved for all the nested calculi $\mathbf{NS}_k \cup X$, for any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso; in these cases, the proof is analogous to that for the nested calculus \mathbf{NS}_k , except that properties of the accessibility relation in Kripke frames need to be taken into account as well, see Brünnler (2006) and (Poggiolesi, 2010b, Section 8).

For each rule \mathcal{R} of \mathbf{NS}_k we consider the *Kleene'd version*, i.e., the rule \mathcal{R}^+ that is the same as \mathcal{R} except that it maintains the main formula of the conclusion in the premiss (analogously to the G3 systems in Kleene, 1952). For example, if \mathcal{R} is the rule $\neg R$, then \mathcal{R}^+ will be:

$$\frac{G\{\alpha, M \Rightarrow N, \neg\alpha\}}{G\{M \Rightarrow N, \neg\alpha\}} \neg^+ R$$

Let \mathbf{NS}_k^+ be the calculus obtained by replacing each rule \mathcal{R} in \mathbf{NS}_k with the corresponding rule \mathcal{R}^+ . It is straightforward to show that the calculi \mathbf{NS}_k^+ and \mathbf{NS}_k are equivalent (see Brünnler, 2006). Thus we can essentially consider nested sequents based on sets instead of multisets:

Definition 3.1 The *set nested sequent* of the nested sequent $M \Rightarrow N, [G_1], \dots, [G_n]$ is the nested sequent $P \Rightarrow Q, [H_1], \dots, [H_n]$, where P and Q are the underlying sets of M and N , respectively, and H_1, \dots, H_n are the set nested sequents of G_1, \dots, G_n , respectively. Clearly the set nested sequent of a nested sequent is still a nested sequent since a set is a multiset.

Definition 3.2 A leaf of a nested sequent (thinking of the nested sequent as a tree-frame of Kripke semantics) is *cyclic* if in its branch there exists a second sequent that contains the same set of formulae.

Definition 3.3 A sequent occurring in a nested sequent is *saturated* for a nested sequent calculus \mathbf{Y} if no rule of that calculus can be applied to one of its formulae to obtain a formula which is not already present in that sequent. A nested sequent is finished for a nested sequent calculus \mathbf{Y} if all sequents that compose it are finished or cyclic.

Definition 3.4 We define a procedure $prove(G, \mathbf{NS}_k^+)$, which takes a nested sequent G and the calculus \mathbf{NS}_k^+ , and builds a derivation tree for G by applying rules from that calculus to non-initial and unfinished derivation leaves in a bottom-up fashion, as follows:

1. keep applying all the rules of \mathbf{NS}_k^+ which are not the rule $(\Box K)^+$;
2. wherever possible, apply the rule $(\Box K)^+$ once.

Repeat this operation until each non-initial derivation leaf of the nested sequent G is finished. If $\text{prove}(G, \mathbf{NS}_k^+)$ terminates and all derivation leaves are initial then it succeeds; otherwise, i.e. if it terminates and there is a non-initial derivation leaf, it fails.

Definition 3.5 The *set of subformulae* of a nested sequent G , denoted $\text{sf}(G)$, is the set of all subformulae of all formulae that compose all sequents that belong to the nested sequent.

Lemma 3.6 For all nested sequents G , the procedure $\text{prove}(G, \mathbf{NS}_k^+)$ terminates after at most $2^{O(|\text{sf}(G)|)}$ iterations.

Proof Consider a sequence of nested sequents along a given branch of the derivation starting from the root. None of the rules that we can apply in accordance with step 1 creates new sequents in the nested sequents, but each of them causes the set of formulae of some sequent belonging to the nested sequent to strictly grow. By the subformula property, only $|\text{sf}(G)|$ many formulae can occur in each side of a sequent, so step 1 terminates. If after step 1 there is an unfinished leaf in a nested sequent, then the size of the nested sequent strictly grows in step 2. Since there are only $2^{|\text{sf}(G)|}$ different sets of formulae that can occur, each unfinished nested sequent leaf has to be cyclic eventually.

In this following definition we exploit the strong analogy between nested sequents and tree-frames of Kripke semantics (see Definition 2.4).

Definition 3.7 A nested sequent H is an *immediate subtree* of a nested sequent G if G is of the form $M \Rightarrow N, [H], [H_1], \dots, [H_n]$. It is a *proper subtree* if it is an immediate subtree either of G or of a proper subtree of G , and it is a *subtree* if it is either a proper subtree of G or $G = H$. The *set of all subtrees* of G is denoted by $\text{st}(G)$.

Lemma 3.8 For all nested sequents G , if $\text{prove}(G, \mathbf{NS}_k^+)$ fails, then there is a countermodel for G .

Proof We give a sketch of the proof by indicating how to construct the countermodel for G after $\text{prove}(G, \mathbf{NS}_k^+)$ has failed. A Kripke model (see Definition 2.5) is composed of (i) a set of worlds W , (ii) an accessibility relation amongst worlds \rightarrow and (iii) an evaluation function v mapping formulae and worlds into truth values. Here is how we construct each of the elements of the Kripke model \mathfrak{M} intended to be a countermodel for G .

(i) Let G^* be the set nested sequent of the non-initial nested sequent obtained by the failure of $\text{prove}(G, \mathbf{NS}_k^+)$. Let Υ be the set of all cyclic leaves in G^* . Then in the model \mathfrak{M} the set of worlds is obtained by considering the set of all subtrees of G^* minus Υ , i.e. $W = \text{st}(G^*) \setminus \Upsilon$. Let $f : \Upsilon \rightarrow W$ be some function which maps a cyclic leaf to a nested sequent in W whose root contains the same set of formulae, and extend f to $\text{st}(G^*)$ by the identity on W .

(ii) The accessibility relation on W is defined in the following way. For all G and H , $G \rightarrow H$ if, and only if, either H is an immediate subtree of G , or G has an immediate subtree H_1 , such that $H_1 \in \Upsilon$ and $f(H_1) = H$.

(iii) Finally the evaluation v from propositional atoms and nested sequents to truth values is defined in the following way. An atom p is true in a world-nested sequent G if it occurs on the left side of a sequent belonging to the nested sequent. It is false, otherwise.

The rest of the proof aims at showing that the model \mathfrak{M} is a countermodel for G , see Brünnler (2006) and (Poggiolesi, 2010b, pp. 173-174). \square

The definition of validity of a nested sequent is tedious but it is a quite straightforward generalization of the notion of validity of a sequent. Its definition can be found in (Poggiolesi, 2010b, p. 165-166).

Theorem 3.9 *For all nested sequents G , if G is valid with respect to the class of all frames, then the nested sequent calculus \mathbf{NS}_k is such that $\vdash G$ in \mathbf{NS}_k .*

Proof It follows from the previous Lemma and the fact that the calculi \mathbf{NS}_k and \mathbf{NS}_k^+ are equivalent. \square

3.2 Decidability

Another notable feature of nested calculi is that most of them are decidable: given a nested sequent G , there is an algorithm determining whether G is provable or not. We dedicate this section to a brief presentation of decidability in nested calculi;⁸ as we have done in the previous sections, for our purposes we will focus on the calculus \mathbf{NS}_k ; however decidability can be proved for (most of)⁹ the nested calculi $\mathbf{NS}_k \cup X$, for any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso; only in the case of the system for S4 a certain attention needs to be dedicated to the interaction between the rules 4 and the rule $\Box K$ which might create endless chain of sequents: however, thanks to the use of the rule t the problem is easily fixed, see (Poggiolesi, 2010b, p. 161).

We start by noticing that in a calculus enjoying the cut-elimination theorem, the *subformula property* - in any of its rules the formulae occurring in the premiss(es) are subformulae of the formulae occurring in the conclusion - and contraction-admissibility, any source of potential non-terminating proof-search seems to be avoided. Although the calculus \mathbf{NS}_k enjoys all these features, because of the rule $\Box L$ where the same formula $\Box \alpha$ occurs in the conclusion as well as in the premiss, we still have potentially endless applications. In order to overcome this problem, a solution is to work with *minimal derivations* namely derivations where shortening are not possible in the sense that every subderivation is of minimal height. Suppose

⁸ Note that decidability for modal logic is a well-known result since the 1960s, e.g. see Segerberg (1968).

⁹ The nested calculi for which a decidability proof is developed are \mathbf{NS}_k , \mathbf{NS}_{kt} , \mathbf{NS}_{kd} , \mathbf{NS}_{kb} , \mathbf{NS}_{kbt} , \mathbf{NS}_{S4} , \mathbf{NS}_{S5}

that the rule $\Box L$ is used twice on the same pair of sequents in a minimal derivations, then we can show that such derivation is no longer minimal and hence that the rule $\Box L$ can at most be used once. Thanks to this device, we can prove that the calculus \mathbf{NS}_k allows for terminating proof-search.

As shown in Poggiolesi (2010b), in the other nested calculi the problem raised by the rule $\Box L$ also arises with the special logical rules. However, minimal derivations represent again the solution to such a problem: in these derivations special logical rules cannot be applied more than once on the same formula or pair of sequents.

Lemma 3.10 *The rule $\Box L$ permutes down with respect to the propositional rules. It also permutes with instances of the rule $\Box R$ in the case where their auxiliary formulae, α and $\Box\alpha$, respectively, are not active in the sequent where the auxiliary formula of $\Box R$ occurs.*

Proof The proof is straightforward. \square

Corollary 3.11 *In a minimal derivation in \mathbf{NS}_k , the rule $\Box L$ cannot be applied more than once on the same pair of formulas belonging to the same pairs of sequents of any branch.*

Proof Let us suppose we have a minimal derivation where the rule $\Box L$ has been applied twice on the same pair of sequents,

$$\frac{G\{\Box\alpha, M' \Rightarrow N', [\alpha, P' \Rightarrow Q']\}}{G\{\Box\alpha, M' \Rightarrow N', [P' \Rightarrow Q']\}} \Box L$$

$$\vdots$$

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L$$

By permuting down $\Box L$ with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, \alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}} \Box L$$

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L$$

By applying the height-preserving admissible rule CL to the two occurrences of the formula α in place of the upper $\Box L$, we obtain a shortened derivation, contrary to the assumption of minimality. \square

We can finally prove that the modal logic \mathbf{K} is decidable by means of showing effective bounds on proof search in the calculus \mathbf{NS}_k .

Theorem 3.12 *The calculus \mathbf{NS}_k allows terminating proof search.*

Proof Place a nested sequent G , for which we are looking for a proof search, at the root of the procedure. Apply first the propositional rules and then the modal rules.

The propositional rules reduce the complexity of the nested sequent. The rule $\Box R$ removes the modal constant \Box , and adds a new sequent, the rule $\Box L$ increases the complexity. However, by the corollary above, the rule $\Box L$ cannot be applied more than once to the same pair of sequents with the same principal formula. Therefore, the number of applications of the rule $\Box L$ is bounded by the number of subformulae of the conclusion and the number of sequents that may appear in the derivation. The latter, in its turn, is bounded by the number of sequents belonging to the nested sequents to prove, and the sequents that can be introduced by applications of the rule $\Box R$. \square

3.3 Interpolation

A logic L is said to enjoy the *Craig Interpolation Property*, for short *CIP*, if whenever, $A \vdash_L B$, there exists a formula C , called the *interpolant*, that stands in between A and B in terms of derivability, namely such that $A \vdash_L C$, $C \vdash_L B$ and C contains only propositional variables common to A and B . Since in many logics where the CIP is proved, it is possible to define an implication enjoying modus ponens and the deduction theorem, a more common formulation of CIP uses $\vdash_L A \rightarrow B$ instead of $A \vdash_L B$. The CIP is a deep feature of a logic: it has notable applications in computer science (e.g. consistency proofs, model checking), it is connected to Beth definability (e.g. see Hodges, 1997) and it can be translated to the amalgamation property of algebraic varieties (e.g. see Day and Jaroslav, 1984). However the CIP might fail even in some reasonable fundamental logics, such as intuitionistic logic with constant domains, see Mints et al. (2013).

The topic of this section is the Craig Interpolation Property for the main systems of modal logic. There are several different ways to prove this result (e.g. see Fitting, 1983); in these pages the focus goes on a constructive method for proving interpolants for modal logic, namely using induction on derivations in nested calculi. The first research in this direction has been initiated by Bílková (2011) for the stronger *uniform* interpolation property. An extensive body of work was further developed starting from Fitting and Kuznets (2015) and Kuznets (2018). In Lyon et al. (2020) the technique adopted by Fitting and Kuznets has been developed in a purely syntactical variant independent of the semantics of the corresponding logics and applied to the realm of logics with converser modalities including tense bi-intuitionistic logics. In what follows, we will mainly rely on Fitting and Kuznets's results; however, given the length and complexity of these results, we will not dwell on the details but only sketch their main features and key-points.

In order to understand how the CIP can be proved via nested calculi, let us start from how it is proved in a simpler framework, namely that of sequent calculi for classical logic. Consider then a theorem of classical logic of the form $A \rightarrow B$; since it is a theorem, there will be a derivation in the sequent calculus **Gcl** of the sequent $A \Rightarrow B$ and the derivation will possibly involve other sequents of the form $M \Rightarrow N$.

