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General Versions of the Ekeland Variational Principle: Ekeland Points and Stop and Go Dynamics

Le Phuoc Hai^{1,2} · Phan Quoc Khanh³ · Antoine Soubeyran⁴

Abstract

We establish general versions of the Ekeland variational principle (EVP), where we include two perturbation bifunctions to discuss and obtain better perturbations for obtaining three improved versions of the principle. Here, unlike the usual studies and applications of the EVP, which aim at exact minimizers via a limiting process, our versions provide good-enough approximate minimizers aiming at applications in particular situations. For the presentation of applications chosen in this paper, the underlying space is a partial quasi-metric one. To prove the aforementioned versions, we need a new proof technique. The novelties of the results are in both theoretical and application aspects. In particular, for applications, using our versions of the EVP together with new concepts of Ekeland points and stop and go dynamics, we study in detail human dynamics in terms of a psychological traveler problem, a typical model in behavioral sciences.

Keywords Ekeland variational principle · Two perturbation bifunctions · Better perturbation · Ekeland points · Partial quasi-metric space · Worthwhile move · Variational trap · Stop and go dynamics

Mathematics Subject Classification 90C30 · 49J52 · 49J53 · 90C26 · 91E45

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1 Introduction

The variational principles are fundamental tools in optimization and related areas. Among the known ones, the following Ekeland variational principle (EVP) is the most important one.

Ekeland's Variational Principle (Ekeland [18]) Let (X, d) be a complete metric space, $\varphi : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ lower semicontinuous and bounded from below, $\varepsilon > 0$, and $x \in X$ with $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$, there exists $z \in X$ such that

- (i) $d(x, z) \leq \lambda$;
- (ii) $\varphi(z) + \frac{\varepsilon}{\lambda}d(x, z) \leq \varphi(x)$;
- (iii) $\varphi(u) + \frac{\varepsilon}{\lambda}d(u, z) > \varphi(z)$ for all $u \in X \setminus \{z\}$.

Briefly and roughly speaking, it says that if X is noncompact, then the minimizer could not exist as the classical Weierstrass theorem asserts for the compact case, but one has an approximate minimizer z which is a strict minimizer of the perturbed function $\varphi(\cdot) + \frac{\varepsilon}{\lambda}d(\cdot, z)$. It is equivalent to many well-known and independently established results such as the theorem on solvability of operator equations [46], the Phelps lemma [40], the drop theorem [14], the Caristi fixed-point theorem [9], the petal theorem [39], the Takahashi theorem on nonconvex minimization [44]), and the induction theorem proved in [26, 28]. But, it inspired remarkably more developments and applications in various areas, including interferences between optimization, operational research, and behavioral sciences, [1, 3–5, 12, 15, 17, 23, 24, 26, 33, 37, 42, 45], among many others. Regarding these references, we would emphasize the important developments of the dynamical approach to extensions of the EVP in metric spaces of [4] and of further results in this direction about variants of the EVP in quasi-metric spaces with variable ordering structures and particular applications to psychological models in [5].

We observe that a major of contributions to developments of the EVP and applications aim at exact minimizers (in certain senses) by applying the EVP together with some limiting process. In this paper, our goal is improving perturbations for using the obtained approximate minimizer when the approximation is good enough in application situations. Namely, we develop general versions of the EVP in terms of two perturbations (not one as usual). These two perturbations are interrelated, one is a local perturbation of the other. To show our goal in improving perturbations, we choose applications in behavioral sciences and a partial quasi-metric space (PQMS, hereafter) for the underlying space. The novelties of the paper are in both mathematical and application aspects. Let us speak first about three features of our results: They involve two perturbations; they are sharper than the generalization of the classical EVP, proved for a PQMS with one perturbation; and their main proof technique is new. Regarding perturbations, several papers already considered various types of perturbations related to the generalized metric d of the underlying space, see e.g., [7, 22, 29]. There, the perturbation is a vector of pseudo-quasi-metrics, a smooth combination related to the norm of the space, a family of ordered metrics, or a set perturbation

$d(x, y)D$ with a convex set D . So, they are related to proximity measures. These perturbations may occur simultaneously, see [11, 13, 22], among others. They may occur only sequentially, as in the present paper. See also the famous smooth variational principle [7, 8]. Our second perturbation, as a local perturbation of the first one, is neither a distance-like nor built from the generalized metric of the space. This requires a striking new proof of the generalized EVP. The newness of our proof technique is shown in Remark 3.3. The comparisons of our versions of the EVP with the known ones and the main mathematical novelties of the paper about building better perturbations are explained in Remark 3.4. Below are some introductory words. We define, in the context of the EVP, what can be the quality of an approximate solution of an optimization program as an exact solution of a perturbed program. Then, we define a “perturbation function better than a given one” and show how our general versions of the EVP can help to build it when the initial one is a partial quasi-metric and then to find a better approximate solution of an optimization program as the unique optimal solution of a perturbed program. Our main theoretical contributions are three general versions of the EVP in Sect. 3.

To make the novelties clear, we pose five questions and provide short answers in the context of the chosen application in behavioral sciences. The detailed answers are given in Sect. 4.

Question 1: Why is a partial quasi-metric (PQM) used? Answer: because, for example, it expresses generalized inconveniences to hire or to fire workers in the context of a model relative to the dynamics of organizations in operations research and behavioral sciences.

Question 2: Why is a family of perturbations employed? Answer: it models a changing work environment (hence the perturbations appear in sequences).

Question 3: What do conditions about the relationship between the two perturbations (later called conditions (A), (B), and (C)) mean? Answer: they model different local perturbations with respect to the previous one.

Question 4: What do the points we introduce (later called the Ekeland points, see Definitions 3.1, 3.3, and 3.4) represent? Answer: they model the three periods “go, stop-stop, and go” of human dynamics.

Question 5: What can a better perturbation be? Answer: there is a trade-off between efficiency and stability.

The (application) motivation of the paper comes from a typical behavioral science model where the recent *variational rationality (VR) approach* (see e.g., [2, 3, 6, 41] and references therein) expresses human dynamics in a locomotion space as a psychological traveler problem. Its main task is to know when to start and when to stop changing internal and external environments. The applications of the results in Sect. 3 presented in Sect. 4 are specific and clear answers to the above five questions from an application view point.

We would expect that this beginning of the approach of developing versions of the EVP with better perturbations in order to employ the obtained approximate minimizers in applications inspires further researches on the EVP and applications.

The layout of the rest of the paper is as follows. Section 2 contains preliminaries. Section 3 is devoted to general versions of the EVP. Section 4 explains in detail the main application motivation of this paper in terms of stop and go worthwhile dynamics in

the context of the variational rationality approach to behavioral sciences. Conclusions and perspectives of further developments are contained in the last short Sect. 5.

2 Preliminaries

For a nonempty set X , a bifunction $q : X \times X \rightarrow \mathbb{R}_+$ is called a *quasi-metric* iff for $x, y, z \in X$, one has two conditions: $q(x, y) = 0$ if and only if $x = y$ and $q(x, y) \leq q(x, z) + q(z, y)$.

Definition 2.1 (*partial quasi-metric*, [25]) Let X be a nonempty set. A bifunction $q : X \times X \rightarrow \mathbb{R}_+$ is said to be a *partial quasi-metric (PQM)* on X iff, for any $x, y, z \in X$,

- (p₁) $q(x, x) = q(y, y) = q(x, y) \Leftrightarrow x = y$ (equality);
- (p₂) $q(x, x) \leq \min\{q(x, y), q(y, x)\}$ (small self-distances);
- (p₃) $q(x, y) \leq q(x, z) + q(z, y) - q(z, z)$ (triangle inequality).

Then, X equipped with q is called a *partial quasi-metric space (PQMS)*.

From now on, X is always a PQMS if not otherwise specified.

Definition 2.2 (*Cauchy sequence, completeness*, [25, 36])

- (i) x_n in X are said: to be convergent to $x \in X$ iff

$$q(x, x) = \lim_{n \rightarrow \infty} q(x, x_n) = \lim_{n \rightarrow \infty} q(x_n, x);$$

to be a Cauchy sequence iff $\lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m)$ and $\lim_{n, m \rightarrow \infty, m \geq n} q(x_m, x_n)$ exist (finitely); to be 0-Cauchy iff $\lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m) = \lim_{n, m \rightarrow \infty, m \geq n} q(x_m, x_n) = 0$.

- (ii) X is termed: complete iff, for every Cauchy sequence $(x_n), x_n \rightarrow x \in X$ such that $q(x, x) = \lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m) = \lim_{n, m \rightarrow \infty, m \geq n} q(x_m, x_n)$; 0-complete iff every 0-Cauchy sequence in X converges to a point x with $q(x, x) = 0$.

Definition 2.3 (*left-convergence, left-completeness*)

- (i) x_n are said: to *left-converge* to $x \in X$ ($x_n \rightarrow x$) iff $q(x, x) = \lim_{n \rightarrow \infty} q(x_n, x)$; to be a *left-Cauchy* (0-left-Cauchy) *sequence* iff $\lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m)$ exists finitely (is 0, resp).
- (ii) X is called *left-complete* (0-left-complete) iff, for every left-Cauchy (0-left-Cauchy, resp) sequence $(x_n), x_n \rightarrow x$ such that $q(x, x) = \lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m)$ ($0 = \lim_{n, m \rightarrow \infty, m \geq n} q(x_n, x_m)$, resp).

Example 2.1 (0-completeness, but not left-completeness) Let $(\mathbb{R}; q)$ be with $q(x; y)$ defined as follows.

$$q(x, y) = \begin{cases} a + x - y & \text{if } x > y, \\ b & \text{if } x = y, \\ b + y - x & \text{if } x < y. \end{cases}$$

Then, it is clear that $(\mathbb{R}; q)$ is a PQMS if $0 \leq b \leq \min\{a, c\}$. Also, if $0 < b$, then $(\mathbb{R}; q)$ is 0- left-competive. Finally $(\mathbb{R}; q)$ is not left-complete with $b < a$.

Definition 2.4 (*left lower semicontinuity*) $\varphi : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to be *left lower semicontinuous (l-lsc)* at x iff $\varphi(x) \leq \liminf_{n \rightarrow \infty} \varphi(x_n)$ whenever x_n left-converge to x .

3 General Versions of the EVP

Aiming at better perturbations for versions of the EVP, we include a second perturbation $k : X \times X \rightarrow \mathbb{R}_+$, besides the first one being the generalized metric q of X . It is the involvement of two perturbations that allows us to drill into the nature of the EVP to find its different versions, which may be better and more appropriate for application.

For the first version of the EVP, let k and q satisfy the following assumption, for all $x, y, z \in X$ (recall that for our chosen application, X is a PQMS),

$$(A) \quad k(x, x) \leq k(x, y) \text{ and} \\ \max\{k(x, z), q(x, z)\} \leq k(x, y) + q(y, z) - \min\{k(x, x), k(y, y)\}.$$

((A) holds for many types of k , including just the possibility that $k := q$.)

