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Symmetric and hierarchical fusion of expert opinion in the Transferable Belief Model, application on a climate sensitivity dataset

Minh Ha-Duong*

December 10, 2007

Abstract

This paper examines the fusion of conflicting and not independent expert opinion in the Transferable Belief Model. Regarding procedures that combine opinions symmetrically, when beliefs are bayesian the non-interactive disjunction works better than the non-interactive conjunction, cautious conjunction or Dempster's combination rule.

Then a hierarchical fusion procedure based on the partition of experts into schools of thought is introduced, justified by the sociology of science concepts of epistemic communities and competing theories. Within groups, consonant beliefs are aggregated using the cautious conjunction operator, to pool together distinct streams of evidence without assuming that experts are independent. Across groups, the non-interactive disjunction is used, assuming that when several scientific theories compete, they can not be all true at the same time, but at least one will remain. This procedure balances points of view better than averaging: the number of experts holding a view is not essential.

This is illustrated with a 16 experts real-world dataset on climate sensitivity from 1995. Climate sensitivity is a key parameter to assess the severity of the global warming issue. Comparing our findings with recent results suggests that, unfortunately, the plausibility that sensitivity is small (below 1.5C) has decreased since 1995, while the plausibility that it is above 4.5C remains high.

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1 Introduction

Is there a single all-purpose aggregation method for expert opinion ? According to Ouchi [2004], the answer is negative. Indeed, there is at least three different classes of aggregation methods, based on different ways to represent mathematically an expert opinion. One is probabilistic risk analysis [Cooke and Goossens, 1999]. Another approach is to use the fuzzy numbers theory to combine opinions represented as possibility distributions [Sandri et al., 1995]. We are interested here in a third approach: Dempster-Shafer theory of evidence [Shafer, 1990].

We will use a variant of the theory of evidence named the Transferable Belief Model, and more specifically examine new operators for information fusion recently proposed by Denœux [2006]. We study the applicability of these operators for the aggregation of expert opinion, using a real-world dataset from Morgan and Keith [1995].

This dataset illustrates four challenges for mathematical aggregation methods. First, it cannot be assumed that opinions are, statistically speaking, independent: that would overestimate the precision of the actual information in the field. Second, there is complete contradiction among experts: aggregation methods that take somehow the intersection of the opinions can not work when the intersection is empty. Third, the disagreement between experts is not a balanced opposition, but rather a dissent minority situation. Some aggregation methods, like averaging, give more weight to views held by a larger number of expert, but this is arguably unbalanced because scientific theories should be evaluated only on their own merits, not by the number of proponents. And fourth, there is no proxy available to calibrate the reliability of experts, so we can't assume that some experts are less reliable than others.

The outline is as follows. Section 2 describes the mathematical theory for information fusion in the Transferable Belief Model. It defines three ways to combine opinions, namely the non-interactive conjunction, non-interactive disjunction and the cautious conjunction. Section 3 discusses theoretically these operators, along with the well known averaging and Dempster's rules.

We argue that all these ways to combine expert opinion do not adequately address the four challenges defined above. To this end, we propose a hierarchical method for the fusion of expert opinion. Experts are not combined symmetrically, but grouped into schools of thought. Within groups, beliefs are combined using the cautious conjunction rule, and across, with the non-interactive disjunction.

These approaches are numerically applied and compared in section 4. The data used represents the opinion of 16 experts on climate sensitivity, a key parameter of the climate change issue. We examine which fusion method works best, showing that the answer is not the same for bayesian beliefs and consonant beliefs. The discussion section 5 analyzes the results' sensitivity, compares them with the more recent literature, and points out at existing social science tools that could be used with the proposed hierarchical approach. Section 6 concludes.

2 Operators of the Transferable Belief Model

2.1 Basic Belief Assignments

The Transferable Belief Model is an elaboration of the Dempster-Shafer mathematical theory of evidence [Dempster, 1967, Shafer, 1976], a theory that represents and combines uncertain beliefs. This section briefly reminds the parts of this model that are relevant for expert aggregation. The reader may refer to Denœux [2006], Smets [2000] for a more complete exposition including the mathematical demonstrations.

As usual, let us denote Ω a frame of reference, that is a set of mutually exclusive states of the world. This paper assumes a finite number of states of the world. Classical probability theory represents uncertainty by allocating a unit mass of belief among states of the world, that is a function $p : \Omega \rightarrow [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Dempster-Shafer theory of evidence represents uncertainty by allocating the unit mass of belief among *subsets* of the frame of reference Ω . Formally, let 2^Ω denote the power set of Ω , that is the set of all its subsets. Elements of 2^Ω will be denoted with upper case letters such as $A \subseteq \Omega$ or $X \subseteq \Omega$. The empty subset $\{\}$ will be denoted \emptyset . A basic belief assignment (BBA) is a function $m : 2^\Omega \rightarrow [0, 1]$ such that:

$$\sum_{A \subseteq \Omega} m(A) = 1 \quad (1)$$

The mass $m(A)$ is the portion of the total belief supporting A which do not support more precisely any specific subset of A . Any subset $A \subset \Omega$ such that $m(A) > 0$ is called a *focal set* of m .

As a classical example, consider a drawing from an urn containing white, black, and red marbles ($\Omega = \{\text{white}, \text{black}, \text{red}\}$). Knowing only that there is 1/3 of white marbles would lead to the basic belief assignment defined as: $m(\{\text{white}\}) = 1/3$, $m(\{\text{black}, \text{red}\}) = 2/3$. This is not the same as drawing from an urn known to have 1/3 of each color, which would be represented with the basic belief assignment defined as: $m(\{\text{white}\}) = m(\{\text{black}\}) = m(\{\text{red}\}) = 1/3$.

For any subset $A \subseteq \Omega$, the BBA that represents the certain belief that the state of the world is in A is the indicator function $\mathbf{1}_A : 2^\Omega \rightarrow [0, 1]$ defined by:

$$\begin{cases} \mathbf{1}_A(A) = 1 \\ \mathbf{1}_A(X) = 0 \quad \text{if } X \neq A \end{cases} \quad (2)$$

The basic belief assignment $\mathbf{1}_\Omega$ is called the *vacuous* BBA. It allocates all belief to Ω itself, and represents the absence of information. Following up the urn example above, the vacuous BBA is defined by $m(\{\text{white}, \text{black}, \text{red}\}) = 1$, $m(X) = 0$ otherwise. Again, this is not the same as the equidistribution. We will call $m\Omega$ the weight of ignorance.

In Shafer's original theory, in addition to equation 1 a BBA must verify the axiom $m(\emptyset) = 0$. The Transferable Belief Model drops this constraint: it allows non-zero belief mass to the empty set, and considers that renormalization, defined as follows, should not be applied systematically. *Renormalizing* a BBA m means replacing it by the BBA m^* defined as:

$$\begin{cases} m^*(\emptyset) = 0 \\ m^*(A) = \frac{m(A)}{1-m(\emptyset)} \end{cases} \quad \text{if } A \neq \emptyset \quad (3)$$

Smets [1992] discusses two reasons for using non renormalized BBAs: incompleteness and conflict. Incompleteness means that $m(\emptyset)$ measures the belief that something out of Ω happens. For example, if $\Omega = \{\text{Head}, \text{Tail}\}$ models a coin toss, then $m(\emptyset)$ is the extend of the belief that the coin could fall sideways, break or otherwise disappear. In what follows, we assume that the states of the world are collectively exhaustive, disregarding incompleteness.

Therefore in this context, $m(\emptyset)$ relates to conflict only. The number $m(\emptyset)$, called weight of conflict, is a measure of internal contradiction which arise when forming belief from information sources pointing in different directions. The extreme case $\mathbf{1}_{\emptyset}$ represents complete contradiction. As opposed to the vacuous BBA $\mathbf{1}_{\Omega}$ which can be adopted when one has no information at all, the complete contradiction BBA $\mathbf{1}_{\emptyset}$ represents a situation of too much inconsistent information.

2.2 Non-interactive fusion operators

The two basic combination rules of the transferable belief model will be denoted \odot and \oplus . They provide a way to compute the “intersection” or the “union” of two experts’ opinions.

Before turning to the formal definitions, these rules will be illustrated on a special case: the fusion of two experts holding certain beliefs. Expert 1 views that the state of the world is in $A \subseteq \Omega$, and expert 2 views that the state of the world is in $B \subseteq \Omega$. Their beliefs are respectively represented with $\mathbf{1}_A$ and $\mathbf{1}_B$.

To start with \oplus , consider what the result of the fusion should be when one thinks that either expert 1 or expert 2 is a reliable information source. In this case, one is led to believe that the state of the world is in A or B , that is in $A \cup B$. The \oplus combination rule is precisely such that $\mathbf{1}_A \oplus \mathbf{1}_B = \mathbf{1}_{A \cup B}$. It is called the non-interactive disjunction rule. This operator can be qualified as a gullible rule, which means it accepts all that it is told.

The non-interactive conjunction rule \odot is meant to be used when one thinks that both expert 1 and expert 2 are reliable information sources. Apparently, there are two cases. When $A \cap B$ is non-empty, the fusion of the two opinions should be the belief that the state of the world is in $A \cap B$. When the experts have no common ground, that is $A \cap B = \emptyset$, then we have a contradiction problem. However, in the transferable belief model this is not a problem, this state of affairs is represented with $\mathbf{1}_{\emptyset}$. So actually in both cases, the operator should be such that $\mathbf{1}_A \odot \mathbf{1}_B = \mathbf{1}_{A \cap B}$. This operator can be qualified as a consensus rule, to mean that all parties accept the result.

For reasons that will become apparent with equation 8, we define below these two combination rules with slightly more general functions than BBAs. Let use the Greek letter μ to denote a real-valued subset function $\mu : 2^{\Omega} \rightarrow \mathbb{R}$ which verifies equation 1, but may or may not be a basic belief assignment, that is take values in $[0,1]$ or not. The non-interactive conjunction of μ_1 and μ_2 is defined as the subset function $\mu_1 \odot \mu_2 : 2^{\Omega} \rightarrow \mathbb{R}$ such that, for any subset X :

\odot	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$
$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$
$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$
$\mathbf{1}_A$	$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$	$\mathbf{1}_A$	$\mathbf{1}_A$
$\mathbf{1}_\Omega$	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$

\odot	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$
$\mathbf{1}_\emptyset$	$\mathbf{1}_\emptyset$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$
$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_{\overline{A}}$	$\mathbf{1}_\Omega$	$\mathbf{1}_\Omega$
$\mathbf{1}_A$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$	$\mathbf{1}_A$	$\mathbf{1}_\Omega$
$\mathbf{1}_\Omega$	$\mathbf{1}_\Omega$	$\mathbf{1}_\Omega$	$\mathbf{1}_\Omega$	$\mathbf{1}_\Omega$

Table 1: The Transferable Belief Model non-interactive conjunction (left) and disjunction (right) operators applied on the elementary cases. $\mathbf{1}_\Omega$ represents vacuous beliefs, $\mathbf{1}_A$ represents certainty that the state of the world is in A , $\mathbf{1}_{\overline{A}}$ represents certainty that the state of the world is not in A , $\mathbf{1}_\emptyset$ represents contradiction.

$$(\mu_1 \odot \mu_2)(X) = \sum_{\substack{A \subseteq \Omega \\ B \subseteq \Omega \\ A \cap B = X}} \mu_1(A) \times \mu_2(B) \quad (4)$$

In the same way, \odot is defined by:

$$(\mu_1 \odot \mu_2)(X) = \sum_{\substack{A \subseteq \Omega \\ B \subseteq \Omega \\ A \cup B = X}} \mu_1(A) \times \mu_2(B) \quad (5)$$

These operators are commutative, associative and if μ_1 and μ_2 are two BBAs then the result is also a BBA. These properties allow to treat the experts symmetrically when combining their opinions.

