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Making diagrams speak,

in Bhāskara I's commentary of the *Āryabhaṭīya*

Agathe Keller

Abstract

This article is a survey of the numerous questions raised by the presence of diagrams in a VIIth century Sanskrit mathematical commentary. Exploring the links between text, manuscripts and edition of Bhāskara I's *Āryabhaṭīyabhāṣya*, our inquiry ranges from the tools employed to draw geometrical figures to the diverse functions that were assigned to drawings in geometry. Whether technical objects disposed on a working surface or testimonies of oral explanations, diagrams deal with that part of mathematical reasoning which was not formulated in written speech.

Keywords: Indian Mathematics, Diagrams, Āryabhaṭa I, Bhāskara I.

About the author: A. Keller is a french post-doctoral researcher whose studies range from medieval sanskrit mathematics to the political uses of history of science in contemporary India. She has recently

published an article with K. Chemla, published in an edited book on mathematical exchanges, on the different practices of irrational numbers in medieval India and China.

Address: Associate researcher, REHSEIS UMR 7596, Centre Javelot, 2 place Jussieu 75251 Paris Cedex 05, France, 33 1 44 27 86 46, kellera@paris7.jussieu.fr.

Introduction

Mathematics is an intellectual activity: what material traces does it leave behind? Historians and epistemologists working on mathematics, as they try to recover the way mathematical objects were thought of in different times and various cultures, are constantly confronted with this question. In this respect, diagrams are especially interesting objects ¹. Indeed, although a written artifact, diagrams are not discursive testimonies of mathematical reasonings. Studying mathematical drawings and their functions may thus be a way to reach aspects of mathematical practices that are not transmitted through written speech. This is the reason why diagrams and pictures have attracted recently the attention of historians and epistemologists of mathematics².

¹In the following we will call ‘figure’ the abstract idea of mathematical objects and ‘diagram’ any drawn representation of such ideas. This contrast may not always be relevant: practicing mathematicians will sometime use mental diagrammatic representations of figures. But this fact will not create any difficulties for our understanding of Bhāskara’s text.

²For recent studies on diagrams in mathematics, one can refer, for instance to [Netz 1999]’s study of Euclidean diagrams, or [Brown 1999] for a philosophical point of view. K. Chemla, in her research seminar ‘history of mathematics, history of text’ in Paris, has for

The specificity of India

Compared to many other civilizations, India has left us with a remarkable amount of mathematical texts³. This huge amount of manuscripts is surprising for a tradition which values the oral transmission of knowledge. Treatises in pre-muslim India were often a versified set of *sūtras* (aphoristic rules) in Sanskrit language. These *sūtras* were usually so condensed that they could not be understood on their own and required a commentary. The more important treatises in India were, the more commentaries they gave rise to. Treatises were usually considered as spoken (*uc-*) whereas commentaries were written affairs (*likh-*). Astronomical treatises were no exception to this rule. And most mathematical texts that have come down to us, were somewhat autonomous chapters of astronomical treatises.

Despite the importance given to oral transmissions, mathematics in India was clearly an activity which required writing. Indeed, rules in treatises were given to note numbers and draw diagrams. One of the recorded names of arithmetics was ‘dust or finger mathematics’⁴, a reference to an existing working surface on which elements were drawn or written.

With India we are thus in an ideal case: the tradition openly admits that the written form is but a fixed trace of a more important oral transmission, but such scanty traces are numerous. In this context, the study of diagrams

the past 7 years been pursuing a historical reflection on this theme.

³See [CESS, Volume I, Introduction] where D. Pingree explains his motivations for his monumental and still incomplete (after almost fifty years) census.

⁴[Datta & Singh 1935; p. 123].

is interesting precisely because of its non discursive nature. In the following we will see how diagrams appear in a specific text. We hope to highlight with this study the limits of what written texts tell us of ancient mathematical activities, and question how to make them ‘speak’ in other manners. We will also have plunged into the mathematical world of a VIIth century Indian astronomer.

Bhāskara’s commentary

This article will concentrate on a VIIth century Sanskrit commentary written by an astronomer called Bhāskara I⁵ on a late Vth century versified astronomical treatise, the *Āryabhaṭīya* (Ab) of Āryabhaṭa. The *Āryabhaṭīya* has four chapters, the second concentrates on *gaṇita* or mathematics. In the following, we will turn to the contents of the mathematical chapter of the *Āryabhaṭīya* and of Bhāskara’s commentary on this part (BAB.2), questioning when and how diagrams appear. We will then attempt to characterize these drawings, and reconstruct how they were drawn. Finally we will turn to the different functions that were assigned to diagrams in this text.

1 Where can Diagrams be found?

Diagrams can be found in Bhāskara’s commentary of the *Āryabhaṭīya*. This means that they can be seen in the preserved manuscripts of this text, and thus in its printed edition. In the following we will examine the editorial

⁵Also called the ‘the elder Bhāskara’ to distinguish him from the XIIth century astronomer bearing the same name, Bhāskara II or ‘the younger Bhāskara’.

work on the diagrams of the printed edition before turning to a description of where and how diagrams appear in the text of Bhāskara’s commentary.

1.1 Edition and Manuscripts

A printed version of Bhāskara’s commentary was published by K.S. Shukla in 1976 for the Indian National Science Academy (INSA)⁶. We have relied on this edition. However, it has a certain number of limitations.

A first problem arises from the state of the sources themselves. Only six manuscripts⁷ of the commentary are known to us. Five were used to elaborate Shukla’s edition. Five belong to the Kerala University Oriental Research Institute and Manuscripts Library (KUORIML) in Trivandrum and one to the British Office in London⁸. All the manuscripts used in the edition prepared by K. S. Shukla have a same source. This means they all have a same basic pattern of mistakes, each version having its own additional ones as well. They are all incomplete. Shukla’s edition of the text has used a later commentary of the text inspired by Bhāskara’s commentary, to provide a gloss of the end of the last chapter of the treatise.

While the lack of primary material is a major difficulty, other problems

⁶[Shukla 1976].

⁷Indian manuscripts are generally made of dried and treated palm leaves which are carved and then inked. They do not keep well.

⁸Shukla has used four from the KUORIML and the one from the BO. A fifth manuscript was uncovered by D. Pingree at the KUORIML. As one of the manuscripts of the KUORIML is presently lost it is difficult to know if the ‘new one’ is the misplaced old one or not. Furthermore, this manuscript is so dark that its contents cannot be retrieved anymore. See [CESS].

arise from the quality of the edition itself. Indeed, no history or dating of the manuscripts was attempted by the editor. We just know that palm leaf manuscripts can not be much older than 500 years old. When differences between the written text and what is found in the manuscripts appear, the lack of historical contextualization and reflection on the relation of the manuscripts with the original text will prevent us from pinpointing with exactitude what belongs to Bhāskara and what belongs to the manuscripts⁹.

Furthermore, editorial choices concerning textual arrangements (such as diagrams and number dispositions) are often, if not systematically, implicit. I have consulted four of the six manuscripts of the text and can testify that dispositions of numbers and diagrams vary from manuscript to manuscript. Discrepancy in between the printed text and the manuscripts further deepen the already existing gap between the written text and the manuscripts themselves.

In 1997, I obtained a photographic copy of one of the manuscripts: Co 1712 of the KUORIML (Manuscript D of Shukla's edition)¹⁰. KUORIML Co 1712 most probably was written in the XIXth century¹¹. In the following, reproductions of this manuscript will occasionally be given with pictures of the printed text in order to compare them. These reproductions will enable

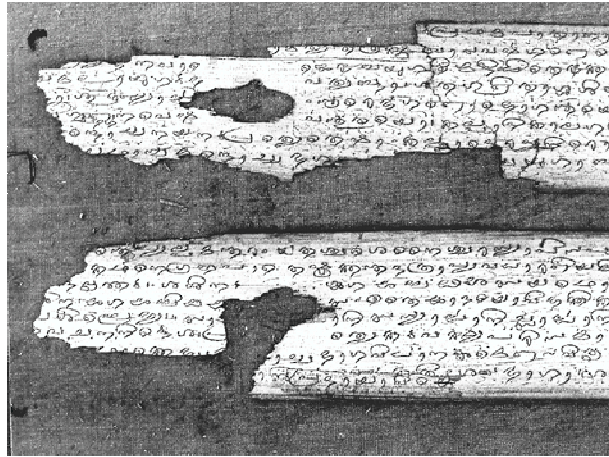
⁹A case study on the dispositions of the Rule of Three has been carried out in [Sarma 2002] which underlines discrepancies between numerical dispositions described in a text and those adopted in manuscripts.

¹⁰The scribe of the library, P. L. Shaji, took them. I would like to thank him here.

¹¹According to an oral information provided by Dr. F. Voegli of the University of Lausanne, using epigraphical evidence. According to P. L. Shaji, the manuscript would be several hundred years older.

us to keep in mind that we do not always know if what we are analyzing is an element of the diagram or an editorial innovation. When there is no picture of the manuscript with a picture of the edition it is just that the manuscript was broken in its place, like in Figure 1.