For constructing the interpolant C , we do not look at the sequents $A \Rightarrow B$ or $M \Rightarrow N$ themselves, but to their *split* version, namely:

$$A; \Rightarrow; B \quad M_1; M_2 \Rightarrow N_1; N_2$$

The need of splitting sequents comes from the fact that certain rules, e.g., for the propositional negation and implication, move formulae around between the antecedent and consequent and thus disqualify them from being the stable left and right side to be interpolated. Because of the split of the sequents, it is not entirely correct to say that Craig interpolation is proved by induction on the sequent derivation. In reality, one finds a suitable generalization of CIP and proves that by induction on the split version of the sequent derivation. Formally, it is proved that, for any derivable sequent $M \Rightarrow N$ and for any partition of $M = M_1 \cup M_2$ and $N = N_1 \cup N_2$ into two parts each, there exists a formula C such that:

- $\vdash_{\text{CL}} M_1 \Rightarrow N_1, C$,
- $\vdash_{\text{CL}} C, M_2 \Rightarrow N_2$,
- each atomic proposition occurring in C occurs both in $M_1 \Rightarrow N_1$ and $M_2 \Rightarrow N_2$.

The fact that C is an interpolant is ensured by the standard interpretation of sequents. Indeed we have (see Section 2):

$$(M_1, M_2 \Rightarrow N_1, N_2)^\tau = \bigwedge M_1 \wedge \bigwedge M_2 \rightarrow \bigvee N_1 \wedge \bigwedge N_2$$

that is equivalent to

$$\neg(\bigwedge M_1 \rightarrow \bigvee N_1) \rightarrow (\bigwedge M_2 \rightarrow \bigvee N_2)$$

The idea of split sequents can be found in Maehara (1961) and is a key feature of what is called the *Maehara method* for constructing interpolants via the Gentzen system. Since such a method is extended and employed by Fitting and Kuznets for nested calculi for modal logics, let us briefly describe it. Given a cut-free sequent calculus for the logic CL, and a theorem of the form $A \rightarrow B$ of CL:

1. find a sequent derivation of $A \Rightarrow B$,
2. split the endsequent to be $A; \Rightarrow; B$,
3. propagate this split from the endsequent to the leaves turning the derivation into a split derivation,
4. find interpolants for each split initial sequent in a leaf,
5. propagate interpolants from the leaves to the endsequent using rules for transforming interpolants for each rule.

Such method comes with tables presenting split initial sequents, split sequent rules, but also with sufficient interpolant transformations. We will not present these tables here, they can be found in (Kuznets, 2018, 1377-1378).

Let us now move to modal logic and to nested calculi. At the first glance it might seem that the Maehara method can be straightforwardly applied to this extension of classical logic given that nested calculi are cut-free and they only contain logical

rules. Unfortunately, this is not so obvious and for the following reason. Consider a quite simple nested sequent of the following form:

$$M \Rightarrow N, [P \Rightarrow Q], [S \Rightarrow T]$$

The splitting should be performed by splitting each of the sequent components:

$$M_1, M_2 \Rightarrow N_1, N_2, [P_1, P_2 \Rightarrow Q_1, Q_2], [S_1, S_2 \Rightarrow T_1, T_2]$$

According to the standard interpretation of a nested sequent (see Definition 2.9), this nested sequent corresponds to the following formula:

$$\begin{aligned} & (\bigwedge M_1 \wedge \bigwedge M_2 \rightarrow \bigvee N_1 \vee \bigvee N_2) \vee \Box(\bigwedge P_1 \wedge \bigwedge P_2 \rightarrow \bigvee Q_1 \vee \bigvee Q_2) \vee \\ & \Box(\bigwedge S_1 \wedge \bigwedge S_2 \Rightarrow \bigvee T_1 \vee \bigvee T_2) \end{aligned}$$

As far as it is known, it seems there is no way to find an equivalent syntactical representation of this formula as an implication having $M_1, N_1, P_1, Q_1, S_1, T_1$ in its antecedent and $M_2, N_2, P_2, Q_2, S_2, T_2$ in its consequent. Does this mean that the Maehara method cannot be applied to modal logics (via nested sequents) or is there another way of splitting a nested sequent? Thanks to a careful use of semantic methods, Fitting and Kuznets propose a novel and fine-grained way to split a nested sequent.

Definition 3.13 Let $\mathfrak{M} = (W, R, \nu)$ be a Kripke model with the set of worlds $W \neq \emptyset$, the accessibility relation R amongst worlds, and the propositional valuation ν . We define \mathfrak{M} -decorated nested objects G^\star as a nested sequent with each sequent node decorated by a world from W in such a way that the world decorating a child of a given sequent node is R -accessible from the world decorating the sequent node itself.

Definition 3.14 Let $\mathfrak{M} = (W, R, \nu)$ be a Kripke model. An \mathfrak{M} -decorated nested object G^\star is *true* if at least one formula from G^\star holds at the world of W that decorates the sequent node of this formula. We write $\models_{\mathfrak{M}} G^\star$ to denote that G^\star is true in the model \mathfrak{M} .

The crucial idea behind this definition is that decorations help us to look at nested sequents as disjunctions of member formulae, similar to the standard interpretation of normal sequents. However, while in the case of standard sequents, all formulae are evaluated at the same world of the Kripke model, in the nested case, formulae from different sequent nodes are evaluated at different worlds of the Kripke model.

Consider again the nested sequent above, namely:

$$M_1, M_2 \Rightarrow N_1, N_2, [P_1, P_2 \Rightarrow Q_1, Q_2], [S_1, S_2 \Rightarrow T_1, T_2]$$

Then appropriately decorate this nested sequent by the three worlds w, w_1, w_2 , such that wRw_1 and wRw_2 . We can finally define the split of the nested sequent in the following adequate way:

$\models_{\mathfrak{M}} M_1 \Rightarrow N_1, [P_1 \Rightarrow Q_1], [S_1 \Rightarrow T_1]$ implies $\models_{\mathfrak{M}} M_2 \Rightarrow N_2, [P_2 \Rightarrow Q_2], [S_2 \Rightarrow T_2]$

We have defined how to split a nested sequent and how to evaluate it by decorating sequent nodes. It remains to define interpolants, which will be intermediaries of implications of the types above. However, implications of this sort are not purely syntactic anymore: they involve sequents as well as multiple worlds where these sequents are evaluated. As a consequence, interpolants will have a similar form: they will not be mere formulae or mere sequents, but *generalized nested sequents*, namely a mix of nested sequents and worlds, with a particular tree-structure analogous to that of the nested sequents they are the intermediaries for. Generalized nested sequents are defined in (Fitting and Kuznets, 2015, p. 284).

In order to prove the CIP via nested calculi, in (Fitting and Kuznets, 2015, p. 284) as well as in Kuznets (2018) (although with different notation and names), tables for splitting initial nested sequents, splitting nested sequent rules, but also tables with sufficient interpolant transformations are given. We are forced to omit all these tables together with the definitions and properties of generalized nested sequents because they are all too long and laborious to be briefly summed up here. However they represent the other key-ingredient to apply the Mahera method to the case of modal logic.

Theorem 3.15 *For any $X \subseteq \{d, t, b, 4\}$ under the 5-closure proviso, the nested calculus $\mathbf{NS}_k \cup X$ allow to prove that the corresponding modal logic enjoys the Craig interpolation property.*

4 Other logics

In line with the predominant historical development of the nested sequent formalism in this article so far we concentrated on the application of this formalism to modal logics, and in particular to normal modal logics. Due to its expressiveness and flexibility the formalisms however also has been applied to other logics. In this section we present a brief overview over the range of such applications. For space reasons we only sketch some examples of calculi and refer to the cited literature for the details. We also do not discuss the applications to substructural logics via linear nested sequents in detail and refer the interested reader instead to Guerrini et al. (1998) and Lellmann et al. (2017).

4.1 Intuitionistic and first order logics

Due to the close connection between normal modal logics and intuitionistic or intermediate logics via Kripke or possible world semantics it is rather natural that the nested sequent formalism was also applied to intuitionistic logic. This has been done first in Postniece (2009) and Postniece (2010), based on the previously developed

calculi from Goré et al. (2008). Perhaps owing to previous works on display calculi, these works consider calculi for *bi-intuitionistic logic*, an extension of intuitionistic logic with a *co-implication*. Restricting to the language without the co-implication however immediately yields a nested sequent calculus for intuitionistic logic, hence for the sake of presentation here we restrict ourselves to this case. In line with the idea that different components in the nested sequent correspond to different worlds in the Kripke model, the nesting operator in the framework of intuitionistic logic now corresponds to the implication instead of the modal box. Thus, the formula translation of a nested sequent in this setting is obtained by deleting the boxes from the modal formula interpretation of Sec. 2. E.g., the nested sequent $(M \Rightarrow N, [P_1 \Rightarrow Q_1], \dots, [P_n \Rightarrow Q_n])$ where N, Q_1, \dots, Q_n only contain formulae is given by

$$\bigwedge M \rightarrow \bigvee N \vee \left(\bigwedge P_1 \rightarrow \bigvee Q_1 \right) \vee \dots \vee \left(\bigwedge P_n \rightarrow \bigvee Q_n \right) .$$

Note that the nesting operator is interpreted by the intuitionistic implication, hence several layers of nestings correspond to nested implications. This corresponds intuitively to the fact that in Kripke semantics for intuitionistic logic the implication is evaluated via the successor worlds. Hence, again, the different nodes of a nested sequent correspond to worlds in a Kripke model. The logical rules for the implication then are (for the sake of clarity in the form where formula contraction is not absorbed into the rule set, i.e., the principal formula is not repeated in the premisses):

$$\frac{G\{M, \beta \Rightarrow N\} \quad G\{M \Rightarrow N, \alpha\}}{G\{M, \alpha \rightarrow \beta \Rightarrow N\}} \rightarrow_L \quad \frac{G\{M \Rightarrow N, [\alpha \Rightarrow \beta]\}}{G\{M \Rightarrow N, \alpha \rightarrow \beta\}} \rightarrow_R$$

The system of Postniece (2009) contains further structural rules to capture the additional properties of Kripke models for intuitionistic logic, but it was shown later in Fitting (2014) that it suffices to add a slight reformulation of one of these rules, sometimes called the *lift rule*:

$$\frac{G\{M \Rightarrow N, [P, \alpha \Rightarrow Q]\}}{G\{M, \alpha \Rightarrow N, [P \Rightarrow Q]\}} \text{ lift}$$

Intuitively, this rule captures the condition that valuations in Kripke models for intuitionistic logic need to be *persistent*, resulting in the fact that truth of a formula is preserved when moving to successor worlds. Completeness of the full calculi was shown by simulating previously existing so-called *shallow sequent systems* in Postniece (2009) (see *op. cit.* for the details) and by syntactic cut elimination in Postniece (2010). For the intuitionistic fragments it was also shown by correspondence to prefixed tableaux in Fitting (2014), by simulating Maehara's standard multi-conclusion sequent calculus in Lellmann (2015), and by translating from a labelled sequent calculus in Lyon (2020a).

The similarity between the truth conditions for the intuitionistic implication and the universal quantifier was further exploited in Fitting (2014) to obtain nested sequent calculi for *first-order intuitionistic logic*, both in the constant-domain and non-

constant domain formulations. In particular the nested sequent system for constant-domain intuitionistic logic is rather elegant, since only the “standard” quantifier rules need to be added. Writing b for a *parameter*, i.e., one of a number of special variables for instantiating quantified variables, in the case of the universal quantifier these are:

$$\frac{G\{M, \forall x \alpha(x), \alpha(b) \Rightarrow N\}}{G\{M, \forall x \alpha(x) \Rightarrow N\}} \forall_L \quad \frac{G\{M \Rightarrow N, \forall x \alpha(x), \alpha(b)\}}{G\{M \Rightarrow N, \forall x \alpha(x)\}} \forall_R$$

b arbitrary b not in conclusion

The fact that the nested sequents in the conclusions of these rules contain all the worlds of a so-far constructed countermodel together with the restriction of the \forall_R rule that b does not occur in the conclusion of this nested sequent rule captures the requirement that the domains are constant across all worlds of the model. In the non-constant domain version these rules need to be adjusted and are more similar to the implication rules. Again, alternative completeness proofs can be given for the different versions by simulating a standard sequent system Lellmann (2015) or by translation from a labelled sequent calculus Lyon (2020a).

In the classical setting, the analogy between quantifiers and modalities was exploited even earlier, to obtain nested sequent calculi for first-order classical logic in Brünnler (2010). In this setting the nesting is seen as a structural connective corresponding to the quantifier, with the advantage that due to the presence of different structural connectives for the quantifiers (i.e., the nesting operator) and the propositional connectives (i.e., the comma and the sequent arrow) it is easier to adjust the system to other logics by varying the rules for the (structural) quantifiers. In particular, in *op. cit.* this gives rise to a nested sequent system for *free logic*, i.e., classical first-order logic where the domain might be empty.