Table 1 illustrates these operators by showing all possible results in a simple case: opinions are only about whether the state of the world is in A or not. Vacuous beliefs $\mathbf{1}_\Omega$ is an absorbing element for disjunction and a neutral element for conjunction. Conversely, contradiction $\mathbf{1}_\emptyset$ is absorbing for conjunction and neutral for disjunction.

Turning to more general example, consider $\Omega = \{A, B\}$, and the BBA m defined by $m(A) = m(B) = 1/2$. Then $(m \odot m)(A) = (m \odot m)(B) = 1/4$, and $(m \odot m)(\emptyset) = 1/2$. Such a large weight of conflict in the result may seem surprising. One way to avoid these surprises is to systematically renormalize, as described by equation 3. The non-interactive conjunction \odot followed by normalization is known as Dempster's combination rule, usually denoted \oplus in the literature :

$$m_1 \oplus m_2 = (m_1 \odot m_2)^* \quad (6)$$

However, in some situations the surprising result is the correct one, and renormalization should not be used. It depends on what is being modeled. Consider for example a setting in which two experts simultaneously replicate a large number of fair coin tosses. Both conclude that $p(\text{Head}) = p(\text{Tail}) = 1/2$. But if the experiments are independent, then experts disagreed half the time. This suggests that the non-interactive conjunction \odot is relevant to combine information sources only when some kind of independence relation can be assumed between information sources. It justifies why this operators is called *non-interactive*.

2.3 Factorization and cautious conjunction

The non-interactive combination rules should not be used to combine experts who share pieces of evidence. To perform information fusion in this kind of situations, Denœux [2006] introduced an operator called *cautious conjunction*. To define it mathematically, it is necessary to introduce first the factorization of BBAs.

For any proper subset $A \subset \Omega$, $A \neq \Omega$ and any real number s , we denote A^s the function $\mu : 2^\Omega \rightarrow \mathbb{R}$ such that:

$$\begin{cases} \mu(\Omega) = e^{-s} \\ \mu(A) = 1 - e^{-s} & (\text{we assumed } A \neq \Omega) \\ \mu(X) = 0 & \text{if } X \neq A \text{ and } X \neq \Omega \end{cases} \quad (7)$$

The letter s stands for “Shafer’s weight of evidence”. This value was previously denoted w by Shafer [1976, Chapter 5]. But the recent literature [Denœux, 2006] uses the letter w to denote the “weight of evidence” defined by $w = e^{-s}$.

Regarding the interpretation of A^s , when $s \geq 0$ the function A^s is a BBA, but when $s < 0$ it is not, so A^s can generally not be interpreted as a state of belief. Smets [1995] has shown that for any BBA m such that $m(\Omega) > 0$ there is a unique function $s : 2^\Omega \setminus \Omega \rightarrow \mathbb{R}$ such that:

$$m = \bigodot_{\substack{A \subset \Omega \\ A \neq \Omega}} A^{s(A)} \quad (8)$$

Any BBA m such that $m(\Omega) > 0$ is the non-interactive conjunction of elementary pieces of the form A^s . The weights of evidence function s may take negative values, in which case the BBA is not *separable* according to Shafer [1976], who did not consider negative weights of evidence.

This unique factorization theorem allows to come back to the interpretation of A^s . It can be seen as the change in one’s beliefs realized when integrating with weight s a piece of evidence stating that the state of the world is in A . Positive infinity for s represents a perfectly convincing proof that the state of the world is in A . This remains excluded in the above definition, for reasons discussed further below. Negative weight $s < 0$ have an algebraic justification similar to that of negative numbers: considering A with weight s exactly counterbalances considering A with weight $-s$, to produce vacuous beliefs $\mathbf{1}_\Omega$. It is more difficult to achieve an intuitive understanding of negative information. Smets [1995] suggested that negative values of s should be used when information come from sources known to be not only unreliable, but deliberately lying.

Let us denote $|X|$ the number of elements (cardinality) of a subset $X \subseteq \Omega$. The weights can be computed as follows, introducing the function q called the commonality function :

$$q(X) = (m \odot \mathbf{1}_X)(X) = \sum_{X \subseteq A} m(A) \quad (9)$$

For any $X \subset \Omega$, remark that $q(X) > m(\Omega)$, therefore $m(\Omega) > 0$ implies $q(X) > 0$, so the logarithm is well defined in the following:

$$s(X) = \sum_{X \subseteq A} (-1)^{|X|-|A|} \ln(q(A)) \quad (10)$$

Using equations 4 and 8, along with commutativity and associativity, it is straightforward to verify that if two BBA m_1 and m_2 admit corresponding weight functions s_1 and s_2 , their non-interactive conjunction can be computed simply by adding those:

$$m_1 \odot m_2 = \bigodot_{\substack{A \subset \Omega \\ A \neq \Omega}} A^{s_1(A) + s_2(A)} \quad (11)$$

This property allows to clarify the intuition behind the \odot operator. The non-interactive conjunction adds up distinct pieces of evidence. For example, when combining two experts which points exactly in the same direction A with the same weight s , the result is $A^s \odot A^s = A^{2s}$. Once again, it is correct to argue that a stream of evidence pointing out in the same direction leads to stronger beliefs only when they are distinct.

To combine experts that share evidence, Denc  ux [2006] defined the cautious conjunction operator, denoted \otimes . It combines any two BBA such that $m_1(\Omega) > 0$ and $m_2(\Omega) > 0$ by taking the maximum of their weight functions as follows:

$$m_1 \otimes m_2 = \bigodot_{\substack{A \subset \Omega \\ A \neq \Omega}} A^{\max(s_1(A), s_2(A))} \quad (12)$$

It can be shown that if m_1 and m_2 are BBAs, then $m_1 \otimes m_2$ is also a BBA (this is immediate only when m_1 and m_2 are separable). This combination rule \otimes is also commutative and associative, it treats experts symmetrically. It is also idempotent, that is $m \otimes m = m$, and distributes over the noninteractive rule $(m_1 \odot m_2) \otimes (m_1 \odot m_3) = m_1 \odot (m_2 \otimes m_3)$.

Distributivity has an interesting interpretation related to the fusion of beliefs. Consider two experts in the following scenario. Expert 1's belief results from the noninteractive conjunction of two pieces of evidence, $m_1 = A^s \odot B^t$. Expert 2 share one piece of evidence with expert 1, and has an independent piece, so that $m_2 = A^s \odot C^u$. Then distributivity implies that in the fusion, the shared evidence A^s is not counted twice $m_1 \otimes m_2 = A^s \odot (B^t \otimes C^u)$.

2.4 Discounting beliefs

A BBA m that verifies $m(\Omega) = 0$ cannot be factorized as described above. Equation 4 allows to see that $(\mu_1 \odot \mu_2)(\Omega) = \mu_1(\Omega) \times \mu_2(\Omega)$, and we defined A^s in equation 7 such that $A^s(\Omega) > 0$ always. Therefore the right hand side of equation 8 cannot be BBA such that $m(\Omega) = 0$.

Various reasons justify to take basic beliefs assignments such that $m(\Omega) = 0$ with a grain of salt:

- No information source is 100% reliable, especially human ones.
- Many philosophers consider that fundamentally, scientific knowledge can never be absolute and definitive. On the contrary, it is necessarily based on a possibly large but finite number of human observations, and is always open to revision in front of new experimental evidence.
- The elicitation of expert's opinions, for example by asking them probability density functions, was necessarily coarse. Experts who allocated no

significant probability weight to extreme outcomes might have agreed that there was a very small possibility.

Shafer [1976, p. 255] proposed a simple way to add doubt to a basic belief assignment, called *discounting*. Let r be a number in $[0, 1]$ called a *reliability factor*. Discounting the BBA m means replacing it by the BBA defined as:

$$\text{discount}(m, r) = rm + (1 - r)\mathbf{1}_\Omega \quad (13)$$

Discounting allows beliefs to be factorized, and therefore combined using the cautious operator. Admittedly, discounting experts beliefs is deliberately blurring the data, a practice to be considered with extreme care if used at all. But the reasons above justify reliability factors, provided they are close enough to unity. The theoretical literature suggests that the fusion operators can be extended by continuity to deal with $m(\Omega) = 0$, and the sensitivity analysis will allow to check that results don't change much when r varies from 0.999 to 0.9999.

To sum up, that section defined a mathematical object used to represent an expert's opinion, denoted m and called a basic belief assignment (BBA). Four operators were defined to combine the opinions of two experts. The cautious conjunction operator \odot is meant to be used when experts share data. Otherwise, the non-interactive disjunction \oslash takes the union of experts beliefs, while non-interactive conjunction \odot takes their intersection. Dempster's rule \oplus is the renormalized non-interactive conjunction.

3 Fusion in the Transferable Belief Model

Having defined the mathematical framework and the binary fusion operators, we discuss now the complete procedures involving pooling the opinions of experts. Experts opinions are typically called for in situations in which there is not enough statistical evidence to support precise probabilities. This motivates our interest in an imprecise probability theory, such as the Transferable Belief Model, to model and combine beliefs. But imprecision has implications along the whole analytical process, not just the fusion of beliefs.

First, we discuss the implications of imprecision for the process' ultimate aim, to facilitate decision making. In our view, it implies to take a step back from the standard expected utility-maximization methodology implicit in probabilistic risk analysis. Second, we discuss the elicitation of opinions, a necessary step before the fusion, and question the validity of asking experts for probability density functions when more imprecise communication instruments can be used. Third, we discuss theoretically alternative ways to fusion beliefs, and fourth, we introduce a hierarchical approach to set the stage for the numerical application that will follow.

3.1 Decision making and uncertainty communication

A reason why decision analysis processes involving the fusion of opinion is important is that when decisions involve different parties and a scientific experts are not unanimous, most policymakers will tend to break the symmetry of the elicitation process by myopically focusing on the results best supporting their

interest. Another risk is that organizations seeking a balanced point of view would overemphasize the most extreme positions in the group, even when they are actually a minority not representative of the experts' general opinion.

