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**CANON AND COMMENTARY:**

**AN OUTLOOK BASED ON MATHEMATICAL SOURCES**

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**First draft. Please do not quote without the author’s permission**

Some decades after the unification of the Chinese Empire, a mathematical book that was to have a singular fate in China started to be compiled. Known as *The nine chapters on mathematical procedures* (*Jiuzhang suanshu*)—a title that I shall abbreviate below as *The nine chapters*—, it was, soon after its completion in the 1<sup>st</sup> century BCE or CE, referred to as a “Canon (*jing*)”, and even, later on, during the Song dynasty, as the most important of all mathematical canons. What did it mean for a book to be perceived as a Canon? How was it read accordingly? These are the questions that constitute the horizon of this paper. They are much too broad to be exhaustively addressed within a few pages. I shall hence restrict myself to tackling them from a specific angle, hoping to demonstrate thereby how fruitful they could be for different directions of research.

In fact, in his *Scripture, Canon and Commentary. A Comparison of Confucian and Western Exegesis*, John Henderson already treated these questions extensively, mainly with reference to the Confucian canons. His survey offers an appropriate background, against which to identify which of the attitudes towards *The nine chapters* documented in our historical sources could stem from perceiving the book as a “Canon”. Two issues appear to me to be at stake when considering these questions again, with respect to a mathematical text. The first one relates to the history of text: we shall have to consider how our understanding of the perception of such a kind of text as a Canon, throughout Chinese history, could benefit from considering the case of mathematical Canons. More generally, virtually all scholarly disciplines identified some texts as being “canons”, and, as John Henderson himself noted, research on this type of text has so far hardly begun to take into account these samples of “canons”. The second issue connects with history of mathematics more specifically: what can our reading of *The nine chapters* gain from situating the book more adequately, within the set of texts to which it belongs?

With these two agenda in mind, in a first part, I shall introduce one recurring expectation towards *The nine chapters* that appears to translate its being approached as a Canon, namely: that its readers regularly assumed that it should encompass the whole of mathematics, which, in ancient China, meant all mathematical procedures. In a second part, for the specific case of mathematics, I shall offer an interpretation of this expectation and describe the modes of reading the text and practicing mathematics that seem to have been put into play in relation to this belief. This may help us understand this statement more generally, as regards other canons.

The reason why we can address such questions is that our sources provide evidence enabling us to observe Chinese readers on quite a large time span. Indeed, as was usually the case for canons, commentaries were composed on *The nine chapters*, some being chosen by the

tradition to be handed down together with the text of the canon itself. This will provide the source materials on the basis of which to tackle our questions. In fact, no extant edition of *The nine chapters* survived that did not include the commentaries completed by Liu Hui in 263 and those composed by a group under Li Chunfeng's supervision and presented to the throne in 656—I shall designate the latter as “Li Chunfeng's comments”. Incidentally, the first extant source that refers to *The nine chapters* as a “Canon” is Liu Hui's preface to his own commentary<sup>1</sup>. After what the available evidence shows as an interruption, the Song dynasty bears witness to a revival of interest in *The nine chapters*. The Department of the Imperial Library (*mishusheng*) printed the Canon with its two traditional commentaries in 1084, the reprint of this edition by Bao Huanzhi in 1213 being the earliest extant edition today. Moreover, new commentaries were composed, such as: Jia Xian's *Detailed procedures of Huangdi's Canon of the Nine chapters on mathematics* (*Huangdi jiuzhang suanjing xicao*), in the first half of the 11<sup>th</sup> century, printed by Rong Qi in 1148, and Yang Hui's *Detailed explanations of The nine chapters on mathematical methods* (*Xiangjie jiuzhang suanfa*) in 1261. Through these commentaries, the historian can thus observe, on a time span of almost 1000 years, specific readers who made explicit their expectations towards, and their interpretation of, *The nine chapters*. Let us hence start by observing some features of their approach to the Canon.

#### THE COMMENTATORS' EXPECTATIONS TOWARDS THE CANON

The striking fact with which one is confronted when observing the attitudes towards *The nine chapters* manifested by the commentators is that, although Liu Hui, Li Chunfeng and Yang Hui operated at very different time periods and although their commentaries present important differences, they, however, share some surprising expectations towards the book, which seem to relate to its status as a Canon. To bring this point to light, let us concentrate on three

declarations made, respectively, by each of them. If, at first sight, they might appear somewhat obscure, they should become progressively clearer, as I develop an interpretation of the key elements they introduce that are all essential for my argumentation.

The first declaration, by Liu Hui, occurs as, in his preface, he describes the genesis of the text of *the nine chapters* and his own commentary. He writes<sup>ii</sup>:

(1) “ (...) Records tell that Li Shou created mathematics (...). It is only when the Duke of Zhou established the Rites that [we know that] the nine parts of mathematics existed. The development (*liu*) of these nine parts hence produced *The nine chapters*. Formerly, the cruel Qin burnt the books. The procedures of the Canon got scattered and damaged. After that time, the Bei Ping Marquis Zhang Cang and the Assistant of the Grand Minister of Agriculture Geng Shouchang both acquired a universal reputation for their excellence in mathematics. On the basis of scraps of the old text (*wen*) that were handed down, Zhang Cang and others made both excisions (*shan*) and completions. This is why, when one examines its sections, at places they differ from the ancient and what is discussed is much in modern terms.