Remark 3.1 Assumption (A) has also a simpler form: $k(x, x) \leq k(x, y) \leq q(x, y)$ and $q(x, z) \leq k(x, y) + q(y, z) - \min\{k(x, x), k(y, y)\}$. Indeed, from (A), it follows that $q(x, z) \geq k(x, z)$ for all $x, z \in X$ (to see this, put $x = y$ in (A)). This simple form may be more convenient in some applications.

Example 3.1 (illustrations of assumption (A)) Let $X = \mathbb{R}_+$, $1 > \gamma > 0$, $a \in \mathbb{R}_+$ and

$$k(x, y) = \begin{cases} y - x & \text{if } x \leq y, \\ \gamma(x - y) & \text{if } x > y, \end{cases} \quad q(x, y) = k(x, y) + a.$$

Then, k is a quasi-metric and satisfies assumption (A). Indeed, to check the triangle inequality, consider two possibilities $x \leq y$ and $x > y$. For details of the first possibility, discuss the three cases as follows.

- Case 1: $x \leq z \leq y$. Then, $k(x, z) = z - x = z - y + y - x \leq y - x + \gamma(y - z) = k(x, y) + k(y, z)$.
- Case 2: $y \geq x > z$. Then, $k(x, z) = \gamma(x - z) = \gamma(x - y + y - z) \leq y - x + \gamma(y - z) = k(x, y) + k(y, z)$.
- Case 3: $x \leq y < z$. Then, $k(x, z) = z - x = z - y + y - x = k(x, y) + k(y, z)$.

For the second possibility $x > y$, we can similarly check that for all $z \in X$, k satisfies the triangle inequality for x, y, z . Hence, k is a quasi-metric. From the formula of q , it is clearly a partial quasi-metric. Assumption (A) is satisfied because, for all x, y, z in X ,

$$\begin{aligned} q(x, z) &= k(x, z) + a \leq k(x, y) + k(y, z) + a \\ &= k(x, y) + q(y, z) - \min\{k(x, x), k(y, y)\}. \end{aligned}$$

Example 3.2 ((A) is satisfied with two different PQMs) Let X be a nonempty set, $k : X \times X \rightarrow \mathbb{R}_+$ be a PQM (then k and $q_1 := k$ satisfy (A)), and a be a nonnegative number. Then, $q_2(x, y) := k(x, y) + a$ is also a PQM and assumption (A) is satisfied.

Definition 3.1 (Ekeland points of type (\tilde{k}, q)) Let $\varphi : X \rightarrow \overline{\mathbb{R}}$, $\varepsilon, \lambda > 0$, and

$$\begin{aligned} \tilde{k}(x, z) &:= k(x, z) - \min\{k(x, x), k(z, z)\}, \\ \mathcal{E}_x^{\tilde{k}} &:= \left\{ z \in X \mid \varphi(z) \leq \varphi(x) - \varepsilon\lambda^{-1}\tilde{k}(x, z) \right\}, \\ \mathcal{G}^q &:= \left\{ z \in X \mid \exists u \neq z, \varphi(u) + \varepsilon\lambda^{-1}q(z, u) \leq \varphi(z) \right\}. \end{aligned}$$

Then, $z \in \mathcal{E}_x^{\tilde{k}}$ is called an *Ekeland point of type (\tilde{k}, q)* of φ (relative to x , ε , and λ) iff $\mathcal{E}_z^{\tilde{k}} \cap \mathcal{G}^q = \emptyset$.

Definition 3.2 (strict-decreasing left lower semicontinuity) Let X be a PQMS. A function $\varphi : X \rightarrow \overline{\mathbb{R}}$ is called *strictly decreasingly left lower semicontinuous (sdl-lsc)* at $x \in X$ iff whenever x_k left-converge to x such that $\varphi(x_{k+1}) < \varphi(x_k)$, then $\lim_{k \rightarrow \infty} \varphi(x_k) \geq \varphi(x)$.

Example 3.3 (being sdl-lsc but not decreasingly l-lsc) Let $X = \mathbb{R}$, $q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$, $\varphi : (\mathbb{R}, q) \rightarrow (\mathbb{R}, |\cdot|)$ be defined by

$$q(x, y) = |y - x| + |x| \quad \text{and} \quad \varphi(x) = \begin{cases} e^x & \text{if } x \geq 0, \\ -2 & \text{if } x < 0. \end{cases}$$

Then, at 0, φ is sdl-lsc, but not decreasingly l-lsc, as for $x_n = -\frac{1}{n} \xrightarrow{\text{left}} 0$, $\varphi(x_n) = -2 < \varphi(0)$ for all n .

Theorem 3.1 (existence of Ekeland points of type (\tilde{k}, q)) Let X be 0-left-complete, q and k satisfy assumption (A), $\varphi : X \rightarrow \overline{\mathbb{R}}$ be proper, sdl-lsc, and bounded from below, and ε, λ be positive numbers. Then, for each $x \in X$, there exists an Ekeland point of type (\tilde{k}, q) of φ (relative to x , ε , and λ).

Proof Note that $x \in \mathcal{E}_x^{\tilde{k}}$ for all $x \in X$. We claim first that if $\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$ for all $w \in \mathcal{E}_x^{\tilde{k}}$, then there exists y in $\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$. If $v \in \mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q$, then

$$\begin{cases} k(w, v) - \min\{k(w, w), k(v, v)\} \leq \frac{\lambda}{\varepsilon} [\varphi(w) - \varphi(v)], \\ \varphi(u) + \frac{\varepsilon}{\lambda} q(v, u) \leq \varphi(v) \text{ for some } u \neq v. \end{cases}$$

Hence, by assumption (A),

$$\begin{cases} \varphi(w) \geq \varphi(v) > \varphi(u), \\ \max\{k(w, u), q(w, u)\} \leq k(w, v) + q(v, u) - \min\{k(w, w), k(v, v)\} \\ \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)]. \end{cases} \quad (1)$$

Using the last inequality in (1) for $\max\{k(w, u), q(w, u)\}$ and the inequality defining $w \in \mathcal{E}_x^{\tilde{k}}$ for the remaining term in the above right-hand side, we have

$$\max\{k(x, u), q(x, u)\} \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(w)] + \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)] \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(u)].$$

Consequently, $u \in \mathcal{E}_x^{\tilde{k}}$. Hence, $\mathcal{E}_u^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$. Pick $a \in \mathcal{E}_u^{\tilde{k}} \cap \mathcal{G}^q$. Then, similar to obtaining $u \neq v$ from v , one gets $b \neq a$ from a with

$$\begin{cases} k(u, a) - \min\{k(u, u), k(a, a)\} \leq \frac{\lambda}{\varepsilon}[\varphi(u) - \varphi(a)], \\ \varphi(b) + \frac{\varepsilon}{\lambda}q(a, b) \leq \varphi(a), \\ \varphi(b) < \varphi(w). \end{cases}$$

Similar to $\max\{k(w, u), q(w, u)\}$, one has $\max\{k(u, b), q(u, b)\} \leq \frac{\lambda}{\varepsilon}[\varphi(u) - \varphi(b)]$. This and (1) imply that

$$\max\{k(w, b), q(w, b)\} \leq k(w, u) + q(u, b) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(b)]. \quad (2)$$

Hence, $b \in \mathcal{E}_w^{\tilde{k}}$. Because $w \in \mathcal{E}_x^{\tilde{k}}$, (2) implies that

$$\begin{aligned} \max\{k(x, b), q(x, b)\} &\leq [k(x, w) - \min\{k(x, x), k(w, w)\}] + \max\{k(w, b), q(w, b)\} \\ &\leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(b)]. \end{aligned}$$

So, $b \in \mathcal{E}_x^{\tilde{k}}$ and hence $b \in \mathcal{E}_x^{\tilde{k}} \cap \mathcal{E}_w^{\tilde{k}}$. By the contradiction assumption, $\mathcal{E}_b^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$. Let $c \in \mathcal{E}_b^{\tilde{k}} \cap \mathcal{G}^q$. Then,

$$\begin{cases} \varphi(c) \leq \varphi(b) - \frac{\varepsilon}{\lambda}[k(b, c) - \min\{k(b, b), k(c, c)\}], \\ \varphi(d) + \frac{\varepsilon}{\lambda}q(c, d) \leq \varphi(c) \text{ for some } d \neq c. \end{cases} \quad (3)$$

Assumption (A) and (3) imply that

$$\max\{k(b, d), q(b, d)\} \leq k(b, c) + q(c, d) - \min\{k(b, b), k(c, c)\} \leq \frac{\lambda}{\varepsilon}[\varphi(b) - \varphi(d)]$$

and $\varphi(b) \geq \varphi(c) > \varphi(d)$. Thus, $\varphi(d) + \frac{\varepsilon}{\lambda}q(b, d) \leq \varphi(b)$ for some $d \neq b$. Hence, $b \in \mathcal{G}^q$ and so $b \in \mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q$. Because $\varphi(w) > \varphi(b)$, $\inf_{\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q} \varphi \leq \varphi(b) < \varphi(w)$

and so $\inf_{\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q} \varphi < \frac{1}{2} \left[\inf_{\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$. Next, take $y \in \mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$ to ensure the claim.

To prove the existence of an Ekeland point of type (\tilde{k}, q) , suppose to the contrary that $\mathcal{E}_z^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$ for any $z \in \mathcal{E}_x^{\tilde{k}}$. We construct sequences (z_n) and (y_n) as follows. In step 1, set $z_0 := x$ and pick $y_1 \in \mathcal{E}_{z_0}^{\tilde{k}} \cap \mathcal{G}^q$ such that $\varphi(y_1) \leq \frac{1}{2} \left[\varphi(z_0) + \inf_{\mathcal{E}_{z_0}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \right]$ according to the above claim. Then,

$$\begin{cases} \varphi(y_1) \leq \varphi(z_0), \\ k(z_0, y_1) - \min\{k(z_0, z_0), k(y_1, y_1)\} \leq \frac{\lambda}{\varepsilon} [\varphi(z_0) - \varphi(y_1)], \\ \varphi(z_1) + \frac{\varepsilon}{\lambda} q(y_1, z_1) \leq \varphi(y_1) \text{ for some } z_1 \neq y_1. \end{cases}$$

Hence,

$$\begin{cases} \varphi(y_1) \leq \frac{1}{2} \left[\varphi(z_0) + \inf_{\mathcal{E}_{z_0}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \right], \\ \varphi(z_0) \geq \varphi(y_1) > \varphi(z_1), \\ \max\{k(z_0, z_1), q(z_0, z_1)\} \leq k(z_0, y_1) + q(y_1, z_1) - \min\{k(z_0, z_0), k(y_1, y_1)\} \\ \leq \frac{\lambda}{\varepsilon} [\varphi(z_0) - \varphi(z_1)]. \end{cases}$$

In the second step, viewing z_1 obtained in step 1 as w in the claim, we have $y_2 \in \mathcal{E}_{z_1}^{\tilde{k}} \cap \mathcal{G}^q$ such that $\varphi(y_2) \leq \frac{1}{2} \left[\varphi(z_1) + \inf_{\mathcal{E}_{z_1}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \right]$. Then, like obtaining $z_1 \neq y_1$ from y_1 in step 1, we get $z_2 \neq y_2$ from y_2 such that

$$\begin{cases} \varphi(y_2) \leq \frac{1}{2} \left[\varphi(z_1) + \inf_{\mathcal{E}_{z_1}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \right], \\ \varphi(z_0) \geq \varphi(y_1) > \varphi(z_1) \geq \varphi(y_2) > \varphi(z_2), \\ \max\{k(z_1, z_2), q(z_1, z_2)\} \leq \frac{\lambda}{\varepsilon} [\varphi(z_1) - \varphi(z_2)], \\ \max\{k(z_0, z_2), q(z_0, z_2)\} \leq k(z_0, z_1) + q(z_1, z_2) \leq \frac{\lambda}{\varepsilon} [\varphi(z_0) - \varphi(z_2)], \end{cases}$$

and thus $\mathcal{E}_{z_2}^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$ ($z_2 \in \mathcal{E}_x^{\tilde{k}}$).