Smets [2005] offers a way to find a balanced point of view for decision making in the Transferable Belief Model. He points out that any BBA m such that $m(\emptyset) \neq 1$ defines a probability function p^m , that he calls the pignistic probability function of m , by:

$$p^m(\omega) = \frac{1}{1 - m(\emptyset)} \sum_{\omega \in X} \frac{m(X)}{|X|} \quad (14)$$

Smets then argues that when beliefs are described by m , a decision-maker should choose actions that maximizes the expected utility, where expectation is computed using the probability distribution p^m .

But offering a single precise probability distribution from which expected utility maximization can provide an optimal answer to all policy issues is problematic. This position has been explicated by Morgan and Keith [1995], who argued that while expert aggregation can help decision making by presenting a simpler picture of the multiplicity of opinions on a given subject, in many cases presenting an aggregate probability is oversimplification and it is better to leave with the decision-maker the task of the combining the judgment of all experts. Keith [1996] discusses in more details why combining experts is rarely appropriate, and suggests instead to use alternative analysis framework such as seeking robust adaptative strategies or using scenario analysis to bound the problem.

Our position is that fusioning expert opinion in the Transferable Belief Model do not necessarily imply using Smets' decision making probabilistic rule. Different interpretations of how to implement precautionary decision making are available. One could, for example, see the Transferable Belief Model as a special kind of imprecise probability theory [Walley, 1991], and consider the decision-making rules in that broader context. In some ways imprecise probabilities induce a partial, but not total, ordering between decisions. As Troffaes [2006] survey shows, these rules allow to find a set of optimal (*maximal* or *E-admissible*) decisions, which can be rather large if uncertainty is wide. The analysis do not prescribe further which action should be selected in that set. Used in this way, the fusion of opinions in the Transferable Belief Model can respect the intuition that when there is a multiplicity of opinions, then decisions cannot be made mechanically using standard expected utility maximization.

Thus, communication of the results obtained by information fusion in the transferable belief model do not have to put forward a single probability distribution. Instead, it can involve the measures of *belief* and *plausibility* associated with a BBA m . The value of the belief function for an event $X \subseteq \Omega$, denoted $bel(X)$, measures the strength of conviction that X must happen. The value of the plausibility function, denoted $pl(X)$, relates to the strength of conviction that X could happen. With the special case $bel(\emptyset) = pl(\emptyset) = 0$, these functions are defined when $X \neq \emptyset$ as :

$$bel(X) = \sum_{\substack{A \subseteq X \\ A \neq \emptyset}} m(A) \quad (15)$$

$$pl(X) = \sum_{\substack{A \subseteq \Omega \\ A \cap X \neq \emptyset}} m(A) \quad (16)$$

An intuitive interpretation of the theory of evidence sees $m(X)$ as a mass of belief that can flow to any subset of X . In this view, $bel(X)$ represents the minimal amount of belief that is constrained to stay within X , while $q(X)$ represents the amount of belief that can flow to every point of X , and $pl(X)$ the maximal amount of belief that could flow into X .

These functions can be used with the ‘calibrated vocabulary’ approach to communicate qualitatively about uncertainty. For example, if the analytical result is $bel(X) > 0.90$, it could be said that X is correct with *very high confidence*. If $pl(X) < 0.33$, it could be said that X is *unlikely*. No calibrated uncertainty vocabulary (probabilistic or otherwise) is universally accepted, and presumably it would depend upon the readers language and culture.

Calibrated vocabularies have often been defined using a probability scale [IPCC, 2005, Weiss, 2003, Wallsten et al., 1986]. Such scales have to be revised, if one wish to account for the multi-dimensionality of uncertainty: a basic belief assignment m allows to express levels of belief, plausibility, and contradiction.

3.2 Elicitation: bayesian or consonant BBAs?

We now turn to the methods for expert elicitation, upstream of the information fusion itself. Approaches in experts elicitation include:

1. Expert’s opinion elicitation in the tradition of risk assessment: asking the experts about probabilities, obtaining subjective probability density functions.
2. Expert’s knowledge elicitation in the tradition of robust control: collecting opinions in natural language, modeling them with fuzzy numbers or possibility distributions [Zadeh, 1978].
3. Qualitative methods: asking the experts to make hypothetical choices. Opinions can then be deducted from elicited preferences, using the assumption that choices follow rationally from beliefs [Baillon and Caban-tous, 2007].

Formally, information fusion in the transferable belief model can deal with these three approaches. We will focus on the first two, because qualitative methods, which could potentially be used to elicit directly BBAs, are also less well developed. There is a natural embedding of probability distributions in the set of BBAs and a natural embedding of possibility functions in the set of BBAs.

Any probability function $p : \Omega \rightarrow [0, 1]$ naturally defines a BBA m by:

$$\begin{cases} m(\{\omega\}) = p(\omega) & \text{for any } \omega \in \Omega \\ m(X) = 0 & \text{if } |X| \neq 1 \end{cases} \quad (17)$$

A BBA m that naturally corresponds with a probability p by the above equation is said to be *bayesian*. A BBA is bayesian when its only focal sets are singletons.

By definition, a normalized possibility distribution is a function $\pi : \Omega \rightarrow [0, 1]$ such that $\max_{\omega \in \Omega} \pi(\omega) = 1$. Given such π , a BBA m naturally associated with π can be computed via its commonality function as follows:

$$q(A) = \min_{\omega \in A} \pi(\omega) \quad (18)$$

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B) \quad (19)$$

In the numerical application (section 4 and following), we will use a dataset where opinions are given as probabilities, using equation 17 to transform them into bayesian belief functions when needed. We will also explore information fusion when beliefs are more imprecise, for two reasons:

- First, probabilities transformed by equation 17 are a very special kind of basic belief assignments, in which all belief mass is supported by singletons. The information fusion methodologies needs to be tested in a more general case.
- Second, while in this specific dataset probabilities have been obtained in the tradition of risk assessment, other elicitation exercises might seek opinions using different approaches. These include providing judgements using natural language, probability bounds or possibility estimates. We argued that presenting a single probability distribution was not justified when statistical data is insufficient, even considering the whole pool of experts. This data scarcity argument is even stronger at the individual level, since each expert holds only a fraction of the data.

Since we only have probabilities, it is necessary to transform each expert's elicited distribution into a consonant belief function. This transformation problem was already discussed by Sandri et al. [1995] in a possibilistic context for the same reason: most of the existing available datasets are probabilistic.

There are many ways to transform a probability p into a BBA m , starting with the natural injection defined equation 17. Since we want to test fusion methods when beliefs are imprecise, a transformation according to a principle of maximal uncertainty is justified. This principle goes as follows. Given p , consider the set M of belief functions consistent with p , for some definition of consistency. Then select m as the member of M which has the most uncertainty in it, having defined an uncertainty-related order relation that admits a single maximum in M .

Smets [2000] suggested to select for M the set of all BBAs m such that $p^m = p$, where p^m is defined equation 14. This set is never empty because it contains the BBA naturally corresponding to p itself. The order relation is defined as follows: for any two BBAs m_1 and m_2 with respective commonality function q_1 and q_2 , if $q_1(A) \geq q_2(A)$ for all $A \subset \Omega$, we write that $m_1 \sqsubseteq_q m_2$.

Smets [2000, Theorem 7.1] states that there is an unique maximum in M with respect to \sqsubseteq_q , which can be computed as follows. Order the states of the world from most to least probable, that is $p(\omega_{n_1}) > \dots > p(\omega_{n_{|\Omega|}})$. Consider the sets $A_k = \{\omega_{n_1}, \dots, \omega_{n_k}\}$ and assign to A_k the belief mass:

$$m(A_k) = |A| \times (p(\omega_{n_k}) - p(\omega_{n_{k+1}})) \quad (20)$$

with the convention that $p_{n_{|\Omega|+1}} = 0$. The procedure is illustrated on figure 3.2, which demonstrates graphically that m is indeed a basic belief assignment, it adds up to unity. Note that the focal sets A_k are nested, that is $A_k \subset A_{k+1}$ for all k . In this case, it is said that m is *consonant*. It can be shown that the result m is naturally associated with a possibility distribution (via equation 19).

For each expert i , we have a method to transform the bayesian belief function (corresponding to the elicited probability distribution p_i) into a consonant belief function (corresponding to a possibility distribution that we will denote π_i).

3.3 Symmetric fusion of experts' opinion

Having discussed opinion elicitation and decision making, we now turn to the fusion of opinions. The literature offers many rules to combine beliefs, see Smets [2007] for a survey. This section examines systematically ten ways to combine opinions symmetrically: five operators defined above, each used with or without discounting.

We will explore two discounting options. The high reliability factor, $r = 0.999$, amounts to practically no discounting at all, but is technically necessary to ensure that beliefs can be factorized and combined using the cautious operator. A medium reliability factor, $r = 0.8$, can be justified as in 2.4. The five operators are: the non-interactive conjunction and disjunction, the cautious conjunction, Dempster's combination rule, and averaging.

Theoretical analysis allows to disregard seven of the ten different ways to fusion opinions, because they can be expected to give mathematically degenerate or otherwise uninteresting results in the context of expert opinion fusion.