As a child, I studied *The nine chapters*; when adult, I again looked at it in detail. I observed the dividing of *Yin* and *Yang*, synthesized the source of mathematical procedures. Having spent much time to fathom its depths, I managed to understand its meaning/intention<sup>iii</sup> (*yi*). This is why I dared (...) compose a commentary on it. The accomplishments and their categories develop in relation to one another<sup>iv</sup>, but they each have that to which they go back/amount (*gui*). Therefore, the reason why, although they divide into branches, they share the same stem is that they emerge from only one of their ends (*duan*). Furthermore, if

the internal constitutions (*li*) are analyzed with statements (*ci*) and if the bodies are dissected with figures (*tu*), (one sees that *The nine chapters*, as restored during the Han dynasty) get close to, though made simple, being able to encompass and, though bringing into communication, not being confusing. (...) Although one speaks of “the nine parts of mathematics”, they have the capacity to exhaust the subtle (*xian*) and to penetrate the minute (*wei*), to fathom what knows no bounds (what has no location —the *shen*)”. (My emphasis)

Note, first, that Liu Hui considers *The nine chapters* from the perspective of its ability to encompass mathematics and, more generally, stresses the unlimited potentiality of the “nine parts of mathematics” that, in his view, developed into *The nine chapters*, contrasting it with the moderation of its size. If one may note a nuance in Liu Hui’s assertion of the comprehensiveness of the Canon (“they get close to being able to encompass”), in my view, it is directed at Zhang Cang’s and the other Han editors’ inability to adequately restore the Canon, damaged by the Qin burning of books. This interpretation is supported by the second part of Liu Hui’s preface, where, after having introduced a problem and evoked some procedures of *The nine chapters*, he states:

(2) “But (in *The nine chapters*) there is nothing of the category (*lei*) of (...the problem introduced...). Therefore the procedures made by Zhang Cang and the others do not yet suffice to exhaust extensively all mathematics (*bo jin qun shu*). Within the nine parts of mathematics, I investigated the one named “double difference”. I examined (*yuan*) its essential points (*zhiqu*) so as to make them extend to/be efficient for (*shi*) this (problem).

(...Here follows the statement of the procedures belonging to this category to solve the problem) I elaborated the “double difference” and wrote a commentary on it so as to explore the meaning (*yi*) of the ancients. I joined it after (the chapter) “base and height” (*gougu*, i.e.: the last chapter).” (My emphasis)

As one can see here, again, the expectation made explicit is that the Canon restored should “exhaust all mathematics”. The blame for the inadequacy of *The nine chapters* in this respect is put on the Han editors, not on the Classic itself. Moreover, Liu Hui expounds the method he follows to deal with what he feels is a lacuna that requires to be filled. He does not claim to have invented the procedures he introduces in his preface to this end, but shows how he obtained them through “investigating”, in a certain way, one of “the nine parts of mathematics”. He hence maintains to be using for this purpose that from which he stated, as we saw (1), that *The nine chapters* derive. It is interesting to note, in this respect, that his last sentence seems to indicate Liu Hui conceives of himself as continuing the work of editing the text of the Canon, on the basis of the expectation that it should be complete. If such were the case, we would have here a concrete description of the editorial procedure, as practiced by Liu Hui, which would indicate the mathematical activities involved in editing.

Other elements in these excerpts are worth stressing. According to the former one (1), the capacity of the Canon to encompass mathematics seems to be brought to light as a result of the commentary. More precisely, it is the analysis of the “internal constitutions” or *li*’s and the dissections of the bodies that highlight the properties of the Canon stated. In addition to this, Liu Hui makes explicit that what prompted the writing of his commentary was the fact of having reached an understanding of the “meaning”, the “intention” (*yi*) of *The nine chapters*. In the same vein, in the latter passage (2), he again gives his commentary as aiming at “exploring the meaning (*yi*)” of the ancients. We shall come back to interpreting these terms below. Lastly,

let us stress that in the same lines of his preface where he introduces the theme of the completeness of the Canon, Liu Hui also alludes to an architecture of mathematical knowledge as deriving, by differentiation, from a unique stem.

Interestingly enough, a comparable combination of elements is to be found in a declaration in which the 7<sup>th</sup> century commentator Li Chunfeng also reveals his belief that *The nine chapters* somehow encompass all mathematical procedures. This declaration, included in the “Monograph on the musical scale and the calendar (*Lilizhi*)” of the *History of the Sui dynasty* (*Suishu*) that was prepared under Li Chunfeng’s supervision<sup>v</sup>, reads as follows:

(3) “As for what is named *lü*, there are nine (parts of mathematics) that flow from them: the first is called “Rectangular fields”... (Li Chunfeng lists here the titles of all chapters of *The nine chapters* and the mention by which Liu Hui makes precise the questions they enable to deal with)... They all multiply to disaggregate them, divide to assemble them, homogenize and equalize to make them communicate, apply the (procedure) of “Suppose” (i.e.: the rule of three) to link them together, hence the methods of the mathematical procedures are exhausted by these.” (My emphasis)

Li Chunfeng thus also both asserts that the Canon is complete in a sense and describes an architecture of mathematical knowledge. The articulation between these two elements is worth stressing: the statement regarding the comprehensiveness of the Canon appears as a *conclusion* based on the architecture described. The reasoning underlying the assertion can be sketched as follows: the nine chapters of the Canon have a common origin —*lü*, a concept to which, again, we shall come back—, and their procedures all make use of the same and limited set of fundamental operations applied to such entities. If, from this notation, Li Chunfeng can

*deduce* that, more generally, these also “exhaust the methods of the mathematical procedures”, this implies that the property established for *The nine chapters* extends to the whole of mathematics. This deduction hence presupposes the expectation that the Canon encompasses, in a way to be analyzed, all mathematical procedures.

Song-Yuan statements are clear-cut in expressing the same expectation. For instance, the 13<sup>th</sup> century commentator Yang Hui asserts:

(4) “Everyone who studies mathematics (*suan*) considers the method for multiplication as capital. Every time one puts (the terms) of the method (on the counting board), one wants that its result be appropriate; if one determines the positions, so that one makes the numbers correspond to each other adequately, one wants that it be not false. In case by division, one does not exhaust (the dividend), one takes the divisor as denominator and the dividend as numerator (i.e.: as becomes clearer below, division is considered from the point of view of what makes it the exact inverse of multiplication). If they are too complex (i.e.: if they have a common divisor), one simplifies them; if, in return, one makes the parts communicate (i.e.: one multiplies the integer by the denominator and adds the numerator), then one comes back to the origin (*huanyuan*). Such are the fundamental tools of multiplication and division (...A list of the crucial points of *The nine chapters* follows, which all amount to diverse uses of multiplication and division...). These are what exhausts the inner constitution (*li*) of mathematical methods. *The nine chapters by Huangdi* are complete and subtle; they encompass all situations (*qing*). They were undeniably written by a Sage.” (My emphasis)

It is remarkable that, again, the same elements are combined in the declaration of exhaustiveness of *The nine chapters*: an architecture of mathematical knowledge is unfolded from a base constituted of multiplication and division, after what Yang Hui asserts the completeness of the Canon. Again, this conclusion is mediated by a statement regarding the “inner constitution” (*li*) of mathematical methods, a point that echoes with Liu Hui’s preface.