Continuing the process, we build strictly decreasing sequences $(\varphi(z_n))$ and $(\varphi(y_n))$ such that

$$\begin{cases} \varphi(y_{n+1}) \leq \frac{1}{2} \left[\varphi(z_n) + \inf_{\mathcal{E}_{z_n}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \right], \\ \varphi(z_{n-1}) \geq \varphi(y_n) > \varphi(z_n), \\ \max\{k(z_n, z_{n+1}), q(z_n, z_{n+1})\} \leq \frac{\lambda}{\varepsilon} [\varphi(z_n) - \varphi(z_{n+1})], \\ \varphi(z_n) + \frac{\varepsilon}{\lambda} q(y_n, z_n) \leq \varphi(y_n), z_n \neq y_n. \end{cases} \quad (4)$$

It is easily seen that if $n < m$, then

$$\max\{k(z_n, z_m), q(z_n, z_m)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(z_m)]. \quad (5)$$

So, $\lim_{n \rightarrow \infty} \varphi(z_n) = \lim_{n \rightarrow \infty} \varphi(y_n)$ and $\lim_{n, m \rightarrow \infty, m \geq n} q(z_n, z_m) = \lim_{n, m \rightarrow \infty, m \geq n} k(z_n, z_m) = 0$. Then, (z_n) is 0-left-Cauchy and so left-converges to a point $z \in X$ with

$$q(z, z) = \lim_{n \rightarrow \infty} q(z_n, z) = \lim_{n, m \rightarrow \infty, m \geq n} q(z_n, z_m) = \lim_{m \rightarrow \infty} q(z_m, z_m) = 0. \quad (6)$$

Thus, $\liminf_{m \rightarrow \infty} k(z_n, z_m) \geq \max\{k(z_n, z), q(z_n, z)\}$ (since $\max\{k(z_n, z), q(z_n, z)\} \leq k(z_n, z_m) + q(z_m, z) - \min\{k(z_n, z_n), k(z_m, z_m)\}$). As φ is sdl-lsc, from (5) one has for all n ,

$$\begin{cases} \max\{k(z_n, z), q(z_n, z)\} \leq \liminf_{m \rightarrow \infty} k(z_n, z_m) \\ \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \lim_{m \rightarrow \infty} \varphi(z_m)] \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(z)], \\ \varphi(z) \leq \varphi(z_n). \end{cases} \quad (7)$$

Hence, $z \in \mathcal{E}_{z_n}^{\tilde{k}}$. As $z_0 = x$, by the contradiction assumption, $\mathcal{E}_z^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$. Pick $t \in \mathcal{E}_z^{\tilde{k}} \cap \mathcal{G}^q$ to see that

$$\begin{cases} \varphi(t) \leq \varphi(z), \\ k(z, t) - \min\{k(z, z), k(t, t)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z) - \varphi(t)], \\ \varphi(w) + \frac{\varepsilon}{\lambda}q(t, w) \leq \varphi(t) \text{ for some } w \neq t. \end{cases}$$

Consequently,

$$\begin{cases} \varphi(z) \geq \varphi(t) > \varphi(w), \\ \max\{k(z, w), q(z, w)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z) - \varphi(w)]. \end{cases} \quad (8)$$

Now applying first assumption (A) and next (7), (8), we have

$$\begin{aligned} \max\{k(z_n, w), q(z_n, w)\} &\leq k(z_n, z) + q(z, w) \\ &\leq \max\{k(z_n, z), q(z_n, z)\} + \max\{k(z, w), q(z, w)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(w)]. \end{aligned}$$

Hence, as $\varphi(z) \leq \varphi(z_n)$,

$$\begin{cases} \max\{k(z_n, w), q(z_n, w)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(w)], \\ \varphi(w) < \varphi(z) \leq \varphi(z_n) \end{cases} \quad (9)$$

for all n and so $w \in \mathcal{E}_{z_n}^{\tilde{k}}$ and $\mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q \neq \emptyset$ as well. Taking $t_1 \in \mathcal{E}_w^{\tilde{k}} \cap \mathcal{G}^q$, one has

$$\begin{cases} \varphi(t_1) \leq \varphi(w) < \varphi(z), \\ k(w, t_1) - \min\{k(w, w), k(t_1, t_1)\} \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(t_1)], \\ \varphi(t_2) + \frac{\varepsilon}{\lambda}q(t_1, t_2) \leq \varphi(t_1) \text{ for some } t_2 \neq t_1. \end{cases}$$

Therefore,

$$\begin{cases} \varphi(t_2) < \varphi(t_1) \leq \varphi(w) < \varphi(z), \\ \max\{k(w, t_2), q(w, t_2)\} \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(t_2)], \\ q(w, t_2) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(t_2)] \text{ for some } t_2 \neq w. \end{cases}$$

Hence, $w \in \mathcal{G}^q$ and so $w \in \mathcal{E}_{z_n}^{\tilde{k}} \cap \mathcal{G}^q$ for all n . Therefore, the first inequality in (4) gives $2\varphi(y_{n+1}) - \varphi(z_n) \leq \inf_{\mathcal{E}_{z_n}^{\tilde{k}} \cap \mathcal{G}^q} \varphi \leq \varphi(w)$. Letting $n \rightarrow \infty$ yields $\varphi(w) \geq \varphi(z)$. This contradiction with (9) completes the proof. \square

Remark 3.2 Notice that when $\inf_{x \in X} k(x, x) > 0$, (X, q) is 0-left-complete because $0 < \inf_{x \in X} k(x, x) \leq \inf_{x \in X} q(x, x)$ (see Remark 3.1). Moreover, in this case the above proof is more simple and shorter because (6) could not happen.

Example 3.4 (existence of Ekeland points of type (\tilde{k}, q)) Let $X = ([-1, 1] \cap \mathbb{Q}, q)$, $\varepsilon = \lambda > 0$, and for the usual metric d ,

$$q(x, y) = d(x, y) + d(0, y), \quad k(x, y) = d(x, y), \quad \varphi(x) = \begin{cases} x & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then, all the assumptions in Theorem 3.1 are verified. Direct computations yield that $\mathcal{E}_x^{\tilde{k}} = [-1, x] \cap \mathbb{Q}$ for $x \geq 0$ and $\mathcal{E}_x^{\tilde{k}} = \{x\}$ for $x < 0$, and $\mathcal{G}^q = [0, 1] \cap \mathbb{Q}$. Therefore, for each $x \in [-1, 1] \cap \mathbb{Q}$, one can always find an Ekeland point of type (\tilde{k}, q) .

Observe that in Example 3.4, both $([-1, 1] \cap \mathbb{Q}, q)$ and $([-1, 1] \cap \mathbb{Q}, d)$ are not left-complete, and φ is not decreasingly lsc at 0 (and so not lsc at 0).

Theorem 3.2 (general version 1 of the EVP) *Let X be 0-left-complete, q and k satisfy assumption (A), $\varphi : X \rightarrow \overline{\mathbb{R}}$ be proper, sdl-lsc, bounded from below, and $\varepsilon > 0$. Consider a point $x \in X$ satisfying $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$, there exists $z \in X$ such that*

- (i) $k(x, z) \leq \lambda + \min\{k(z, z), k(x, x)\}$;
- (ii) $\varphi(z) \leq \varphi(x) - \frac{\varepsilon}{\lambda}[k(x, z) - \min\{k(z, z), k(x, x)\}]$;
- (iii) $\varphi(u) + \frac{\varepsilon}{\lambda}q(z, u) > \varphi(z)$ for all $u \in X \setminus \{z\}$.

Proof By Theorem 3.1, there exists $z \in \mathcal{E}_x^{\tilde{k}}$ such that $\mathcal{E}_z^{\tilde{k}} \cap \mathcal{G}^q = \emptyset$. Then, the conclusion (ii) is checked because $z \in \mathcal{E}_x^{\tilde{k}}$. As $z \in \mathcal{E}_z^{\tilde{k}}$, $z \notin \mathcal{G}^q$, i.e., (iii) is fulfilled. (i) is satisfied as $\tilde{k}(x, z) \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \inf_X \varphi] \leq \lambda$. \square

Remark 3.3 (1) If $k(x, x) = 0$ for all $x \in X$, then the minimum expressions in the conclusions (i) and (ii) disappear and these conclusions become stronger. If $k = q$ is a QM, then Theorem 3.2 collapses to the classical EVP for the quasi-metric case; see e.g., [3].

(2) We need to note some differences between the proof technique for Theorems 3.1 and 3.2 and the usual proof of the variants of the EVP (see the origin of the latter in the proof of Theorem 1 of Ekeland [19]) for the case (X, d) is a metric space and $k = q = d$. Namely the usual proof builds a sequence (z_n) depending on a set called S_n as follows: starting with $z_0 := x$, assume that $(z_n) \subset X$. Now either

(a) $\forall z \neq z_n, \varphi(z) > \varphi(z_n) - \varepsilon d(z_n, z)$, then set $z_{n+1} = z_n$; or

(b) $\exists z \neq z_n : \varphi(z) \leq \varphi(z_n) - \varepsilon d(z_n, z)$, then take S_n as the set of such z .

This usual technique builds a sequence $z_{n+1} \in S_n$ by the rule $\varphi(z_{n+1}) \leq \frac{1}{2} [\inf_{S_n} \varphi + \varphi(z_n)]$.

Meanwhile, in the proof of Theorem 3.1, we build simultaneously both (z_n) and (y_n) , which are related to each other and depend on $\mathcal{E}_{z_n}^{\tilde{k}} \cap \mathcal{G}^q$, that is two sequences (y_n) and (z_n) with $y_{n+1} \in \mathcal{E}_{z_n}^{\tilde{k}}$ by the relation $\varphi(y_{n+1}) \leq \frac{1}{2} [\inf_{\mathcal{E}_{z_n}^{\tilde{k}} \cap \mathcal{G}^q} \varphi + \varphi(z_n)]$. Our sequences $(\varphi(z_n))$ and $(\varphi(y_n))$ are strictly decreasing (that is why the sdl-lsc assumption is enough), while $(\varphi(z_n))$ in the above common proof is only decreasing. Moreover, with the new approach, we can take the advantage of having two perturbations to develop several versions of the EVP with different characteristics that may be suitable for different application situations (we will see this clearly in the subsequent theorems).