Consider first averaging, also called the linear opinion pool. It is mathematically equivalent to discount the opinions before averaging, or to discount after averaging. But there is no reason to discount the average opinion, once it is computed. That only adds unjustified imprecision to the result. This explains why, in the next section, we will only check averaging with $r = .999$. More precisely, denoting m_i the BBA associated with expert i and denoting n the number of experts, we will compute :

$$m_{average} = \frac{1}{n} \sum_{i=1 \dots n} \text{discount}(m_i, 0.999) \quad (21)$$

On the contrary, using Dempster's combination rule \oplus without discounting can give counter-intuitive results. Consider for example three states of the world, $\Omega = \{A, B, C\}$, and the problem of combining bayesian beliefs corresponding to the two probability distributions p_1 and p_2 , defined respectively by $p_1(A) = 0.9$, $p_1(B) = 0$, $p_1(C) = 0.1$, and $p_2(A) = 0$, $p_2(B) = 0.9$, $p_2(C) = 0.1$. The result according to Dempster's rule has a belief weight 0.85 to the state of the world C , which is paradoxical since both information sources agree that this is the least probable outcome. In the same example, if opinions are taken with a reliability factor $r = 0.8$ before combination, the weight going to state of the world C is only 0.105, which is much more intuitive. This is why we will only examine Dempster's rule with the medium reliability factor:

$$m_{dDempster} = \bigoplus_{i=1 \dots n} \text{discount}(m_i, 0.8) \quad (22)$$

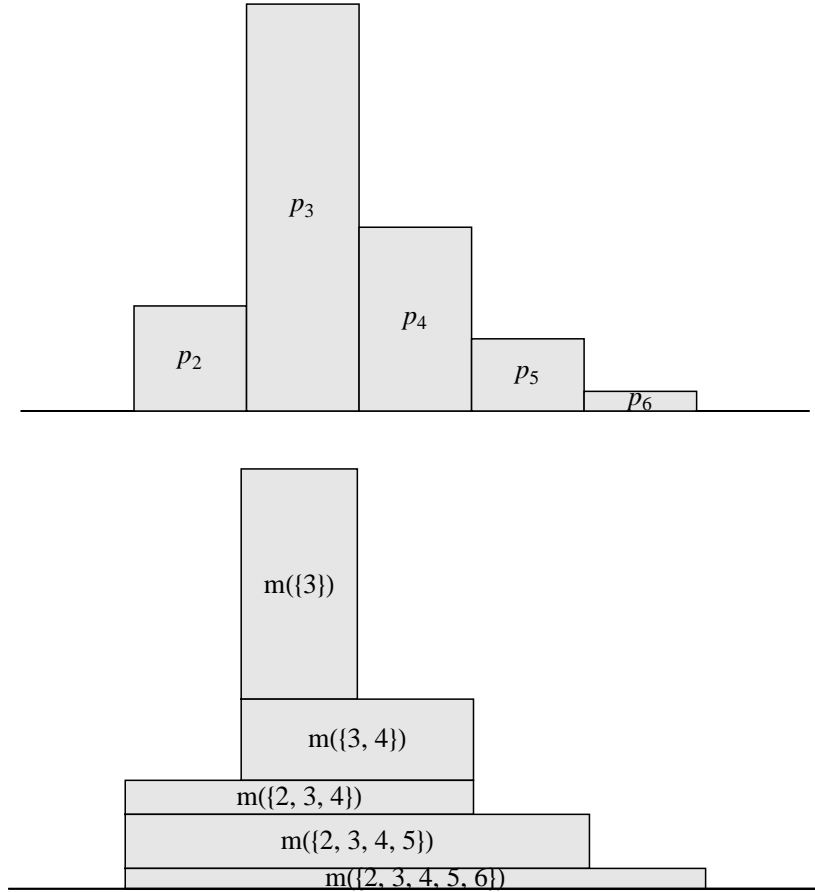


Figure 1: From bayesian to consonant beliefs. Top, bayesian beliefs (from expert 1 in the dataset). Assuming that the width of each rectangle is 1, and its height is proportional to the probability, the area of the rectangle denoted p_3 is p_3 , and the sum of all rectangles' area is 1. Bottom, the corresponding consonant belief function. The area of the rectangle denoted $m(\{3, 4\})$ is the belief mass going to the focal set $\{3, 4\}$. The slices are cut horizontally, but the outline remains the same. The total area remains 1, meaning that m is a BBA.

Turning now to the non interactive disjunction \odot , this operator tends to produce very uninformative beliefs. Adding imprecision to the input by discounting leads even faster to a vacuous result $\mathbf{1}_\Omega$. This goes against the purpose of information fusion, so we will only consider the fusion with the very almost no discount:

$$m_{niDisjunction} = \bigcup_{i=1\dots n} \text{discount}(m_i, 0.999) \quad (23)$$

The non interactive conjunction operator \odot and the cautious operator \odot produce a trivial result when the information sources conflict completely. In this case, the fusion falls into pure contradiction $\mathbf{1}_\emptyset$. As with Dempster’s rule, discounting could be used to decrease conflict before the fusion. This would technically allow to recover more informative results. But discounting is not justified for these operators, since in the transferable belief model $\mathbf{1}_\emptyset$ is accepted as a theoretically correct result. Worse, the non-interactive conjunction finds conflict when combining a bayesian belief with itself. As seen previously, when combining the fifty-fifty probability with itself, the belief mass of \emptyset is 0.5.

The introduction enumerated four challenges for mathematical aggregation methods: non-independence, complete contradiction, minority views, and discounting. Contradiction between experts ruled out conjunction operators, but is not a problem for the remaining three approaches. None of these, however, completely answers the other challenges. Dempster’s rule needs discounting, but there is little evidence to determine reliability factors. Contrary to the cautious conjunction, Dempster’s rule and the non-interactive operators assume that experts are, statistically speaking, independent. This can lead to artificially over-precise results, by counting the same pieces of evidence multiple times.

With averaging and Dempster’s rule, the weight of an opinion increases with the number of experts holding it. This can be seen as a problem, as scientific arguments should be evaluated on their own merits, not by *argumentum ad populum* (Latin: “appeal to the people”). It is only at the social decision-making stage that the quality and number of people behind each view should matter. Groupthink and bandwagon effects are known dangers when pooling opinions. Thus, all other things being equal, a fusion method that gives equal attention to the minority and the majority views is preferable.

3.4 A hierarchical approach

To address these challenges, we propose a hierarchical fusion procedure. This procedure aims to be relevant when science is not yet stabilized, and the notion of ‘competing theories’ can be used. Sociology of science suggests that at some moments in the progress of science, in front of a big unexplained problem, scientists tend to group into schools of thought, which correspond to alternative candidate theories [Kuhn, 1962]. Within each group, experts share an explanation of the way the world works. But only time can tell which theory will emerge, and only one will be adopted in the end.

This suggests to use different operators across and within groups. Across groups, we will use a non-interactive disjunction operator, assuming that at least one theory, but not all theories, is a reliable information source. This deals with

the challenge of representing equally minority views because all theories are treated equally, regardless of the number of experts in the group.

Within groups, beliefs will be combined using a cautious conjunction operator. This assumes that experts are all reliable information sources, but not statistically independent. Discounting is needed if the beliefs verify $m_i(\Omega) = 0$, but this is only a technical operation, the reliability factor can be as close to 1 as desired: we will use $r = 0.999$. This method deals with contradiction as far as within groups, the degree of conflict between experts is low. Denoting G_1, \dots, G_N the groups of experts, we will compute:

$$m_{Hierarchical} = \bigcup_{k=1 \dots N} \bigcap_{i \in G_k} \text{discount}(m_i, 0.999) \quad (24)$$

In that equation, using the \odot operator amounts to assume that schools of thought are non-interactive, that is somewhat independent. This assumption could be discussed, but the disjunctive combination rule corresponding to the cautious conjunction has been published by Denœux [2007] too recently to be examined here.

At this point, we have defined four ways to combine beliefs: the simple linear opinion pool (equation 21), the discounted Dempster’s combination rule (eq. 22), the non-interactive disjunction (eq. 23), and a hierarchical disjunctive-cautious fusion based on the notion of competing theories (eq. 24). These four methods will be applied both to the elicited bayesian beliefs (eq. 17), and to the consonant beliefs (eq. 20). This defines theoretically eight distinct ways to perform opinion fusion in the transferable belief model. The next section examines how they perform on a real-world dataset.

4 Application on climate sensitivity

4.1 Data used

Climate sensitivity is a measure the climate change problem. It is denoted $\Delta T_{2\times}$, and defined as the equilibrium global mean surface temperature change following a doubling of atmospheric CO_2 concentration, compared to pre-industrial levels [Randall et al., 2007]. Over the last two decades, climate sensitivity has become one of the main communication anchors between the scientists and policymakers to quantify the seriousness of the climate change issue, as discussed by van der Sluijs [1997], Boia [2003].

The value of this parameter is not known precisely. For a long time, the $[1.5^\circ\text{C}, 4.5^\circ\text{C}]$ interval has been regarded as the canonical uncertainty range of $\Delta T_{2\times}$ [National Research Council, 1979]. Yet knowing better climate sensitivity is critical for climate policy. According to current trends, humankind is well on track to double the CO_2 concentration in the Earth’s atmosphere, not to mention other greenhouse gases. IPCC [2001a] estimated that 2°C of global warming raises serious concerns such as risks to many unique and threatened ecosystems (such as coral reefs or the arctic ice sheet), plus a large increase in the frequency and magnitude of extreme climate events (such as coldwaves, heatwaves, droughts and storms).

If climate sensitivity was around 1.5°C , one could argue that doubling the CO_2 concentration would not lead immediately to a dangerous interference with

the climate system. But if climate sensitivity was at the upper end of the canonical uncertainty range, 4.5°C, then doubling the CO₂ concentration would certainly be a very dangerous interference with the climate system.

Morgan and Keith [1995] conducted structured interviews using expert elicitation methods drawn from decision analysis with 16 leading U.S. climate scientists. The authors obtained quantitative, probabilistic judgments about a number of key climate variables, including the climate sensitivity parameter.

This dataset received significant interest in the climate change literature, as in the late nineties there were very few other estimates for this parameter's probability distribution. For example Webster and Sokolov [2000, 4.1] derived a climate sensitivity probability distribution by taking the median (across the 16 experts) of each of the fractiles (0.05, 0.25, 0.5, 0.75, 0.95), and using the median fractile values to fit a beta distribution. According to this distribution, $p(\Delta T_{2\times} \leq 1.5^\circ\text{C}) = 0.24$, $p(1.5^\circ\text{C} \leq \Delta T_{2\times} \leq 4.5^\circ\text{C}) = 0.67$, $p(\Delta T_{2\times} \geq 4.5^\circ\text{C}) = 0.09$.

But Reichert and Keith [2001] noted that combining these opinions into a single judgment on climate sensitivity, like Webster and Sokolov [2000], raised methodological issues. The 16 experts are not independent, they are part of a research community regularly sharing data, models and ideas. And yet opinions on climate sensitivity are widely different in qualitative terms. The authors confirm that there is an interest in finding an aggregation technique where the combined probability distribution does not necessarily narrow as the number of experts increases, and which is more robust with respect to extreme experts judgments than previously published techniques.

4.2 Implementation

The Transferable Belief Model was implemented in *Mathematica* version 6 using matrix calculus as described by Smets [2002]. The whole notebook file used to create the published figures and tables is available as an electronic supplement to this manuscript.

In the dataset, no probability is allocated to climate sensitivity lower than -6°C, or larger than 12°C. For the sake of numerical tractability, this range was subdivided in seven ranges:

$$\begin{aligned}\Omega &= \{\omega_1, \dots, \omega_7\} \\ &= \{[-6, 0], [0, 1.5], [1.5, 2.5], [2.5, 3.5], [3.5, 4.5], [4.5, 6], [6, 12]\}\end{aligned}$$

Each expert's probability distribution on Ω was computed from the elicited probability density function P_i :

$$\begin{aligned}p_i(\omega_1) &= P_i(-6 \leq \Delta T_{2\times\text{CO}_2} < 0) \\ p_i(\omega_2) &= P_i(0 \leq \Delta T_{2\times\text{CO}_2} < 1.5) \\ &\dots\end{aligned}$$

The procedure described in section 3.2 (see equation 20 and figure 3.2) was used to transform the bayesian beliefs into consonant beliefs. We computed

an implicit possibility distribution π_i associated with each expert's probability distribution p_i . Figure 2 represent p_i and π_i for the 16 experts.