In conclusion, even if their declarations display differences, the three commentators concur in believing that *The nine chapters* do, or should, encompass all mathematical procedures. In contrast to Li Chunfeng and Yang Hui who both assert the completeness without restraint, Liu Hui’s statement is less affirmative. However, as I tried to show, his restriction regards the text of the Canon as edited by previous generations and, in his view, the genuine Canon should be comprehensive. Hence, the nature of the expectation remains the same.

As a side remark, note that, if the commentators believed *The nine chapters* encompassed mathematical reality, no wonder that their mathematical activity could take the shape of composing a commentary on the Canon. More importantly, the key point here is that, as John Henderson (1991, pp. 100 *ff.*) has shown, this feature constitutes a very common expectation that commentators exhibit with respect to “Canons” throughout Chinese history. One early expression of this belief can be found, for example, in the “Great commentary” (*Xici dazhuan*) with respect to the *Book of changes*: “The *Changes (Yi)* is broad, great and all-encompassing. There are in it the Way of heaven, the Way of man, and the Way of earth”<sup>vi</sup>. The corresponding expectation encountered with respect to *The nine chapters* hence probably only expresses that the book was perceived as a Canon, comparable to, among others, the Confucian ones. We do have evidence that such a comparison was made by the actors themselves. One of its clearest expressions can be found in a passage by the 13<sup>th</sup> century commentator, Yang Hui, who is himself quoting the preface composed by Rong Qi when he had Jia Xian (11<sup>th</sup> c.)’s commentary printed in 1148. He states: “When the government instituted the examinations in

mathematics to select officials, they chose *The nine chapters* to be the most important of the mathematical Canons, since, indeed, it is like the six Canons of the Confucians, the (*Canon of difficulties*) and the (*Grand Simplicity*) of the medical schools, the *Book of Master Sun* of military art!” This piece of evidence, incidentally, legitimates that we may rely on studying the attitudes towards *The nine chapters* to approach more broadly the phenomenon of canonicity.

A second point, common to our three commentators, deserves some attention: they all describe an architecture of mathematical knowledge in connection to stating the comprehensiveness of the Canon. As regards *how The nine chapters* encompass mathematics or in relation to which architecture of mathematical knowledge this belief is expressed, again the commentators diverge. Liu Hui’s assertion that mathematical realities “share the same stem” may be interpreted as expressing views that are quite close to Li Chunfeng’s. Indeed, in his commentary on the addition of fractions (*hefenshu*), Liu Hui states:

(5) “Multiply to disaggregate them, simplify to assemble them, homogenize and equalize to make them communicate, how could those not be the key-points (*gangji*) of computations/mathematics (*suan*)?”

Without yet discussing the meaning of this assertion, one can notice that it focuses on three of the four fundamental operations that lay at the core of Li Chunfeng’s declaration quoted above (3). These operations may hence constitute the stem from which, according to Liu Hui’s description of the architecture of mathematics, the procedures “diverge into branches”. In fact, when considering his entire commentary, this is indeed the most plausible interpretation<sup>vii</sup>. As a result, the architectures described by our two commentators would hence partially coincide. Two main differences can be noticed, though. First, Li Chunfeng diagnoses that a fourth fundamental operation, the “rule of three”, should be added to the list of key-points

suggested by Liu Hui. Moreover, the architecture described by the 7<sup>th</sup> century commentator does not only consist in a stem of operations to which all procedures go back, but develops from a concept—the *lǚ*—that is given as its root. Yet, besides these differences, both architectures share fundamental features.

In contrast to them, Yang Hui (4) claims that a couple of opposed, but complementary, operations, namely: multiplication and division, are at the basis of mathematics. From there, the fundamental operations at play in the various chapters of the Canon are derived, and these are those from which mathematical procedures are exhausted, in a sense still to be made clear.

However, despite the divergences, the fact remains common to our three commentators that their formulation of the expectation of completeness always went along with making explicit a conception of an architecture of mathematics. Furthermore, in fact, the various architectures described share the feature of displaying how mathematics develops from a limited list of fundamental operations.

These facts raise several questions relating to “Canons” as a type of text. First of all, how are we to interpret that a book may be all encompassing? And, in our case, more specifically, how does this belief relate to providing a description of the structure of mathematics that shares the features outlined above? Dealing with the issues will lead us to the main question of this paper: how was such a kind of book as a Canon read? How did the kind of exegesis carried out by our commentators connect to these issues? In what follows, I shall address, though briefly, these questions, relying mainly on the commentary ascribed to Liu Hui.

#### MODES OF READING THE CANON

Indeed, as already noted, the expectation that a book is all encompassing, that it contains all mathematical procedures may disconcert a modern reader. This astonishment may become even deeper when skimming through *The nine chapters*. In fact, the Canon is composed of 246

particular problems, their numerical answers and algorithms to solve them. Let us quote some of these elements to give an idea of the whole.

The ninth chapter, “ base and height ” (*gougu*), which is devoted to the right-angled triangle, opens as follows<sup>viii</sup>:

(6) “SUPPOSE THAT THE BASE (*GOU*) IS WORTH 3 *CHI* AND THE HEIGHT (*GU*) 4 *CHI*.  
ONE ASKS HOW MUCH THE HYPOTENUSE MAKES.