Figure 1: Palm leaf manuscript breaking into pieces



KUORIML Co 1712 Folio 45b/46a

Thus, all analysis of diagrams should be subject to great care.

1.2 Diagrams and the structure of Bhāskara's commentary

All the diagrams that can be found in the text belong to the commentary, there is no diagram in the treatise¹². All the diagrams belong to the mathematical chapter, no diagrams can be found in the commentaries to the other chapters of the *Āryabhaṭīya*. Diagrams are therefore mathematical objects in this case¹³. The mathematical chapter of the *Āryabhaṭīya* is made of 33 verses.

Table 1 on page 41 gives an idea of the diversity of the problems that the chapters as per the interpretation given in Bhāskara's commentary¹⁴.

We can apply our own classification of mathematics to sort the subjects treated in this chapter. For instance, we would consider that verses 6 to 10 deal with geometry while verse 30 is algebra and verse 11 trigonometry. Bhāskara gives his own definition of the subsubjects that form mathematics¹⁵. This is how he classifies the different topics treated by Āryabhaṭa, ascribing them to the 'mathematics of quantities' (*rāśiganita* or 'arithmetics') and to

¹²As we will see below, verse 13 of the mathematical chapter alludes to methods to construct diagrams. However, the treatise, which would be an oral text, does not possess any drawn figure.

¹³In the *Māhabhāskarīya* (MBh.5.60 [Shukla 1960; p. 64]) our commentator refers to diagrams representing eclipses. Diagrams were certainly of current use in astronomy. This is therefore a specific instance.

¹⁴As underlined in [Hayashi 1997], other interpretations of Āryabhaṭa's verses were made by different commentators.

¹⁵A first attempt at understanding these different subdivisions can be found in [Keller 2000, Volume I, 2.1] and a second description in [Keller forthcoming]. Both are written in French.

the ‘mathematics of fields’¹⁶ (*kṣetragaṇita* or geometry) :

*apara āha: gaṇitaṃ rāśikṣetraṃ dviddhā’/ (...) gaṇitaṃ dviprakāram
rāśigaṇitaṃ kṣetragaṇitam/ anupātakutṭākārādayo gaṇitaviśeṣaḥ
rāśigaṇite ’bhihitāḥ, średdhīcchāyādayaḥ kṣetragaṇite/*

Another says-‘Mathematics is two fold: quantity (*rāśi*) and field (*kṣetra*) (...) Mathematics (*gaṇita*) is of two kinds: mathematics of fields and mathematics of quantities. Proportions, pulverizers, and so on, which are specific ⟨subjects⟩ of mathematics, are mentioned in the mathematics of quantities; series, shadows, and so on, ⟨are mentioned⟩ in the mathematics of fields.

Even though Bhāskara’s statement is elusive, we can understand that the following subjects belong to what he considers is geometry: measuring segments, areas and volumes belonging to abstract objects such as squares, trilaterals and spheres; considering succession of numbers and evaluating their sum or their number (series or *średhi*); measuring the length of the shadow of a sun dial or gnomon (*śaṅku*) or measuring other related segments; applying such procedures to astronomical problems such as eclipses or to traditional problems such as the breaking of a bamboo rod and the sinking of a lotus.

All of the diagrams appearing in the commentary of the mathematical chapter appear when dealing with subjects that Bhāskara understand as

¹⁶Geometrical figures are generically called fields (*kṣetra*).

belonging to *kṣetraganita* or geometry.

Bhāskara's commentary follows a systematical pattern. The structure of each verse commentary is summarized in Table 2 on page 42.

This structure can be found in other mathematical commentaries as well¹⁷. Bhāskara glosses Āryabhaṭa's verses in the order in which they appear. He can comment half a verse, whole verses or two verses at a time.

The commentary of a verse starts by an introductory sentence, followed by a quotation of the text at stake. It is followed by what we call a 'general commentary' of the verse. This commentary is syntactical and grammatical essentially. Its aim is to lift all the ambiguities that arise from the verse because of its elliptic short form. Thus all the different steps of a general procedure are spelled out in this 'general commentary'. This is also the place where debates are staged, and the validity of algorithms and definitions is discussed.

It is followed by a succession of solved examples whose function is to unravel the different realms of application of the algorithm and to give to it substance (the kind of problem it gives an answer to or the larger procedure it can be integrated within, the different type of results it can yield, and the different interpretations it can lead to, etc.). Solved examples also have a standard pattern. After announcing an example (*uddeśaka*), a versified problem is exposed. It is followed by a 'setting down' (*nyāsaḥ*) of the elements given in the problem on a working surface. This is precisely the

¹⁷See for instance [Jain 1995].

part of the text which is a window opened on mathematical practices that are not discursive, numbers are noted on it in a tabular fashion, and diagrams are drawn.

This is followed by a reasoning (*karaṇam* , we have translated this by ‘procedure’) showing how the example is solved.

78% of the diagrams that can be found in the printed edition are in the ‘setting down’ part of the solved examples, 7% belong to their ‘resolution’, and 16% belong to the ‘general commentary’.

Let us now turn to the diagrams themselves.

2 Looking at diagrams

Diagrams are referred to in the text of Bhāskara’s commentary.

2.1 Vocabulary

Chedyaka is the Sanskrit word of the commentary that we have translated as ‘diagram’¹⁸. Etymologically it means ‘what can be cut’. This may be a reference to the process by which several segments are drawn from the outline of a geometrical figure¹⁹. Another word is used once, *ālekhyā* . It literally

¹⁸It is used only in relation to a specific diagram, the one used in the commentary of the verse 11, illustrated in Figure 11 on page 33. But since this word is also used in the *Mahābhāskarīya* , an astronomical treatise written by Bhāskara, on a totally different subject we can gather that it was a generic name for such technical drawings.

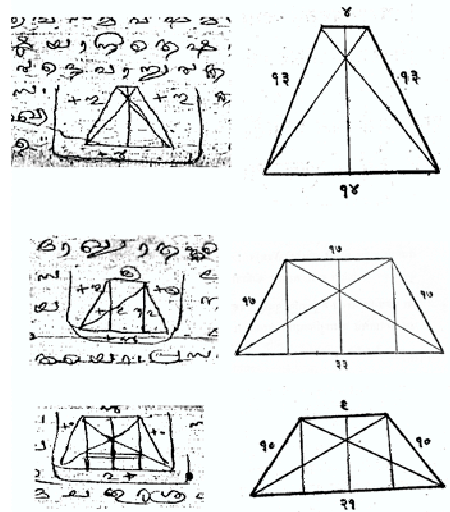
¹⁹This interpretation is suggested by the description of the construction of the diagram in the commentary of verse 11. See BAB.2.11, [Shukla 1976;p. 78, line 40 sqq.] for the sanskrit and [Keller 2000, Volume II, BAB.2.11] for an English translation. K. Chemla

means ‘what should be written, drawn’ and consequently ‘drawing’. In most cases, diagrams are referred to by a composite expression, such as ‘a drawn field’, or ‘a field ⟨which⟩ is set down’. The verb used for drawing (*likh-*) is the same as for writing and means ‘to dig, scratch’, maybe an allusion to the way one writes on palm-leaves.

2.2 Diversity

As we go through the commentary we can distinguish several types of diagrams. Some represent simple geometrical figures, as those reproduced in Figure 2 and in Figure 10 on page 31.

Figure 2: Trapezium



KUORIML Co 1712 Folio 38/39a

Representations can be more complex and include several geometrical

has also suggested that this could refer to the ‘cutting out’ of a shape, as when we use scissors to cut a piece of paper.

figures, as in Figure 6 on page 24 and Figure 11 on page 33. Some represent tridimensional-dimensional objects, as in Figure 9 on page 29. Others are drawn from concrete-like situations, as in Figure 5 on page 16. Numbers are noted in these drawings specifying the lengths of certain segments. In the manuscripts, the interior segments do not come with numbers. In the printed edition, on the contrary, numbers are sometimes ascribed to these inner segments. Some diagrams are endowed with letters in the edition, but not in the manuscript, as in Figure 4 on page 15, Figure 5 on page 16 and Figure 11 on page 33. There are other striking differences between Manuscript KUORIML Co 1712 and the printed edition. The diagrams of the manuscript are drawn in little boxes that separate them from the printed text. Strikingly, they are not drawn with exactitude, they do not have any titles and are not numbered. In the printed edition however, a specific space distinctly separated from the text is allotted to diagrams. As specified in the Introduction of the printed edition²⁰, they have been labelled and numbered. We will point out other differences, when they occur, between the diagrams of the manuscript and those of the edition.

In both cases, never are letters (or syllables) used to label segments or tips of figures²¹. This has some consequence on the differentiation of diagrams.