Four qualitatively different groups of distributions can be identified. The widest distributions come from experts 2, 3 and 6, they allow a positive probability both to cooling and to climate sensitivity well above 6°C. Distributions from experts 4, 7, 8, 9 do not give weight to cooling, but have an upper bound above 8°C. Experts {1,10–16} disallow extreme cases, the width of the range supporting their probability distributions is between 4.2 and 5.5°C. Expert's 5 probability distribution lies completely in the range $\omega_2 = [0^\circ\text{C}, 1.5^\circ\text{C}]$.

The 0.80 reliability factor used for Dempster's rule is arbitrary. Discounting is also necessary to compute the cautious conjunction, as all experts except {2,3,6} give a zero probability to some outcomes. We used a reliability factor 0.999. Since results will be shown only to 2 digits, that is presumably close enough to 1, an assumption that will be tested in the sensitivity analysis.

We used the four qualitative groups outlined above for the hierarchical approach: $G_1 = \{2, 3, 6\}$, $G_2 = \{4, 7, 8, 9\}$, $G_3 = \{1, 10, 11, 12, 13, 14, 15, 16\}$, $G_4 = \{5\}$. Better ways to group experts together will be discussed in section 5.3, but this heuristic is sufficient to illustrate the method.

We further assumed that within a school of thought, all experts are reliable but not independent information sources. Their beliefs were combined using a cautious conjunction operator: $m_{G_k} = \odot_{i \in G_k} m_i$. The second stage combined the four groups together using the non-interactive disjunction operator.

4.3 Results

Figure 3 and table 2 present the results, in two different ways. The figure shows the results obtained by combining bayesian beliefs in the left column, and those obtained with consonant beliefs in the right column. Correspondingly, the table is divided in a top half showing the fusion of bayesian beliefs and a bottom half, for the consonant beliefs. In each half, we compare the results obtained using the four ways to combine opinions: averaging (equation 21), discounted Dempster's rule (equation 22), non-interactive disjunction (equation 23) and the hierarchical approach (equation 24). Numbers are shown with two significant digits.

On each plot in figure 3, the vertical axis goes from 0 to 1, and horizontally the numbers (from 1 to 7) denote the states of the world ω_1 to ω_7 . The legend at the bottom defines these states of the world in terms of climate sensitivity. Finally, there are three series of points on each plot. The top one is labelled pl , while the middle one is labelled p and the bottom bel . They display respectively the plausibility $pl(\omega_i)$, the pignistic probability $p^m(\omega_i)$, and the belief $bel(\omega_i)$. Labels are sometimes superposed. The lines are drawn for readability, they do not mean that we plot continuous densities.

Showing these three functions only on the ω_i does not represent completely the results, except when the result is bayesian. Since there are 7 states of the world, a general BBA m is defined with $2^7 = 128$ numbers. As an example of what the full results look like, the basic belief assignment resulting from the hierarchical fusion of the consonant beliefs is completely tabulated in table 6 (see Annex) with 5 decimals. It has 18 focal sets.

Table 2, each line describes aspects of the BBA obtained using a different fusion method. Lines 1 to 4 show the combination of bayesian beliefs, lines 5 to 8

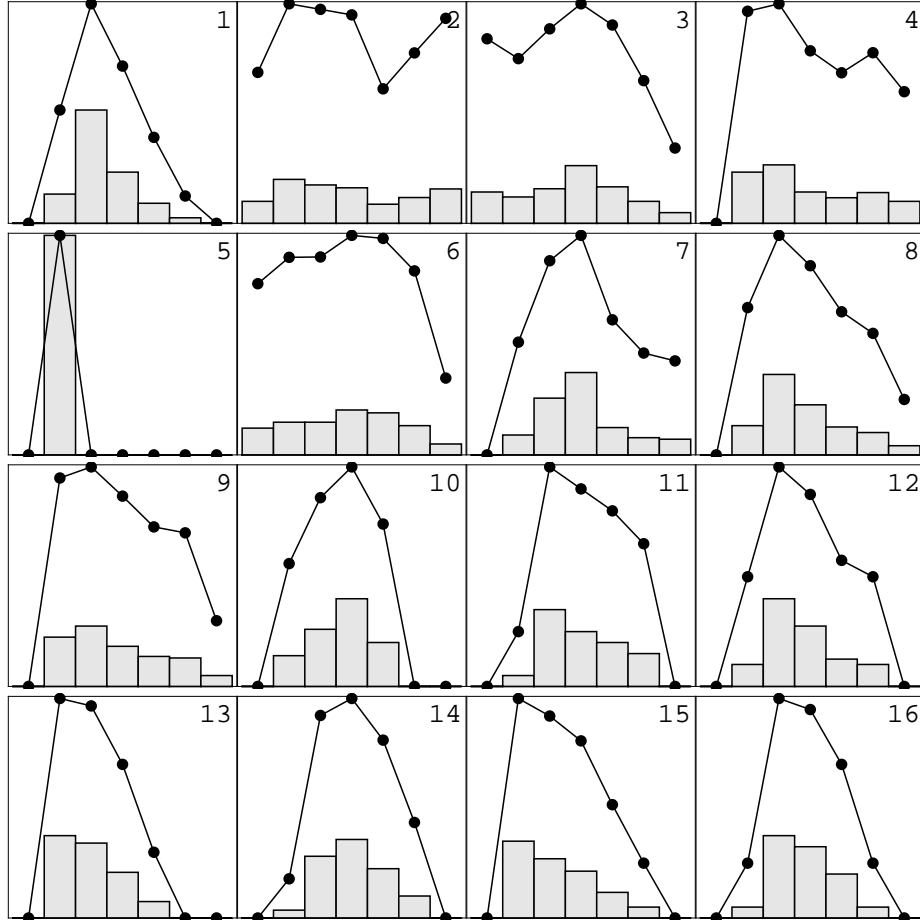


Figure 2: The probability (grey histograms) and implicit possibility (dotted lines) for the 16 experts in [Morgan and Keith, 1995]. The vertical axis goes from 0 to 1. The horizontal axis discretizes the $[-6^{\circ}\text{C}, 12^{\circ}\text{C}]$ climate sensitivity range into seven intervals using a non-uniform subdivision at $-6, 0, 1.5, 2.5, 3.5, 4.5, 6$ and 12°C . Four qualitatively different groups of distributions can be seen: Experts 2,3,6 allow cooling, 4,7,8,9 allow high outcomes but no cooling, 1,10–16 disallow extreme cases, and 5 is concentrated on $[0^{\circ}\text{C}, 1^{\circ}\text{C}]$. Data given numerically in Table 7.

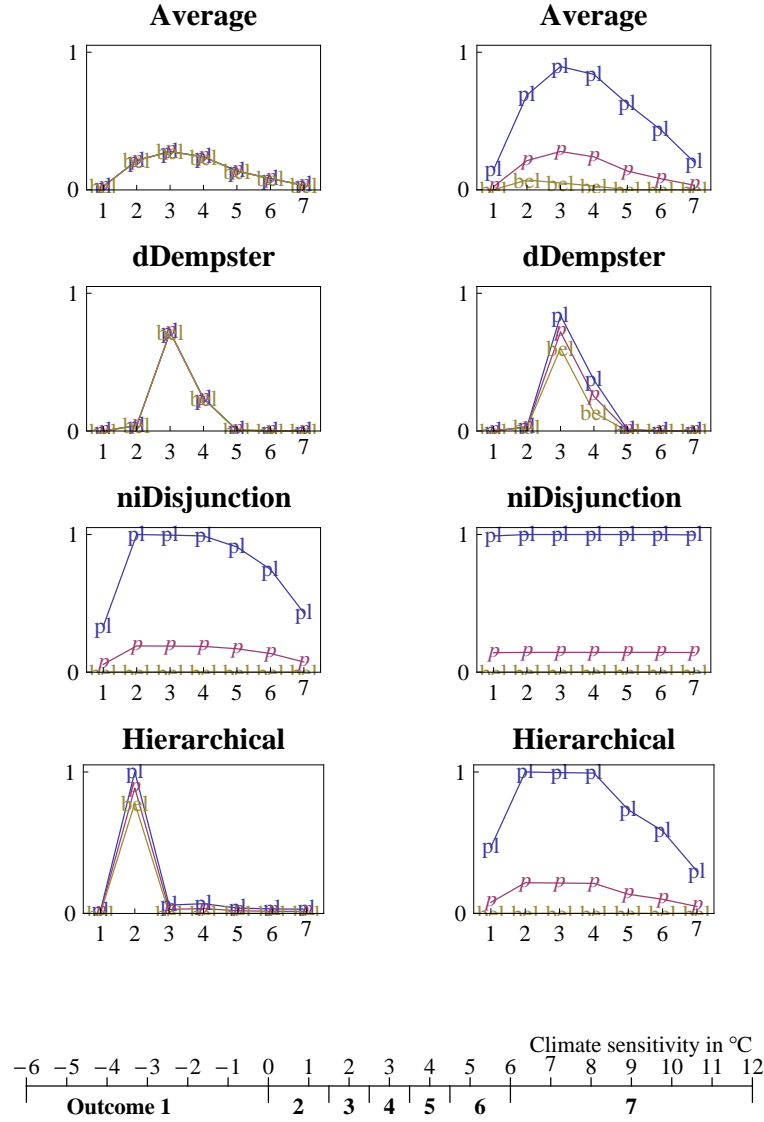


Figure 3: Result of the fusion, using different operators. Left column using bayesian beliefs, right column using consonant beliefs.

	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
$m_{average}$	0.	0.00	0.23–0.24	0.65–0.65	0.11–0.11
$m_{dDempster}$	0.	0.	0.04–0.04	0.96–0.96	0.–0.
$m_{niDisjunction}$	0.	0.08	0.–1.	0.–1.	0.–0.86
$m_{Hierarchical}$	0.	0.00	0.79–1.	0.–0.16	0.–0.06
	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
$m_{average}$	0.	0.08	0.07–0.69	0.27–0.93	0.–0.45
$m_{dDempster}$	0.	0.	0.02–0.03	0.97–0.98	0.–0.
$m_{niDisjunction}$	0.	0.99	0.–1.	0.–1.	0.–1.
$m_{Hierarchical}$	0.	0.18	0.–1.	0.–1.	0.–0.61

Table 2: The fusion of expert opinion on climate sensitivity. Top table using bayesian beliefs, bottom table using consonant beliefs.

of consonant beliefs. There are five columns. The first column shows the degree of conflict $m(\oslash)$, while the second column shows $m(\Omega)$. Heuristically, smaller numbers in these columns are better, since they correspond to intuitively more interesting or informative results. The last three columns show the values of the belief and plausibility functions. They refer to coarsened frame of reference: states of the world have been grouped into three policy relevant cases. The less worrying case is $\{\omega_1, \omega_2\}$, that is sensitivity below 1.5°C. The historical canonical range is represented by $\{\omega_3, \omega_4, \omega_5\}$. The worst case groups together outcomes for which climate sensitivity is above 4.5°C, that is $\{\omega_6, \omega_7\}$.