(3) *Relation of the version 1 of the EVP to a generalized contraction mapping principle* The role of Banach's contraction principle in various fields of mathematics is enormous; in particular it is applied in denotational models for programming languages (e.g., [16]) and in the PMS/PQMS case, aiming mainly to study theoretical computer science (see, e.g., [10, 34, 35, 38]). More recently, based on weakening the completeness and the contraction condition, some variants of the contraction principle, which remain to be appropriate for program verification, were provided in [21, 36, 43]. The EVP is equivalent or closely related to fixed point theorems of mappings satisfying inwardness conditions, in the form of a generalized contraction mapping principle (see, e.g., [9, 27]). Our version 1 of the EVP implies a simple result of fixed points of that type as follows. Let (X, q) be a 0-left-complete PQMS and $f : X \rightarrow X$ satisfies a generalized contraction $q(f(x), f(y)) \leq \alpha q(x, y)$ for all $x, y \in X$, where $0 \leq \alpha < 1$. Assume that $\varphi(\cdot) := q(\cdot, f(\cdot))$ is sdl-lsc. Then, with some $\gamma \in (0, 1 - \alpha)$, Theorem 3.1 yields a $z \in X$ such that $\varphi(z) \leq \varphi(x) + \gamma q(z, x)$ for all $x \in X$. Substituting $x = f(z)$ in this inequality, we get

$$q(z, f(z)) \leq q(f(z), f(f(z))) + \gamma q(z, f(z)) \leq (\gamma + \alpha)q(z, f(z)).$$

Since $\gamma + \alpha < 1$, we have $q(z, f(z)) = 0$. By (p₂) in the definition of q , $q(z, z) = q(f(z), f(z)) = 0$, and so by (p₁), $f(z) = z$, i.e., z is a fixed point of f . Also, note that with the given contraction condition, if (X, q) is a metric space, then function $\varphi(\cdot)$ is Lipschitz-continuous (since $|q(x, f(x)) - q(y, f(y))| \leq q(x, y) + q(f(x), f(y)) \leq (\alpha + 1)q(x, y)$).

We now propose and use other types of Ekeland points to get versions better than Theorem 3.2 in certain aspects. Replace (A) by the following weaker assumption

$$(B) \quad \max\{q(x, z), k(x, z)\} \leq k(x, y) + q(y, z).$$

Example 3.5 (with incomparable k, q , (B) is satisfied, (A) not) Let $X = \mathbb{R}$, $k(x, y) = |y-x|+|x|$, and $q(x, y) = |y-x|+|y|$. Then, both k and q are PQM, but incomparable, and assumption (B) is fulfilled, but (A) is not (to see this, take $x = 3$, $y = 4$, and $z = 0$).

Example 3.6 (with $k \geq q$, (B) is satisfied, (A) not) Let $X = \mathbb{R}$, $k(x, y) = |y-x|+|x|$, and $q(x, y) = |x-y|$. Then, for all $x, y, z \in X$, $k(x, y) \geq q(x, y)$ (and so Theorem 3.4 is stronger in some cases) and (B) is fulfilled, but (A) not (for $(x, y, z) = (1, 2, 3)$).

To obtain the next version of the EVP, we propose and use the following.

Definition 3.3 (*Ekeland points of type (k, q)*) For $\varphi : X \rightarrow \overline{\mathbb{R}}$ and $\varepsilon, \lambda > 0$, let

$$\mathcal{E}_x^k := \left\{ z \in X \mid \varphi(z) \leq \varphi(x) - \varepsilon\lambda^{-1}k(x, z) \right\}$$

and \mathcal{G}^q be as in Definition 3.1. Then, $z \in \mathcal{E}_x^k$ is called an *Ekeland point of type (k, q)* of φ (relative to x, ε , and λ) iff $\mathcal{E}_z^k \cap \mathcal{G}^q = \emptyset$ or $z \notin \mathcal{G}^q$.

Theorem 3.3 (existence of Ekeland points of type (k, q)) *Let X be 0-left-complete, assumption (B) satisfied, $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, bounded from below, and ε, λ positive real numbers. Then, for each $x \in X$ such that $\mathcal{E}_x^k \neq \emptyset$, there exists an Ekeland point of type (k, q) of φ (relative to x, ε , and λ).*

Proof Start from a given $x \in X$ which satisfies the hypotheses of the above theorem. Without loss of generality, assume that $\mathcal{E}_x^k \cap \mathcal{G}^q \neq \emptyset$ (when this intersection is empty, any $z \in \mathcal{E}_x^k$ is a desired Ekeland point of type (k, q) since $z \notin \mathcal{G}^q$) and $\mathcal{E}_z^k \neq \emptyset$ for all $z \in \mathcal{E}_x^k$ (if $\mathcal{E}_z^k = \emptyset$ for some $z \in \mathcal{E}_x^k$, then z is a desired Ekeland point of type (k, q)).

Claim that if $\mathcal{E}_w^k \cap \mathcal{G}^q \neq \emptyset$ for all $w \in \mathcal{E}_x^k$ including $w := x$, then there exists y in $\mathcal{E}_w^k \cap \mathcal{G}^q$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^k \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$. Applying assumption (B) instead of assumption (A) and arguments similar to those in the proof of Theorem 3.1, we verify the claim as follows. If $v \in \mathcal{E}_w^k \cap \mathcal{G}^q$, then

$$\begin{cases} k(w, v) \leq \frac{\lambda}{\varepsilon} [\varphi(w) - \varphi(v)], \\ \varphi(u) + \frac{\varepsilon}{\lambda} q(v, u) \leq \varphi(v) \text{ for some } u \neq v. \end{cases}$$

Hence, by assumption (B),

$$\begin{cases} \varphi(w) \geq \varphi(v) > \varphi(u), \\ \max\{k(w, u), q(w, u)\} \leq k(w, v) + q(v, u) \leq \frac{\lambda}{\varepsilon} [\varphi(w) - \varphi(u)]. \end{cases} \quad (10)$$

Again by assumption (B),

$$\max\{k(x, u), q(x, u)\} \leq k(x, w) + \max\{k(w, u), q(w, u)\}.$$

Using the last inequality in (10) for $\max\{k(w, u), q(w, u)\}$ and the inequality defining $w \in \mathcal{E}_x^k$ for the remaining term on the above right-hand side, we have

$$\max\{k(x, u), q(x, u)\} \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(w)] + \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)] \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(u)].$$

Consequently, $u \in \mathcal{E}_x^k$. Hence, $\mathcal{E}_u^k \cap \mathcal{G}^q \neq \emptyset$. Pick $a \in \mathcal{E}_u^k \cap \mathcal{G}^q$. Then, similar to obtaining $u \neq v$ from v , one gets $b \neq a$ from a with

$$\begin{cases} k(u, a) \leq \frac{\lambda}{\varepsilon}[\varphi(u) - \varphi(a)], \\ \varphi(b) + \frac{\varepsilon}{\lambda}q(a, b) \leq \varphi(a) \quad \text{and} \quad \varphi(b) < \varphi(w). \end{cases}$$

Similar to $\max\{k(w, u), q(w, u)\}$, one has $\max\{k(u, b), q(u, b)\} \leq \frac{\lambda}{\varepsilon}[\varphi(u) - \varphi(b)]$. This and (10) imply that

$$\max\{k(w, b), q(w, b)\} \leq k(w, u) + q(u, b) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(b)]. \quad (11)$$

(Hence, $b \in \mathcal{E}_w^k$.) Because $w \in \mathcal{E}_x^k$, the above inequality implies that

$$\max\{k(x, b), q(x, b)\} \leq [k(x, w)] + \max\{k(w, b), q(w, b)\} \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(b)].$$

So, $b \in \mathcal{E}_x^k$ and hence $b \in \mathcal{E}_x^k \cap \mathcal{E}_w^k$. By the contradiction assumption, $\mathcal{E}_b^k \cap \mathcal{G}^q \neq \emptyset$. Let $c \in \mathcal{E}_b^k \cap \mathcal{G}^q$. Then,

$$\begin{cases} \varphi(c) \leq \varphi(b) - \frac{\varepsilon}{\lambda}[k(b, c)], \\ \varphi(d) + \frac{\varepsilon}{\lambda}q(c, d) \leq \varphi(c) \quad \text{for some } d \neq c. \end{cases} \quad (12)$$

Assumption (B) and (12) imply that

$$\max\{k(b, d), q(b, d)\} \leq k(b, c) + q(c, d) \leq \frac{\lambda}{\varepsilon}[\varphi(b) - \varphi(d)]$$

and $\varphi(b) \geq \varphi(c) > \varphi(d)$. Thus, $\varphi(d) + \frac{\varepsilon}{\lambda}q(b, d) \leq \varphi(b)$ for some $d \neq b$. Hence, $b \in \mathcal{G}^q$ and so $b \in \mathcal{E}_w^k \cap \mathcal{G}^q$. Because $\varphi(w) > \varphi(b)$, $\inf_{\mathcal{E}_w^k \cap \mathcal{G}^q} \varphi \leq \varphi(b) < \varphi(w)$

and so $\inf_{\mathcal{E}_w^k \cap \mathcal{G}^q} \varphi < \frac{1}{2} \left[\inf_{\mathcal{E}_w^k \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$. Next, take $y \in \mathcal{E}_w^k \cap \mathcal{G}^q$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^k \cap \mathcal{G}^q} \varphi + \varphi(w) \right]$ to ensure the claim.