ANSWER: 5 *CHI*

(...<sup>ix</sup>)

PROCEDURE OF THE BASE AND THE HEIGHT:

BASE AND HEIGHT BEING EACH MULTIPLIED BY ITSELF, ONE ADDS (THE RESULTS)

AND DIVIDES THIS BY SQUARE ROOT EXTRACTION, WHICH GIVES THE

HYPOTENUSE”.

This problem is specific as regards its data, but the situation it involves is abstract.

However, most of the other problems are specific in both respects, such as the following one<sup>x</sup>:

(7) “SUPPOSE THAT A GOOD WALKER WALKS 100 *BU* WHILE A BAD WALKER WALKS  
60 *BU*. IF, NOW, THE BAD WALKER FIRST WALKS 100 *BU*, BEFORE THE GOOD  
WALKER [STARTS] PURSUING HIM, ONE ASKS IN HOW MANY *BU* HE WILL CATCH  
HIM UP.

ANSWER: 250 *BU*”

And the algorithm given to solve it is expressed in terms relating to the situation described and with concrete numbers.

Understanding how 246 such problems and the procedures to solve them could be conceived of as encompassing all mathematical procedures, here is the challenge confronting us.

Observing how commentators read *The nine chapters* and how they apprehended mathematics will provide some clues. This implies that, if we did not take such precautions and if, as modern readers, we were anachronistically reading the Canon as the set of its problems and algorithms, we would run the danger of completely missing the import of *The nine chapters* for its Chinese readers of the past.

The commentaries attest to several distinct modes of reading that were put into play and that, each in a different way, developed the text as given in *The nine chapters* beyond its simple appearance. I have already dealt with some of them elsewhere, and I shall simply summarize here some results.

A first mode of reading illustrates how one item can be taken to stand for a multitude. Indeed, it can be shown that, as one may expect, the commentators do not read a problem as a particular case, but as standing for a whole category of problems. Interestingly enough, Liu Hui explicitly relates this reading of problems to the way in which, according to the *Analects*, Confucius expects his disciples to develop his teachings. In our case, it can be shown more precisely that it is the procedure given to solve a given problem that provides the support on the basis of which the category for which it stands is determined. This category consists in all the similar problems that the same procedure can solve. It does happen, though, that the procedure given by *The nine chapters* is not as general as it could be, and hence that the problem does not stand for as large a class as it could: Liu Hui indicates such cases and reformulates a more general algorithm in his commentary. This reveals that the commentator expects that a procedure be general and a problem stand for a class<sup>xi</sup>. Let us stress that Liu Hui does not seem to particularly value formulating an *abstract* procedure that would *apply* to various particular

cases. In contrast to this, he rather prizes the *generality* of an algorithm that *circulates* from problem to problem as far as possible. This fits with his reading the Canon as expressing the general in terms of the particular. This interest in the generality of a procedure, as we shall see, extends quite far. Moreover, such examples show that Liu Hui, seemingly, does not manifest any undue awe towards *The nine chapters*, despite its status, and, in this case, simply indicates its limitations.

Observing Liu Hui's commentary reveals a second mode of reading. The commentator appears to read meanings in the way in which the sequence of problems are organized within *The nine chapters*. This can be argued in several ways as regards chapter 8 "Measures in square" (*Fangcheng*), which deals with systems of  $n$  linear equations with  $n$  unknowns. Liu Hui thus progressively extends the range of systems to which the basic algorithm can be applied, through filling the gaps between the successive problems. He hence appears to "read" the arrangement of the sequence of problems. Moreover, in this case, his interpretation of chapter 8 can be shown to develop from reading in two distinct ways the terms of the first problem<sup>xii</sup>.

Chapter 9 also provides an example where Liu Hui appears to read a meaning in the arrangement of the problems. We mentioned above (6) the first and abstract problem with which this chapter devoted to the right-angled triangle opens. The sequel of the chapter consists in a sequence of seemingly more concrete problems. Liu Hui comments on the title "PROCEDURE OF THE BASE AND THE HEIGHT" as follows:

(8) "(...) (base, height, hypotenuse and the procedure) are about to be extended (*shi*) to all the algorithms, this is why this procedure is set out first so as to make their origin (*yuan*) appear." (My emphasis)

This comment is interesting in several respects. First, Liu Hui's point aims at *explaining* the position of the first set of problem and procedure in the chapter: he reads a meaning, an intention, in its being placed at the beginning of the chapter.

Secondly, the way in which this location in the chapter is justified is most interesting and it takes us closer to essential aspects of the commentary. The “procedure of the base and the height”, an algorithm amounting to what we call “Pythagorean theorem”, is described as the “origin” of the following algorithms in the chapter. This relates to the fact that, in his comments where he accounts for the correctness of these algorithms, Liu Hui brings to light that each of the procedures solving a problem of chapter 9 puts into play the “procedure of the base and the height”. Therefore, a first connection appears, with respect to procedures, between “going back to their origin”, to the source —a recurring theme in Liu’s preface— and proving through revealing that a general algorithm is put into play.

Moreover, the relationship of this fundamental procedure to the following ones is described as an “extension”, in the sense that its use spreads to settling the situations listed in the sequel of the chapter. We recognize the interest in the circulation of procedures, or in generality, mentioned above. This remark by Liu Hui thus reveals a second connection between the fact that a procedure be the “origin” of others and its capacity to spreading widely. In Liu Hui's understanding, putting the “procedure of the base and the height” at the beginning of the chapter amounts to highlighting these facts.

It is interesting that a similar connection between “going upstream” towards the “essential points”, or the “origin”, of a procedure and being able to spread their efficiency further can be noticed in Liu Hui's preface (2), where we read: “ I examined (yuan) its essential points (zhiqu) so as to make them extend to/be efficient for (shi) this (problem)”.

Could this be what Liu Hui refers to when, twice in his preface, he speaks of his commentary as offering an understanding of “the meaning” or “the intention (*yi*)” of the

Ancients or the Canon? I believe it may be so in several ways. And, in order to capture from yet another angle Liu Hui's ways of reading *The nine chapters*, I suggest to turn now to what his own commentary tells us of his conception of the “*yi*” of the Canon, a term which seems central in his own representation of his approach to the text.