We now discuss each operator successively. Results obtained by averaging are shown in figure 3 on the top row. Top left, the three curves bel , p and pl are superposed: when all beliefs are bayesian, the average is also bayesian. The top right graphic represents the average of consonant beliefs. Plausibility and beliefs are now very different. For a decision-maker only focused on pignistic probabilities (curve labelled p), the left and right results would seem very close. But from an evidence theory perspective, the left plot under-represents scientific controversies and the need for precautionary decision-making.

Consider for example what the results say about $\Delta T_{2\times} < 1.5$. As table 2 shows, averaging in the bayesian case leads to the conclusion that $bel(\{\omega_1, \omega_2\}) = 0.23$ and $pl(\{\omega_1, \omega_2\}) = 0.24$ (the small difference between belief and plausibility levels is explained by the reliability factor 0.999 we introduced.). Yet the bottom half of the table shows that averaging in the consonant case leads to $bel(\{\omega_1, \omega_2\}) = 0.07$ and $pl(\{\omega_1, \omega_2\}) = 0.7$.

These results are qualitatively different. The first could be stated as ‘There is a low confidence that $\Delta T_{2\times} < 1.5$ ’, to mean a probability around 0.2. But the second result could be stated as ‘ $\Delta T_{2\times} < 1.5^\circ\text{C}$ has a low degree of belief but a significant level of plausibility’. That second, more imprecise statement describes more accurately the state of scientific controversies.

The graphics in the second row allow to see that Dempster’s rule with discounting produces very focused beliefs around ω_3 . This rule considerably reduces the plausibility of states of the world that are outside the canonical range. The issues with discounting, shared evidence and the bandwagon effect discussed

section 3.3 suggest that much of this precision is unwarranted.

Results with the non-interactive disjunction are shown third. With consonant beliefs, the result is almost completely uninformative: line 7, column 2 in table 2 shows indeed that $m(\Omega) = 0.99$. With bayesian beliefs, the figure shows that the results are well shaped. In that case, while the levels of belief are close to zero, the levels of plausibility are shaped interestingly, with lower values for the extreme cases. This suggests empirically that the non-interactive disjunction rule produces degenerate results when used to combine consonant beliefs, but works better when combining bayesian beliefs.

Lastly, let us consider the results of the hierarchical fusion. In the case of bayesian beliefs, almost all the weight goes to ω_2 , and the plausibility of $\{\omega_3, \omega_4, \omega_5\}$ is only 0.16. This can be explained by looking at the cautious conjunction within groups (table 8 in the annex). At this stage, the degree of conflict is high, respectively 0.86, 0.86 and 1 within G_1 , G_2 and G_3 . That contradiction problem does not arise when combining consonant beliefs: the degree of conflict within groups is only 0.01, 0.03 and 0.14. The non-interactive disjunction rule across groups gives a more intuitive image of the opinion pool.

Figure 3 shows that some combinations of operators and input data produce degenerate results, while other give more interesting BBAs. Designing a study to formally elicit and aggregate information using expert opinion involves several heuristic choices: which experts to interview, how to elicit their opinions and how to fusion them. To avoid degenerate results, one has to balance these choices.

Conjunction operators move the belief masses ‘down’, into smaller focal sets, but too much precision is a problem given conflicting opinions. Disjunction operators move the belief masses ‘up’, to larger focal sets, but too much generality produces useless results. To avoid extreme results, a proper heuristic may be to apply a precision increasing operator (conjunction) to imprecise data (consonant beliefs), or conversely an imprecision-increasing operator (disjunction) to precise data (bayesian beliefs). Averaging is neutral here.

This may contribute to explain the qualitative results on figure 3. For example, when opinions are represented with bayesian beliefs, it is interesting to combine them with the non-interactive disjunction rule: this operator produces beliefs that are more imprecise than its inputs. Conversely, this operator gives the trivial vacuous beliefs when opinions are represented using consonant beliefs, because it adds imprecision to a pool of already imprecise data.

5 Discussion

5.1 Sensitivity analysis

Tables 3 and 4 shows the result of the fusion under alternative operators (also represented in the annex, figures 5 and 6). First, we examine the sensitivity of the discounted Dempster’s rule to the reliability factor. Decreasing the reliability factor means adding doubt to the beliefs to be combined. This spreads around the belief weights, so the result becomes less focused compared to the previous case with $r = 0.8$. The magnitude of change in the results can be significant. Consider for example the ‘below 1.5°C’ case when combining bayesian

	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
Dempster $r=0.5$	0.	0.01	0.16–0.17	0.8–0.81	0.03–0.04
cautious	1.	0.	0.–0.	0.–0.	0.–0.
cautious $r=0.8$	0.96	0.00	0.01–0.02	0.01–0.02	0.00–0.01
niConj.	1.	0.	0.–0.	0.–0.	0.–0.
niConj. $r=0.8$	1.	0.	0.–0.	0.–0.	0.–0.
	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
Dempster $r=0.5$	0.	0.	0.06–0.12	0.88–0.94	0.–0.01
cautious	0.95	0.	0.05–0.05	0.–0.	0.–0.
cautious $r=0.8$	0.71	0.	0.1–0.13	0.16–0.19	0.–0.01
niConj.	1.	0.	0.–0.	0.–0.	0.–0.
niConj. $r=0.8$	0.87	0.	0.00–0.00	0.13–0.13	0.–0.

Table 3: Sensitivity analysis: the fusion of expert opinion using alternative symmetric operators. Top half, using bayesian beliefs; bottom half using the corresponding consonant beliefs.

	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
Hierarchical $r=.99$	0.	0.01	0.74–1.	0.–0.2	0.–0.08
Hierarchical $r=.9999$	0.	0.00	0.79–1.	0.–0.16	0.–0.06
Hierarchical 3-way	0.	0.00	1.–1.	0.–0.00	0.–0.00
average within	0.	0.00	0.01–1.	0.–0.96	0.–0.39
	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
Hierarchical $r=.99$	0.	0.2	0.–1.	0.–1.	0.–0.62
Hierarchical $r=.9999$	0.	0.18	0.–1.	0.–1.	0.–0.61
Hierarchical 3-way	0.	0.00	0.01–1.	0.–0.99	0.–0.15
average within	0.	0.58	0.–1.	0.–1.	0.–0.95

Table 4: Sensitivity analysis: the fusion of expert opinion using alternative hierarchic procedures. Top half, using bayesian beliefs; bottom half using the corresponding consonant beliefs.

beliefs. Between $r = 0.8$ (table 2, line 2, column 3) and $r = 0.5$ (table 3, line 1, column 3), its probability increases by a factor 4.

The other four lines in table 3 illustrate the problem of contradiction. In the non-interactive conjunction and in the cautious conjunction of all experts, the degree of conflict $m(\odot)$ is very high. We check that \odot is less sensitive to conflict than \odot , and that adding doubt, either by discounting or by transforming bayesian into consonant beliefs, decreases conflict. This only confirms the theoretical reasons why in section 3.3 we disqualified these operators.

Table 4 present variants of the hierarchical approach. Using a reliability factor $r = 0.99$ do not change the results much compared to the case with $r = 0.999$. Adding more doubt to the input data mechanically increases $m(\Omega)$ in the output, this in turn increases plausibility levels. Moving to $r = 0.9999$, the results do not change visibly, as the display is rounded to 2 digits.

Merging G_1 and G_2 together allows to check that results are significantly sensitive to the clustering of experts: the plausibility of the ‘above 4.5°C ’ case, with consonant beliefs, drops from 0.61 to 0.15 (table 2, line 7, column 5). Finally, we examined a hierarchic fusion where the first step is averaging, rather than the cautious conjunction. The plausibility function levels are generally greater with averaging. This is less desirable, since it means the result is less informative.

5.2 Existing results on climate sensitivity

In its third assessment published in 2001, the Intergovernmental Panel on Climate Change [IPCC, 2001b, Technical Summary F.3] stated that “climate sensitivity is likely to be in the range of 1.5 to 4.5°C . This estimate is unchanged from the first IPCC Assessment Report in 1990”. This estimate can be traced back even earlier [National Research Council, 1979]. The $[1.5, 4.5^\circ\text{C}]$ was not offered as a 90% confidence interval, but as a “likely” range. The word “likely” had a formally defined meaning, it was used to indicate a judgmental estimate of confidence of 66 to 90% chance. Since this report, several studies have estimated probability density functions for climate sensitivity based on models and observations.

Hall et al. [2007] combined a set of 7 such distributions in an imprecise probability framework. The result, given as upper probability bounds, suggests that $p(\Delta T_{2\times} \leq 1.5) \leq 0.10$ and $p(\Delta T_{2\times} \geq 4.5) \leq 0.60$ (determined graphically from Figure 5 in Hall et al. [2007]).

Kriegler [2005, section 3.2.3] conducted a deeper analysis of the combination of these distributions with imprecise probabilities. Four of the six estimates examined show a 90% confidence interval in the range between 1.3 and 6.3°C . In the other two studies, these ranges are $[1.4, 7.7]$ and $[2.2, 9.3]$. The author then estimated a prior imprecise distribution based on the literature, and then updated it using a climate model and observational data for 1870–2002. Updating was done using both Dempster’s rule and the Generalized Bayes Rule, but only Dempster’s rule produced meaningful results. Table 5 summarizes them. For example, the posterior results suggest that the probability of climate sensitivity being less than 1.5°C is very small (0.00 meaning less than 1 per thousand). In the posterior, the probability that climate sensitivity falls in the $[1.5, 4.5]$ range is between 0.53 and 0.99.

	$T_{2\times} \in [0^\circ\text{C}, 1.5^\circ\text{C}]$	$[1.5^\circ\text{C}, 4.5^\circ\text{C}]$	$[4.5^\circ\text{C}, 10^\circ\text{C}]$
Prior	[0, 0.07]	[0.31, 0.98]	[0.02, 0.62]
Posterior	[0, 0.00]	[0.53, 0.99]	[0.01, 0.47]

Table 5: Probabililty bounds on climate sensitivity $\Delta T_{2\times}$ according to Kriegler [2005, table 4.2]. The prior summarizes the literature, the posterior is updated by Dempster’s rule, using the results of a simulation model.

According to these results, there is a large possibility that climate sensitivity lies above 4.5°C . The relatively high range’s upper bound has been contested by Hegerl et al. [2006, Figure 3], who recently estimated that the 5–95 per cent confidence range of climate sensitivity was about $1.5\text{--}6.2^\circ\text{C}$. Still, this does not refute the idea that the 90% confidence interval has its upper bound above 4.5°C .