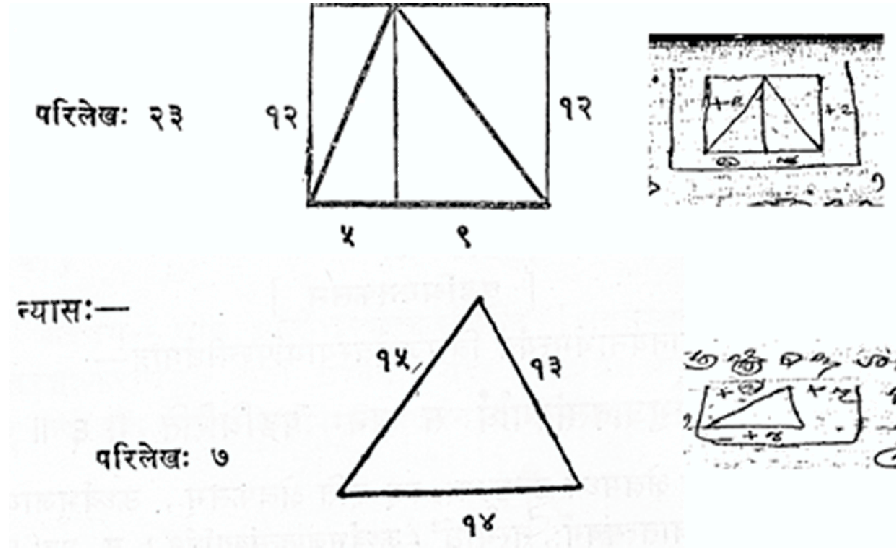
²⁰[Shukla 1976; Introduction, 10.2.iii, p. cxv].

²¹We will see below, that the syllables found in the printed edition are used to indicate cardinal directions.

2.3 Differentiating figures

Indeed, Bhāskara does not seem to discriminate in between fields that are mirror images of one another. This at least appears from an apparent ‘mistake’ in the diagrams of the printed edition. In Figure 3, one can see two representations of the same triangle (a scalene triangle whose sides measure respectively, 14, 13, 15): one is drawn when computing its area according to the rule given in verse 6, the other when ‘verifying’ its area according to the rule given in verse 9. They are mirror images of one another in the printed edition.

Figure 3: Two mirror diagrams of a same triangle



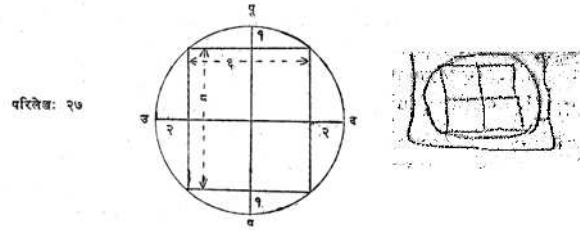
Because of the vagueness with which the diagrams are drawn in the manuscript, it is not possible for one to deduce from the reproductions if these triangles were indifferently, for the scribe, mirror images of one an-

other. However, measures of areas and of segments, which are the purpose of Bhāskara's geometry, are not altered by such transformations. It is therefore possible that Bhāskara, like the scribe who wrote the manuscript, considered these two triangles were the same. We do not know to whom this non discrimination should be ascribed: it may just as well be a far fetched interpretation of an editorial carelessness.

2.4 Orientation

The absence of letters does not, however, prevent Bhāskara from giving an orientation to geometrical figures when he needs it. He uses cardinal directions for this purpose. In Sanskrit, the East (*pūrva*) is 'in front'; the West (*paścāt*) is 'behind'; North (*uttara*) is 'left'; South (*dakṣina*) is 'right'. Figure 11 on page 33, Figure 4 and Figure 5 on the following page use such an orientation.

Figure 4: A circle, its bow fields and inner rectangle

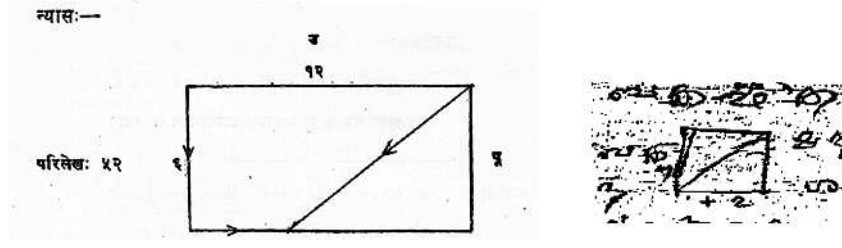


KUORIML Co 1712 Folios 44/45b

These figures are oriented *within the text*, as Bhāskara evokes for instance their eastern part. In the printed edition, these references appear in the

diagram itself, as abbreviations for given cardinal points. These letters are not to be found in the diagrams of manuscript D. In Figure 4 on the page before this is extremely striking, as the printed edition not only presents initials for the four cardinal points outside of the circle, but also inserts arrows inside the drawing and numbers indicating measures of length, Manuscript D has no such notations at all. In the diagram reproduced in Figure 5, two cardinal directions are noted above and on the right.

Figure 5: Fish and Hawk problem



KUORIML Co 1712 Folio 60/61b

The North is represented above as conceived of in Europe in the printed edition. No such letters appear in the diagram of the manuscript. Therefore, letters to indicate cardinal points may, once again, be an artifact created by the edition.

Besides this straight forward orientation, the names given to the sides of a geometrical figure, seem to confer an implicit positioning to the figure in space. Indeed, the word ‘earth’ (*bhū*, *dhatrī*, etc.) used for the base of a triangle and a trapezium always refers to a horizontal segment, the lowest possible in a given diagram. Conversely, a ‘tilted trapezium’ considered in

the commentary of the first half of verse 9²², does not have an earth but a face (*mukha*) and a para-face (*paramukha*). In the commentary of verse 16²³ the source of light on a height is called the base (*bhujā*) of a right-triangle. The ‘upright side’ (*koṭi*) which appears lying down on the other hand is referred to as ‘the earth’ (*bhūmi*).

Thus by observing the diagrams, we have already pinpointed specific ways of working with them. Let us turn to how they were constructed.

3 How were diagrams drawn?

The diagrams of the manuscript, this is one of their remarkable features, seem to have been sketched without much care for exactitude. This is surprising in regard to the techniques that are described in the commentary of verse 13 concerning the construction of trilaterals, quadrilaterals and circles with the help of strings and a pair of compasses²⁴.

Indeed, the first half of verse 13 of the mathematical chapter of the treatise, the *Āryabhaṭīya*, lists tools that can be used for sketching diagrams:

Ab.2.13.ab *ṛttaṃ bhramaṇa sādhyam tribhujam ca caturbhujam
ca karṇābhyām |*

A circle should be brought about with a pair of compasses, and a trilateral and a quadrilateral each ⟨are brought about⟩ with two

²²[Shukla 1976; p. 69] also reproduced in [Keller 2000; Volume II, BAB.2.9.ab].

²³[Shukla 1976; p. 93 sqq].

²⁴BAB.2.13, see Appendix B on page 48.

diagonals|

3.1 Compasses

Āryabhata's name for compasses is *bhrama* 'a rolling ⟨object⟩'. Bhāskara, his commentator, calls it a *karkaṭa* or *karkaṭaka*, literally a 'crab'. In his commentary on verse 13, Bhāskara gives only a brief explanation on this object²⁵:

*bhramaśabdena karkaṭakaḥ parigṛhyate| tena karkaṭakena samavṛttaṁ
kṣetraṁ parilekhāpramāṇena parimāyate|*

With the word *bhrama* a pair of compasses (*karkaṭa*) is understood. With that pair of compasses an evenly circular field is delimited by the size of the outline (*parilekhā*).

Elsewhere he is slightly more specific. Thus in his commentary of the latter half of verse 9 of the chapter on mathematics, he writes²⁶:

asmin ca viracitamukhadeśasitavartyaṅkurakarkaṭena ālikhite chedyake...

And in this diagram, which is drawn with a compass (*karkaṭa*) for which a sharp stick (*vartyaṅkura*) secured (*sita*) at the mouth spot (*mukhadeśa*) has been arranged....

²⁵[Shukla 1976; p.85].

²⁶[Shukla 1976, p.71].

According to the meanings we give to *vartī* (or *vartikā*; usually the wick of a lamp, a paint-brush or chalk) and to *sita* (has been fastened, white color), different readings of this description are possible, and hence different images of compasses appear. We also do not know what is the compasses' 'mouth spot' (*mukhadeśa*). The same difficulties arise when we read the short description in Bhāskara's commentary of verse 11²⁷:

*tathā ca paridhiniṣpannaṃ kṣetraṃ karkaṭakena viracitavartikāmukhena
likhyate*

And thus a field produced by a circumference is drawn with a pair of compasses whose opening (*mukha*) has a sharpened stick (*viracitavartikā*).

We have adopted the improbable reading of *vartī* (or *vartikā* that we have read as a synonym of the first) as 'stick' by accepting Parameśvara's interpretation of the compound *vartikāṅkura*. Parameśvara is a well known, prolific, astronomical commentator of the XVth century²⁸. He wrote commentaries on Bhāskara II's works as well as on the *Āryabhaṭīya*²⁹. He also wrote a direct and a super commentary on Bhāskara I's *Mahābhāskarīya* and a direct commentary on the same author's *Laghubhāskarīya*³⁰. Appendix A on page 43 provides the translations of two texts where he describes the construction

²⁷[Shukla 1976; p.79].

²⁸See [CESS,].

²⁹The first edition of the *Āryabhaṭīya* was published with his commentary: [Kern 1874].