4.2 Intuitionistic and constructive modal logics

Considering modal logics based on intuitionistic instead of classical propositional logic leads to the area of *intuitionistic and constructive modal logics*. Semantically, these logics are typically given by Kripke-frames with two accessibility relations \leq and R , with the intuitionistic implication interpreted using the accessibility relation \leq as usual, and the box modality understood as truth in all R -successors of all \leq -successors of the current world. The presence of two accessibility relations then gives rise to a number of different possible interaction principles and in turn to a number of different possible logics (see, e.g., Simpson (1994) for an introduction). Unfortunately, from the perspective of nested sequents, many of these interaction principles are similar to “diamond” or “confluence” principles such as $\forall x, y, z (xRy \wedge xRz \rightarrow \exists w.(yRw \wedge zRw))$ which gave rise to the extension of nested sequents to indexed nested sequents (see Sec. 5.6 below). Hence it is not clear whether they could be captured by combining the systems for modal logic from Sec. 2 with those

for intuitionistic logic from Sec. 4.1, e.g., by considering two nesting operators for the modal and intuitionistic accessibility relations respectively.

Using a different approach, however, nested sequent systems for intuitionistic and constructive modal logics were proposed successfully by considering essentially a *single-conclusion* version of nested sequents. While the literature on nested sequents for these logics mainly makes use of the single-sided formulation of nested sequents, extended with a marker for a single designated *output formula*, i.e., the formula on the right hand side of the corresponding two-sided nested sequent, for the sake of consistency of notation here we stick to the two-sided notation. The system from Straßburger (2013) then reads as follows.

Definition 4.1 An *intuitionistic nested sequent* is a nested sequent where the right hand side of exactly one component is non-empty and contains exactly one formula, called its *output formula*. The *output pruning* of an intuitionistic nested sequent G is obtained by removing its (unique) output formula and is written as G^\downarrow .

It is worth noting that the modalities \Box and \Diamond are not interdefinable in intuitionistic and constructive modal logics. This together with the fact that an intuitionistic nested sequent contains only one output formula, and hence only one formula should appear on the right hand side of its formula interpretation leads to a slightly more involved definition of the latter:

Definition 4.2 The *formula interpretation* of an intuitionistic nested sequent G is written as G^t and defined as follows, where G_1, \dots, G_n do not contain the output formula but O does:

$$\begin{aligned} (M \Rightarrow [G_1], \dots, [G_n])^t &= \bigwedge M \wedge \Diamond(G_1^t) \wedge \dots \wedge \Diamond(G_n^t) \\ (M \Rightarrow [G_1], \dots, [G_n], \alpha)^t &= \bigwedge M \wedge \Diamond(G_1^t) \wedge \dots \wedge \Diamond(G_n^t) \rightarrow \alpha \\ (M \Rightarrow [G_1], \dots, [G_n], [O])^t &= \bigwedge M \wedge \Diamond(G_1^t) \wedge \dots \wedge \Diamond(G_n^t) \rightarrow \Box(O^t). \end{aligned}$$

Thus the nesting operator is interpreted as the modal box on the right hand side of the implication whenever the structure inside contains the output formula, and as the modal diamond on the left hand side of the implication otherwise. The rules for the modal box then are essentially the same as in the classical setting:

$$\frac{G\{M, \Box\alpha \Rightarrow N, [P, \alpha \Rightarrow Q]\}}{G\{M, \Box\alpha \Rightarrow N, [P \Rightarrow Q]\}} \Box_L \quad \frac{G\{M \Rightarrow N, [\Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box_R$$

Since the modal operators are not inter-definable, in addition we also need the following rules for the modal diamond:

$$\frac{G\{M \Rightarrow N, [\alpha \Rightarrow]\}}{G\{M, \Diamond\alpha \Rightarrow N\}} \Diamond_L \quad \frac{G\{M \Rightarrow N, [P \Rightarrow Q, \alpha]\}}{G\{M \Rightarrow N, [P \Rightarrow Q], \Diamond\alpha\}} \Diamond_R$$

The intuitionistic character of the system is then captured on the level of propositional logic by adjusting the implication rules in the same way as is done for single-

conclusion standard sequent systems for intuitionistic logic:

$$\frac{G^{\downarrow}\{M, \alpha \rightarrow \beta \Rightarrow N, \alpha\} \quad G\{M, \beta \Rightarrow N\}}{G\{M, \alpha \rightarrow \beta \Rightarrow N\}} \rightarrow_L \quad \frac{G\{M, \alpha \Rightarrow N, \beta\}}{G\{M \Rightarrow N, \alpha \rightarrow \beta\}} \rightarrow_R$$

Notice that the left premiss of the \rightarrow_L rule contains α as the output formula, so the output formula of G needs to be removed by considering the output pruning G^{\downarrow} . This in turn may drastically change the formula interpretation of the subtree which contained the output formula before the output pruning. The remaining propositional rules are the standard ones. The resulting nested sequent system is shown to be sound and complete for intuitionistic modal logic IK in Straßburger (2013) via a cut elimination argument. The same work further contains extensions to all other logics of the intuitionistic version of the modal cube. Fully modular versions of these calculi are introduced in Marin and Straßburger (2014), and the systems are adapted to constructive modal logics in Arisaka et al. (2015). An implementation is presented in Girlando and Straßburger (2020). A Maehara-style multi-conclusion version of the calculi was introduced in Kuznets and Straßburger (2018) and used to obtain countermodels from failed proof search. Calculi for the more general class of logics extending IK with seriality and so-called *Horn Scott-Lemmon axioms*, i.e., axioms of the shape $(\diamond^n \Box A \rightarrow \Box^k A) \wedge (\diamond^k A \rightarrow \Box^n \diamond A)$ were proposed recently in Lyon (2021a). The single-conclusion nested sequent calculi for the logics of the intuitionistic modal cube were also developed independently in Galmiche and Salhi (2018) under the name of *tree sequents*, based on the natural deduction systems of Galmiche and Salhi (2010).

4.3 Other non standard normal modal logics

Modal logic was firstly introduced to formalize the concepts of necessity and possibility; however, soon after the introduction of Kripke possible worlds semantics modal logic was used to formalize other notions such as knowledge, belief or obligation. In accordance with these new interpretations of the modal operator, new logical systems have been developed. We use this subsection to briefly present the application of nested calculi to some of these alternative interpretations of \Box and \diamond .

An important interpretation of the modalities is given in the context of *modal tense logics*, where $\Box\alpha$ is interpreted as “at every point in the future it will be the case that α ”. Dually, $\diamond\alpha$ is interpreted as “at some point in the future it will be the case that α ”. Analogously to these *forward looking* modalities, modal tense logics also contain the *backward looking* modality \blacksquare and its dual \blacklozenge , interpreted as “at every point in the past it was the case that” and “at some point in the past it was the case that”, respectively. As usual, \diamond and \blacklozenge can be taken to be defined in terms of \Box and \blacksquare , respectively. The tense analogue of modal logic K is denoted K_t and given by Hilbert-style axiomatisations of modal logic K for both the forward and the

backward looking modalities together with the axioms $\alpha \rightarrow \Box\Diamond\alpha$ and $\alpha \rightarrow \blacksquare\Diamond\alpha$ for the connection between the forward and backward looking modalities.

Historically, modal tense logics were one of the areas which sparked the creation of the nested sequent framework in Kashima (1994) (originally formulated in the one-sided setting, see Sec. 2.1). For capturing both the forward and backward looking modalities, nested sequents for modal tense logics contain a *forward* or *future* nesting operator $^F[.]$ as well as a *backward* or *past* nesting operator $^P[.]$. The rules for the basic logic K_t are essentially the nested sequent rules for K for both modalities \Box and \blacksquare together with the following two rules for their interaction:

$$\frac{G\{\Gamma, \alpha \Rightarrow \Delta, ^F[\Sigma \Rightarrow \Pi]\}}{G\{\Gamma \Rightarrow \Delta, ^F[\Sigma, \blacksquare\alpha \Rightarrow \Pi]\}} \quad \frac{G\{\Gamma, \alpha \Rightarrow \Delta, ^P[\Sigma \Rightarrow \Pi]\}}{G\{\Gamma \Rightarrow \Delta, ^P[\Sigma, \Box\alpha \Rightarrow \Pi]\}}$$

Nested sequent calculi for basic extensions of modal tense logic with frame properties like reflexivity, transitivity, connectedness or totality were also already considered in Kashima (1994). However, very general completeness results for large classes of modal tense logics were given later. In particular, Goré et al. (2011) obtain cut-free deep inference calculi for modal tense logics given by *Scott-Lemmon axioms*, i.e., axioms of the form $\Diamond^h\Box^i\alpha \rightarrow \Box^j\Diamond^k\alpha$ for $h, i, j, k \geq 0$ (see also Sec. 5.6 below), some of which also yield cut-free nested sequent calculi, and for modal tense logics given by *path axioms*, i.e., axioms of the form $\langle \rangle_1 \dots, \langle \rangle_n \alpha \rightarrow \langle \rangle \alpha$, where $n \geq 0$ and each of $\langle \rangle_1, \dots, \langle \rangle_n, \langle \rangle$ is either \Diamond or \blacksquare . Since normal modal logics can be understood as the restriction of modal tense logic to the forward looking modalities, by appropriate conservativity results shown in *op. cit.* the calculi for the Scott-Lemmon axioms also give rise to nested sequent calculi for the corresponding normal modal logics. It should be noted, however, that these calculi might still contain the backward looking structural connective, and hence their structures might not have a formula interpretation in the language of standard (non-tense) normal modal logic. Even further, general completeness results for *grammar logics*, i.e., multimodal tense logics, where loosely speaking the connections between the modalities are given by rewriting rules, were shown in Tiu et al. (2012). For space reasons we do not consider all the details here. The interested reader is referred to the mentioned literature.

The second non-alethic interpretation of modal logic that we present in this section is the epistemic interpretation, namely the interpretation of the \Box in terms of “it is known that.” More precisely, in order to express the fact that different agents may have different knowledges, we add indexes to the \Box : one index for each agent of the group of agents considered in the language. The system that seems to best represent an epistemic multiagent interpretation of modal logic is S5. We can implement the multi-agent version of S5 by adding a new operator \boxtimes for common knowledge, where common knowledge is standardly defined as the infinite conjunction “all agents know α , and all agents know that all agents know α and so on.” We will denote the multi-agent version of the Hilbert system S5 plus common knowledge by the name CS5. In CS5 we capture common knowledge by means of a fixed point axiom, which states that common knowledge is a fixed point, and an induction rule that states that this fixed point is the greatest fixed point. From a semantic point of view, the

common knowledge operator is formally defined as the modality of reachability that uses accessibility edges corresponding to any of the knowledge operators for the agents.¹⁰

In order to get a nested calculus for CS5, the first mandatory step is to index nested sequents with agents. However, as explained in Poggiolesi (2010c), there are two nested calculi for S5: one that simply uses hypersequents (if we look at hypersequents as nested sequents where the accessibility relation has collapsed, see Section 5.2.) and one with full nested sequents apparatus that include rules for reflexivity, transitivity, and asymmetry (see Section 2). Since the former is simpler, it seems the more adequate to work with in the epistemic context. Hence, first of all, the hypersequents for S5 are indexed with indexes denoting agents,¹¹ and then rules for the operator of common knowledge are introduced.

The difficulty of dealing with the common knowledge operator is that it has deep infinitary features; hence at the Gentzen level, it has been taken into account in either of the following two ways: either with finitary calculi (that often do not enjoy cut-elimination, e.g. see Alberucci and Jäger, 2005), or with infinitary rules (e.g. see Tanaka, 2003). In the nested framework both ways have been explored. We will briefly present them.

Definition 4.3 We consider a language \mathcal{L}_h^\square with a set Φ of agents $\{a, b, c, \dots\}$. Propositions S are atoms. Formulae are given by the following grammar:

$$\alpha ::= S \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid \Box_z\alpha \mid \boxtimes\alpha$$

where $z \in \Phi$, the formula $\Box_z\alpha$ is read as “agent z knows α ” and the formula $\boxtimes\alpha$ is read as “ α is common knowledge”. We will use the formula $\Box\alpha$ as an abbreviation for “everybody knows α ”:

$$\Box\alpha = \Box_a\alpha \wedge \dots \wedge \Box_z\alpha$$

In Brünnler and Stüder (2009) the following two rules for common knowledge in the framework of nested calculi are considered:¹²

$$\frac{G\{A : \Box^k\alpha, M \Rightarrow N\}}{G\{A : \boxtimes\alpha, M \Rightarrow N\}} \boxtimes L \qquad \frac{G\{A : M \Rightarrow N, \Box^k\alpha\} \text{ for all } k \geq 1}{G\{A : M \Rightarrow N, \boxtimes\alpha\}} \boxtimes R$$

Although these rules allow a cut-elimination procedure, the rule $\boxtimes R$, being infinitary, vanish any proof-search attempt. A more fine-grained result can then be found

¹⁰ For the Hilbert-system and the semantics for CS5, e.g. see Fagin et al. (2004).

¹¹ See also Section 5.6.