Let us start from an occurrence of the term that is highly revealing of both an assumption the commentator makes with respect to *The nine chapters* and the mathematical practice relating to searching the “meaning (*yi*)” of the Canon. The context is the procedure given for the so-called “extraction of the spherical root”, in which a given volume is considered to correspond to a sphere whose diameter is sought-for. The Canon reads as follows<sup>xiii</sup>:

(9) “SUPPOSE AGAIN ONE HAS A NUMBER—PRODUCT OF 1644866437500 *CHI*. ONE ASKS HOW MUCH THE DIAMETER OF THE SPHERE MAKES.

ANSWER: 14300 *CHI*.

PROCEDURE FOR EXTRACTING THE SPHERICAL ROOT:

ONE PUTS THE QUANTITY OF *CHI* OF THE NUMBER—PRODUCT, MULTIPLIES IT BY 16, AND DIVIDES IT BY 9. TO DIVIDE WHAT IS OBTAINED BY EXTRACTION OF THE CUBE ROOT GIVES THE DIAMETER OF THE SPHERE.”

Liu Hui's commentary first formulates a reasoning that may have produced the procedure. Let us follow it, before drawing some conclusions. First, he identifies that the procedure “probably (*gai*)” rests on taking the ratio of the circumference of a circle to its diameter as the one of 3 to 1. These values are used throughout *The nine chapters*. Liu Hui qualifies them as *lü*, thereby indicating that they can be both multiplied, or divided, by a same number without the relationship between the circumference and the diameter they express to be

changed. More generally, Liu Hui will qualify as *lū* any set of values sharing this property.

Afterwards, he goes on:

(10) “ If one hence supposes that the surface (*mi*) of the circle fills  $\frac{3}{4}$  of the surface (*mi*) of the square, the circular cylinder thus also fills the  $\frac{3}{4}$  of the cube.

If, furthermore, one supposes that, the cylinder being represented by the *lū* of the square, 12, what represents the *lū* of the sphere is 9, then, in addition to this, the sphere fills  $\frac{3}{4}$  of the circular cylinder.

(...computation on fractions...)

Therefore the sphere fills  $\frac{9}{16}$  of the cube. This is why, when one multiplies its volume by 16 and divides by 9, one obtains the volume of the cube.

The diameter of the sphere and the side of the cube are equal, hence, if one divides this by extraction of the cube root, one obtains the diameter.

But this meaning/reasoning (*yi*) is wrong. How to prove (*yan*)<sup>xiv</sup> it?”

The most important fact here is that, in a first part of his commentary on this problem, Liu Hui develops a reasoning that accounts for the procedure given, and concludes that this “*yi*” is wrong, a fact that he sets out to prove. The idea of the reasoning runs as follows: the procedure amounts to giving the sphere as  $\frac{9}{16}$  of the circumscribed cube. If the ratio to the circumference to the diameter is 3 to 1, the inscribed cylinder fills  $\frac{3}{4}$  of the cube. Then, if the sphere is taken as filling  $\frac{3}{4}$  of the cylinder, the given procedure is obtained. This is the *yi* that is said to be wrong, and, in order to criticize it, Liu Hui will bring to light that the solid in which the sphere occupies  $\frac{3}{4}$  of the volume is not the cylinder, but the intersection of two inscribed cylinders. However, he will lament of being unable to push the reasoning forward and establish the relationship between the volumes of this solid and the cube.

This passage is interesting for us here in several respects. Liu Hui seems to rely on the context of *The nine chapters* as well as on the form of the procedure to formulate a reasoning producing the procedure. The key point for me here is that he refers to this reasoning that may have accounted for the correctness of the procedure with the term “*yi*”. The “*yi*” of the Canon, from the understanding of which he says his commentary derives, could hence have taken the form of a reasoning that yields the procedure recorded in the Canon. This would imply that Liu Hui assumes there are proofs “meant” in the statement of the procedures of the Canon, and his commentary makes them explicit. This fits with his conceiving of a commentary as “exploring the meaning (*yi*) of the ancients” (passage 2)<sup>xv</sup>. Li Chunfeng’s commentary attests to the same use of the term *yi*. After having quoted the alternative procedure elaborated by Zu Gengzhi for solving the same problem, he introduces the proof that establishes its correctness by the question: “What is its meaning (*yi*)?”.

However, in this case, after having brought to light a “*yi*”, Liu Hui discards it, through proving its mathematical inadequacy. It is difficult for me to determine whether he discards what he conceives of as being here “the *yi*” of the Canon, or whether he rejects this first interpretation of the Canon. I would be tempted to opt for the first explanation, which would have important consequences as regards our understanding of the practice of exegesis in this case. Whatever the case, it is interesting that we see mathematical reasoning intervening in two ways for the exegesis. Interpreting the Canon requires writing down a would-be proof. Accepting it as *yi* also supposes that reasonings be put into play.

This example reveals some general points concerning proof as carried out in the context of the commentaries to *The nine chapters*. The deduction quoted above and formulated to account for the “procedure for extracting the spherical root” combines a geometrical reasoning and the writing of the procedure to which it corresponds and that yields the result sought-for. The end of the proof consists in using valid operations to transform this procedure

that is now known to be correct into the procedure the correctness of which is to establish<sup>xvi</sup>.

The proof hence makes use of algorithms at various levels. This explains why, as we shall see, there are tight relations, and exchanges, between proof and algorithm.