Hegerl et al. [2007, 718–727] offers a comprehensive assessment of the literature on climate sensitivity. In this more recently published Fourth Assessment Report, IPCC continues to formulate uncertainty statement literally, with an explicit correspondence on a probability scale [IPCC, 2005]. The conclusion is that, in spite of new research, the result is not changed much since the previous report: the likely range is $[2, 4.5]$, where “likely” means a probability between 66 and 90 percent. IPCC also states that it is “very unlikely” that climate sensitivity lies below 1.5, meaning a less than 10% probability.

Andronova et al. [2007, Figure 1.1a] also published an historical perspective on climate sensitivity. They conclude that recent studies based on observations indicate that there is more than a 50% likelihood that $\Delta T_{2\times}$ lies outside the canonical range of 1.5°C to 4.5°C , with disquietingly large values not being precluded. They combined the 16 experts opinions in terms of their mean estimation and variance into a single cumulative density function, under the assumption that each of the 16 estimations is normally distributed, but this was mostly for historical comparison.

Results presented table 2 can be compared to this more recent literature. Only the non-degenerate cases in lines 1, 3, 5 and 8 need to be considered. The plausibility that climate sensitivity lies below 1.5 appears to be low in the recent literature. But it is high in our results (respectively 1, 0.7 and 1 in lines 3, 5 and 8). In the linear pooling case line 1, the probability is 0.23 which can also seen as rather significant. The fusion results are not in line with the more recent literature here.

Given that the dataset included one opinion certain that $\Delta T_{2\times} \leq 1.5^\circ\text{C}$, this discrepancy can hardly be seen as a mathematical artefact. A more intuitive explanation is that the scientific consensus has evolved since 1995, to revise downward the likelihood of that event. The increase in the IPCC lower bound from 1.5°C to 2°C can be taken as a sign of this change.

Consider now the last column in table 2, related to the case in which climate sensitivity lies above 4.5°C . The recent literature finds that this case is rather plausible. Lines 3, 5 and 8, this event’s plausibility is respectively 0.88, 0.45 and 0.62. Line 1, the probability is 0.11. Thus, the fusion results are in better agreement with the more recent findings here. If the visibility given to the higher than 4.5°C case has increased in the recent publications, its subjective

weight was already present in experts' minds back in 1995.

5.3 Remarks on the hierarchical approach

Clemen and Winkler [1999] dichotomize ways to summarize the opinion of a variety of experts in two classes: behavioral approaches and mathematical methods. In behavioral approaches, also called interactive expert aggregation methods, experts exchange information with each other. In mathematical approaches, each expert is interviewed separately in a first phase, and then opinions are combined afterward according to some algorithmic aggregation method.

Behavioral approaches have many interesting advantages over algorithmic methods. The group judgment is more legitimate since it comes from the experts themselves and collective deliberation is a natural social process. The way scientific panels such as the Intergovernmental Panel on Climate Change (IPCC) write their reports is an interactive expert aggregation method. However, behavioral approaches also have drawbacks. Any group of experts is subject to the social dynamics inherent in any group of humans. There are known biases towards conservatism and overconfidence in group-thinking. More importantly, managing all the interactions between the experts is complicated, time consuming and thus costly.

Mathematical methods aim to simplify and rationalize the procedure by separating in time the expert opinion elicitation step from the aggregation step, and performing the later without the experts. The simplest aggregation method we have seen is linear pooling, that is averaging. It works with bayesian as well as with consonant beliefs. As an alternative, we have seen that the non-interactive disjunction produces meaningful and non-trivial results, when beliefs are bayesian. Finally, we examined a hierarchical approach. Using consonant beliefs, it gave results comparable to those obtained with the non-interactive disjunction.

The hierarchical approach is based on a partition of experts into a small numbers of schools of thought. Contrary to symmetric fusion operators, it requires to structure the pool of experts. Thus, it requires to put back some sociology aspects in a mathematical aggregation framework. While in this paper we determined the groups from the elicited probability distribution, social sciences offer much better procedures:

- The network of experts can be analyzed through publications. Experts who have published together have seen the same data, they are more likely to share evidence. Newman [2004] shows that bibliometrics can help determine the patterns of scientific collaborations.
- Expert elicitation techniques involve semi-structured interviews. That material is prime experimental data for social scientists. Working from transcripts is a classical method to analyze how a group of people is organized. Such analysis is usually conducted without mathematical tools. There are more formal content analysis methods, often based on the written rather than oral production of the subjects.
- The experts themselves know their community. They can help to discover how it is organized, and they can validate the results of the sociological analysis.

Note that the expert selection step, in an elicitation exercise, has to make sure that no major point of view is omitted. This shows that sociological considerations on the population of experts cannot be avoided, even in a mathematically oriented study. When it is clear from the start what the different schools of thought are, one can select a single expert to represent each position, and then pool the opinions symmetrically. Otherwise, it is only after analyzing the interviews transcripts that the population of experts can be organized around a small number of archetypes.

Representing the diversity of viewpoints by a small numbers of schools of thought is admittedly a strong simplification of complex social reality. But it is less simplistic than treating all experts symmetrically. Looking at the detailed structure of epistemic communities, and explaining the differences between theories can be very informative in itself. Bringing forward that qualitative analysis is a valuable advantage of the hierarchical approach.

6 Conclusion

This paper compared several procedures to aggregate expert opinion in the Transferable Belief Model. We considered both bayesian beliefs and consonant beliefs. The former corresponds naturally with probabilities, the latter with possibilities. Regarding the procedures that combine opinions symmetrically, results show that:

- Taking either the non-interactive conjunction or the cautious conjunction of all opinions produces degenerate results, indicating only that experts contradict each other.
- Dempster’s rule of combination, even after discounting, led to excessively narrow results (overconfidence).
- Averaging always produces non-degenerate results, but there are two problems with that method. First, when beliefs are bayesian, the result is bayesian too. In the Dempster-Shafer theory of evidence, bayesian beliefs under-represents scientific controversies. Second, averaging is essentially a way to allocate more weight to views held by a larger number of experts. This is a problem because scientific theories should be assessed only on their own merit.
- The non-interactive disjunction rule produces a degenerate (uninformative) result when beliefs are consonant. The intuition is that consonant beliefs are vague to start with, and the result of the disjunction is more imprecise than its inputs. The non-interactive disjunction of bayesian beliefs represents more appropriately scientific controversies their average.

Then a hierarchical fusion procedure was assessed. This procedure is built around a simple model of experts’ social relations: it divides them into schools of thought. Social science methods are available to determine the fine structure of epistemic communities, and knowing this structure may be as interesting as knowing an aggregate opinion. Within each school, beliefs are aggregated using the cautious conjunction operator. Across the groups, beliefs are combined using the non-interactive disjunction rule. Hierarchical fusion in the Transferable

Belief Model offers a solution to several theoretical problems regarding opinion aggregation:

- It allows to represent the issue of precautionary decision making due to scientific controversies in ways that purely probabilistic methods are not able to, beyond standard expected utility maximization.
- Disjunction allows to deal with complete contradiction among opinions without falling into degenerate results or paradoxes. When several scientific theories compete to explain the same observations, it should not be assumed that both are true at the same time (conjunction), but that at least one will remain (disjunction).
- Within groups, cautious conjunction does pool together distinct streams of evidence to make beliefs firmer. But it is not assumed that opinions are, statistically speaking, independent: this would overestimate the precision of actual information.
- Pooling opinions across schools of thoughts, rather than across individual experts, is arguably a more balanced procedure. Contrary to averaging, where the number of experts holding a view is essential, with hierarchical fusion minority views are equally taken into account.
- This hierarchical fusion procedure does not use discounting, thus avoiding the issue of expert calibration.

This study was conducted using a real-world dataset on climate sensitivity, published in 1995. The fusion of expert opinion was compared to the more recent stochastic results on climate sensitivity, some of them based on model simulations. That comparison suggest that since 1995, the plausibility that climate sensitivity will remain below 1.5°C has decreased. The plausibility that climate sensitivity is above 4.5°C was significant in the community's opinion in 1995. It remains so today.

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7 Annex: Additional tables and figures

S	$m(S)$
$\{2\}$	0.00014
$\{2, 3\}$	0.00765
$\{2, 4\}$	0.00340
$\{2, 3, 4\}$	0.16386
$\{1, 2, 3, 4\}$	0.00663
$\{2, 4, 5\}$	0.00114
$\{2, 3, 4, 5\}$	0.13432
$\{1, 2, 3, 4, 5\}$	0.07212
$\{2, 3, 4, 6\}$	0.02748
$\{1, 2, 3, 4, 6\}$	0.01334
$\{2, 3, 4, 5, 6\}$	0.08973
$\{1, 2, 3, 4, 5, 6\}$	0.18316
$\{2, 3, 4, 7\}$	0.02128
$\{2, 3, 4, 5, 7\}$	0.00624
$\{2, 3, 4, 6, 7\}$	0.01364
$\{1, 2, 3, 4, 6, 7\}$	0.01063
$\{2, 3, 4, 5, 6, 7\}$	0.06289
$\{1, 2, 3, 4, 5, 6, 7\}$	0.18234

Table 6: The basic belief assignment resulting of the hierarchical fusion. Cautious conjunction within groups, non interactive disjunction across, consonant beliefs.

Climate sensitivity (°C)	-6-0	0-1.5	1.5-2.5	2.5-3.5	3.5-4.5	4.5-6	6-12
Expert 1	0.	0.1333	0.5167	0.2333	0.0917	0.025	0.
Expert 2	0.1	0.2	0.175	0.1625	0.0875	0.1179	0.1571
Expert 3	0.1429	0.1203	0.1579	0.2632	0.1667	0.1002	0.049
Expert 4	0.	0.2333	0.2667	0.1429	0.1171	0.14	0.1
Expert 5	0.	1.	0.	0.	0.	0.	0.
Expert 6	0.1217	0.1488	0.1494	0.205	0.1917	0.1333	0.05
Expert 7	0.	0.0909	0.2591	0.375	0.125	0.0786	0.0714
Expert 8	0.	0.1333	0.3667	0.2286	0.127	0.1023	0.0421
Expert 9	0.	0.225	0.275	0.1833	0.1367	0.13	0.05
Expert 10	0.	0.14	0.26	0.4	0.2	0.	0.
Expert 11	0.	0.05	0.35	0.25	0.2	0.15	0.
Expert 12	0.	0.1	0.4	0.275	0.125	0.1	0.
Expert 13	0.	0.375	0.3417	0.2083	0.075	0.	0.
Expert 14	0.	0.0357	0.281	0.3583	0.225	0.1	0.
Expert 15	0.	0.35	0.27	0.2133	0.1167	0.05	0.
Expert 16	0.	0.05	0.375	0.325	0.2	0.05	0.
Climate sensitivity (°C)	-6-0	0-1.5	1.5-2.5	2.5-3.5	3.5-4.5	4.5-6	6-12
Expert 1	0.	0.5167	1.	0.7167	0.3917	0.125	0.
Expert 2	0.6875	1.	0.975	0.95	0.6125	0.7768	0.9339
Expert 3	0.8409	0.7506	0.886	1.	0.9035	0.6499	0.3427
Expert 4	0.	0.9667	1.	0.7857	0.6857	0.7771	0.6
Expert 5	0.	1.	0.	0.	0.	0.	0.
Expert 6	0.7804	0.9005	0.9022	1.	0.9867	0.8384	0.35
Expert 7	0.	0.5136	0.8841	1.	0.6159	0.4643	0.4286
Expert 8	0.	0.6714	1.	0.8619	0.6524	0.5538	0.2526
Expert 9	0.	0.95	1.	0.8667	0.7267	0.7	0.3
Expert 10	0.	0.56	0.86	1.	0.74	0.	0.
Expert 11	0.	0.25	1.	0.9	0.8	0.65	0.
Expert 12	0.	0.5	1.	0.875	0.575	0.5	0.
Expert 13	0.	1.	0.9667	0.7	0.3	0.	0.
Expert 14	0.	0.1786	0.9226	1.	0.8107	0.4357	0.
Expert 15	0.	1.	0.92	0.8067	0.5167	0.25	0.
Expert 16	0.	0.25	1.	0.95	0.7	0.25	0.