³⁰See [Sastri 1957].

of compasses³¹. Almost 800 years separate Parameśvara's and Bhāskara's commentaries. Most probably compasses underwent technical changes during this lapse of time. But Parameśvara has left us a quite precise testimony. Bhāskara, on the other hand, never seems to have been prolific on this subject. We have therefore, rather than letting our imagination run free, echoed Parameśvara's compasses in our translation of Bhāskara's descriptions.

3.2 Ropes or strings

The oldest mathematical texts known of in the Indian subcontinent, the *śulbasūtras*³², described delimitation of sacrificial areas having specific geometrical shapes with the help of sticks and ropes. Bhāskara describes computations with strings (*sūtras*³³). The question of the posterity of the *śulba* mathematics in Bhāskara's commentary remains open to further research. His commentary of verse 13 is quite straight forward when describing the construction of isosceles triangles and rectangles. A translation of this commentary can be found in Appendix B on page 48. These techniques, however, seldom seem to be used or referred to in other parts of the commentary. In most cases, indeed, such methods could not be applied because they required the knowledge of the length of a height or a diagonal, from which the whole figure was then drawn. Versified problems usually did not readily provide

³¹Pr. T. Hayashi of Doshisha University has brought these texts to my knowledge, may he be thanked here.

³²[Bag and Sen 1983].

³³This is indeed the same word as the one used for aphoristic rules, which are as tight and small as a tensed string.

these inputs, which were computed during the resolution. Why, then, are those techniques described? This puzzle remains unsolved.

3.3 Tridimensional objects

Reference is also made to three dimensional artifacts, whether practical objects or abstract figures. Verticality is indeed a very important feature of the gnomon. Bhāskara in his commentary of verse 14 describes many different gnomons and the way they may be constructed³⁴.

Solids such as pyramids and spheres described in the treatise are represented as plane figures in diagrams. But Bhāskara while describing a cube, adds (p. 51; line 5) :

aśrayo yasya mṛdānyena vā pradarśayitavyāḥ/

Or its side should be shown with earth or something else

This could be a reference to clay representations of a cube.

Similarly concerning the height of a triangular based regular pyramid he writes (p. 59; lines 25-26) :

tatra ūrdhvabhujā sūtrakaiḥ śalākādibhir vā pradaśayitavyā/

In this case one should explain the ‘upward side’ with strings, sticks, etc.

³⁴For astronomical instruments one can refer to [Ōhashi 1994]. His description of Āryabhaṭa’s gnomons is discussed in [Keller 2000, Volume II, Annex of BAB.2.14].

We can also note that a sphere is sometimes referred to as an iron ball (*ayoguda*). No archeological excavation, to my knowledge, has unearthed any such artifact.

We have seen that we did not know if diagrams needed to be constructed with precision or not. This uncertainty can also be found concerning three dimensional artifacts. Indeed, in some instances, precise verticality is crucial (for the gnomon). In other cases, however, it does not seem so important (when visualizing three dimensional figures). The paradox deepens further when we observe what the text tells us about the skill required to draw figures.

3.4 Expertise

The construction of few diagrams are described and then drawn. More often than not, diagrams are drawn with no comment on the way they were constructed, or referred to without even being drawn³⁵.

Constructions are always elusive, and seldom seem to use the strings referred to in the commentary of verse 13³⁶. Therefore, it is difficult to know what method was used.

³⁵Neither Shukla's edition nor the manuscripts consulted have diagrams. This is for instance the case in BAB.2.8, as quoted below.

³⁶Two constructions of diagrams are described with some precision in Bhāskara's commentary. One can refer to BAB.2.11 ([Shukla 1976; p. 78 sqq] for the Sanskrit, [Keller 2000; Volume II, BAB.2.11] for an English translation) whose diagram is reproduced in Figure 11 on page 33, and, in BAB.2.3.ab ([Shukla 1976; p. 48] for the Sanskrit, and [Keller 2000; Volume II, BAB.2.3.ab] for an English translation), whose diagram is reproduced in Figure 6 on page 24. They do not use the methods described in BAB.2.13.

However, the expertise necessary to construct diagrams is mentioned sometimes. For instance in his commentary of verse 8, Bhāskara comments on the computation, in a trapezium, of the two segments of the height defined by the point of intersection of its diagonals. These two segments are called by the treatise, *svapātalekhā*, the ‘lines on their own falling’. Bhāskara writes³⁷ :

*samyagādiṣṭena ālikhite kṣetre svapātalekhāpramāṇaṃ traīrāsīkagaṇitena
pratipādayitavyam/*

The size of the ‘lines on their own fallings’ is explained with a Rule of Three in a field drawn by a properly instructed person.

But at the end of the textual description of the diagram reproduced in Figure 6 on the following page, in his commentary of the first half of verse 3, Bhāskara writes³⁸:

durvidagdhapratyāyanāya ca kṣetram ālikhyate

And to convince the dull minded, a field is drawn

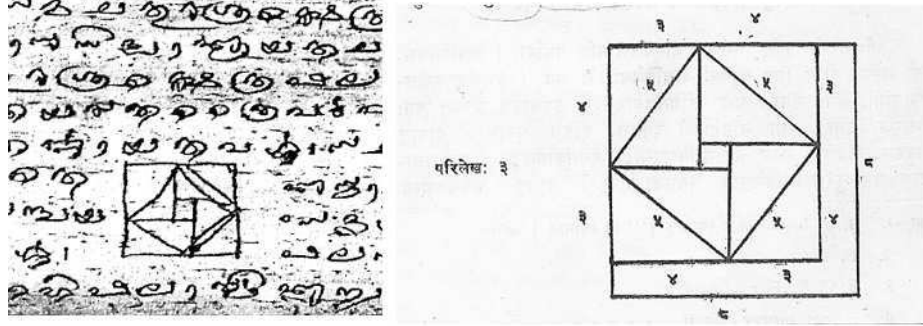
Would it then be as if some diagrams had to be drawn with accuracy, while others could be sketched, without much care for exactitude, or even represented mentally?

We are indeed, confronted with contradictory information. Verse 13 describes the construction of trilaterals and quadrilaterals, but these methods

³⁷[Shukla 1976; p. 63; line 17].

³⁸[Shukla 1976; p. 48; line 16].

Figure 6: A diagram for the dull minded



cannot be used to construct the diagrams of the ‘setting-down’ part of solved examples. In fact, most of the constructions are not described at all. Similarly, if Bhāskara recognizes that the construction of diagrams requires expertise, he also seems to prefer mental representation of figures over their actual tracing. This may explain the apparent ‘inexactitude’ of the manuscripts, which could also have its origin in the discrepancy in between the practices at the time of Bhāskara and those of the later times of which manuscripts would testify. These paradoxes could also spring from the different functions bestowed to diagrams. They could have been constructed with accuracy at the moment one had to work with them, but such precision may have not been required when diagrams were transmitted via manuscripts. More generally, as we will see below, when diagrams were needed for an explanation it was not necessary for them to be drawn to their exact dimensions and proportions. In other instances, however, if one had to conduct a procedure within a diagram, more accuracy may have been required. We thus need to turn to the functions of diagrams in geometry. In return we may then be

able to specify if they needed to be precisely constructed and drawn.

4 What was the diagram's role?

Most (78%) of the diagrams can be found in the ‘setting-down’ part of solved examples. Consequently, the function they assume, what one is supposed to do with them, is not stated explicitly in these cases. Diagrams indeed appear as common mathematical objects whose status does not require any explanation. By analyzing the context in which diagrams are found, we will show that numerous functions were ascribed to diagrams: useful tools to specify a definition as well as the place where a process is summarized, they can be the object in which a procedure is carried on or the very locus of a proof.

4.1 Specifying a definition

We have mentioned drawings of triangles, trapeziums, etc. Are diagrams representations of geometrical figures? Definitions of geometrical figures can be found in Bhāskara’s commentary, where they belong to the ‘general commentary’ of a verse. A definition is elaborated when the word used in the treatise to name them is glossed by Bhāskara. He then attempts to determine whether the ‘right word’ is used to designate a given figure³⁹. For instance, Āryabhaṭa, when providing a rule to compute the area of triangles in the

³⁹[Keller 2000; Volume I, 1.7, p.91-101].

first half of verse 6, uses the word *samadalakoti* , lit. ‘equally halving height’ for their heights. Does this mean that he only considers equilateral and isosceles triangles? Should one understand this word as technically meaning the height of any triangle? All of these questions are raised and discussed by Bhāskara in his commentary to this half verse⁴⁰. By thus arguing to determine if *samadalakoti* is the right word to name the height of a triangle, Bhāskara in fact defines the object at stake.