The elements of interpretation of the term *yi* provided by this example are confirmed by another interesting occurrence in the commentary<sup>xvii</sup>. The problem quoted in passage (7) is solved by a procedure the correctness of which the commentator establishes. More precisely, he formulates a reasoning that explains what the algorithm carries out and he simultaneously highlights that, in fact, it formally amounts to putting into play a rule of three. The following problem in *The nine chapters* is different, even though the situation described is of the same type. And, accordingly, the algorithm given to solve it differs from the previous one. Again, Liu Hui expounds a reasoning showing why this procedure works and highlighting that it too has the structure of a rule of three. However, what is most striking is that the two reasonings exhibited bring to light the fact that both procedures make use of similar features in both situations in the same way. In other terms, Liu Hui comments both procedures from an angle chosen in such a way as to disclose that the same reasoning accounts for both. This is what the commentator stresses by saying, when dealing with the second one: “Its *yi* is like the one for the previous procedure”. Again, here, the two procedures have a *yi*, “a meaning”, which is made explicit by the commentator. This *yi* is a reasoning accounting for the correctness of the procedure. This is what can be deduced from what is being compared and declared similar. The procedures appear different. It is only at the level of their “meaning/*yi*” that their likeness is exhibited and stated.

In conclusion, to sum up, Liu Hui seems to assume that the *yi* of *The nine chapters* — the understanding of which, his preface tells, was what prompted the writing of the commentary— can take the form of the reasonings that produced the various algorithms. As a consequence, making reasonings explicit constitutes one of the mathematical practices relating

to inquiring into the “meaning (*yi*)” of the Canon. This agrees with what can be found in Liu Hui’s and Li Chunfeng’s commentaries: after virtually every procedure of the Canon, they formulate proofs of its correctness. Incidentally, this also agrees with the more general use of the term *yi* throughout the commentaries. *Yi* refers to the “meaning” of an operation, in the sense of an interpretation of that which the operation computes formulated in the terms of the situation, and making these *yi* explicit proves to be a key aspect in the proof of algorithms.

Another argument supports this interpretation of *yi*. In proving algorithms that evaluate geometrical magnitudes, Liu Hui regularly makes use of figures (*tu*) for plane geometry or blocks (*qi*) for space geometry. As we saw (1), he introduced the *tu* as a major tool for his commentary: “(...) and if the bodies are dissected (*jie*) with figures (*tu*) (...)”. When, for proving the correctness of the algorithm of cube root extraction, he first introduces the second type of visual auxiliary, the blocks, he justifies it by quoting the “Great commentary” (*Xici dazhuan*) and referring to his preface: “ ‘Speech cannot exhaust the “meaning” (*yi*) (*yan bu jin yi*)<sup>xviii</sup>, hence to dissect/analyze (*jie*) this (volume), one must use blocks, this is the only way to get to understanding (the procedure)”. Among the many links that this statement establishes, let us stress two. Blocks were just used to formulate the proof of the correctness of the procedures carried out by Liu Hui. Their introduction, in addition to words, is thus connected with making the “meaning (*yi*)” explicit. Moreover, the production of the proof, with words and blocks, is given as aiming at “understanding” this part of the Canon. Both points confirm the link we established between looking for the *yi* and producing proofs.

But the previous example involving the rule of three allows us to go further and go back to the questions with which we started, by revealing a crucial fact: writings down proofs, or, in other terms, exhibiting the *yi*, leads to establish connections between procedures that, at first sight, seemed unrelated. Indeed, the fact that, on the basis of the proofs, unseen

connections between procedures can be unveiled appears to be a key motivation for inquiring into the *yi*.

The relation exhibited between the two problems evoked above establishes a link between them at two levels. First, at the level of the situations for which the procedures are provided, the proofs bring to light that, in fact, the reasonings make use of intermediaries that have the same meaning and use them in the same way. From a semantic point of view, the procedures thus appear to rely on the same strategy. Secondly, at a more formal level, the proofs simultaneously highlight that both procedures are instantiations of a more general algorithm, that is: the rule of three. In other terms, they now appear to also share the same formal strategy. And this connects them with the variety of procedures for which the commentators show that they, too, formally amount to putting into play rules of three. Such a situation evokes the relationship of all the procedures of chapter 9 to the “procedure of the base and the height”.

This is the very point where we can go back to the questions addressed in this paper, since, here, a link is established between what is brought to light by the proof and Li Chunfeng’s declaration (3). Indeed, the rule of three that proofs reveal is at play in various procedures happens to be one of the four fundamental operations that the 7<sup>th</sup> century commentator lists. What is striking here is that if we now turn to all the other operations that appear in the same list—and in Liu Hui’s list (5)—, i.e.: multiplication, division, homogenization and equalization, we realize that these operations present themselves in proofs too, and for some of them like homogenization only in proofs. Furthermore, in the proofs, they play exactly the same part as what we sketched for the rule of three. I cannot develop this point in detail here, I shall only allude to it<sup>xix</sup>: While proving the correctness of different procedures, Liu Hui regularly interprets the “meaning”, the “intention” (*yi*) of their key operations as “equalizing” (*tong*) some quantities and, in correlation with this, “making” other quantities

“homogeneous” (*qi*). This interpretation allows him to establish that these different procedures are correct. The quantities that are made equal differ in each of these contexts, and the reasons why, once they are equalized, the procedure can be shown to be correct, vary. However, at the formal level, like in the previous example, the proofs reveal an operational pattern common to all of them. They bring to light that, in fact, the procedures follow the same formal strategy in their way of relying on the situation in which they operate, to yield the result sought-for. This is expressed by the recurring of the pair of operations “equalize and homogenize”. As a consequence, in the same way as above, through proving, Liu Hui hence unveils links between procedures of *The nine chapters* that, at first sight, seemed unrelated. Procedures given by the Canon and apparently unconnected are revealed, through the proof, as being mere instantiations of the same fundamental procedures. It is interesting that the commentators specifically used another term, *yi*, to designate, for a procedure, the kind of “meaning” that comes from bringing to light the fundamental procedure underlying it. One hence regularly reads about a procedure that it has “the *yi*’ of homogenization and equalization” or the “*yi*’ of the rule of three”.