¹² In the rules, the notation A stands for a set of agents that satisfy certain features. We omit to describe these features here for the sake of brevity; they can be found in Hill and Poggiolesi (2015). Note also that in Brünnler and Stüder (2009) the rules for common knowledge are added to the multiagent version of the nested calculus for \mathbf{NS}_k . However, for the sake of uniformity, they can also be thought as added to the multiagent version of the hypersequent system for S5.

in Hill and Poggiolesi (2015), where common knowledge is treated by means of the following three rules in an hypersequent framework:

$$\frac{G \mid A : \boxtimes\alpha, \Box\alpha, M \Rightarrow N}{G \mid A : \boxtimes\alpha, M \Rightarrow N} \boxtimes L_1$$

$$\frac{G \mid A : \boxtimes\alpha, M \Rightarrow N \mid B : \boxtimes\alpha, P \Rightarrow Q}{G \mid A : \boxtimes\alpha, M \Rightarrow N \mid B : P \Rightarrow Q} \boxtimes L_2$$

$$\frac{G \mid A : M \Rightarrow N, \beta \quad \beta \Rightarrow \Box\alpha \quad \beta \Rightarrow \Box\beta}{G \mid A : M \Rightarrow N, \boxtimes\alpha} \boxtimes R$$

The rules are finitary, the calculus they belong to is cut-free; moreover, in order to control proof-search, it has been shown that the rule $\boxtimes R$ can be restricted to a certain type of formulae only, i.e. it has been shown that the formula β occurring in the premiss can only be of a certain form. Details can be found in Hill and Poggiolesi (2015).

Let us now move to a third different interpretation of the modal operator and related logic, incidentally one of the first treated by the nested sequent method in Bull (1992). Propositional dynamic logic, or PDL for short, is a (modal) logic based on the idea of associating with each program term a of a programming language a modality $[a]$. This means that in PDL we still deal with boxed formulae as we do in modal logic, but the box is no longer empty but filled with program terms.

Several different programs can fill the box: atomic programs (a_0, a_1, a_2, \dots), but also more complex ones, which are denoted by the capital letters A, B, C, \dots . These latter are constructed by means of the following program operators: the union operator, $A \cup B$, that should be interpreted as: “do A or B non-deterministically;” the composition operator, $A \otimes B$, that should be interpreted as: “first do A and then do B ;” the test operator, $\alpha?$, that should be interpreted as: “verify that α is true;” and finally the iteration operator, A^* , that should be interpreted as: “repeat A a finite number of times.” In PDL, we thus deal with formulae of the following form: $[a]\alpha$, $[A \cup B]\alpha$, $[A \otimes B]\alpha$, $[\beta?]\alpha$, $[A^*]\alpha$, each of which should be read as: “ α is true after every terminating execution of the program that is in the box.”

From the point of view of Hilbert systems, propositional dynamic logic is well-defined. Indeed, the system for PDL is obtained by adding to classical propositional logic: (i) the distribution axiom schema, that has the form: $[A](\alpha \rightarrow \beta) \rightarrow ([A]\alpha \rightarrow [A]\beta)$, for each program α ; (ii) modus ponens and the rule of necessitation; and (iii) at least one axiom schema or inference rule for each program operator. As for the Gentzen perspective, the steps and difficulties to formulate a satisfactory nested calculus are analogous to those for common knowledge.¹³ On the one hand, just as in the epistemic case nested sequents were indexed by agents, now nested sequents are indexed by (atomic) programs; on the other hand, just as the common knowledge operator involved an infinitary rule, so it is the case in PDL for the program operator $*$. Hence the nested calculus for PDL is obtained by adding to the usual initial

¹³ Note that if PDL is considered without the program operator $*$, then the first nested sequent calculus for it has been formulated by Bull (1992).

nested sequents and nested rules for the propositional and modal connectives,¹⁴ the following rules:¹⁵

$$\frac{G\{[B]\alpha, [C]\alpha, M \Rightarrow N\}}{G\{[B \cup C]\alpha, M \Rightarrow N\}} \cup L \quad \frac{G\{M \Rightarrow N, [B]\alpha\} \quad G\{M \Rightarrow N, [C]\alpha\}}{G\{M \Rightarrow N, [B \cup C]\alpha\}} \cup R$$

$$\frac{G\{[B][C]\alpha, M \Rightarrow N\}}{G\{[B \otimes C]\alpha, M \Rightarrow N\}} \otimes L \quad \frac{G\{M \Rightarrow N, [B][C]\alpha\}}{G\{M \Rightarrow N, [B \otimes C]\alpha\}} \otimes R$$

$$\frac{G\{M \Rightarrow N, \alpha\} \quad G\{\beta, M \Rightarrow N\}}{G\{[\alpha?]\beta, M \Rightarrow N\}} ?L \quad \frac{G\{\alpha, M \Rightarrow N, \beta\}}{G\{M \Rightarrow N, [\alpha?]\beta\}} ?R$$

$$\frac{G\{[B^*]\alpha, [B]^n \alpha, M \Rightarrow N\}}{G\{[B^*]\alpha, M \Rightarrow N\}} *L \quad \frac{G\{M \Rightarrow N, [B]^n \alpha\} \text{ for each } n < \omega}{G\{M \Rightarrow N, [B^*]\alpha\}} *R$$

In the calculus so obtained all structural rules are (height-preserving) admissible, the logical rules are invertible and the calculus is cut-free, see Hill and Poggiolesi (2010).

Let us end the section by briefly mentioning some further applications of the nested sequent method. Nested sequent calculi for a class of multimodal logics called *simply dependent multimodal logics* can be obtained from the linear nested sequent calculi (see also Sec. 5.3) for these logics constructed in Lellmann and Pimentel (2019). However, since these logics all have standard sequent calculi already, we do not consider the details here but refer the interested reader to *op. cit.* instead. Another application concerns the modal logic of provability - also called the logic GL from the initials of Gödel and Löb - which is the modal logic resulting from the interpretation of the \Box in terms of “it is provable in Peano Arithmetic that.” The Hilbert system GL is obtained by adding to the system K4 the GL axiom: $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$. The nested sequent method can be successfully applied to the system GL. Since this logic provides a good showcase of the connection between the standard and nested sequent frameworks we will discuss the resulting system in Sec. 5.1 below. A system for the logic GLP, a different flavor of provability logic related to ordinal analysis of arithmetic, is developed in Shamkanov (2015). Nested sequents have also been applied to justification logic, the logic introduced by Artemov (2008), where the \Box is substituted by terms or justifications that have the form t, f, r, \dots . Whilst in Fitting (2012) one may find a nested calculus for justification logic which is cut-free but not analytic, in Hill and Poggiolesi (2019) hypersequents are used to provide a cut-free and analytic calculus for justification logic. However, the process to obtain analyticity is quite long and laborious so we completely omit it here.

¹⁴ Modal rules are adapted to the fact that the \Box contains a program.

¹⁵ Note that in both the rules $*L$ and $*R$, we use the notation $[B]^n \alpha$, which is inductively defined in the following way:

- $[A]^0 \alpha := \alpha$
- $[A]^{k+1} \alpha := [A][A]^k \alpha$

4.4 Non-normal modal logics and conditional logics

All the modal logics considered so far share a common property, whether based on classical or intuitionistic propositional logic: they are *normal* in the sense that they satisfy the necessitation rule, or alternatively, that the formula $\Box\top$ is a theorem. However, already in Kripke (1965), the Kripke semantics was extended to cover also *non-normal* modal logics. In fact, historically many of the first systems proposed for modal logic were non-normal (see the references in *op. cit.*). Thus it is an interesting question whether the nested sequent formalism can be extended to cover non-normal modal logics as well. Unfortunately, for non-normal modal logics the parallel between the tree structure of the models and the tree structure of nested sequents breaks down to some extent, since non-normality of the modal logic necessitates additional structure on the models. This could be the addition of *non-normal worlds*, where the formula $\Box\top$ is false in Kripke (1965), or the more fine-grained analysis of the accessibility relation in what is known under the alternative names of *neighbourhood semantics*, *Scott-Montague semantics* or *minimal model semantics*, see, e.g., Chellas (1980). Perhaps even more problematic is the fact that in every nested sequent system which contains the standard (local) propositional rules of Sec. 2 and interprets the nesting operator $[\cdot]$ as a connective \circ using rules analogous to the standard left and right rules for \Box we have the following derivations:

$$\frac{\frac{\frac{}{\Rightarrow [\Rightarrow \top]} \circ_R}{\Rightarrow \circ \top}}{\circ p, \circ q \Rightarrow [p, q \Rightarrow p]} \quad \frac{\frac{\frac{}{\circ p, \circ q \Rightarrow [p, q \Rightarrow q]} \wedge_R}{\circ p, \circ q \Rightarrow [p, q \Rightarrow p \wedge q]} \circ_L, \circ_L}{\circ p, \circ q \Rightarrow [\Rightarrow p \wedge q]} \circ_R}{\circ p, \circ q \Rightarrow \circ(p \wedge q)}$$

Hence in every such system we can derive the axioms $\circ\top$ as well as $\circ p \wedge \circ q \rightarrow \circ(p \wedge q)$. Similarly, every such system also derives the converse of the latter, i.e., $\circ(p \wedge q) \rightarrow \circ p$. These three axioms together, however, characterise \circ as a normal modality. Hence it is not possible to simply reinterpret the nesting operator as a non-normal modality while keeping the standard propositional rules intact.

While the development of a general proof theory for non-normal modal logics in the nested sequent framework and in particular a general link to the semantics is still an open problem, first steps were taken in Lellmann (2019). The main idea in that approach is to make use of the fact from Brown (1988) that neighbourhood structures can be seen to give rise to both a non-normal and a normal modality at the same time. On the level of nested sequents this means that in order to capture certain non-normal modal logics it is enough to add a second nesting operator $\langle \cdot \rangle$ which acts as a structural connective for the non-normal modality. To model both the normal and the non-normal modality arising from neighbourhood structures, we extend the language to include the (non-normal) modality $\langle \cdot \rangle$ as well as the normal modality \Box . Note that due to the inherently normal nature of any interpretation of the standard nesting operator $[\cdot]$ as considered above we can only obtain a formula interpretation of nested sequents with this operator if we include a normal modality

in the language, or if in the considered logic it is definable in terms of the non-normal modality. The latter is in general not the case.

Definition 4.4 A *non-normal nested sequent* is a finite object

$$M \Rightarrow N, \langle P_1 \Rightarrow Q_1 \rangle, \dots, \langle P_n \Rightarrow Q_n \rangle, [G_1], \dots, [G_m]$$

where $M \Rightarrow N$ as well as $P_i \Rightarrow Q_i$ for $i \leq n$ is a sequent and where G_j for $j \leq m$ is a non-normal nested sequent. Its formula interpretation G^τ is given by:

$$G^\tau := \bigwedge M \rightarrow \bigvee N \vee \bigvee_{i \leq n} \langle \bigwedge P_i \rightarrow \bigvee Q_i \rangle \vee \bigvee_{j \leq m} \Box(G_j)^\tau .$$

The crucial point is that in a non-normal nested sequent other nested sequents can occur only in the scope of the normal nesting operator $[.]$, not the non-normal one $\langle . \rangle$. Further, in order to prevent the derivation of the axioms for normality above, the application of all rules needs to be restricted not to occur in the scope of the non-normal nesting operator $\langle . \rangle$. The resulting modal rules then are the standard rules \Box_L and \Box_R for the normal operator together with the left and right rules for the non-normal modality

$$\frac{G\{M \Rightarrow N, [P, \alpha \Rightarrow Q]\}}{G\{M, \langle \Box \alpha \Rightarrow N, \langle P \Rightarrow Q \rangle\}} \langle \Box \rangle_L \quad \frac{G\{M \Rightarrow N, \langle \Rightarrow \alpha \rangle\}}{G\{M \Rightarrow N, \langle \Box \alpha \rangle\}} \langle \Box \rangle_R$$

and a single *interaction rule* for the relation between the normal and non-normal modalities:

$$\frac{G\{M \Rightarrow N, [P \Rightarrow Q]\}}{G\{M \Rightarrow N, \Box \alpha, \langle P \Rightarrow Q \rangle\}} I .$$

This rule corresponds to the interaction axiom $\Box \beta \rightarrow \Box \alpha \vee \langle \Box \beta \rangle$ and essentially states (seen bottom-up) that whenever there is any boxed formula on the right hand side of the conclusion, then a non-normal nesting operator can be converted into a normal one.

Note that since no rule can be applied inside $\langle . \rangle$, and the only rules which mention this operator in the conclusion change it to the normal nesting operator in the premiss, essentially the only property which can be derived for the language with only the non-normal modality $\langle \Box \rangle$ is $\langle \Box \alpha \rightarrow \langle \Box \beta \rangle$ for derivable $\alpha \rightarrow \beta$. Correspondingly, this system restricted to this language yields a nested sequent system for non-normal *monotone modal logic*. The system was further modified to cover some extensions in *op. cit.* Purely syntactic systems exploiting the idea of translating standard sequent calculi to a (linear) nested framework (see Sec. 5.1 below) were also considered in Lellmann and Pimentel (2019). This approach yields modular systems for a non-normal analogue of the modal cube, the *modal tesseract*, but has the disadvantage that in absence of a second normal modality the (linear) nested sequent structures do not seem to have a formula translation.

Finally, a different flavour of non-normal modal logics is given by *conditional logics*, or logics of *counterfactual implication*. The weakest of these logics extend

classical propositional logic with a binary modality ($\alpha > \beta$) for *conditional implication*, read as “if α were the case, then also β would be the case”. This modality behaves like normal modal logic K in the second argument, but only contains the *congruence rule* for the first argument: if $\alpha \leftrightarrow \beta$ is derivable, then so is $(\alpha > \gamma) \rightarrow (\beta > \gamma)$. Nested sequents for the basic conditional logic CK and a number of extensions were introduced in Alenda et al. (2013), making use of a dyadic nesting operator $[\cdot : \cdot]$, where the first argument contains a single formula, and the second argument contains a nested sequent.