To prove the existence of an Ekeland point of type (k, q) , suppose to the contrary that $\mathcal{E}_z^k \cap \mathcal{G}^q \neq \emptyset$ for any $z \in \mathcal{E}_x^k$. Similar to the proof of Theorem 3.1, we construct sequences (z_n) and (y_n) as follows. For step 1, set $z_0 := x$ and pick $y_1 \in \mathcal{E}_{z_0}^k \cap \mathcal{G}^q$ such that $\varphi(y_1) \leq \frac{1}{2} \left[\inf_{\mathcal{E}_{z_0}^k \cap \mathcal{G}^q} \varphi + \varphi(z_0) \right]$ according to the above claim. Then,

$$\begin{cases} \varphi(y_1) \leq \varphi(z_0), \\ k(z_0, y_1) \leq \frac{\lambda}{\varepsilon} [\varphi(z_0) - \varphi(y_1)], \\ \varphi(z_1) + \frac{\varepsilon}{\lambda} q(y_1, z_1) \leq \varphi(y_1) \text{ for some } z_1 \neq y_1. \end{cases}$$

Hence,

$$\begin{cases} \varphi(y_1) \leq \frac{1}{2} \left[\varphi(z_0) + \inf_{\mathcal{E}_{z_0}^k \cap \mathcal{G}^q} \varphi \right], \\ \varphi(z_0) \geq \varphi(y_1) > \varphi(z_1), \\ \max\{k(z_0, z_1), q(z_0, z_1)\} \leq k(z_0, y_1) + q(y_1, z_1) \leq \frac{\lambda}{\varepsilon} [\varphi(z_0) - \varphi(z_1)]. \end{cases}$$

For the second step, the argument for obtaining y_2 and z_2 is similar to the counterpart for Theorem 3.1. Continuing the process, we obtain strictly decreasing sequences $(\varphi(z_n))$ and $(\varphi(y_n))$ such that

$$\begin{cases} \varphi(y_{n+1}) \leq \frac{1}{2} \left[\varphi(z_n) + \inf_{\mathcal{E}_{z_n}^k \cap \mathcal{G}^q} \varphi \right], \\ \varphi(z_{n-1}) \geq \varphi(y_n) > \varphi(z_n), \\ \max\{k(z_n, z_{n+1}), q(z_n, z_{n+1})\} \leq \frac{\lambda}{\varepsilon} [\varphi(z_n) - \varphi(z_{n+1})], \\ \varphi(z_n) + \frac{\varepsilon}{\lambda} q(y_n, z_n) \leq \varphi(y_n), z_n \neq y_n, \end{cases} \quad (13)$$

where (z_n) is a 0-left-Cauchy sequence and so left-converges to a point $z \in \mathcal{E}_{z_n}^k$ for $n \in \mathbb{N}$. Then, similar to Theorem 3.1, we obtain w, t_1, t_2 such that

$$\begin{cases} \varphi(t_2) < \varphi(t_1) \leq \varphi(w) < \varphi(z), \\ \max\{k(w, t_2), q(w, t_2)\} \leq \frac{\lambda}{\varepsilon} [\varphi(w) - \varphi(t_2)], \end{cases} \quad (14)$$

and $w \in \mathcal{E}_{z_n}^k \cap \mathcal{G}^q$ for all n . So, inequality in (13) gives $2\varphi(y_{n+1}) - \varphi(z_n) \leq \inf_{\mathcal{E}_{z_n}^k \cap \mathcal{G}^q} \varphi \leq \varphi(w)$. Letting $n \rightarrow \infty$, we get $\varphi(w) \geq \varphi(z)$ in contradiction to (14). \square

Remark 3.4 We explain some specific items in the assumptions of Theorem 3.3. Since \mathcal{E}_x^k is not always nonempty (while $\mathcal{E}_x^{\tilde{k}}$ is, clearly). The hypothesis that $\mathcal{E}_x^k \neq \emptyset$ and $\mathcal{E}_z^k \neq \emptyset$ for all $z \in \mathcal{E}_x^k$ reflects the essence of probably all the variants of the proof of the EVP. For the classical EVP, the completeness of the underlying metric space guarantees the

non-emptiness of similar but simpler sets (in particular, clearly \mathcal{E}_x^k is always nonempty when $k(x, x) = 0$). But in our setting with some relaxed assumptions (including in the structures) and aiming at more conclusions, the 0-left-completeness is not enough). Namely, we start with an initial point z_0 to find a better alternative z_1 satisfying a threshold level and from z_1 to z_2 , etc., for going further in the process to a final approximate minimizer. Setting $z_0 := x \in X$, our assumption $\mathcal{E}_x^k \neq \emptyset$ is interpreted as a condition for the existence of such a $z_1 \in \mathcal{E}_x^k$ with the better “alternative” meaning that $\varphi(z_1) \leq \varphi(z_0) - \varepsilon\lambda^{-1}k(z_0, z_1)$, where the quantity “ $\varepsilon\lambda^{-1}k(z_0, z_1)$ ” may be understood as the “satisfaction level” for the change from the position z_0 to the new one z_1 . The additional hypothesis $\mathcal{E}_z^k \neq \emptyset$ for all $z \in \mathcal{E}_x^k$ guarantees a possibility of building a successive approximation sequence (z_n) of better alternatives (see the proof of Theorem 3.3 for more details).

Example 3.7 (Ekeland points of type (k, q)) Let $X = ([0, 1] \cap \mathbb{Q}, q)$ and, for the usual distance d ,

$$k(x, y) = d(x, y) + d(0, x), \quad q(x, y) = d(x, y) + d(0, y), \quad \varphi(x) = \begin{cases} x & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Then, 0 is an Ekeland point of type (k, q) (here k and q are incomparable).

Theorem 3.4 (general version 2 of the EVP) *Let X be 0-left-complete, assumption (B) satisfied, $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, bounded from below, and $\varepsilon > 0$. Consider a point $x \in X$ such that $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$ such that $\mathcal{E}_x^k \neq \emptyset$, there exists $z \in X$ such that*

- (i) $k(x, z) \leq \lambda$;
- (ii) $\varphi(z) \leq \varphi(x) - \frac{\varepsilon}{\lambda}k(x, z)$;
- (iii) either $\mathcal{E}_z^k \cap \mathcal{G}^q = \emptyset$ or $\varphi(u) + \frac{\varepsilon}{\lambda}q(z, u) > \varphi(z)$ for all $u \in X \setminus \{z\}$.

Proof All conclusions of the theorem follow immediately from Theorem 3.3 (note that both (i) and (ii) follow because $z \in \mathcal{E}_x^k$). \square

Remark 3.5 Note that in the case $\mathcal{E}_z^k \cap \mathcal{G}^q = \emptyset$, it happens possibly $\mathcal{E}_z^k = \emptyset$ (i.e., $\varphi(u) + \frac{\varepsilon}{\lambda}k(z, u) > \varphi(z)$ for all $u \in X \setminus \{z\}$) or $\mathcal{E}_z^k \neq \emptyset$. In the latter case, one picks $v \in \mathcal{E}_z^k$ and $v \notin \mathcal{G}^q$ (cf. the conclusion (iii) of the classical EVP).

Corollary 3.1 (a parametrized version of the EVP) *Let (X, d) be a complete metric space, $a \in X$, $k_\gamma(x, y) := \max\{d(x, y), d(a, y) - \gamma d(a, x)\}$ for $\gamma \in \Gamma$ with nonempty $\Gamma \subset \mathbb{R}$, $x, y \in X$, $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, bounded from below, and $\varepsilon > 0$. Assume that a point $x \in X$ satisfies $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$ and $\gamma \in \Gamma$ such that $\mathcal{E}_x^{k_\gamma} \neq \emptyset$, there exists $z \in X$ such that*

- (i) $\max\{d(x, z), d(a, z) - \gamma d(a, x)\} \leq \lambda$;
- (ii) $\varphi(z) \leq \varphi(x) - \frac{\varepsilon}{\lambda} \max\{d(x, z), d(a, z) - \gamma d(a, x)\}$;
- (iii) either $\mathcal{E}_z^{k_\gamma} \cap \mathcal{G}^q = \emptyset$ or $\varphi(u) + \frac{\varepsilon}{\lambda}d(u, z) > \varphi(z)$ for all $u \in X \setminus \{z\}$.

Proof Assumption (B) is satisfied for k_γ and $q := d$ because, for any fixed $\gamma \in \Gamma$,

$$\begin{aligned} k_\gamma(x, y) + q(y, z) &= \max\{d(x, y), d(a, y) - \gamma d(a, x)\} + d(y, z) \\ &\geq \begin{cases} d(x, y) + d(y, z) \geq d(x, z), \\ d(a, y) - \gamma d(a, x) + d(y, z) \geq d(a, z) - \gamma d(a, x). \end{cases} \end{aligned}$$

Hence, applying Theorem 3.4, all conclusions of this corollary are verified. \square

Note that when $\gamma = 1$, this corollary collapses to the classical EVP.

Example 3.8 (an illustration of Corollary 3.1) Let $X = ([0, 1], q)$, d be the usual distance, $\Gamma \subset \mathbb{R}_+$, and

$$k_\gamma(x, y) = \max\{d(x, y), d(0, y) - \gamma d(0, x)\}, \quad q(x, y) = d(x, y), \quad \varphi(x) = 2x.$$

Then, (B) is satisfied and φ is lsc. Consider $\varepsilon_n = \frac{1}{n^2}$ and $\lambda_n = \frac{1}{n}$ (depending on n). Then, for all $x \geq 0$,

$$\mathcal{E}_x^{k_\gamma} = \{z \in [0, 1] \mid \varphi(z) + n^{-1} \max\{d(x, z), d(0, z) - \gamma d(0, x)\} \leq \varphi(x) = 2x\} \neq \emptyset.$$

Moreover, for $x = \frac{1}{2n^2}$, $z = 0$ fulfills the conclusions of Corollary 3.1 (and so one can let $n \rightarrow \infty$ to get a limit statement if it is wanted).

Example 3.9 (Theorem 3.2 is applicable but Theorem 3.4 not) Let $X = \{[a, b] \mid a, b \in [0, 2], a \leq b\}$, $q([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ for $[a, b], [c, d] \in X$, and $k = q$. Then, q is a partial metric (this q is considered in computer science, see, e.g., [34, 38]). Furthermore, $\varphi : X \rightarrow \mathbb{R}$ defined by $\varphi([a, b]) = -b^3$ is lsc and $\inf_X \varphi = -8$. For $\varepsilon = 5$, $\varphi([1.7, 2]) \leq \inf_X \varphi + \varepsilon$. The assumptions of Theorem 3.2 are satisfied. For $\lambda = 1/2$, $z = [1.7, 2]$ satisfies the conclusions of Theorem 3.2. However, \mathcal{E}_x^k in Theorem 3.4 is empty, causing that the conclusions (i) and (ii) of Theorem 3.4 are not true.

Example 3.10 (Theorem 3.4 is applicable) Let $X = \mathbb{Q} \cap [0, 2]$, d be the usual metric, $q(x, y) = d(x, y) + d(0, y)$, and $\varphi(x) = -x^3$. Then, (X, q) is a 0-left-complete PQMS (but not complete). If $k := q$, $\varepsilon = 8$, and $\lambda = 4$, then $\mathcal{E}_2^k = \emptyset$ and $\mathcal{E}_x^k = \{2\}$ for $x \neq 2$. Moreover, the conclusions of Theorem 3.4 are satisfied. If $k := d$, then for $\varepsilon = 7$, $x = 1$, and $\lambda = 1$, $\mathcal{E}_x^k \neq \emptyset$ ($2 \in \mathcal{E}_x^k$). $z = 2$ fulfills the conclusions of Theorem 3.4.

Remark 3.6 Notice that \mathcal{E}_x^k is always nonempty if $k(x, x) = 0$ for all $x \in X$. So, this assumption in Theorem 3.4 is not restrictive.

In the rest of this section, to improve conclusions (ii) and (iii) of Theorems 3.2 and 3.4, we replace (A) and (B) by the following assumption (C) incomparable with (A) and (B), including an explicit comparison of the two perturbations $k(x, y) \leq q(x, y)$ for all $x, y \in X$,

(C) $k(x, y) \geq 0$, $k(x, y) \leq q(x, y) \quad \forall x, y \in X$, $k(x, y) \neq 0$ if $x \neq y$, and $q(x, z) \leq q(x, y) + k(y, z)$.