Table 7: Top half, the elicited probability distributions, corresponding to histograms in Figure 2. Bottom half, possibility distributions derived from these, represented as dotted lines in Figure 2

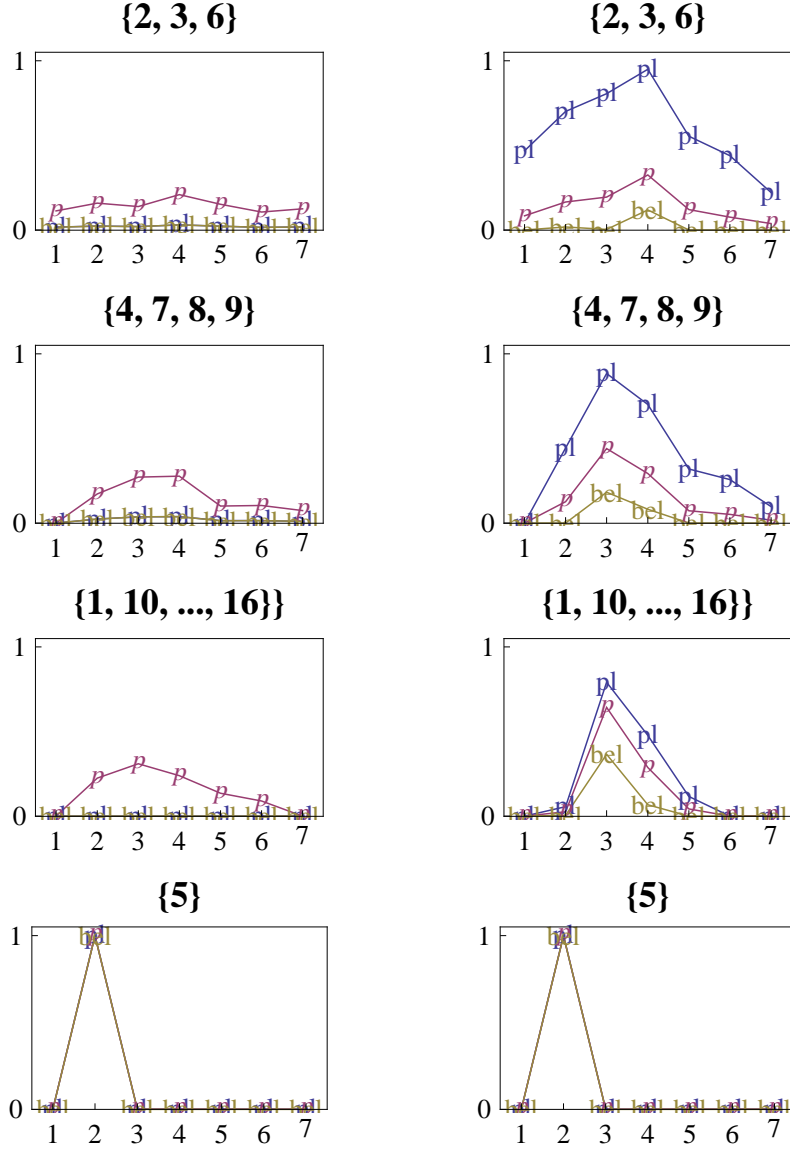


Figure 4: Result of the hierarchical fusion's first stage, the cautious conjunction within each group. Left column using bayesian beliefs, right column using consonant beliefs. See also table 8.

	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
2, 3, 6	0.86	0.	0.04–0.04	0.07–0.07	0.03–0.03
4, 7, 8, 9	0.86	0.	0.02–0.02	0.09–0.09	0.02–0.02
1,10, ..., 16	1.	0.	0.–0.	0.–0.	0.–0.
5	0.	0.00	1.–1.	0.–0.00	0.–0.00
	Conflict $m(\oslash)$	Ignorance $m(\Omega)$	Below 1.5°C $bel-pl$	In range $bel-pl$	Above 4.5°C $bel-pl$
2, 3, 6	0.01	0.14	0.02–0.75	0.24–0.97	0.–0.48
4, 7, 8, 9	0.03	0.	0.–0.44	0.52–0.97	0.–0.26
1,10, ..., 16	0.14	0.	0.00–0.06	0.81–0.86	0.–0.
5	0.	0.00	1.–1.	0.–0.00	0.–0.00

Table 8: The cautious conjunction within groups of expert opinion on climate sensitivity. Top table using bayesian beliefs, bottom table using consonant beliefs. See also figure 4.

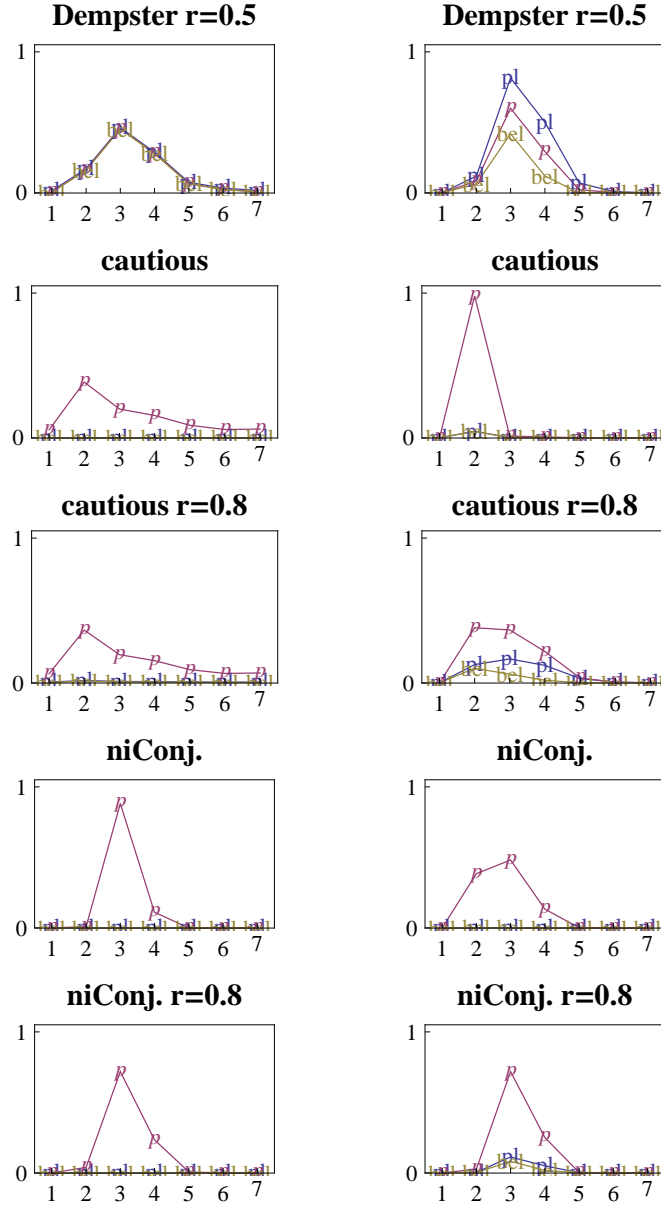


Figure 5: Sensitivity analysis, alternative symmetric fusion operators. Left column using bayesian beliefs, right column using consonant beliefs.

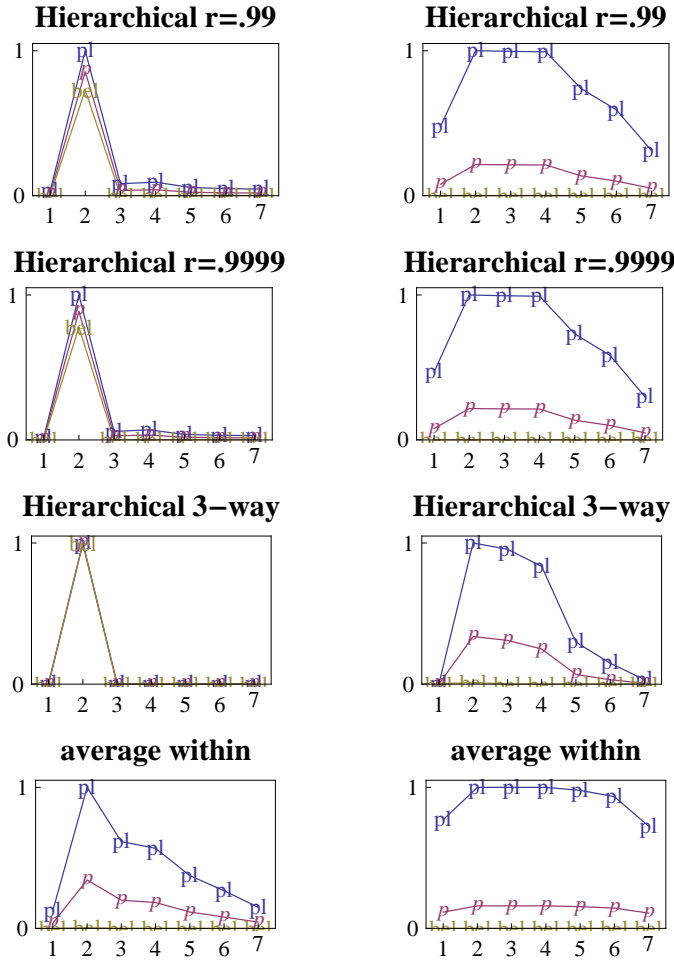


Figure 6: Sensitivity analysis, alternative hierarchic fusion. Left column using bayesian beliefs, right column using consonant beliefs.