Definition 4.5 A *conditional nested sequent* is a finite structure

$$M \Rightarrow N, [\alpha_1 : G_1], \dots, [\alpha_n : G_n]$$

where $M \Rightarrow N$ is a sequent, the α_i are formulae and the G_i are nested sequents. Its formula interpretation G^τ is $\bigwedge M \rightarrow \bigvee N \vee (\alpha_1 > (G_1)^\tau) \vee \dots \vee (\alpha_n > (G_n)^\tau)$.

Thus, again only the “normal” part of the nesting operator can contain further nesting operators. The rules for the propositional connectives then are standard (restricted to the second argument of the nesting operators). The basic modal rules, reformulated from the original one-sided framework to the two-sided framework of this paper, are given by:

$$\frac{G\{M \Rightarrow N, [\alpha : P, \beta \Rightarrow Q]\} \quad \alpha \Rightarrow \gamma \quad \gamma \Rightarrow \alpha}{G\{M, \alpha > \beta \Rightarrow N, [\gamma : P \Rightarrow Q]\}} >_L \quad \frac{G\{M \Rightarrow N, [\alpha : \Rightarrow \beta]\}}{G\{M \Rightarrow N, \alpha > \beta\}} >_R$$

For the sake of clarity we omitted copies of the principal formulae in the premisses. This system along with a number of extensions was used in Alenda et al. (2013) in a syntactical cut elimination procedure and yielded complexity results via backward proof search.

A nested sequent system for a much stronger conditional logic, rephrased as a logic of conditional belief, is given in Girlando et al. (2019). Since in this logic a knowledge operator is definable, it contains a multi-agent version of modal logic S5. On the nested sequent level this is mirrored by the fact that the system essentially contains a multi-agent version of the nested sequent calculus with the rules t , 4 and 5 from Sec. 2.

5 Relation to other formalisms

In order to get a clearer understanding of the benefits and shortcomings of the nested sequent framework in general, it is helpful to situate it with respect to other important proof-theoretic frameworks. In the following we consider the relations to those well-known frameworks which to the best of our knowledge are most often used in the context of modal logics. In particular, these are the frameworks of ordinary sequents (Sec. 5.1), hypersequents (Sec. 5.2), prefixed tableaux (Sec. 5.4), labelled

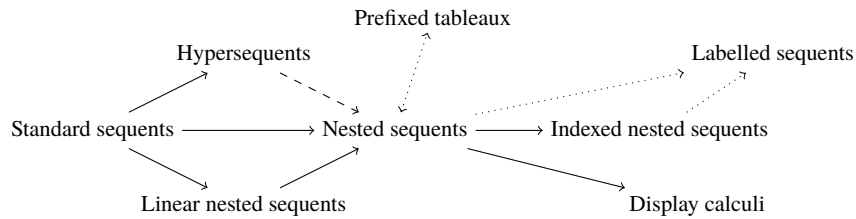


Fig. 9 The proof-theoretic frameworks considered in this section with their relations. Solid arrows indicate that the structures manipulated in one framework can also be found in the other framework. Dotted arrows indicate that the structures of one framework can be translated into structures in the other framework. The dashed arrow indicates that there is an interpretation (see Sec. 5.2 for details).

sequents (Sec. 5.5) and display calculi (Sec. 5.7). In addition we also consider two modifications of the nested sequent framework in the form of linear nested sequents and indexed nested sequents (Sec. 5.3 and Sec. 5.6, respectively). These frameworks are interesting in that they provide a middle-ground in expressivity between standard and nested sequents (linear nested sequents), and extend the expressivity of nested sequents toward that of labelled sequents (indexed nested sequents). The relative size of the different sections of course by no means reflects the relative volume of work in the corresponding areas or their importance. The relations between the frameworks are summarized in Figure 9. For space reasons unfortunately we have to omit the interesting connections to frameworks like *multiple sequent calculi* (e.g., Indrzejczak, 1997), *higher-arity sequent calculi* (e.g., Blamey and Humberstone, 1991), *modal tree-sequents* (e.g., Cerrato, 1993) and *indexed sequent calculi* (e.g., Mints, 1997). The reader is referred to Poggiolesi (2010b) for more details. To avoid unnecessary and potentially confusing notation in this section we omit names for rules where they are only mentioned or used locally. For space reasons we also do not give all the details on soundness and completeness results about specific logics in the different proof-theoretic frameworks in the form of theorems, but refer to the literature instead.

5.1 Sequents

As mentioned above, the nested sequent framework is a direct extension of the original *sequent* framework, introduced in Gentzen (1935). In particular, standard sequents can be seen as nested sequents without occurrences of the nesting operator $[.]$. Because of this, every standard sequent calculus can also be seen as a nested sequent calculus. However, in such calculi none of the additional structures and features of nested sequents are used. Hence doing so means relinquishing the benefits of the nested sequent framework, in particular modularity, and so does not gain much over staying in the original sequent framework.

A deeper connection stems from the fact that the modal rules in the standard sequent framework can be seen as *transitional rules* (see, e.g., Goré, 1999), which in the context of countermodel construction from a failed proof search correspond to moving to a successor world. Since a successor world in the Kripke semantics corresponds directly to a successor node in a nested sequent, this suggests a straightforward simulation of the modal or transitional rules for standard sequents in terms of the modal rules for nested sequents as shown in the following example.

Example 5.1 The application of the modal rule in a standard sequent system for modal logic K (below on the left) is simulated by the sequence of rule applications in the nested sequent system \mathbf{NS}_k^+ for K (below on the right).

$$\frac{\alpha_1, \dots, \alpha_n \Rightarrow \beta}{\Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta} k \quad \rightsquigarrow \quad \frac{\frac{\Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta, [\alpha_1, \dots, \alpha_n \Rightarrow \beta]}{\Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta, [\Rightarrow \beta]} \Box L^+}{\Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta} \Box R^+$$

Here the double lines in the derivation on the right stand for multiple applications of the rule $\Box L$. Note that in the premiss of the derivation on the right the root sequent contains exactly the conclusion of the application of the modal rule on the left, while the successor node in this nested sequent contains exactly the premiss of the application of the sequent rule.

The same principle can be applied to all the logics of the modal cube (Figure 2) which have a cut-free standard sequent system, i.e., the logics K, KD, KT, K4, KD4, S4, K45, KD45. Given that the latter calculi are complete for the respective logics, this yields straightforward alternative completeness proofs for these systems:

Proposition 5.2 *For any $X \subseteq \{d, t, 4\}$ the nested sequent system $\mathbf{NS}_k^+ \cup X$ is cut-free complete for the corresponding modal logic. The same holds for $X = \{4, 5\}$ and $X = \{d, 4, 5\}$.*

Proof By simulating the sequent rules at the appropriate node in the nested sequent. \square

Note that this approach can also be used to convert standard sequent calculi into nested sequent calculi: Identify the part of the standard sequent rule which is responsible for the noninvertible behaviour (the decision on the boxed formula on the right in the rule k above) and turn this into a successor-creating rule. Then, build up the rest of the premiss(es) by unpacking/transferring one formula at a time into the successor node (the boxed formulae on the left in the rule k above). See also, e.g., Lellmann (2015); Lellmann and Pimentel (2019).

The idea of simulating sequent derivations using the nested sequent seems to have been mentioned first in Brünnler (2009) for modal logic KT. It was systematically explored in Lellmann (2015); Lellmann and Pimentel (2019) and used also in Parisi (2017) and Parisi (2020). While the latter references are in the intermediate framework of *linear nested sequents* (see Sec. 5.3), the results apply to the full nested sequent framework as well. A similar approach is also taken in Marin et al. (2016), where derivations in a number of different frameworks are simulated in a focused

labelled sequent system (see also Sec. 5.5. In the form presented here, the idea of simulating sequent derivations by nested sequent derivations leads to a number of observations, listed in Lellmann (2015); Lellmann and Pimentel (2019).

First, while, e.g., in the standard sequent system for modal logic K4, the modal rule is changed to accommodate the boxed formulae in the premiss, in the nested sequent system it is enough to just add a single rule which can be used to copy the boxed formulae into the successor node one by one. This is due to the fact that the standard sequent rules are essentially broken down into smaller parts in the nested sequent framework. Hence nested sequent systems for these logics can be seen as a *modularization* of standard sequent systems.

Second, since the simulation of a standard sequent derivation does not introduce any branching in the resulting nested sequents, for completeness with respect to these simpler rules it is enough to consider non-branching nested sequents. This idea is further explored in the framework of *linear nested sequents*, see also Sec. 5.3.

Third, since the simulation of standard sequent rules always works on a leaf node of the corresponding nested sequent, the corresponding nested sequent calculi can be restricted to their *end-active* version, where in every rule application one of the active nodes of the nested sequent in the premiss(es) must be a leaf node.

Fourth, given that a node and its successor node in a nested sequent resulting from translating a sequent rule can be seen as the conclusion and premiss of that rule, the branches of such a nested sequent can be viewed essentially as *traces* or *histories* of depth-first backward proof search in the corresponding sequent system. Here, the conclusion of a noninvertible (i.e., modal or transitional) standard sequent rule is stored in the nested sequent before moving on to the premiss, i.e., the successor node in the nested sequent. This in turn suggests that creating branching in a nested sequent by applying a successor-creating rule such as $\Box R$ to a non-leaf node of the nested sequent loosely corresponds to the creation of a *backtracking point* in the backward proof search for the standard sequent system: One of the noninvertible rules is applied (backward), and its premiss can be further expanded. Backtracking to this backtracking point in the proof search and applying a different noninvertible rule then roughly corresponds to applying a different successor-creating rule to the node. In particular, the branching in the nested sequent then corresponds to *OR-branching* in the proof search tree. An advantage of the nested sequent formulation is however, that the current branch is not discarded when applying such a rule, as would be the case for backtracking in depth-first backward proof search. In fact, the procedure $prove(G, \mathbf{NS}_K^+)$ of Sec. 3 for proof search in nested sequents, which first applies all rules which do not create successor nodes, and then creates an additional layer by applying $(\Box R)^*$ wherever possible once, corresponds to *breadth-first* proof search in the corresponding standard sequent system: First, all invertible rules are applied to the sequent, then every possible noninvertible rule is applied. The latter applications yield a set of sequents corresponding essentially to the leaves of the nested sequent obtained by applying the rule $(\Box R)^*$ wherever possible. Then the process is repeated on this set of sequents, i.e., the new leaf nodes of the nested sequent. This suggests that the nested sequent framework can also be used as a tool to analyse different

forms of proof search in the standard sequent framework, an area of research which apart from the basic notions has not yet been fully explored.

The previous observations are based on enhancing or studying existing standard sequent calculi, using translations and simulations in calculi in the nested sequent framework. Moreover, the fact that the nested sequent framework can capture logics for which no cut-free sequent calculus is known, such as modal logics S5 or K5, suggests that the nested sequent framework is stronger than the standard sequent framework. In light of this it is also interesting to investigate the fragment of nested sequent calculi corresponding to standard sequent calculi, and in particular which kinds of nested sequent calculi can be converted back into standard sequent calculi.

In this spirit, some sufficient criteria for converting a nested sequent calculus back into a standard sequent calculus were introduced in Pimentel et al. (2019). The main idea again stems from the connection between the nesting operator in the nested sequent framework and the transitional rules in the standard sequent framework, and is based on reordering nested sequent derivations by rule permutations so that the modal rules are grouped in *blocks* consisting (when seen bottom-up from conclusion to premisses) of a single application of a *creation rule*, i.e., a nested sequent rule introducing a new nesting operator, followed by a number of *upgrade rules*, i.e., rules which move (possibly modified) formulae into the newly created node. The properties of a nested sequent system which ensure that this can be done then are:

1. the creation rules introduce exactly one nesting operator;
2. the upgrade rules never move a formulae toward the root of the nested sequent;
3. the upgrade rules move a formula inside exactly one nesting operator.

Nested sequent calculi satisfying these criteria are called *basic nested sequent calculi*. These properties together entail that such a block consisting of a creation rule and a number of upgrade rules can be converted into a standard sequent rule. Moreover, the resulting rule follows the general format of *basic sequent calculi* of Lahav and Avron (2013) (see *op. cit.* for the definition). Hence, we have:

Theorem 5.3 (Pimentel et al., 2019, Cor. 30) *For every basic nested sequent system, there is an equivalent standard sequent system.*

It is instructive to consider how, e.g., nested sequent calculi for modal logics like S5 and KB, which are notoriously difficult in the (cut-free) standard sequent setting, fail these criteria. E.g., the rules

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}} \quad 5 \quad \text{or} \quad \frac{G\{\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}} \quad b$$

considered above move the formula $\Box\alpha$ toward the root of the nested sequent and hence fail property 2 above.