Example 3.11 ((C) is satisfied, (A) and (B) not) Let $X = \mathbb{R}$, $q(x, y) = |y - x| + |x|$, and $k(x, y) = |y - x|$. Then, (C) is satisfied. But, neither (A) nor (B) is fulfilled (to see this, consider $|y| < |x|$ and $y = z$).

Definition 3.4 (*Ekeland points of a mixed type*) Let $\varphi : X \rightarrow \overline{\mathbb{R}}$, $\varepsilon, \lambda \geq 0$, and

$$\begin{aligned} \mathcal{E}_x^q &:= \left\{ z \in X \mid \varphi(z) \leq \varphi(x) - \varepsilon\lambda^{-1}q(x, z) \right\}, \\ \mathcal{G}^{k,q} &:= \left\{ z \in X \mid \exists u \neq z, \varphi(u) + \varepsilon\lambda^{-1}k(z, u) \leq \varphi(z) \right\} \cup \left\{ z \in X \mid \exists u \neq z, \right. \\ &\quad \left. \varphi(u) + \varepsilon\lambda^{-1}[q(z, u) - q(z, z)] \leq \varphi(z) \right\}. \end{aligned}$$

Then, $z \in \mathcal{E}_x^q$ is called an *Ekeland point of a mixed type* of φ iff $\mathcal{E}_z^q \cap \mathcal{G}^{k,q} = \emptyset$.

Theorem 3.5 (existence of Ekeland points of a mixed type) *Let X be 0-left-complete, assumption (C) satisfied, $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, and bounded from below, and ε, λ positive numbers. Then, for each $x \in X$ such that $\mathcal{E}_x^q \neq \emptyset$, there exists an Ekeland point of a mixed type of φ (relative to x , ε , and λ).*

Proof Observe that if $\mathcal{E}_z^q = \emptyset$ for some $z \in \mathcal{E}_x^q$, then z is a desired Ekeland point of a mixed type. Hence, we assume that $\mathcal{E}_z^q \neq \emptyset$ for all $z \in \mathcal{E}_x^q$. Also without loss of generality, we can assume that $\mathcal{E}_x^q \cap \mathcal{G}^{k,q} \neq \emptyset$. Otherwise any $z \in \mathcal{E}_x^q$ is a desired Ekeland point of a mixed type since $\mathcal{E}_x^q \neq \emptyset$ and $\mathcal{E}_z^q \subset \mathcal{E}_x^q$. Indeed, since $z \in \mathcal{E}_x^q$, for any $u \in \mathcal{E}_z^q$, we have

$$\varphi(u) + \varepsilon\lambda^{-1}q(z, u) \leq \varphi(z) \leq \varphi(x) - \varepsilon\lambda^{-1}q(x, z).$$

This implies that $\varphi(u) \leq \varphi(x) - \varepsilon\lambda^{-1}q(x, z) - \varepsilon\lambda^{-1}q(z, u) \leq \varphi(x) - \varepsilon\lambda^{-1}q(x, u)$, and thus $u \in \mathcal{E}_x^q$, i.e., $\mathcal{E}_z^q \subset \mathcal{E}_x^q$.

Now let $x \in X$ satisfy the hypotheses of the theorem. We claim first that if $\mathcal{E}_w^q \cap \mathcal{G}^{k,q} \neq \emptyset$ for all $w \in \mathcal{E}_x^q$ (even $w := x$), then there exists y in $\mathcal{E}_w^q \cap \mathcal{G}^{k,q}$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^q \cap \mathcal{G}^{k,q}} \varphi + \varphi(w) \right]$. The proof of this claim is similar to the corresponding proof for Theorem 3.1, applying assumption (C) and the triangle inequality instead of assumption (A). The main difference is that in the process we usually have two options (not one as for Theorem 3.1). We write in detail only some parts relatively different from the counterparts for Theorem 3.1. If $v \in \mathcal{E}_w^q \cap \mathcal{G}^{k,q}$, then by assumption (C) and the triangle inequality, one has

$$\begin{cases} \varphi(w) \geq \varphi(v) > \varphi(u), \\ \max\{q(w, u), k(w, u)\} \leq q(w, v) + k(v, u) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)] \\ \text{or } \max\{q(w, u), k(w, u)\} \leq q(w, v) + q(v, u) - q(v, v) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)]. \end{cases}$$

Hence,

$$\begin{cases} \varphi(w) \geq \varphi(v) > \varphi(u), \\ \max\{q(w, u), k(w, u)\} \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)]. \end{cases} \quad (15)$$

Again in view of (C), $\max\{q(x, u), k(x, u)\} \leq q(x, w) + \max\{q(w, u), k(w, u)\}$. Using the inequality defining $w \in \mathcal{E}_x^q$ for the right-most side of (15), we have

$$\max\{q(x, u), k(x, u)\} \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(w)] + \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(u)] \leq \frac{\lambda}{\varepsilon}[\varphi(x) - \varphi(u)].$$

Consequently, $u \in \mathcal{E}_x^q$. So, $\mathcal{E}_u^q \cap \mathcal{G}^{k,q} \neq \emptyset$. Then, we obtain b such that $b \in \mathcal{E}_w^q \cap \mathcal{G}^{k,q}$, $\varphi(w) > \varphi(b)$, and $\inf_{\mathcal{E}_w^q \cap \mathcal{G}^{k,q}} \varphi \leq \varphi(b) < \varphi(w)$ and so $\inf_{\mathcal{E}_w^q \cap \mathcal{G}^{k,q}} \varphi < \frac{1}{2} \left[\inf_{\mathcal{E}_w^q \cap \mathcal{G}^{k,q}} \varphi + \varphi(w) \right]$. To ensure the claim, take $y \in \mathcal{E}_w^q \cap \mathcal{G}^{k,q}$ such that $\varphi(y) < \frac{1}{2} \left[\inf_{\mathcal{E}_w^q \cap \mathcal{G}^{k,q}} \varphi + \varphi(w) \right]$.

Now we prove the existence of a required Ekeland point. Arguing by contradiction, suppose to the contrary that $\mathcal{E}_z^q \cap \mathcal{G}^{k,q} \neq \emptyset$ for any $z \in \mathcal{E}_x^q$. To construct sequences (z_n) and (y_n) , for step 1, set $z_0 := x$ and pick $y_1 \in \mathcal{E}_{z_0}^q \cap \mathcal{G}^{k,q}$ such that $\varphi(y_1) \leq \frac{1}{2} \left[\varphi(z_0) + \inf_{\mathcal{E}_{z_0}^q \cap \mathcal{G}^{k,q}} \varphi \right]$ according to the claim. Then,

$$\begin{cases} \varphi(y_1) \leq \varphi(z_0), & q(z_0, y_1) \leq \frac{\lambda}{\varepsilon}[\varphi(z_0) - \varphi(y_1)], \\ \varphi(z_1) + \frac{\varepsilon}{\lambda}k(y_1, z_1) \leq \varphi(y_1) & \text{for some } z_1 \neq y_1 \text{ or} \\ \varphi(z_1) + \frac{\varepsilon}{\lambda}[q(y_1, z_1) - q(y_1, y_1)] \leq \varphi(y_1) & \text{for some } z_1 \neq y_1. \end{cases}$$

Hence,

$$\begin{cases} \varphi(y_1) \leq \frac{1}{2} \left[\varphi(z_0) + \inf_{\mathcal{E}_{z_0}^q \cap \mathcal{G}^{k,q}} \varphi \right], \\ \varphi(z_0) \geq \varphi(y_1) > \varphi(z_1), \\ \max\{q(z_0, z_1), k(z_0, z_1)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_0) - \varphi(z_1)]. \end{cases}$$

A similar second step yields y_2 and z_2 . Continuing the process, we receive strictly decreasing sequences $(\varphi(z_n))$ and $(\varphi(y_n))$ such that

$$\begin{cases} \varphi(y_{n+1}) \leq \frac{1}{2} \left[\varphi(z_n) + \inf_{\mathcal{E}_{z_n}^q \cap \mathcal{G}^{k,q}} \varphi \right], \\ \varphi(z_{n-1}) \geq \varphi(y_n) > \varphi(z_n), \\ \max\{q(z_n, z_{n+1}), k(z_n, z_{n+1})\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(z_{n+1})]. \end{cases} \quad (16)$$

If $n < m$, then

$$\max\{q(z_n, z_m), k(z_n, z_m)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(z_m)], \quad (17)$$

and so (z_n) is a 0-left-Cauchy sequence and left-converges to a point $z \in X$. Moreover, $q(z_n, z) \geq k(z_n, z)$ (see (C)) and thus $\lim_{n \rightarrow \infty} k(z_n, z) = 0$. Hence, $\liminf_{m \rightarrow \infty} q(z_n, z_m) \geq \max\{k(z_n, z), q(z_n, z)\}$. Since φ is sdl-lsc, from (17) we have

$$\begin{cases} \max\{k(z_n, z), q(z_n, z)\} \leq \liminf_{m \rightarrow \infty} q(z_n, z_m) \\ \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \lim_{m \rightarrow \infty} \varphi(z_m)] \leq \frac{\lambda}{\varepsilon}[\varphi(z_n) - \varphi(z)], \\ \varphi(z) \leq \varphi(z_n). \end{cases} \quad (18)$$

Hence, $z \in \mathcal{E}_{z_n}^q$ for all n . As $z_0 = x$, by the contradiction assumption, $\mathcal{E}_z^k \cap \mathcal{G}^{k,q} \neq \emptyset$. Then, pick $t \in \mathcal{E}_z^q \cap \mathcal{G}^{k,q}$ to see that

$$\begin{cases} q(z, t) \leq \frac{\lambda}{\varepsilon}[\varphi(z) - \varphi(t)], \\ \varphi(w) + \frac{\varepsilon}{\lambda}k(t, w) \leq \varphi(t) \text{ for some } w \neq t \text{ or} \\ \varphi(w) + \frac{\varepsilon}{\lambda}[q(t, w) - q(t, t)] \leq \varphi(t) \text{ for some } w \neq t. \end{cases}$$

Consequently,

$$\begin{cases} \varphi(z) \geq \varphi(t) > \varphi(w), \\ \max\{k(z, w), q(z, w)\} \leq \frac{\lambda}{\varepsilon}[\varphi(z) - \varphi(w)]. \end{cases} \quad (19)$$

Now applying first assumption (C) and next (18), (19), we have

$$\begin{aligned} \max\{k(z_n, w), q(z_n, w)\} &\leq q(z_n, z) + k(z, w) \\ &\leq \max\{k(z_n, z), q(z_n, z)\} + \max\{k(z, w), q(z, w)\} \leq \lambda\varepsilon^{-1}[\varphi(z_n) - \varphi(w)]. \end{aligned}$$