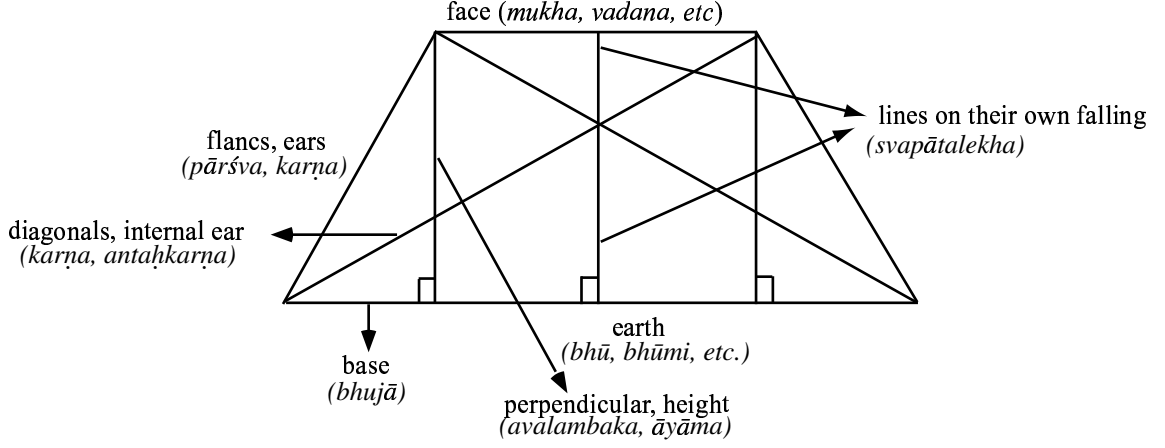
This reveals that Bhāskara considers that there is an illustrative quality to Āryabhaṭa’s and his own technical vocabulary. This picturesque quality is indeed striking for the contemporary reader⁴¹. Indeed, each figure bears specific names for the segments that outline it. For instance, a trapezium is defined by the the earth (*bhū*) parallel to the face (*mukha*), and its lateral sides. They are called ‘flanks’ (*pārśva*) by Āryabhaṭa, ‘ears’ or ‘diagonals’ (*karṇa*) by Bhāskara. It is distinguished from any quadrilateral by the fact that its heights (*āyāma*) are equal. As mentioned above, Āryabhaṭa gives a rule to compute the length of the two segments of the height having the point of intersection of the diagonals for extremity. These segments are called the ‘lines on their own falling’ (*svapātalekha*). This is illustrated in Figure 7 on the next page.

Similarly, there are three classes of trilaterals: those whose sides are all equal (*sama* e.g. equilaterals), those who have two equal sides (*dvisama* ,

⁴⁰Refer to [Shukla 1976; p. 55, line 4 sqq] for the Sanskrit, [Keller 2000; Volume II, BAB.2.6.ab] for an English translation.

⁴¹This has been also underlined in [Filliozat 1988; p. 257-258].

Figure 7: Names of segments in a Trapezium

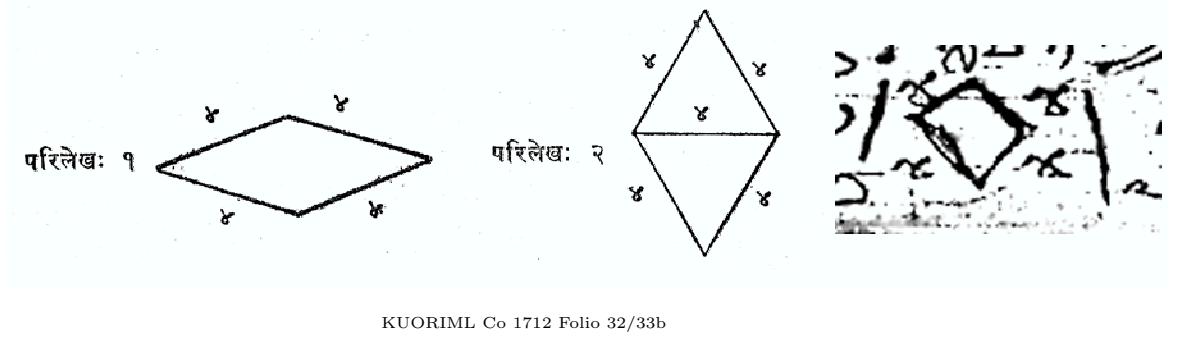


e.g. isosceles), and scalene ones (*viṣama*). The base (*bhujā*) or earth (*bhū*) is distinguished from the other two sides (*pārśva*, flanks or *karṇa* ear, diagonal) by the fact that the height (*avalambaka*) falls on it.

Thus, the text seems sometimes to implicitly refer to a virtual diagram, that one can easily imagine because of the picturesque quality of the vocabulary used. If names of geometrical figures and segments suggest a drawing, they can sometimes be deceitful. Bhāskara offers us one instance where diagrams are used in infirming the virtual representation of a figure that words suggests. Indeed, the word used for geometrical squares in Āryabhaṭa's treatise is *samacaturaśra* or 'equi-quadrilateral'. Bhāskara produces diagrams of 'equi-quadrilaterals' that are not squares, as seen in Figure 8 on the following page⁴².

⁴²BAB.2.3.ab, [Shukla 1976; p. 48], it is also reproduced in [Keller 2000; Volume II, BAB.2.3.ab].

Figure 8: Two ‘equi-quadrilaterals’ which are not squares



Manuscript D has not preserved the second of the two diagrams described in Bhāskara’s commentary. Bhāskara, then, qualifies squares as equilateral quadrilaterals with equal diagonals. Definitions could thus use diagrams to specify (and correct) representations that words suggest. In this case, the diagrams infirm a mental visualizing that could be induced by the picturesque quality of the vocabulary.

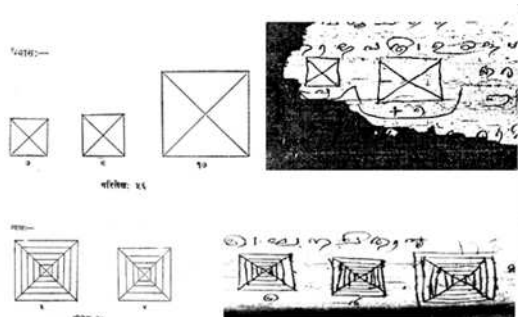
On the contrary, in his commentaries on the verses on series, Bhāskara illustrates the geometrical quality of Āryabhaṭa’s vocabulary in diagrams⁴³, as one can see in Figure 9 on the next page.

Note how the top reproduction of Manuscript D is broken off, a testimony of its bad state. Here, the top diagrams present piles of squares and the bottom diagrams piles of cubes. Series, in Bhāskara’s interpretation of them, are both arithmetical and geometrical objects⁴⁴. What could be deemed an arithmetical computation, a sum of squared or cubed numbers, takes a

⁴³See BAB.2.21, [Shukla 1976p. 109-110]; [Keller 2000; Volume II, BAB.2.21], and BAB.2.22, [Shukla 1976; p. 110-112]; [Keller 2000, Volume II, BAB.2.22].

⁴⁴[Keller 2000; Volume I, 2.4.4, p.95-97].

Figure 9: Piles of squares and cubes



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geometrical, almost concrete, aspect with these diagrams.

Thus, Bhāskara resorts to diagrams either to precise the representation of a figure or to help his reader in discriminating between figures that vocabulary could heap up together.

Diagrams and characterizations of geometrical figures are also intimately linked in an other way. As noted above, the differences between different types of quadrilaterals and trilaterals is constructed from their inner segments: a square has equal diagonals where a trapezium equal heights, equilateral and isosceles triangles's heights are mediators which is not the case for scalene ones... Āryabhaṭa in the first half of verse 13, indicates that trilaterals and quadrilaterals should be constructed from 'diagonals' (*karṇa*). If a *karṇa* is a hypotenuse or a side in a triangle, or a trapezium, in other instances and most generally it names the inner diagonals of a quadrilateral figure. Bhāskara's commentary of this part of verse 13 (translated in Appendix B on page 48) describes construction of fields which rest on their

inner segments. Indeed, the construction of an isosceles or equilateral trilateral is described, using its base and a height. The lengths of both of these segments is thus needed in that case. Similarly, a rectangle is constructed from its diagonals (implicitly, the fact that the diagonals intersect in their middle is used in the construction). Could it be that trilaterals and quadrilaterals are defined, and thus drawn, with their heights and diagonals, in the same way that a circle is always defined by its semi-diameter (*vyāsārdha*) and circumference (*pariṇāha*)? This is but a hypothesis on the conception Bhāskara would have had of geometrical figures.