To sum up, our commentators’ practice of proof can be characterized as follows: through accounting for the correctness of the procedures, their proofs bring to light that, in the end, the various procedures put into play a limited number of formal strategies or, in other terms, fundamental operations. Thereby, the proofs reduce what *The nine chapters* gave as a diversity of procedures. And, a crucial connection emerges between the search for the *yi*, as carried out through proving, and the elements entering the architectures of mathematical knowledge described by our commentators: the formal strategies in a limited number that are brought to light by the proofs are the very operations listed in Liu Hui’s and Li Chunfeng’s list of fundamental procedures. Note that these operations can be identified through writing down the proofs not separately, but as a whole, and through continuously confronting them. We provided evidence that clearly demonstrates that Liu Hui compared procedures on the basis of

their *yi*. Also note the convergence between what comes out here through confronting the proofs and the meaning the commentator read, as we saw above, in the organization of chapter 9: the identification of general operations the efficiency of which extends as widely as possible appears to be a concern that permeates his reading of the Canon right through.

At this point, we are in a position to offer an interpretation of the declarations of our commentators and to elucidate the expectation that *The nine chapters* may be all encompassing. Moreover, we can also suggest an understanding of how this expectation relates to describing an architecture of mathematical knowledge.

Let us start with Li Chunfeng's statement (3). The previous interpretation of the link between the nature of the commentators' proofs and the emergence of fundamental operations fits with the structure and the wording of this declaration (3). All the procedures of *The nine chapters*, he claims, make use of four fundamental operations —this, in my view, is what writing down the proofs, or exhibiting the *yi* of the Canon, brings to light. Li Chunfeng lists the fundamental general procedures that appeared in the proofs the commentators made explicit for the whole Canon.

From this, Li Chunfeng *deduces* that this is what exhausts the methods of the mathematical procedures. Notice that his claim does not bear on the mathematical algorithms themselves, but on their “methods”. I suggest to understand that, by this term, Li Chunfeng refers to the procedure that, as I explained above, follows the reasoning producing, that is to say: accounting for, the algorithm. We shall see below other pieces of evidence supporting this interpretation. Li's claim could hence be understood as follows: the proofs of *The nine chapters* bring to light all the fundamental procedures that could appear in the proof of any mathematical algorithm. This leads me to an interpretation of the expectation that the Canon is complete. *The nine chapters* would be all encompassing in the sense that the procedures it contains reveal, through proving, all the fundamental operations needed in mathematics. This interpretation

would explain why the statement of comprehensiveness is always articulated onto the description of an architecture of mathematics. The commentator, by means of proving, establishes a list of the fundamental operations indirectly indicated by the Canon. They constitute the basis of the structure of mathematical knowledge, which explains why these architectures all display the same features. In addition to this, Li Chunfeng distinguishes himself by placing at the root of the structure a kind of object, the *li*, which he identifies as that which explains the efficiency of the fundamental operations<sup>xx</sup>.

The Canon would indicate these fundamental operations, by displaying some of the diverse manifestations they can take. The task of the commentator would then be to understand how these manifestations reveal the basic patterns of mathematical procedures, the “source” from which all mathematics flow. And we saw that the commentaries diverge as regards their list. Moreover, the Canon would indicate how the basic patterns present themselves in a diversity of situations, showing the way to extend their efficiency to virtually any situation.

If we understand that bringing to light how the fundamental operations are at play in a given procedure amounts to exhibiting the *li*, the “inner constitution”, of a procedure, then we can also interpret Liu Hui’s preface (1) and Yang Hui’s statement (4) in the same way.

The interpretation is straightforward in Yang Hui’s case. As regards Liu Hui, if the “stem” from which mathematical procedures diverge is understood as being the list of fundamental operations (5), then we can interpret in the same way that he moves from “analyzing the *li*” of the procedure to considering the property of *The nine chapters* to be all-encompassing. Again, the “meaning/*yi*” of the procedures would reveal that their *li* consists in articulating only the same limited amount of fundamental operations. Therefore, stating that the Canon is complete would amount to believing that any mathematical procedure would be produced with the same building bricks.

This line of interpretation receives confirmation from a statement included in his postface to *The nine chapters* by the 1213 editor of the Tang collection *Ten canons of mathematics*, Bao Huanzhi. He states:

(11) “Among the books of mathematical procedures, there are altogether ten schools. One can only take *The nine chapters* as the first of the Canons. With the methods of its nine parts of mathematics (*jiu shu*), there is nothing which is not complete. Although the methods established by the various schools present variation, when one looks for the original meaning (*yi*), they all come from it”  
(My emphasis)

If we follow Bao Huanzhi, *The nine chapters* are not all encompassing in the sense that the Canon would contain all mathematical procedures. Bao explicitly refers to procedures that may not be included in it. However, when it comes to their original meaning, an expression that combines the idea of source and that of *yi*, the connection between which was already noted, then it should necessarily fall within the scope of *The nine chapters*. One may understand that the procedure thereby would appear either as an instantiation of a general fundamental operation or amounts to a combination of them. It would hence be the proof of the procedures that reveals how, in fact, they derive from the Canon.

## CONCLUSION

Let us summarize the main points made in this paper. I have argued that the commentators’ explicit attempt to explore the *yi* of the Canon led them to systematically bring to light proofs of

the correctness of the procedures it contained. However, the practice of mathematical proof put into play in relation to exegesis was quite specific. Such proof seems to have been the tool through which connections between procedures of *The nine chapters* apparently unrelated were inquired into, at a formal level. This offered the commentators the main means to identify fundamental algorithms from which all the procedures of the Canon derive and the efficiency of which would extend the farthest away. Valuing the search for such algorithms can be best understood in a context where generality is granted much weight.

Such a practice of exegesis reveals a conception of how the Canon makes sense, which I would formulate as follows: *The nine chapters* display a variety of procedures that, though simple, are rich enough to, by confrontation of their proofs, indicate the “origin”, the “essential points” that can extend to virtually any situation. The canon would reveal the source through a variety of procedures that derive from it.

This may explain how exegesis led the commentators to describe a structure of the world of mathematical procedures, in which all procedures are given to flow from a limited number of fundamental and most general operations.

This may also help us understand in which ways the Canon could be expected to be all encompassing. *The nine chapters* were believed to highlight the fundamental operations with which to produce any procedure. In Bao Huanzhi’s terms, even if a procedure could appear to exceed the scope of the Canon, looking at its “original meaning” would reveal how, in fact, by extension or combination of fundamental operations, it still fell within the range of *The nine chapters*.