Hence, as $\varphi(z) \leq \varphi(z_n)$,

$$\begin{aligned} \max\{k(z_n, w), q(z_n, w)\} &\leq \lambda\varepsilon^{-1}[\varphi(z_n) - \varphi(w)], \\ \varphi(w) &< \varphi(z) \leq \varphi(z_n) \end{aligned} \quad (20)$$

for all n and so $w \in \mathcal{E}_{z_n}^q$ and $\mathcal{E}_w^q \cap \mathcal{G}^{k,q} \neq \emptyset$ as well. Taking $t_1 \in \mathcal{E}_w^q \cap \mathcal{G}^{k,q}$, one has

$$\begin{cases} \varphi(t_1) \leq \varphi(w) < \varphi(z), \\ q(w, t_1) \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(t_1)], \end{cases}$$

and $\varphi(t_2) + \frac{\varepsilon}{\lambda}k(t_1, t_2) \leq \varphi(t_1)$ for some $t_2 \neq t_1$ or $\varphi(t_2) + \frac{\varepsilon}{\lambda}[q(t_1, t_2) - q(t_1, t_1)] \leq \varphi(t_1)$ for some $t_2 \neq t_1$. Therefore,

$$\varphi(t_2) < \varphi(t_1) \leq \varphi(w) < \varphi(z), \quad \max\{k(w, t_2), q(w, t_2)\} \leq \frac{\lambda}{\varepsilon}[\varphi(w) - \varphi(t_2)].$$

Hence, $w \in \mathcal{G}^{k,q}$ and so $w \in \mathcal{E}_{z_n}^q \cap \mathcal{G}^{k,q}$ for all n . Thus, the first inequality in (16) gives

$$2\varphi(y_{n+1}) - \varphi(z_n) \leq \inf_{\mathcal{E}_{z_n}^k \cap \mathcal{G}^{k,q}} \varphi \leq \varphi(w).$$

Letting $n \rightarrow \infty$ gives $\varphi(w) \geq \varphi(z)$, which contradicts (20). \square

Example 3.12 (existence of Ekeland points of a mixed type) Let $X = ([0, 1] \cap \mathbb{Q}, q)$, d be the usual distance, and

$$q(x, y) = d(x, y) + d(0, x), \quad k(x, y) = d(x, y), \quad \varphi(x) = \begin{cases} x & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Then, 0 is an Ekeland point of a mixed type.

Similar to Theorem 3.4, but using Theorem 3.5 instead of Theorem 3.3, we have the following.

Theorem 3.6 (general version 3 of the EVP) *Let X be 0-left-complete, q and k satisfy assumption (C), $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, bounded from below, and $\varepsilon > 0$. Assume that $x \in X$ satisfies $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$ such that $\mathcal{E}_x^q \neq \emptyset$, there exists $z \in X$ such that*

- (i) $q(x, z) \leq \lambda$;
- (ii) $\varphi(z) \leq \varphi(x) - \frac{\varepsilon}{\lambda} q(x, z)$;
- (iii) *either $\mathcal{E}_z^q = \emptyset$ (i.e., $\varphi(u) + \frac{\varepsilon}{\lambda} q(z, u) > \varphi(z)$ for all $u \in X$) or $\varphi(u) + \frac{\varepsilon}{\lambda} k(z, u) > \varphi(z)$ and $\varphi(u) + \frac{\varepsilon}{\lambda} [q(z, u) - q(z, z)] > \varphi(z)$ for all $u \in X \setminus \{z\}$.*

Proof By Theorem 3.5, there exists $v \in \mathcal{E}_x^q$ such that $\mathcal{E}_v^q \cap \mathcal{G}^{k,q} = \emptyset$. In this situation, one has two cases as follows.

- Case 1: $\mathcal{E}_v^q = \emptyset$. Then, the conclusions are verified by setting $z := v$ (both (i) and (ii) follow because $z \in \mathcal{E}_x^q$, while the first option in (iii) is clear as $\mathcal{E}_z^q = \emptyset$).
- Case 2: $\mathcal{E}_v^q \neq \emptyset$. Then, there exists $z \in \mathcal{E}_v^q \subset \mathcal{E}_x^q$ (see the proof of Theorem 3.5) and $z \notin \mathcal{G}^{k,q}$. Therefore, all conclusions of this theorem are verified. \square

Observe that conclusion (ii) of Theorem 3.6 is stronger than (ii) of Theorems 3.2 and 3.4 because $k(x, y) \leq q(x, y)$ for all $x, y \in X$. Conclusion (iii) also improves the corresponding statement in the latter theorems as its second option contains two inequalities including k and q , respectively. Therefore, the following trivial implication, with an assumption ensuring this option, has conclusions stronger than those of Theorem 3.6, and may be convenient in applications.

Corollary 3.2 (a strong version of the EVP) *Let X be 0-left-complete, assumption (C) fulfilled, $\varphi : X \rightarrow \overline{\mathbb{R}}$ proper, sdl-lsc, bounded from below, and $\varepsilon > 0$. Assume that $x \in X$ satisfies $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, for any $\lambda > 0$ such that $\mathcal{E}_x^q \neq \emptyset$ and $\mathcal{E}_y^q \neq \emptyset$ for all $y \in \mathcal{E}_x^q$, there exists $z \in X$ such that*

- (i) $q(x, z) \leq \lambda$;
- (ii) $\varphi(z) \leq \varphi(x) - \frac{\varepsilon}{\lambda} q(x, z)$;
- (iii) $\varphi(u) + \frac{\varepsilon}{\lambda} k(z, u) > \varphi(z)$ and $\varphi(u) + \frac{\varepsilon}{\lambda} [q(z, u) - q(z, z)] > \varphi(z)$ for all $u \in X \setminus \{z\}$.

Example 3.13 (an illustration of Corollary 3.2) Let X, q, k , and φ be as in Example 3.12. For $x = \frac{1}{2}$ and $\varepsilon = \lambda > 0$ with $-1 + \varepsilon \geq \frac{1}{2}$, all the assumptions of Corollary 3.2 are verified. We see directly that $z = 0$ fulfills the conclusions of Corollary 3.2.

Note that in comparison with Theorem 3.4, in this example, Corollary 3.2 provides stronger conclusions (i)–(iii) because $k \leq q$ and $k \neq q$.

4 Application: The Psychological Traveler Problem

4.1 A New Look at Variational Principles

In this section, we apply the obtained results in the preceding sections to behavioral sciences, based on the variational rationality (VR) approach of human dynamics. To see the main ideas of this approach, we refer the reader to the good review in [20, Sect. 1.2 and Sect. 3] and references therein. In mathematics, its main contribution is to show how the EVP and its theory and algorithm developments can be seen as “stop and go” dynamics in a locomotion space of positions. This finding helps a lot to meet the celebrated Lewin’s vision of “topological psychology” [30–32].

Definition of a worthwhile move Let us start with the simplest formulation of the VR approach which is based on the unique concept of worthwhile move. It drives, at the cost of several simplifications, a long list of much more complex formulations of the VR approach.

The first step is to define for an individual/traveler a *utility function* $g : X \rightarrow \mathbb{R}$ where $g(x)$ is the utility of doing the bundle of situated activities $x \in X$ and $g^* := \sup \{g(u), u \in X\} < +\infty$ is an *aspiration level*, which is the highest level of utility the traveler can hope to reach. In his/her initial position, the individual does usually not do the best he/she can do; hereafter we view the traveler as a man. Thus, at the status quo x , he suffers from dissatisfaction, i.e., some *frustration* feeling $\varphi(x) := g^* - g(x) \geq 0$. This dissatisfaction at x pushes him to move from x to y to improve his utility from $g(x)$ to $g(y) \in [g(x), g^*]$ in order to reduce his frustration.

The second step defines a *cost* $C(x, y) \in \mathbb{R}_+$ of the move from x to y . The cost is not symmetric because, in most cases, $C(y, x) \neq C(x, y)$ and C represents a generalized distance.

The third step considers the *payoff of moving* from x to y , i.e., $P(x, y) := g(y) - \xi C(x, y)$ where $\xi > 0$ models the importance given to costs of moving. The *payoff to stay* at x is $P(x, x) = g(x) - \xi C(x, x)$. From $g(y) = g^* - \varphi(y)$, we get $P(x, y) = g^* - Q(x, y)$, where $Q(x, y) := \varphi(y) + \xi C(x, y)$ represents the *dissatisfaction payoff* to move, which is the sum of the residual dissatisfaction $\varphi(y)$ at the end y (lower, if possible, than at the status quo x) plus the cost of moving from x to y .

The individual must compare the two payoffs $P(x, y)$ and $P(x, x)$. This is why he considers the *worthwhile balance* $B(x, y) := P(x, y) - P(x, x) = Q(x, x) - Q(x, y)$.

Let us define the following two quantities.

- $A(x, y) := g(y) - g(x) = \varphi(x) - \varphi(y)$ represents the *advantage to move*, i.e., a variation of the utility or a variation of the dissatisfaction feeling.
- $I(x, y) := C(x, y) - C(x, x)$ refers to the *inconvenience to move*, that is, a variation of the cost to move. $I(x, y)$ may be of different types of generalized distances, e.g., a quasi-distance (see, e.g., [3]), a w-distance [3], a cone pseudo-quasi-distance [6], a pseudo-quasi-distance [2], and a set-valued quasi-distance [41].

Then, $B(x, y) := [g(y) - g(x)] - \xi [C(x, y) - C(x, x)] = A(x, y) - \xi I(x, y)$ is the balance.

If the balance $B(x, y)$ is nonnegative, i.e., the individual improves his payoff by the move from x to y , denoted by $x \curvearrowright y$, this move worthwhile because the advantage

to move $A(x, y)$ is high enough with respect to the inconvenience to move $I(x, y)$: $A(x, y) \geq \xi I(x, y)$.

Hence, the set of the worthwhile moves starting from x is $W(x) := \{y \in X \mid B(x, y) \geq 0\}$.

Stop and go dynamics The following situations may occur in a pair of periods.

(a) A case within the current period. When $B(x, y) < 0$ for all $y \neq x$, it is not worthwhile to move from x . In this case the VR approach defines x as a *stationary trap* “not worthwhile to leave”.

(b) Another case within the current period. When $B(x, z) \geq 0$ for some z , the move $x \curvearrowright z$ is worthwhile.

(c) A case within the current and the next periods. When $B(x, z) \geq 0$ for some z and $B(z, u) < 0$ for all $u \neq z$, there is a worthwhile move $x \curvearrowright z$ in the current period and it is reasonable to stay at z in the next period. In this situation, the VR approach defines z as a *variational trap* “worthwhile to reach, not worthwhile to leave”. In this case a variational trap describes a two-period “go and stop” worthwhile dynamics: $z \in W(x)$ and $W(z) = \{z\}$.