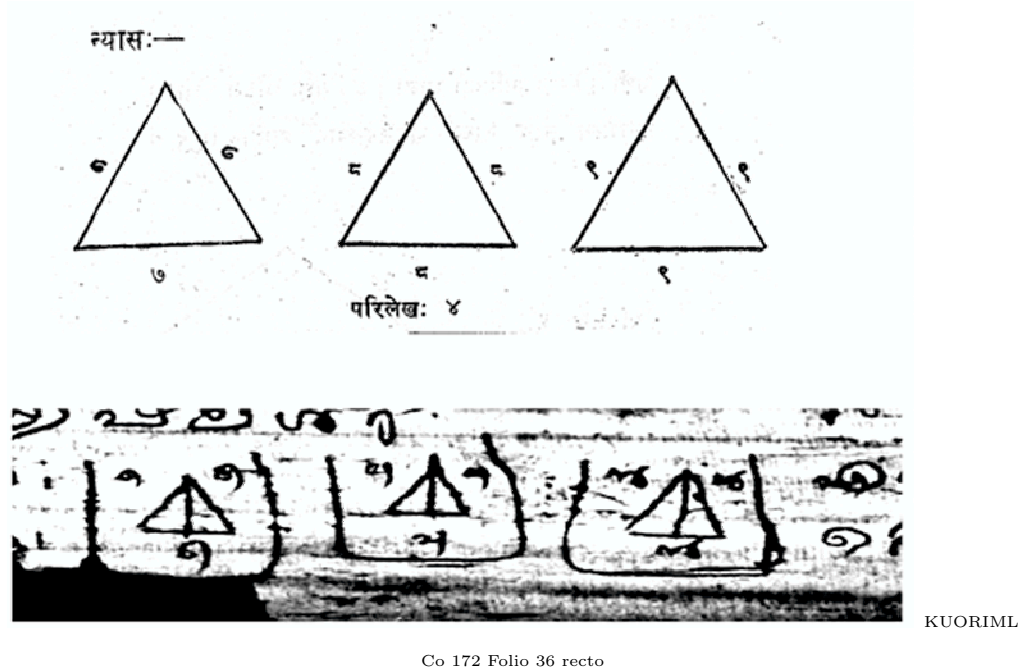
Definitions of geometrical figures and diagrams, implicitly, seem to complete and confirm one another. We have seen that diagrams could rectify definitions, and reversibly definitions enable the construction of correct diagrams. But other functions seem to have been ascribed to diagrams as well.

4.2 Summarizing processes

Solved examples in the commentary of a verse do not only illustrate the procedure that such a verse states, they also give a specific meaning to what is said in an abstract and general manner in the verse⁴⁵. In the ‘setting-down’ part of solved geometrical (in Bhāskara’s sense) examples, diagrams are drawn. Similarly these diagrams, can be ascribed with several functions. Of course, they summarize the problem to be solved. Look at Figure 10.

⁴⁵[Keller 2000; Volume I, 2.6; p.80-91].

Figure 10: Triangles



It shows the ‘setting-down’ part of a problem which consists in computing the area of three equilateral triangles knowing the length of their sides. What is known is stated with numbers. Thus, in the edition as in the manuscripts, the size of the sides is indicated by numbers noted within the diagram. But there is more to the diagram of the manuscript, then just a summary of the problem to be solved: in fact the whole process is illustrated here. Indeed, to compute the area of these triangles one needs the length of the heights which are unknown. The diagrams of the manuscript represent this height without any number. Thus it shows simultaneously what is known and what is sought. It summarizes the problem but also illustrates each step of the

reasoning. This in fact is probably the function of most of the diagrams of the commentary as found in Manuscript D. Concerning this particular diagram, one can note also that this drawing of a triangle could confirm the fact that triangles were thought of as including their heights.

Diagrams of the ‘setting-down’ part of solved examples are therefore not simple, transparent transcriptions of a written problem. We have seen that to draw them one could need technical tools and maybe special methods to draw particular figures. Furthermore, at least if we consider what is suggested by the diagrams of manuscript D, they do not seem to restrict themselves to a reinterpretation of the inputs given in the problem. They also illustrate or summarize the process that will be followed. This is also the case of one diagram, which cannot really be qualified as belonging to the ‘setting-down’ part of a solved example.

4.3 An object to work with

In his commentary to verse 11, Bhāskara describes the construction of a diagram that will be used to derive sines. It is reproduced in Figure 11.

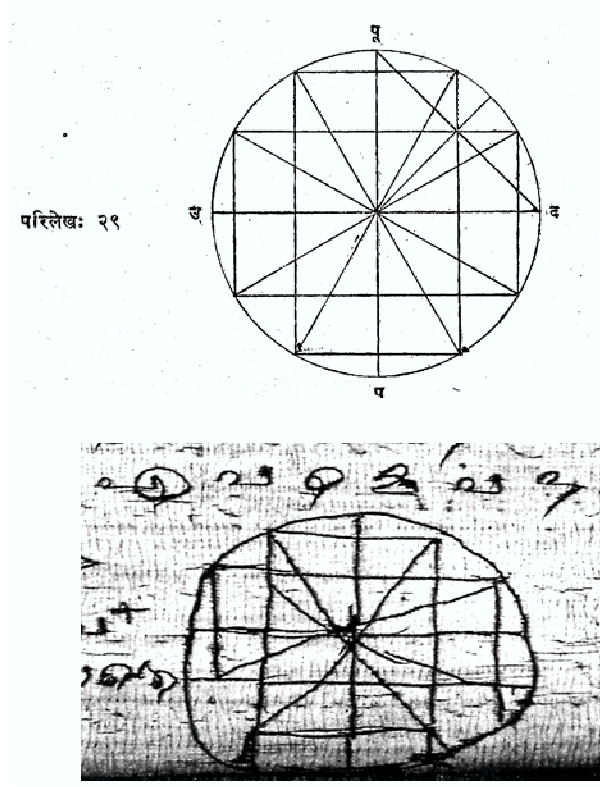
Bhāskara then spells out a computation to derive half-chords within it. He writes indeed⁴⁶:

evam ālikhite kṣetre sarvaṃ pradarśayitavyam

In the field drawn in this way all is to be shown.

⁴⁶[Shukla 1976; p. 79]; [Keller 2000; Volume II, BAB.2.11].

Figure 11: Derivation of half-chords



The verb used to express ‘to show’, *pradṛṣ-*, possesses the same ambiguities as its English counterpart: it can mean to see with the eyes, but also to explain. We can then understand that the function of this diagram is to be a visual aid, a place where the process is explained and understood as it is worked out. However, the diagram that he has just constructed and which is reproduced in Figure 11 does not represent directly the computations described. A classical measure unit is used here within a circle: it measures 1/12th of the circumference and is called a *rāśī*. In the diagram whose construction is described by Bhāskara and which is reproduced above, whole

$rāśis$ are represented. However, in the process that Bhāskara describes, pair subdivisions of $rāśis$ are used in the process. Thus immediately after presenting the diagram, Bhāskara describes a process where the arcs considered are multiples of half a $rāśi$. He then considers the same process, but with arcs multiples of a quarter of a $rāśi$, and then one eighth of a $rāśi$. In this sense the diagram presented in this commentary is like a general model from which all other specific cases could be understood. It does not show directly the computation to be carried on. Furthermore, as the entities computed are geometrical (chords and half-chords in relation to given unit-arcs), it seems almost necessary that the reasoning be verified and understood with the help of a diagram. Consequently, in this case, the diagram needs to be sufficiently well drawn so that right angle triangles and their relation to 1/12th of the circumference of the circle can be seen, although it does not need to be drawn with a high degree of exactitude.

Therefore, diagrams have partly an illustrative function. They can illustrate definitions, or processes to be carried on. But because the frontier separating an illustration to an explanation is hard to define, this does not mean that they are second, unimportant to the reasoning itself. In fact, according to Bhāskara, diagrams are places where everything can be seen and therefore understood.

4.4 Where one understands

Indeed, sometimes diagrams of the ‘setting-down’ part of solved examples not only seem to summarize the problem to be solved but also seem to give an explanation. For instance, the diagrams illustrating the examples of the commentary to verse 22, reproduced in Figure 9 on page 29, provide immediately the explanation of the geometrical aspect of series: piles of squares and cubes are considered and represent their sum.

In a very different way, verse 9 of the mathematical chapter, Āryabhaṭa proposes a ‘verification’ (*pratyayakaraṇa* , lit. ‘producing conviction’⁴⁷) of the areas of all geometrical fields. The commentary to this verse, for each verification, and at each step of the reasoning, uses diagrams. Let us observe one of these diagrams in its context. Bhāskara considers the area of a triangle whose sides are given as 13, 15, 14. This triangle is reproduced in Figure 3 on page 14. Bhāskara provides two verifications of the area of this triangle. Concerning the second method, he writes:

*athavā āyatacatuśrakṣetrayor ardhakṣetraphalasaṃyogo ’sya
phalam| tayor dvayor pañcavistārasya dvādaśāyāmasyaikasya, dvitīyasyāpi
navavistārsya dvādaśāyāmasyārdhakṣetraphalasaṃyogo ’sya phalam|
tayor dvayor pañcavistārasya dvādaśāyāmasyaikasya dvitīyasyāpi
navavistārasya dvādaśāyāmasya nyāsaḥ–*

Or else, its area is the sum of half the areas of two rectangular

⁴⁷A first attempt in analyzing this mode of reasoning can be found in [Keller 2000; Volume 1, I].

fields. This ⟨trilateral's⟩ area is the sum of half the areas of these two ⟨rectangles⟩, the one whose width is five and length twelve, and the second one also, whose width is nine and length twelve.

Setting down these two fields, the first one whose width is five and length twelve, and also the second whose width is nine and length twelve:

(the top figure reproduced in Figure 3 on page 14 is presented here)

The top diagram, reproduced in Figure 3 on page 14, summarizes the problem: it gives the lengths of the known sides. It shows the rectangles whose areas will be computed. It also shows how half the areas of each of these rectangles can add up to give the area of the triangle⁴⁸. Thus the diagrams of the ‘setting-down’ part of the solved examples can also be used to explain a reasoning.