As a more explicit example for the relation between standard and nested sequent systems, consider modal provability logic GL. As already mentioned in Sec. 4.3 above, this logic is given by adding the axiom $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$ to normal modal logic K4. The (Kleene'd version of the) nested sequent calculus CSGL for provability logic GL from Poggiolesi (2009) is given by the modal rules in Figure 10. Since the

$$\begin{array}{c}
\frac{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q, \Box\alpha]\} \quad G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box A_{gl} \\
\frac{G\{M \Rightarrow N, \Box\alpha, [\Box\alpha \Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box K_{gl} \quad \frac{G\{\Box\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} 4
\end{array}$$

Fig. 10 The modal rules of the nested sequent system for provability logic GL.

rules satisfy all the criteria for sequentialisation from Pimentel et al. (2019), we can rearrange derivations in the calculus CSGL into blocks consisting of an application of the creation rule $\Box K_{gl}$ at the bottom to create a new component $[\Box\alpha \Rightarrow \alpha]$ in the nested sequent, followed by applications of the rules $\Box A_{gl}$ and 4 to populate this component. The sequent rule corresponding to such a block is (using the set notation to denote the set of premisses given by sets R_1, R_2 with $R_1 \cup R_2 = R$ and R as defined below):

$$\frac{\{ R_1, \Box S, \Box\alpha \Rightarrow \Box R_2, \alpha : R_1 \cup R_2 = R \}}{\Box M, P \Rightarrow N, \Box\alpha}$$

where $\Box R \subseteq \Box M$ is the multiset of formulae to which the nested sequent rule $\Box A_{gl}$ was applied in the block, and $\Box S \subseteq \Box M$ is the multiset of formulae to which the nested sequent rule 4 was applied. The large number of premisses for this rule (given by all the possibilities for sets R_1, R_2 with $R_1 \cup R_2 = R$) stems from the fact that the nested sequent rule $\Box A_{gl}$ has two premisses, and hence every application of that rule doubles the number of premisses of the corresponding sequent rule. Thus a proof of cut-free completeness for the nested sequent calculus for GL automatically yields cut-free completeness of the corresponding standard sequent calculus, i.e., the sequent calculus with the standard propositional rules and this rule. Note that considering $R_2 = \emptyset$ in the rule above, that rule contains in particular the premiss $R, \Box S, \Box\alpha \Rightarrow \alpha$. But then we can further replace any application of this rule in a derivation by an application of the simpler rule

$$\frac{\Box M, N, \Box\alpha \Rightarrow \alpha}{\Box M, \Box N \Rightarrow \Box\alpha} \text{GLR}$$

(possibly modulo contraction) just by deleting the derivations of the remaining premisses. Since this is (a slight variant of) the standard modal sequent rule for GL (see, e.g., Leivant, 1981; Sambin and Valentini, 1982), cut-free completeness also transfers to the standard sequent system for that logic.

Conversely, using the conversion from standard sequent calculi to nested sequent calculi outlined above, we can decompose the standard sequent rule GLR above into the following nested sequent rules:

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L \quad \frac{G\{\Box\alpha, M \Rightarrow N, [\Box\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} 4$$

$$\frac{G\{M \Rightarrow N, \Box\alpha, [\Box\alpha \Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box K_{gl}$$

Note that these are just the nested sequent rules of Figure 10 where in the rule $\Box A_{gl}$ only the rightmost premiss is kept. Using the simulation of sequent derivations in this calculus as in Prop. 5.2, in particular the simulation of the sequent rule for GL by an application of $\Box K_{gl}$ followed by multiple applications of $\Box L$ and 4, cut-free completeness of the standard sequent system then automatically yields cut-free completeness of this nested sequent system.

5.2 Hypersequents

As mentioned above, the problems of the standard sequent framework in capturing prominent modal logics, in particular modal logic S5, early on led to the development of the *hypersequent* framework, introduced independently in Mints (1971), Pottinger (1983) and Avron (1996), and developed extensively by the latter author. As mentioned above, in this framework the standard sequent structure is extended to collections of sequents, i.e., lists, sets or multisets, each intuitively corresponding to a separate world in a Kripke-model.

Definition 5.4 A *hypersequent* is a finite multiset of standard sequents, written

$$M_1 \Rightarrow N_1 \mid \dots \mid M_n \Rightarrow N_n .$$

The standard *formula interpretation* of such a hypersequent is given by the formula

$$(M_1 \Rightarrow N_1 \mid \dots \mid M_n \Rightarrow N_n)^{\tau} := \Box \left(\bigwedge M_1 \rightarrow \bigvee N_1 \right) \vee \dots \vee \Box \left(\bigwedge M_n \rightarrow \bigvee N_n \right) .$$

While other formula interpretations have also been considered (see, e.g., Kurokawa, 2014; Lellmann, 2016), for modal logics the one given above is by far the most common. Moreover, in virtually all hypersequent calculi the hypersequent bar \mid is interpreted as some kind of disjunction. In line with this, essentially all such calculi include some form of the following *external structural rules* of *external weakening*, *external contraction* and *external exchange*:

$$\frac{G}{G \mid M \Rightarrow N} \quad \frac{G \mid M \Rightarrow N \mid M \Rightarrow N}{G \mid M \Rightarrow N} \quad \frac{G \mid P \Rightarrow Q \mid M \Rightarrow N}{G \mid M \Rightarrow N \mid P \Rightarrow Q}$$

Note that strictly speaking in our formulation of hypersequents as multisets the external exchange rule is already absorbed into the definition, and hence superfluous. As an example of a hypersequent calculus we consider a calculus for modal logic S5. The most commonly used hypersequent calculus for this logic is the one found in Avron (1996). However, the alternative calculus introduced in Mints (1971);

Poggiolesi (2008a) is better suited to illustrating the relation to nested sequents. The modal rules of the latter calculus are given by:¹⁶

$$\frac{G \mid M \Rightarrow N, \Box\alpha \mid \Rightarrow \alpha}{G \mid M \Rightarrow N, \Box\alpha} \quad \frac{G \mid \Box\alpha, M \Rightarrow N \mid \alpha, P \Rightarrow Q}{G \mid \Box\alpha, M \Rightarrow N \mid P \Rightarrow Q} \quad \frac{G \mid \Box\alpha, \alpha, M \Rightarrow N}{G \mid \Box\alpha, M \Rightarrow N}$$

Note that in this case the modal rules are essentially the modal rules of the nested sequent calculus for modal logic KT with the nesting operator replaced with the hypersequent bar. As explained by Poggiolesi (2010c), this leads to an interpretation of a hypersequent as a nested sequent, where the internal accessibility structure is collapsed into a (multi-)set, i.e., where the accessibility relation is universal.

A slightly different connection between hypersequents and nested sequents is given by the standard formula interpretation τ above. Under this interpretation, a hypersequent $M_1 \Rightarrow N_1 \mid \dots \mid M_n \Rightarrow N_n$ has the same interpretation as the nested sequent $\Rightarrow [M_1 \Rightarrow N_1], \dots, [M_n \Rightarrow N_n]$. Hence, hypersequents can be seen as a “layer” of a Kripke model, i.e., number of worlds “side by side” all accessible from a common root. This works reasonably well, e.g., for hypersequent calculi for confluent or linear extensions of modal logic S4. As an example, consider the calculus for modal logic S4.2 (obtained by axiom $\Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$ to an axiomatisation of modal logic S4) from Kurokawa (2014) given by the following characteristic modal rules:

$$\frac{G \mid \alpha, M \Rightarrow N}{G \mid \Box\alpha, M \Rightarrow N} \quad \frac{G \mid \Box M \Rightarrow \alpha}{G \mid \Box M \Rightarrow \Box\alpha} \quad \frac{G \mid \Box M, \Box N \Rightarrow}{G \mid \Box M \Rightarrow \mid \Box N \Rightarrow}$$

Informally, the first rule is the standard (local) rule for reflexivity, and the second rule is the standard rule for K4, corresponding to moving from the world represented by the component $\Box M \Rightarrow \Box\alpha$ to a successor world, represented by the component $\Box M \Rightarrow \alpha$. Taking the interpretation of a hypersequent as a layer of a nested sequent, applying the first rule stays local in a component of the nested sequent, while applying the second advances this component of the layer or slice of the nested sequent one step forward. The third rule, then, informally takes two worlds corresponding to the components $\Box M \Rightarrow$ and $\Box N \Rightarrow$ and replaces them with their common successor world, which corresponds to the component $\Box M, \Box N \Rightarrow$ (recall that the logic is an extension of K4, hence also in this rule the boxes on the left hand side of the sequent arrow carry over to the successor, i.e., premiss of the rule). At this point, however, to present the relations between the components accurately in the nested sequent setting, we would need to introduce a common successor to two nodes in the nested sequent, resulting in an accessibility structure which is no longer a tree (see also Sec. 5.6). In general, it seems that the hypersequent framework is perhaps better suited for capturing such “global” properties dealing with two arbitrary worlds in a model and their successors, while the nested sequent framework is based on a more “local” approach and is more suitable for capturing properties which can

¹⁶ A slightly different version of this hypersequent calculus can be found in Restall (2007).

be expressed by referring to a single world in a Kripke model and its immediate successors.

An intermediate framework between hypersequents and nested sequents is considered under the name of *grafted hypersequents* in Kuznets and Lellmann (2016), where a root node is “grafted” onto a hypersequent, the rules governing the hypersequent layer are those of a hypersequent system, and those governing the connection between the root and the hypersequent layer are those of a nested sequent system.

5.3 Linear nested sequents

An intermediate framework between standard sequents and nested sequents is given by restricting the tree structure of nested sequents to single branches. Originally introduced in a different notation under the name of *2-sequents* Masini (1992), it was also investigated under the names of *non-commutative hypersequents* Indrzejczak (2016), *(relational) hypersequents* Burns and Zach (2020); Parisi (2020) or *linear nested sequents* Lellmann (2015); Goré and Lellmann (2019). Due to its more immediate connection to the nested sequent framework, here we chose to use the latter name. Given that a linear nested sequent is just a branch of a nested sequent, we can equivalently reformulate the linear nested sequent structure as a *list* of sequents:

Definition 5.5 A *linear nested sequent* is a branch of a nested sequent, i.e., a finite list of sequents, written as follows:

$$M_1 \Rightarrow N_1 // \dots // M_n \Rightarrow N_n$$

The *formula interpretation* of such a linear nested sequent is that of the corresponding nested sequent, i.e., $\bigwedge M_1 \rightarrow \bigvee N_1 \vee \Box(\dots \vee \Box(\bigwedge M_n \rightarrow \bigvee N_n) \dots)$.

Note that the structures of a hypersequent and a linear nested sequent are almost the same, but the formula interpretation is different. In semantic terms, while a hypersequent corresponds to a set or multiset of worlds in a Kripke-model, a linear nested sequent corresponds to a branch of worlds. Since linear nested sequents are restrictions of nested sequents, every linear nested sequent calculus can also be seen as a nested sequent calculus. In fact, e.g., the modal rules of the original calculi from Masini (1992) are exactly the linear versions of the standard nested sequent modal rules for the corresponding logics. Hence completeness immediately transfers from linear nested sequents to nested sequents. If moreover, some care is taken in formulating the nested sequent calculus corresponding to a linear nested sequent calculus, so that in the application of the rules (seen bottom-up) never more than one successor node is created in the nested sequent, and the zero-premiss rules including the initial sequents are restricted to linear nested sequents, then also soundness transfers immediately, and hence the calculi are equivalent. Thus we have:

Proposition 5.6 *Let \mathcal{L} be a logic. If there is a linear nested sequent calculus which is sound and complete for \mathcal{L} , then there is also a nested sequent calculus which is sound and complete for \mathcal{L} .*

Thus every logic which can be captured by a linear nested sequent calculus can also be captured by a nested sequent calculus, including, e.g., the logics in the above mentioned literature as well as, e.g., the simply dependent multimodal logics considered in Lellmann and Pimentel (2019). Hence the nested sequent framework is at least as strong as the linear nested sequent framework.

For the reverse direction, a set of sufficient criteria for obtaining a linear nested sequent calculus from a nested sequent calculus is given by the above mentioned criteria for converting *basic nested sequent calculi* to standard sequent calculi from Pimentel et al. (2019), see Sec. 5.1:

Proposition 5.7 *Let $\mathbf{NS}_{\mathcal{L}}$ be a basic nested sequent calculus which is sound and complete for logic \mathcal{L} . Then there is a linear nested sequent calculus which is sound and complete for \mathcal{L} .*

Proof It was shown in (Pimentel et al., 2019, Thm. 23), that every basic nested sequent calculus can be restricted to its *end-active variant*, i.e., the variant where every rule is restricted so that one of the active components in the premisses is a leaf of the nested sequent. Since, moreover, by definition every premiss of a modal basic nested sequent rule only mentions one nesting operator, we can restrict the end-active variant of $\mathbf{NS}_{\mathcal{L}}$ to linear nested sequents, simply by restricting all rules to linear nested sequents. Call the resulting linear nested sequent calculus $\mathbf{LNS}_{\mathcal{L}}$. Since every derivation in $\mathbf{LNS}_{\mathcal{L}}$ is also a derivation in $\mathbf{NS}_{\mathcal{L}}$, soundness follows immediately. To show completeness of $\mathbf{LNS}_{\mathcal{L}}$ we simply follow the completeness proof for the corresponding standard sequent system in Sec. 4 of *op. cit.* by rearranging nested sequent derivations in the end-active variant of $\mathbf{NS}_{\mathcal{L}}$ into blocks consisting of a successor creating (*creation*) rule followed by upgrade rules to populate the newly created successor, and restricting the nested sequents to linear nested sequents using a suitable form of the disjunction property (Lem. 28 in *op. cit.*). \square

While the form of a basic nested sequent system is sufficient for restricting to a linear nested sequent system, it is not necessary: In particular they are not satisfied by the linear nested sequent calculi for linear tense logics of Indrzejczak (2016) and those for modal logic KB and modal tense logic K_t from Goré and Lellmann (2019), and hence by the corresponding nested sequent calculi. It is still an open problem to formulate more general criteria for restricting nested sequent calculi to linear nested sequent calculi, which capture systems for logics like the ones above which do not have a cut-free sequent calculus already.