The Ekeland variational principle as a specific stop and go dynamics

Assume that the frustration feeling $\varphi(x) \equiv g^* - g(x) \geq 0$. In the context of the VR approach, the classical EVP can be read as follows. Let (X, d) be a complete metric space. Let $\varphi : X \rightarrow \mathbb{R}$ be a “to be decreased” payoff function, which is proper, bounded from below, and lsc. Let $\varepsilon > 0$ and $\varphi_* := \inf\{\varphi(y) \mid y \in X\}$. For x with $\varphi(x) \leq \varphi_* + \varepsilon$, $\lambda > 0$, and $\xi = \varepsilon/\lambda > 0$, there exists z such that

- (i) $d(x, z) \leq \lambda$;
- (ii) $B(x, z) = \varphi(x) - \varphi(z) - \xi d(x, z) = g(z) - g(x) - \xi d(x, z) \geq 0$;
- (iii) $B(z, u) = \varphi(z) - \varphi(u) - \xi d(z, u) = g(u) - g(z) - \xi d(z, u) < 0$ for all $u \neq z$.

The set of stationary traps is $\mathcal{T} := \{z \in X \mid W(z) = \{z\}\}$. Then, conditions (i), (ii), and (iii) read:

- (i) z must be not too far away from x , i.e., $d(x, z) \leq \lambda$;
- (ii) first period: $z \in W(x)$, i.e., the move $x \curvearrowright z$ is worthwhile, the inconvenience to move is $d(x, z)$;
- (iii) second period: $W(z) = \{z\}$, i.e., it is not worthwhile to move from z to any $y \neq z$, where the inconvenience to move is $d(z, u)$. This means that, starting from x , there exists a variational trap $z \in \mathcal{T}$, which is worthwhile to reach in the first period but not worthwhile to leave in the second period.

However, in this classical setting, inconveniences to move are symmetric, because d is a distance. This is not realistic in behavioral sciences, since as seen before, inconveniences to move must be some generalized distances.

4.2 Several Translocation Processes (Ways to Move) for the Traveler

This paper adds five points to the most simple model of stop and go worthwhile dynamics.

Point 1 Perturbations are partial quasi-metrics An inconvenience to move is expressed by $q(x, y) = I(x, y) = h(x, y) + F$. It includes not only a variable cost of

moving $h(x, y)$ (a quasi-distance) but also a fixed cost of moving $F > 0$ (including a cost to stop and a cost to start traveling). The sum of these two kinds of costs is a PQM $q(x, y)$.

Point 2 We consider a family of perturbations These perturbations express that inconveniences to move are changing, when moving from the first period to the second one. Let us consider only one case as an example, say the case with assumption (A). Let $\tilde{k}(x, z) = k(x, z) - \min\{k(x, x), k(z, z)\}$ be the inconvenience to move in the first period and $q(z, u)$ be the one in the second period.

Then, the conclusions of Theorem 3.2 read as

(i) $\tilde{k}(x, z) \leq \lambda$;

(ii) $\varphi(z) + \xi \tilde{k}(x, z) \leq \varphi(x)$, i.e., $B^{\tilde{k}}(x, z) \geq 0$;

(iii) $\varphi(u) + \xi q(z, u) > \varphi(z)$ for all $u \neq z$, i.e., $B^q(z, u) < 0$ for all $u \neq z$.

This theorem shows that, starting from the status quo x : (ii) it is worthwhile to move $x \curvearrowright z$, and (iii) it is not worthwhile to move $z \curvearrowright u$ for any $u \neq z$ if the inconvenience to move change from $\tilde{k}(x, z)$ in the first period to $q(z, u)$ in the second period. This shows the existence of a go and stop worthwhile dynamics, that is, the existence of a variational trap even if the shape of inconveniences to move changes from the first period to the second one.

Point 3 We show what conditions (A), (B), and (C) mean Let us only explain condition (A). It models different local perturbations with respect to the previous one. Namely, Condition (A) means that perturbation k is lower than the given PMQ perturbation q , but not too much. That is, $h(x, y) = q(x, y) - r(x, y) \leq k(x, y) \leq q(x, y)$, where $l(x, y) = \min\{k(x, x), k(y, y)\}$ and $r(x, y) := q(y, y) - l(x, y) \geq 0$ for all $x, y \in X$.

Proof The significance of condition (A) comes from an answer to the following question. Given the PQM perturbation q , find another perturbation k that verifies condition (A):

(H1) $k(x, y) \geq k(x, x)$;

(H2) q is a PQM;

(H3) $\max\{k(x, z), q(x, z)\} \leq k(x, y) + q(y, z) - l(x, y)$ for all $x, y, z \in X$.

Then, the result comes from the following two observations.

Observation 1. If condition (A) is true, then we have (H1), (H2), and

(H4) $k(x, z) \leq q(x, z) \leq k(x, y) + q(y, z) - l(x, y)$ for all $x, y, z \in X$ (see Remark 3.1).

Observation 2. If k and q satisfy condition (A), then we have

(H5) $h(x, y) = q(x, y) - r(x, y) \leq k(x, y) \leq q(x, y)$ for all $x, y \in X$.

Indeed, take $z = y$ in (H4). Then, $k(x, y) \leq q(x, y) \leq k(x, y) + q(y, y) - l(x, y)$ for all $x, y, z \in X$. Hence, $k(x, y) \geq h(x, y) = q(x, y) - r(x, y)$. Furthermore, it is easy to see that $r(x, y) = q(y, y) - l(x, y) \geq 0$ for all $x, y, z \in X$. This is because we have

$$r(x, y) = \begin{cases} q(y, y) - k(x, x) & \text{if } k(x, x) \leq k(y, y), \\ q(y, y) - k(y, y) & \text{if } k(x, x) > k(y, y), \end{cases}$$

with $q(y, y) - k(x, x) = [q(y, y) - k(y, y)] + [k(y, y) - k(x, x)] \geq 0$ if $k(x, x) \leq k(y, y)$ and $q(y, y) - k(y, y) \geq 0$ (from Observation 1 and (A)). \square

Point 4 We explain what Ekeland points represent Ekeland points appear in three periods of stop and go dynamics, where if an individual can make two successive worthwhile moves $x \curvearrowright z$ and $z \curvearrowright u$, then u must be a stationary trap. Let us go in more detail for the case (A).

If, in the first period, $B^{\tilde{k}}(x, z)$ (the balance written for the case (A), with the inconvenience to move $\tilde{k}(x, z)$) is nonnegative, the individual can make the worthwhile move $x \curvearrowright z$. Then, if in the second period, $B^{\tilde{k}}(z, u) \geq 0$, he can make the second worthwhile move $z \curvearrowright u$. Next, if in the third period, $B^q(u, v) < 0$ for all $v \neq u$, this individual must prefer to stay at u .

Point 5 We define what can be better perturbations We can point out the mathematical novelties of the results obtained in Section 3 as follows. If $k = q = d$ is the metric of X , then Theorems 3.2, 3.4, and 3.6 almost collapse to the classical EVP (the additional assumptions (A), (B) and (C), and the required non-emptiness of the involved sets are automatically satisfied), except the slight relax of the lower semicontinuity of φ . Moreover, from the theoretical view point, assertion (iii) of Theorems 3.4 and 3.6 has two options. Hence, our results are stronger than the version of the classical EVP proved for the PQMS case. The paper focuses on the introduction of a second perturbation k .

Let us define what is a good perturbation function for a variational principle in the context of a PQMS (X, q) . Surprisingly, this will differ from what can be a good perturbation function for the classical EVP. We come into details. Let the (initial non-perturbed) optimization program (OP) be: $\min \varphi(y)$ s.t. $y \in X$, where $\varphi : X \rightarrow \overline{\mathbb{R}}$ is bounded from below. Let $\underline{\varphi} = \inf\{\varphi(y) \mid y \in X\}$. An initial approximate solution x_0 of OP is good enough if $\underline{\varphi} \leq \varphi(x_0) \leq \underline{\varphi} + \varepsilon$, where $\varepsilon > 0$ defines “how x_0 is good enough”. Using the perturbation function q , we call $Q(\cdot/x) = \varphi(\cdot) + \xi q(x, \cdot)$, for $\xi > 0$ and $x \in X$, a *perturbed payoff* and $\min\{Q(y/x) \mid y \in X\}$ a *perturbed optimization program (POP)*. Then, for every $x \in X$, the perturbed payoff $Q(y/x)$ is higher than the non-perturbed payoff $\varphi(y)$ for all $y \neq x$. It is natural to define x^* as a weak optimal solution of POP if $Q(y/x^*) > Q(x^*/x^*) - \delta(x^*)$ for all $y \neq x^*$, where $\delta(x^*) := \xi q(x^*, x^*)$. Also naturally for the EVP, call x^* a strict solution of the perturbed program if $Q(y/x^*) > Q(x^*/x^*)$ for all $y \neq x^*$. In this case, $q(x, x) = 0$ for all $x \in X$. Then, a perturbation function q is “good enough”, if

- (i) the strict solution x^* of POP is close enough to the starting point x_0 ;
- (ii) the payoff $\varphi(x^*)$ is lower enough than the initial payoff $\varphi(x_0)$, that is, the non-perturbed payoff φ is improved enough;
- (iii) the flatter is the perturbation and the lower is $\delta(x^*) = q(x^*, x^*)$, the better is x^* as an approximate minimizer of the non-perturbed function φ .

We are now in a good position to see how our generalized versions of the EVP can offer some better perturbation functions than the initial one q . Consider case (C). Let us consider the QM

$h' : X \times X \rightarrow \mathbb{R}_+$ defined by $h'(x, z) = q(x, z) - q(x, x)$ associated with q . Then, given the interpretation of condition (C) (see Sect. 3), we can show that every

perturbation function k , which satisfies condition (C), must be higher than h' and lower than q , i.e.,

$$h'(x, z) = k(x, z) - k(x, x) \leq k(x, z) \leq q(x, z) \text{ for all } x, z \in X, \text{ with } k(x, z) > 0 \text{ if } z \neq x.$$

Thus, the lowest perturbation is $k = h'$. Then, it is easy to see that, on one side, such a perturbation k is better than q because it provides a better Ekeland cone condition. But, on the other side, this perturbation $k = h'$ does not provide a higher increase in the advantage to move $A(x, z) = g(z) - g(x)$ because $h'(x, z) \leq k(x, z) \leq q(x, z)$ for all $x, z \in X$. All such perturbations k , not necessarily generalized distances, meet the required conditions. The best one is just h' . This is a striking result!

5 Conclusions and Perspectives

In this paper, we establish general versions of the EVP with improved perturbations in order to apply approximate minimizers provided by the principle, not use limiting processes to get exact minimizers as usual. Hence, we introduce several types of the second perturbation function besides the first one being the generalized metric of the underlying space. We have to create new proof techniques to obtain three versions of the EVP. As applications, these versions are applied to the traveler problem, which is the representative model of human dynamics in behavioral sciences.

We think that this beginning of the approach of developing versions of the EVP with better perturbations in order to employ the obtained approximate minimizers is encouraging and possible to open the way for further studies. As examples of the perspectives, besides direct developments of versions with two perturbations, other types of generalized perturbations may be taken into account. Applications to different practical problems can also be expected. Then, the underlying space of the principle can also be chosen differently from the partial quasi-metric space we use in this paper suitably for the traveler problem in behavioral sciences.

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