4.5 Where one can see, show or prove

As for the verification evoked above, Bhāskara’s commentary refers to procedures justifying the correctness of Āryabhaṭa’s rules⁴⁹. Later Sanskrit commentaries will sometimes include systematical proofs. This is not the case of Bhāskara’s commentary, where proofs are more often alluded to than elaborated. But, all the elements of proof that are found in Bhāskara’s com-

⁴⁸Bhāskara in the resolution that follows uses the expression *anupraviṣṭa*, e.g. says that the areas *enter* within the triangle.

⁴⁹[Keller 2000; Volume I, 1.8; p. 101-117].

mentary have a step where the mathematical properties are represented and ‘shown’ (with all the ambiguity of such an expression) within a diagram.

The commentary to the second half of verse 9, which states the equality of the chord of one sixth of the circumference of a circle with its radius, for instance, evokes a diagram where an explanation is carried out⁵⁰:

*etām eva śadbhāgajyām pratipādayiṣatā vṛttakṣetre ṣaṭ sama-
tryaśrikṣetrāṇi prasaṅgena pradarśitāni |*

In a circular field, six equi-trilateral fields have been shown incidentally (*prasaṅgena*) by one who wishes to explain this very chord of the sixth part.

In his commentary to the second half of verse 17, similarly, Bhāskara announces an explanation relating a procedure computing segments within a circle with the traditional problem of ‘Hawks and rats’. He announces this explanation and then immediately produces a diagram⁵¹:

*tat tu pradarśyate
nyāsaḥ*

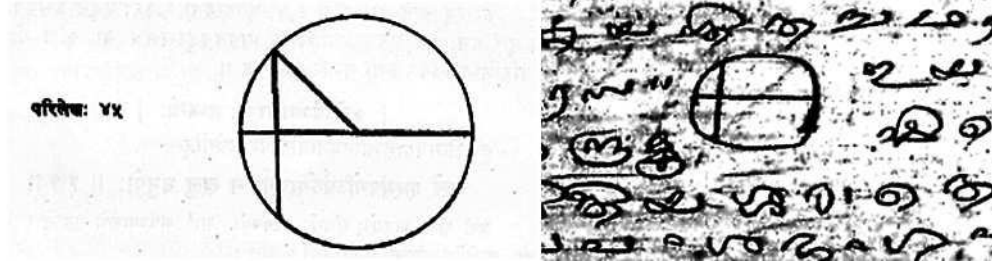
And that is explained:

Setting down:

Note that the diagram here reflects its explanatory aspect, for it bears nor numbers or letters. It just represents the geometrical situation at stake. The

⁵⁰[Shukla 1976; p. 71; lines 12-13]; [Keller 2000; Volume II; BAB.2.9cd].

⁵¹[Shukla 1976, p. 98]; [Keller 2000; Volume II; BAB.2.17.cd]. In this case, the diagram is followed by a reasoning which has been studied in [Keller 2000; Volume I. 1.8.4].



only sentence where the word proof (*upapatti*) is used refers to a diagram where it should be carried out. Bhāskara considers two similar triangles and wanting to compute the length of one of the sides of these triangles with a Rule of Three, he ads⁵²:

trairāśīkopapattipradarśanārthaṃ kṣetranyāsaḥ

In order to show the proof of ⟨that⟩ Rule of Three, a field is set-down: (followed by the diagram)

We cannot say more about the diagrams or the proofs here, without fictionalizing the process. Indeed, in most cases, these diagrams are not followed by any reasoning. It seems that what is -for us- the crucial moment of explanation in geometry, belonged for Bhāskara to the oral sphere. In geometry this oral reasoning would have been based on a diagram. If this was the case, the diagram would not need to be accurate to fulfill its role.

⁵²[Shukla 1976; p. 59, line 3]; [Keller 2000; Volume II, BAB.2.6.cd].

Conclusion

We have seen that diagrams were common objects of Bhāskara's mathematical practice. They are used matter of factly, mostly in the 'setting-down' part of solved examples. As such their functions are not explicitly given by the commentator.

Diagrams are not, however, a transparent, unambiguous transpositions of a written process. We do not know with what degree of precision diagrams would have had to be drawn. Indeed, there is an apparent contradiction in between specific rules of constructions given in the commentary to verse 13 on the one hand and the fact that, in most cases, these methods could not be applied.

Furthermore, we saw that the diagrams not only summarized a problem, they could also indicate the process to be carried out. In this sense, the frontier separating the illustrative function of diagrams and their role as tools explaining a process cannot be specifically pinned down. Diagrams for Bhāskara, indeed have an explicative, even a demonstrative function. However, because these explanations or even proofs were probably oral, the diagram is finally the only trace that we are left with of this explanation. If indeed, diagrams were meant to visualize a process, understand it or provide an explanation, they did not need to be drawn with exactitude. But this leaves open then, the reason why a precise but ineffectual process to draw diagrams is given in the treatise and expounded in the commentary. One

answer to this problem could be to consider that the type of window opened by the ‘setting down’ parts of solved examples as represented in manuscript, differed from what was effectively done on the working surface itself. In the manuscripts, the diagram was used for transmission, not directly as a working object. As such, it could be less precise than the working object itself.

We have noted the distance of diagrams with the words of the problems it was associated to. This distance is but an aspect of the complex relation that the diagrams of Bhāskara’s commentary seem to have with speech. Thus the vocabulary associated with geometrical figures is picturesque, but can induce false representations of them. Diagrams are then used to infirm such false mental visualizations. We have also seen how difficult it was to separate the definition of a geometrical figure from the fact it is represented in diagrams. One and the other complete each other.

Diagrams appear as windows opening the mathematical text onto the larger context of the people and place in which it was used. Who was the person who could explain within a diagram? To whom was the explanation given? On what surfaces did people work? within what institution? Some questions, among the many more, that remain to be explored.

Table 1: Contents of the Chapter on Mathematics (*gaṇitapāda*)

Verse 1	Prayer
Verse 2	Definition of the decimal place value notation
Verse 3	Geometrical and arithmetical definition of the square and the cube
Verse 4	Square root extraction
Verse 5	Cube root extraction
Verse 6	Area of the triangle, volume of an equilateral tetrahedron
Verse 7	Area of the circle, volume of the sphere
Verse 8	Area of a trapezium, length of inner segments
Verse 9	Area of all plane figures and chord subtending the sixth part of a circle
Verse 10	Approximate ratio in a circle, of a given diameter to its circumference
Verses 11-12	Derivation of sine and sine differences tables
Verse 13	Tools to construct circle, quadrilaterals and triangles, verticality and horizontality
Verses 14-16	Gnomons
Verse 17	Pythagoras Theorem and inner segments in a circle
Verse 18	Intersection of two circles
Verses 19-22	Series
Verses 23-24	Finding two quantities knowing their sum and squares or product and difference
Verse 25	Commercial Problem
Verse 26	Rule of Three
Verse 27	Computations with fractions
Verse 28	Inverting procedures
Verse 29	Series/First degree equation with several unknowns
Verse 30	First degree equation with one unknown
Verse 31	Time of meeting
Verses 32-33	Pulverizer (Indeterminate analysis)

Table 2: Structure of a verse commentary

Introductory sentence
Quotation of the half, whole, one and a half or two verses to be commented
‘General Commentary’
Word to word gloss, staged discussions, general explanations and verifications
‘Solved examples’ (<i>uddeśaka</i>)
Versified Problem
‘Setting-down’ (<i>nyāsa</i>)
‘Procedure’ (<i>karaṇa</i>)

APPENDIX

A Parameśvara's description of compasses

The following excerpt has been extracted and translated from Parameśvara's own commentary to verse 13 of the mathematical chapter of the *Āryabhaṭīya* ⁵³:

“With a *bhrama* , that is, with an instrument (*yantra*) called a *karkaṭa* a circle should be brought about. This is what has been stated:

Having acquired any straight stick (*yāṣṭi*), having bound it, firmly, with a cord on its upper-part at the throat-spot (*kaṇṭhapradeśa*), having also split ⟨it, vertically⟩ from the lower tip to the throat, ⟨and thus⟩ having made two sticks (*śalākā*), one should make their two tips sharp ones. In this way is produced a *karkaṭa* instrument having an under mouth (or opening *adhomukhaṃ*). 5
Having further fixed a stick in the space between the ⟨previous⟩ two sticks one should make a pair of compasses having a revolving opening (*vivṛttāśya*). Having made the *karkaṭa* 's opening equal to the semi-diameter of the desired circle by moving up and down the stick which lies in the intermediate space, having laid the tip

⁵³[Kern 1874; p. 32]

of one stick on the central spot of the circle to be brought about,
 having laid the other tip on the spot on circumference of the circle
 one should turn the *karkaṭa* . That is the desired circle.”