5.4 Prefixed Tableaux

In classical logic as well as in modal logic, it is a well-known fact that standard sequent calculi correspond to *semantic tableaux calculi* (see, e.g., Smullyan (1968)

and Fitting, 1983). This correspondence can be easily explained. For the technical details, see *op. cit.* The idea is to try to construct a countermodel for (the formula interpretation of) a given sequent by applying the semantic tableaux rules (see below for details) backward until we reach an initial sequent (a *closed branch* in tableaux terminology) or every possible rule has already been applied, possibly taking into account some loop checking conditions (an *open branch*). Since the emphasis is on the countermodel construction, the presentation of sequents is also more semantic: A sequent $\alpha_1, \dots, \alpha_n \Rightarrow \beta_1, \dots, \beta_m$ is replaced with the *signed formulae* $T : \alpha_1, \dots, T : \alpha_n, F : \beta_1, \dots, F : \beta_m$, corresponding to the fact that in a countermodel for the interpretation $\alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \beta_1 \vee \dots \vee \beta_m$ of this sequent, the formulae $\alpha_1, \dots, \alpha_n$ on the left hand side of the sequent must be true, whereas the formulae β_1, \dots, β_m on the right hand side must be false. *Branch extension rules*, which specify how to deconstruct the principal formula of the sequent rule and add its components to a branch, correspond to sequent rules. E.g., the (classical, Kleene'd) conjunction left and right rules in the sequent framework and their corresponding tableaux rules are:

$$\frac{M, \alpha \wedge \beta, \alpha, \beta \Rightarrow N}{M, \alpha \wedge \beta \Rightarrow N} \wedge_L \qquad \frac{T : \alpha \wedge \beta}{T : \alpha} \\ \frac{T : \alpha \wedge \beta}{T : \beta} \\ \frac{M \Rightarrow N, \alpha \wedge \beta, \alpha \quad M \Rightarrow N, \alpha \wedge \beta, \beta}{M \Rightarrow N, \alpha \wedge \beta} \wedge_R \qquad \frac{F : \alpha \wedge \beta}{F : \alpha \mid F : \beta}$$

The way to read tableaux rules is that the rule corresponding to the conjunction left rule \wedge_L can be applied to a branch containing the signed formula $T : \alpha \vee \beta$ by adding both signed formulae $T : \alpha$ and $T : \beta$ to the bottom of this branch. The rule corresponding to the conjunction right rule \wedge_R can be applied to a branch containing the signed formula $F : \alpha \wedge \beta$ by splitting it into two branches at the bottom, one with the additional signed formula $F : \alpha$, the other with $F : \beta$. Note that we considered the Kleene's version of the sequent rules, i.e., the version where the principal formula is copied into the premisses, to correspond to the fact that signed formulae are added to a branch of a tableau without removing the original signed formula. Branches in the process of constructing a tableau thus correspond directly to sequents at the corresponding stage of backward proof search in the sequent system. In particular, a branch is *closed* if for some formula δ it contains both $T : \delta$ and $F : \delta$, and hence closed branches correspond to initial sequents with the formula δ occurring both on the left and right hand side of the sequent. A tableau starting from a single node containing only a formula $F : \gamma$ in which every branch is closed thus corresponds to a sequent derivation of the sequent $\Rightarrow \gamma$. Conversely, given the direct connection of formula signs to truth values, a countermodel can be read off immediately from an open branch to which every possible rule (modulo loop checking) has been applied by assigning every variable in the branch the truth value corresponding to its sign.

Where semantic tableaux calculi can be seen as the upside-down version of standard sequent calculi, the analogous upside-down version of nested sequent calculi is given by *prefixed tableaux* calculi, a framework going back already to Fitting

(1972). The correspondence was first shown in Fitting (2012) based on the single-sided formulation of nested sequent calculi. To align better with the two-sided formulation used here we adapt the two-sided notation from Fitting (2014), used there for intuitionistic logic. The main idea is to capture the tree-structure of nested sequents and Kripke-models in an additional prefix to every formula, essentially a name for the component of the nested sequent which contains this formula and the corresponding world in a Kripke-model. Formally, a *prefix* is a non-empty and finite sequence of positive numbers starting with 1, written using periods as separators. Examples are given by the prefixes 1.1.2.1 and 1.2.3.2. A set of prefixes which is closed under taking initial segments thus yields a tree-structure: the successors of a prefix σ are the prefixes $\sigma.n$ for positive integers n . A *prefixed signed formula* then has the form $\sigma : T : \alpha$ or $\sigma : F : \alpha$ for some prefix σ and formula α . Using the ideas of the classical case but extended with prefixes as names for components of a nested sequent, we can then convert back and forth between nested sequent calculi and prefixed tableaux calculi. E.g., the modal rules of the Kleene's calculus \mathbf{NS}_k^+ for modal logic K from Sec. 3 together with the corresponding prefixed tableaux rules are as follows:

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L^+ \quad \frac{\sigma : T : \Box\alpha}{\sigma.n : T : \alpha} \quad \sigma.n \text{ not new}$$

$$\frac{G\{M \Rightarrow N, \Box\alpha, [\Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box R^+ \quad \frac{\sigma : F : \Box\alpha}{\sigma.n : F : \alpha} \quad \sigma.n \text{ new}$$

Here the additional restrictions on the prefix state that in the rule corresponding to $\Box L^+$ the prefix $\sigma.n$ is *not new* on the branch, i.e., there is a formula $\sigma.n : F : \delta$ or $\sigma.n : T : \delta$ already on the branch, corresponding to the fact that the component $[P \Rightarrow Q]$ exists in the nested sequent. In contrast, for the prefixed tableau rule corresponding to the nested sequent rule $\Box R^+$ the prefix $\sigma.n$ must be *new* in the branch, corresponding to the fact that the component $[\Rightarrow \alpha]$ is freshly created by the nested sequent rule.

In the same spirit, all the logical variants of the nested sequent calculi from Sec. 2 can be seen essentially as notational variants of prefixed tableaux calculi and vice versa. The details of the proof for equivalence of the (single-sided) nested sequent calculi and the corresponding prefixed tableaux calculi are given in Fitting (2012). As also noted in *op. cit.*, under this connection the extensive body of work on prefixed tableaux thus gives rise to a wealth of nested sequent calculi. See, e.g., Massacci (2000); Castilho et al. (1997) or Goré (1999) for an overview.

5.5 Labelled Sequents

A different semantically motivated extension of the sequent framework is given by the framework of *labelled sequents*, investigated prominently, e.g., in Simpson (1994); Viganò (2000); Negri (2005) and Miller and Volpe (2015). Here, similarly to the

framework of prefixed tableaux, formulae are prefixed with a *label* corresponding to the world in the Kripke-model in which the formula is supposed to be true/false. The difference is however, that the relational structure of the model is captured explicitly in the language. This is achieved by the use of *relational atoms*, i.e., formulae xRy for x and y labels, which are interpreted as explicitly stating that the world denoted by y is a successor of the world denoted by x . Using this explicit representation, a wide ranging class of semantical properties can be captured and the corresponding logics automatically are endowed with a cut-free sequent system (see *op. cit.*). Compared to the tree structure used in nested sequents, the structure given by an arbitrary relation on the worlds is of course much more general. The subclass of labelled sequents corresponding to nested sequents was characterised in Goré and Ramanayake (2012) by restricting the relational atoms to spell out the structure of a tree:

Definition 5.8 A *labelled tree sequent* is a sequent $\mathcal{R}, M \Rightarrow N$ where

- \mathcal{R} is a *treelike relation set*, i.e., a set of relational atoms of the form xRy which is empty or such that the induced relation on the set of variables occurring in \mathcal{R} is a tree
- M and N consist of *labelled formulae*, i.e., formulae of the form $x : \alpha$ for a world variable or *label* x
- if $\mathcal{R} = \emptyset$, then all formulae in $M \Rightarrow N$ have the same label
- if $\mathcal{R} \neq \emptyset$, then every label occurring in a formula in $M \Rightarrow N$ also occurs in \mathcal{R} .

Similarly to the prefixes in the prefixed tableaux framework, the labels of formulae can then be taken as names for components of a nested sequent, with the successor relation in the nested sequent corresponding to the relation induced by the relation set. The restriction to treelike relation sets ensures that the resulting structure is a tree and hence a nested sequent. Thus as shown in *op. cit.*, nested sequent calculi can be translated into labelled tree sequent calculi and vice versa.

As an example, the modal nested sequent rules of the nested sequent calculus NS_K (see Figure 4) below left correspond to the labelled tree sequent rules below right:

$$\frac{G\{\Box\alpha, M \Rightarrow N, [\alpha, P \Rightarrow Q]\}}{G\{\Box\alpha, M \Rightarrow N, [P \Rightarrow Q]\}} \Box L \quad \frac{\mathcal{R}, xRy, X, x : \Box\alpha, y : \alpha \Rightarrow Y}{\mathcal{R}, xRy, X, x : \Box\alpha \Rightarrow Y} \Box R$$

$$\frac{G\{M \Rightarrow N, [\Rightarrow \alpha]\}}{G\{M \Rightarrow N, \Box\alpha\}} \Box R \quad \frac{\mathcal{R}, xRy, X \Rightarrow Y, y : \alpha}{\mathcal{R}, X \Rightarrow Y, x : \Box\alpha}$$

Here the label y in the rule corresponding to $\Box R$ is a name for the new component $[\Rightarrow \alpha]$, and hence must not occur in the conclusion of that rule. The label x in that rule is the name for the component $[M \Rightarrow N, \Box\alpha]$, and the relational atom xRy specifies that the component $[\Rightarrow \alpha]$ is one of its successors. Note that these are exactly the rules of the labelled sequent calculus **G3K** for modal logic K from Negri (2005). Thus, as in the case of prefixed tableaux, the two calculi can be seen as notational variants (see also (Poggiolesi, 2010b, §5.4) and Poggiolesi, 2010a). Similarly, the logical variants of the calculi from Sec. 2 can be seen as labelled tree

sequent calculi. As it was shown in Goré and Ramanayake (2012) using the example of provability logic, this correspondence between nested sequent calculi and labelled tree sequent calculi automatically transfers results such as cut-free completeness. It should be pointed out, that the labelled sequent rules corresponding directly to the standard nested sequent rules for extensions of K do not match the standard labelled sequent rules for these logics. An emulation of these nested sequent calculi in a focused version of the standard labelled sequent calculi for these logics is given in Marin et al. (2016).

As with prefixed tableaux, certain structural rules like the rule \tilde{b} are easier formulated in the nested sequent framework, since there it is not necessary to simultaneously rename all labels in a sequent. For labelled tree sequents this however works slightly better than for prefixed tableaux, since the tree structure is local to a sequent and not distributed along a branch. Hence this structure can be changed by modifying only a single sequent instead of potentially all formulae along the branch of a tableau down to the root. The latter operation in particular might result in conflicts with the other branches of the tableau.

Conversely, in the labelled setting it is very natural to capture properties of the models using rules which only work on the relation sets, and to keep the rules working on formulae of the form $x : \Box\alpha$ constant. E.g., the rules for transitivity and reflexivity are naturally formulated as:

$$\frac{\mathcal{R}, xRy, yRz, xRz, X \Rightarrow Y}{\mathcal{R}, xRy, yRz, X \Rightarrow Y} \text{ trans} \qquad \frac{\mathcal{R}, xRx, X \Rightarrow Y}{\mathcal{R}, X \Rightarrow Y} \text{ refl}$$

Since such rules very easily destroy the tree structure of the relation set, converting labelled sequent calculi with such rules into labelled tree sequent calculi and hence nested sequent calculi is often impossible. Where it is possible, the conversion is often rather involved and non-trivial, see, e.g., Simpson (1994); Lyon (2020a,b). For a more thorough overview and an exploration of a rather general method for such a conversion, see also the recent Lyon (2021b).

5.6 Indexed nested sequents

As mentioned above, nested sequents are trees of sequents, and hence when seen semantically essentially correspond to Kripke models based on trees. Due to this close connection to trees, the nested sequent formalism tends to struggle with modal logics characterised by frame properties which require a different structure, such as *confluence*, stating that for all worlds x, y, z in a Kripke frame with xRy and xRz there is a common successor w with xRw and yRw . To alleviate this problem, nested sequents were generalized in Fitting (2015) to *indexed nested sequents*.¹⁷ Intuitively,

¹⁷ To prevent confusion about the terminology: Structures called *indexed nested sequents* were also independently introduced in Poggiolesi (2013). However in Poggiolesi's terminology, they denote nested sequents with indices for agents, and hence do not break the tree structure of nested

every component in a nested sequent is extended with an *index*, loosely speaking a name for the corresponding world in a Kripke model. In particular, different components with the same index correspond to the same world of a model. This way it is possible to mirror properties like confluence in the syntactic framework by adding successors to two components in the tree structure of a nested sequent, and identifying the successors by giving them the same index. To keep in line with the two-sided notation in this article, we follow the notation from Ramanayake (2018).