The specific practice of proof carried out by the commentators may hence appear to be the key element linking the statement of the completeness of the Canon and the conceptions of the architecture of mathematical reality.



## **BIBLIOGRAPHY**

### **[Chemla, Karine 1991]**

“Theoretical Aspects of the Chinese Algorithmic Tradition (first to third century)”,  
*Historia Scientiarum*, 42, pp. 75-98 +errata in the following issue.

### **[Chemla, K. 1997]**

“Qu'est-ce qu'un problème dans la tradition mathématique de la Chine ancienne ?  
Quelques indices glanés dans les commentaires rédigés entre le 3<sup>ème</sup> et le 7<sup>ème</sup> siècles au  
classique Han *Les neuf chapitres sur les procédures mathématiques*”, *Extrême-Orient, Extrême-Occident*, 19, pp. 91-126.

### **[Chemla, K. 1997a]**

“What is at Stake in Mathematical Proofs from Third Century China?”, *Science in Context*, 10, 2, 1997, pp. 227-51

### **[Chemla, K. 1997/8]**

“Fractions and irrationals between algorithm and proof in ancient China”, *Studies in History of Medicine and Science*, 15, n° 1-2, New Series, 1997/98, pp. 31-54.

### **[Chemla, K. 2000]**

“Les problèmes comme champ d'interprétation des algorithmes dans *Les neuf chapitres sur les procédures mathématiques* et leurs commentaires. De la résolution des systèmes d'équations linéaires”, *Oriens-Occidens*, 3, pp. 189-234.

### **[Chemla, K. forthcoming]**

“Algorithmes et histoire de la démonstration”, in A. Allard, R. Morelon, C. Sasaki,  
*Mélanges*, forthcoming.

### **[Chemla, K. & Guo Shuchun (forthcoming)]**

Les neuf chapitres sur les procédures mathématiques *et les commentaires de Liu Hui et de Li Chunfeng*, édition critique, traduction française et présentation, (to appear).

**[Henderson, John B. 1991]**

*Scripture, Canon and Commentary. A Comparison of Confucian and Western Exegesis*, Princeton University Press, 248 p.

**[Qian Baocong 1963]**

*Suanjing shishu. Qian Baocong jiaodian* (Critical punctuated edition of *The Ten Classics of Mathematics*), Zhonghua Shuju, 2 volumes.

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<sup>i</sup> [Qian 1963, p. 91]. Unless otherwise stated, I shall refer the reader to Qian Baocong's edition of *The nine chapters* —[Qian 1963]. Together with Prof. Guo Shuchun (Institute for the history of natural sciences, since 1984, I have been working on a critical edition and a French translation of *The nine chapters* and the commentaries. My ideas on these sources were deeply influenced by our collaboration.

<sup>ii</sup> For many points or statements that cannot be developed here, I refer the reader to my footnotes to the preface as well as to my glossary of mathematical terms in [Chemla & Guo (to appear)]. In the paper, I shall number the passages quoted so as to refer the reader back to them more conveniently.

<sup>iii</sup> As is suggested by the development on this term below, this may also be understood in the plural: “its meanings/intentions”.

<sup>iv</sup> In this context, “accomplishment” probably refers to mathematical procedures, objects as well as problems contained in *The nine chapters*. One can also interpret “The categories of the accomplishments develop in relation to each other”. This can be understood in relation to fractions (*hefenshu*), to volumes (*kailiyuanshu*) or to procedures (*chengfenshu*). Realities may

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be transformed according to those with which they enter in a procedure, or to those with which they are compared, and hence change category (*lei*), but that “to which they amount” (*suogui*, in the next part of the sentence) does not change. The concepts *lei* and *suogui* form, I believe, a contrasting pair.

<sup>v</sup> *Zhongguo tianwen lifa shiliao*, Dingwen shuju, 1977, volume 3, p. 1859.

<sup>vi</sup> *Xici, xia*, chap. 10. Trans. quoted in J. Henderson [1991, p. 101].

<sup>vii</sup> Perhaps, the procedure for the comment of which these operations are brought to light—the addition of fractions— could then be the unique “end” from which mathematical realities are said to emerge. But there is no evidence I can think of that would confirm this hypothesis.

<sup>viii</sup> [Qian 1963, p. 241], problem 1 of chapter 9. Throughout the paper, I quote the Canon in capital letters so as to distinguish it from the commentaries.

<sup>ix</sup> This first problem is followed by two other ones, deduced by applying a circular permutation to the data and the unknown. And the first procedure quoted below is followed by similar procedures allowing given any two sides of a right-angled triangle to yield the third one.

<sup>x</sup> [Qian 1963, p. 191-191], problem 12 of chapter 6.

<sup>xi</sup> See [Chemla 1997]. Note that, in passage (2) quoted above, Liu Hui mentions looking for a problem in the Canon that would have the same category as the problem he wants to solve. He concludes that the procedures gathered by Zhang Cang and others do not suffice, which reveals the relation between the procedure and the category.

<sup>xii</sup> See [Chemla 2000].

<sup>xiii</sup> [Qian 1963, p. 155-6], problem 24 of chapter 4.

<sup>xiv</sup> In Liu Hui’s commentary and later mathematical texts, *yan* refers to a proof that makes use of visual auxiliaries, “figure” *tu* or blocks (*qi*), as will be here the case [Chemla 1997].

<sup>xv</sup> In [Chemla 1991], I gave arguments in favor of this hypothesis.

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<sup>xvi</sup> This component of the proofs appears to me as a kind of “algebraic proof within an algorithmic context”, see [Chemla 1997/8 and forthcoming].

<sup>xvii</sup> See [Chemla forthcoming] for a detailed analysis of the passage alluded to here.

<sup>xviii</sup> *Xici, shang*, chap. 12.

<sup>xix</sup> See [Chemla 1991, 1997a, 2000].

<sup>xx</sup> For more detail, see [Chemla forthcoming].