An even more detailed description of the making of a *karkaṭa* can be found in Parameśvara’s super commentary to Govindasvāmin’s commentary of the *Mahābhāskarīya* . When glossing on verse 1 of the 3rd chapter of this treatise, which describes the circular, flat setting where a gnomon should be placed, Govindasvāmin writes⁵⁴:

*evam dharātalasya samatvam avagamyā mukhavinyastavartikāṅkuraśobhinā
 karkaṭena vṛttam ālikhet|*

Having, in this way, brought evenness to the ground’s surface, one
 should draw a circle with a pair of compasses (*karkaṭa*) beautiful
 with a sharp stick (*vartikāṅkura*) inlaid at its opening.

Notice that Govindasvāmin uses the compound *vartikāṅkura* which is almost the same expression that we have found difficult to read in Bhāskara’s commentary⁵⁵ *Āryabhaṭṭīyabhāṣya* : the commentator uses the compound *vartyaṅkura* , and once the word *vartikā* , probably as a synonym of *vartī* , (a paint brush, the wick of a lamp). Parameśvara glosses the compound used by Govindasvāmin extensively⁵⁶:

⁵⁴[Sastri 1957; p.103-104]

⁵⁵Refer to Section 3.1 on page 18 on compasses.

⁵⁶[Sastri 1957; *op. cit.*]

“With the word *karkaṭa* an instrument fit for bringing about the out-line (*parilekhana*) of a circle is meant. In this case, having acquired any evenly circular stick (*yaṣṭi*), having bound ⟨it⟩ firmly above its middle at the throat spot (*kaṇṭhapradeśa*) with a string (*rajju*), and so on, having furthermore split ⟨it⟩ at its root, one should make it in such a way that below the throat (*ākaṇṭha*) there are two equal sticks (*śalākā*). Afterwards one should make the tip of ⟨each⟩ stick a sharp tip (*tikṣṇāgra*). This is called a ‘*karkaṭaka* ’.

The intermediate space between is called the the ‘compasses mouth (or opening)’ (*karkaṭāśya*). Afterwards, having taken another stick (*śalākā*) whose width is bigger than the compasses’ sticks (*śalāke*), and whose length is several *aṅgulas* ⁵⁷, having cut its two tips, with a knife, one should make a revolving opening (*vivṛttāśya*). In this way, a stick having a mouth (*mukha*) at its two ⟨tips⟩ is called a *vartikāṅkura* (a sharp stick).

Furthermore, having made a revolving-opening-pair of compasses, having placed transversally the sharp stick in its opening, one should place the two sticks of the compasses on the two mouths (*āśya*) of the sharp stick. In this way, having acquired an instrument called a *karkaṭa* adorned with a sharp stick placed at ⟨its⟩ mouth (*mukha*) one should draw a circle with it. Having made

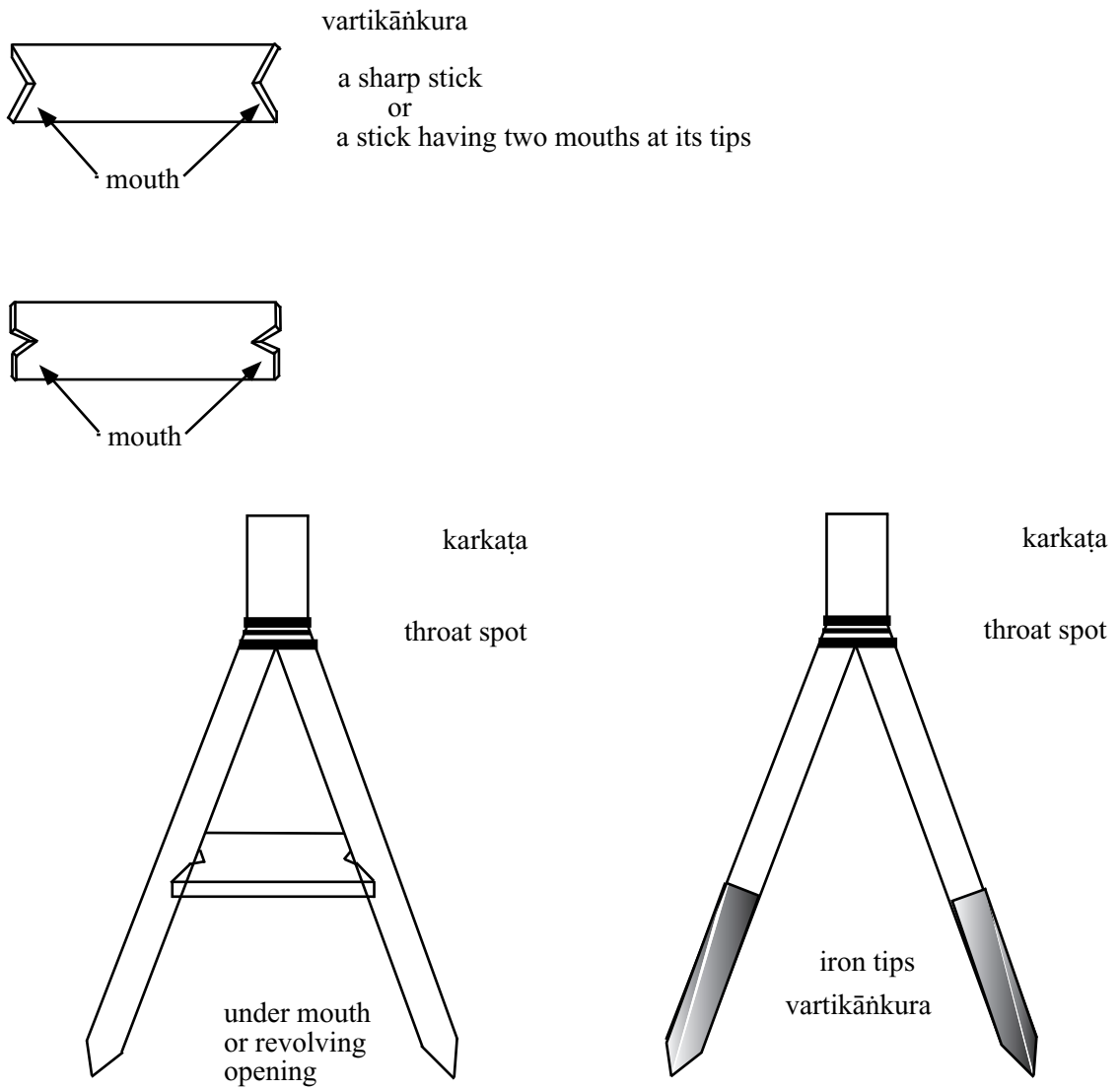
⁵⁷This is a common small unit of length. Literally, an *aṅgula* is a ‘finger, a thumb’.

the compasses' opening equal to the semi-diameter by moving the sharp stick up and down, having fixed one stick (*śṛṅga*) in the middle of the circle one should turn the other one all around. When made in this way, the desired circle appears.

Or else, with the word *vartikāṅkura* another instrument is meant. When one has placed two iron sticks on the tips of the two sticks of a pair of compasses, that is *vartikāṅkura* . A line is made with that.”

We have given a tentative illustration of Parameśvara's two representations of compasses in Figure 12 on the next page.

Figure 12: A pair of compasses as described by Parameśvara



B Bhāskara's description of the construction of trilaterals and quadrilaterals

In his commentary to verse 13 of the mathematical chapter of the *Āryabhaṭīya*, Bhāskara describes two methods to construct geometrical figures⁵⁸.

Having stretched a string (*sūtra*) on level ground one should make a line (*rekha*). And that is:

Figure 30⁵⁹ _____

Here, with a pair of compasses (*karkaṭaka*) which is placed on both tips ⟨of the line⟩, a fish should be produced.

A perpendicular is a second string which goes from the mouth to the tail of this ⟨fish⟩:

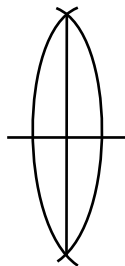


Figure 31

5

Having appointed one tip of a string on the extremity ⟨of the fish⟩, having appointed the second tip ⟨of the string⟩ firmly on

⁵⁸[Shukla 1976;p. 84-87]

the tip of the base, one should make a line. On the second tip ⟨of the base⟩, too, it is just in that way. In this way, there are two diagonal strings. With those two diagonal strings a trilateral is brought about:



Figure 32

In ⟨the case of⟩ a quadrilateral, one should stretch obliquely a string which is equal to [the diagonal of] the desired quadrilateral. And that string is:

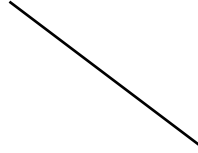


Figure 33

One should stretch obliquely the second ⟨string⟩ too, a cross (*svastika*) is produced from the middle of that ⟨first string⟩. And therefore there are two diagonal strings:

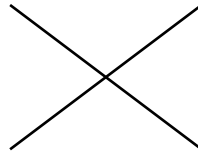


Figure 34

The sides (*pārśva*) of these two ⟨strings⟩ are filled in, ⟨and⟩ a quadrilateral field is produced:

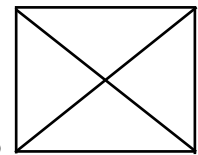


Figure 35

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