Definition 5.9 An *indexed sequent* is obtained by adding an *index* $n \in \mathbb{N}$ to a sequent $M \Rightarrow N$, written $M \xRightarrow{n} N$. An *indexed nested sequent* is of the form

$$M \xRightarrow{n} N, [G_1], \dots, [G_m]$$

where n is an index, $m \geq 0$ and G_1, \dots, G_m are indexed nested sequents.

Other notions like that of a *context* then are defined analogously to the nested sequent framework. The modal rules of an indexed nested sequent system for normal modal logic K then are essentially the nested sequent rules with additional indices:

$$\frac{G\{\Box\alpha, M \xRightarrow{a} N, [\alpha, P \xRightarrow{b} Q]\}}{G\{\Box\alpha, M \xRightarrow{a} N, [P \xRightarrow{b} Q]\}} \Box L \qquad \frac{G\{M \xRightarrow{a} N, \Box\alpha, [\xRightarrow{b} \alpha]\}}{G\{M \xRightarrow{a} N, \Box\alpha\}} \Box R$$

where the rule $\Box R$ has the side condition that the index b does not occur in the conclusion. However, since two components with the same index correspond to the same world in the Kripke model, the calculus also includes rules to communicate between such components:

$$\frac{G\{\alpha, M \xRightarrow{a} N\}\{\alpha, P \xRightarrow{a} Q\}}{G\{\alpha, M \xRightarrow{a} N\}\{P \xRightarrow{a} Q\}} \text{fc-l} \qquad \frac{G\{M \xRightarrow{a} N, \alpha\}\{P \xRightarrow{a} Q, \alpha\}}{G\{M \xRightarrow{a} N, \alpha\}\{P \xRightarrow{a} Q\}} \text{fc-r}$$

$$\frac{G\{M \xRightarrow{a} N, [P \xRightarrow{b} Q]\}\{R \xRightarrow{a} S, [\xRightarrow{b}]\}}{G\{M \xRightarrow{a} N, [P \xRightarrow{b} Q]\}\{R \xRightarrow{a} S\}} \text{sc}$$

Here the notation $G\{ \} \{ \}$ denotes the obvious extension of an indexed nested sequent with a context to an indexed nested sequent with *two* (possibly identical) contexts. We refer to, e.g., Ramanayake (2018) for the full definition. Intuitively, the rules fc-l and fc-r state that formulae can move freely between components with the same index. Rule sc intuitively states that a component with index a has the same successors, irrespective of where in the indexed nested sequent it occurs.

The frame property of confluence mentioned above then is captured by the following additional rule:

sequents. They are used to capture proof-theoretically multi-agent epistemic logic. In Bull (1992), nested sequents have also been indexed with programs. See Section 4.3.

$$\frac{G\{M \overset{a}{\Rightarrow} N, [P \overset{b}{\Rightarrow} Q, [\overset{d}{\Rightarrow}]], [R \overset{c}{\Rightarrow} S, [\overset{d}{\Rightarrow}]]\}}{G\{M \overset{a}{\Rightarrow} N, [P \overset{b}{\Rightarrow} Q], [R \overset{c}{\Rightarrow} S]\}}$$

This rule rather directly captures the frame condition that if the worlds corresponding to the indices b and c are successors of the one corresponding to a , then they have a common successor world corresponding to the index d . More generally, the indexed nested sequent formalism was used in Fitting (2015) to introduce calculi for modal logics axiomatised by *Geach formulae*, also known as *Scott-Lemmon axioms*, i.e., formulae of the form $\diamond^k \Box^\ell \alpha \rightarrow \Box^m \diamond^n \alpha$ for $k, \ell, m, n \geq 0$. Syntactic completeness for these calculi via direct cut elimination and their extension to the intuitionistic setting were investigated in Marin and Straßburger (2017).

The precise relation of indexed nested sequents to nested sequents is clarified in Ramanayake (2018) via a correspondence to a restricted form of labelled sequents (see Sec. 5.5 above). In particular, in *op. cit.* it was shown that indexed nested sequents correspond to *labelled tree sequents with equality*, an extension of the labelled tree sequents of Def. 5.8 to labelled sequents of the form $\mathcal{R}, \mathcal{E}, M \Rightarrow N$ where \mathcal{R} again is a tree-like relation set, M and N contain labelled formulae, and \mathcal{E} now contains *equality terms* of the form $x = y$. Again, labelled tree sequents with equality are subject to certain restrictions ensuring that every variable occurring in M, N or \mathcal{E} occurs also in \mathcal{R} if the latter is non-empty. In case \mathcal{R} is empty, every variable occurring in M, N or \mathcal{E} is the same. Thus, the difference in expressive power between the frameworks of nested sequents and indexed nested sequents is essentially the difference in expressive power between the frameworks of labelled tree sequents and labelled tree sequents with equality.

5.7 Display Calculi

Considering only the syntactic shape of nested sequents, it might seem a bit strange that the nesting operator is allowed to occur only on the right hand side of the sequent arrow. After all, the comma as a structural connective might appear on both sides. Moreover, while in the nested sequent framework we have structural connectives corresponding to implication (the sequent arrow), conjunction (the comma on the left), disjunction (the comma on the right), and the modal box (the nesting operator), there are no structural connectives corresponding to negation \neg , falsum \perp or verum \top . So it would seem a natural step to consider sequent structures which permit the nesting operator $[.]$ on both sides of the sequent arrow and include structural connectives for every logical connective in the language. In fact, this is essentially what happens in the framework of *display calculi*, introduced in Belnap (1982). While adding structural connectives for the propositional connectives works rather smoothly, though, in order to give a formula interpretation for the nesting operator on the left hand side of such structures we need to extend the language to that of *tense logics* (see also Sec. 4.3), by introducing the modal connective \blacklozenge for the residual of

the modal box (see Wansing, 2002). Semantically, this is interpreted as a backward looking diamond: a formula $\blacklozenge\alpha$ is true in a world of a Kripke frame if the formula α is true in at least one of its predecessors.

Definition 5.10 A (modal) *display sequent* is a tuple $X \Rightarrow Y$, where X and Y are *display structures* given by the following grammar:

$$X ::= \alpha \mid \mathbf{I} \mid *X \mid \bullet X \mid (X \circ X)$$

where α is a formula of modal tense logic. The interpretation of the left and right hand sides of a display sequent is given by the two functions ℓ and r as follows:

$$\begin{array}{ll} \ell(\alpha) = \alpha & r(\alpha) = \alpha \\ \ell(\mathbf{I}) = \top & r(\mathbf{I}) = \perp \\ \ell(*X) = \neg r(X) & r(*X) = \neg \ell(X) \\ \ell(\bullet X) = \blacklozenge \ell(X) & r(\bullet X) = \Box r(X) \\ \ell(X \circ Y) = \ell(X) \wedge \ell(Y) & r(X \circ Y) = r(X) \vee r(Y) \end{array}$$

The formula interpretation of a display sequent $X \Rightarrow Y$ is $\ell(X) \rightarrow r(Y)$.

Note that the comma of standard or nested sequents is replaced with a binary structural connective \circ , making properties like associativity of conjunction/disjunction explicit. This has the benefit that the logical rules are applied only to an isolated or *displayed* formula. E.g., the modal rules of a display calculus for tense logic K_t , the tense analogue of modal logic K are:

$$\frac{\alpha \Rightarrow X}{\Box \alpha \Rightarrow \bullet X} \Box_L \quad \frac{X \Rightarrow \bullet \alpha}{X \Rightarrow \Box \alpha} \Box_R \quad \frac{\bullet \alpha \Rightarrow X}{\blacklozenge \alpha \Rightarrow X} \blacklozenge_L \quad \frac{X \Rightarrow \alpha}{\bullet X \Rightarrow \blacklozenge \alpha} \blacklozenge_R$$

However, in order to be able to display a formula so that a logical rule can be applied to it, display calculi require the addition of *display rules*. E.g., for the structural connective \bullet these are

$$\frac{X \Rightarrow \bullet Y}{\bullet X \Rightarrow Y} \quad \frac{\bullet X \Rightarrow Y}{X \Rightarrow \bullet Y} \quad \frac{\mathbf{I} \Rightarrow Y}{\bullet \mathbf{I} \Rightarrow Y} \quad \frac{X \Rightarrow \mathbf{I}}{X \Rightarrow \bullet \mathbf{I}}$$

A detailed introduction to display calculi including the display rules for all the structural connectives is beyond the scope of this article - we refer the reader to, e.g., Belnap (1982); Wansing (1998). From the perspective of nested sequents the two display rules above left are interesting, though: Intuitively, while the operator \bullet on the right hand side of a display sequent corresponds to the nesting operator $[.]$ in a nested sequent, and hence to moving to a successor node in the underlying tree, the operator \bullet on the left hand side of a display sequent corresponds to moving to a predecessor node. Hence, moving between premiss and conclusion in the display rules essentially corresponds to moving between different nodes in a Kripke-model. Viewing a Kripke-model as a nested sequent, then, intuitively, the display rules can be used to shift the root of a nested sequent to a different node, so that logical rules can be applied to formulae there. This intuition is in fact already present in

Kashima (1994), one of the works which introduced the nested sequent framework (in the single-sided formulation, the display rules above correspond to the *turn* rules introduced in *op./cit.*). Using this is it possible to simulate nested sequent derivations using the display rules. E.g. the application of the nested sequent \Box_R rule below left in a successor of the root node can be simulated by the display derivation below right, where the applications of the display rules in the first and last step correspond to shifting the root node so that the logical rule \Box_R can be applied.

$$\frac{\alpha \Rightarrow [\Rightarrow [\Rightarrow \beta]]}{\alpha \Rightarrow [\Rightarrow \Box \beta]} \Box_R \qquad \frac{\frac{\frac{\alpha \Rightarrow \bullet \bullet \beta}{\bullet \alpha \Rightarrow \bullet \beta}}{\bullet \alpha \Rightarrow \Box \beta} \Box_R}{\alpha \Rightarrow \bullet \Box \beta}$$

Making this connection formally precise is somewhat cumbersome and heavily relies on the display rules for the remaining connectives. Hence we refer the reader to Poggiolesi (2010a) for the details.

Given the previous remarks it should be clear that display calculi are at least as expressive as nested sequent calculi. Since there are very general results about how to automatically obtain cut free display calculi for logics given by a large class of axioms (see, e.g., Kracht, 1996), it is then of course interesting to see which kinds of display calculi can be converted into nested sequent calculi. Some very general results in this spirit, albeit based on slightly different display-style calculi, are shown in Goré et al. (2011), where the authors use a transformation from display-like *shallow inference* calculi to nested sequent (or *deep inference*) calculi to uniformly obtain cut-free modular systems for modal tense logics given by *Scott-Lemmon axioms* (see also Sec. 5.6) or by *path axioms*, i.e., axioms of the form $\langle \rangle_1 \dots, \langle \rangle_n \alpha \rightarrow \langle \rangle \alpha$, where $n \geq 0$ and each of $\langle \rangle_1, \dots, \langle \rangle_n, \langle \rangle$ is either \diamond or \blacklozenge . A further extension of these results is presented in the very recent Ciabattoni et al. (2021).

It is worth noting, however, that the greatly increased expressivity of display calculi compared to nested sequent calculi comes at a price: First, owing to the fact that the presence of the display rules essentially necessitates working with equivalence classes of display sequents, proof search in display calculi is a notoriously difficult problem, rendering such calculi suboptimal for decidability or complexity results. Moreover, the display rules for the nesting operator \bullet as well as the formula interpretation for this operator on the left hand side heavily rely on the presence of the residuum \blacklozenge for the modal operator \Box in the language. While in the modal case often it can be shown that the display calculi are conservative for the fragment of the language not containing the operator \blacklozenge , in general this can be complicated, and for some logics such as non-normal modal logics it might also not be clear what the residuum for the modal box is.

6 Conclusion

Modal logic is the logic that was originally introduced to treat the concepts of necessity and possibility, whose basic principles were known since Aristotle. It is to a large extent thanks to the introduction of Kripke semantics, in the early 1960, that modal logic has become a powerful, fruitful and very much used non-classical logic. However, given the difficulties encountered in the search for cut-free sequent calculi for various important modal logics such as $S5$, until the end of the last century there was a general skepticism about the possibility of a thriving general proof theory for modal logic. The tree-hypersequent, deep sequent or nested sequent method is amongst the methods that helped changing this perspective. The main aim of this paper has been to offer a reasonably complete, faithful and harmonious panorama of nested sequents and associated results. After having introduced nested sequents, we have shown the main theorems that can be obtained with it, e.g. cut-elimination, decidability and interpolation; moreover, we have illustrated its relation with other well-known proof-theoretic formalisms and outlined the many logics it can be applied to. With this we encourage the reader to dive deeper into the provided references and hope to contribute to the study of nested sequent calculi as a vibrant and growing area of research.

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