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**The wedge of arbitrage free prices : anything goes**

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# The wedge of arbitrage free prices: anything goes\*

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**ABSTRACT:** We show that if  $K$  is a closed cone in a finite dimensional vector space  $X$ , then there exists a one-to-one linear operator  $T: X \rightarrow C[0, 1]$  such that  $K$  is the pull-back cone of the positive cone of  $C[0, 1]$ , i.e.,  $K = T^{-1}(C_+[0, 1])$ . This problem originated from questions regarding arbitrage free prices in economics.

**Keywords and Phrases:** Closed cones in finite dimensional spaces, pull-back cones, securities markets, arbitrage free prices

**AMS Classification Numbers:** 46A40, 47B60, 47B65, 91B28

## 1 Introduction

This work deals with cones and wedges of vector spaces. For terminology and notation regarding ordered vector spaces and not explained below we refer the reader to [11], [12] and [8]. For topological vector spaces, we refer to [1] and [10].

A nonempty subset  $W$  of a vector space is said to be a **wedge** if it satisfies the following two properties:

1.  $W + W \subseteq W$ ,
2.  $\alpha W \subseteq W$  for all  $\alpha \geq 0$ .

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\*This paper is dedicated to our late friend and colleague H. H. Schaefer, whose pioneering works as summarized in [13] and [14] laid down the foundations of the modern theory of ordered vector spaces. The research of C. D. Aliprantis is supported in part by the NSF grants SES-0128039 and DMS-0437210.

If, in addition,  $W \cap (-W) = \{0\}$ , then  $W$  is called a **cone**.

Clearly, wedges and cones are convex sets. They are associated with respectively vector pre-orderings and vector orderings of vector spaces. An **ordered vector space** is a vector space  $X$  equipped with a cone  $X_+$ . The cone  $X_+$  induces a vector ordering  $\geq$  on  $X$  by letting  $x \geq y$  whenever  $x - y \in X_+$ . An operator  $T: X \rightarrow Y$  between ordered vector spaces is said to be **positive** if  $T(X_+) \subseteq Y_+$ , i.e., if  $x \geq 0$  implies  $Tx \geq 0$ .

Let  $T: X \rightarrow Y$  be an operator between two vector spaces and let  $W$  be a wedge of  $Y$ . It is easy to see that the inverse image of  $W$  under  $T$  is a wedge of  $X$ . That is, the set

$$T^{-1}(W) = \{x \in X: T(x) \in W\}$$

is a wedge of  $X$ . If  $T$  is also one-to-one and  $W$  is a cone, then the wedge  $T^{-1}(W)$  is also a cone of  $X$ .

We start with a simple lemma.

**Lemma 1.** *Let  $T: X \rightarrow Y$  be a one-to-one operator between two vector spaces. If  $K$  is a cone of  $Y$ , then  $T^{-1}(K)$  is a cone of  $X$  and the operator  $T: (X, T^{-1}(K)) \rightarrow (Y, K)$  is a positive operator.*

A cone  $K$  of a vector space  $X$  is called the **pull-back cone** of the cone of an ordered vector space  $L$  if there exists a one-to-one operator  $T: X \rightarrow L$  such that  $K = T^{-1}(L_+)$ . Alternatively,  $K$  is the pull-back of the cone of an ordered vector space  $L$  if and only if the ordered vector space  $(X, K)$  is order-embeddable in  $L$ .

Likewise, a cone  $K$  of a topological vector space  $X$  is called the **continuous pull-back cone** of the cone of an ordered topological vector space  $L$  if there exists a continuous one-to-one operator  $T: X \rightarrow L$  such that  $K = T^{-1}(L_+)$ . Alternatively,  $K$  is the continuous pull-back cone of the cone of a topological ordered vector space  $L$  if and only if the topological ordered vector space  $(X, K)$  is topologically order-embeddable in  $L$ .

As mentioned in the abstract, the objective of this paper is to establish the following basic result. (As usual,  $\mathbf{1}$  will denote the constant function one on  $[0, 1]$ , i.e.,  $\mathbf{1}(t) = 1$  for all  $t \in [0, 1]$ .)

**Theorem 2.** *Every closed cone of a finite dimensional vector space is the pull-back cone of the (standard) cone of  $C[0, 1]$ .*

*Moreover, if  $K$  is a closed and generating cone of a finite dimensional vector space  $X$ , then  $K$  can be taken to be the pull-back cone of a one-to-one operator  $T: X \rightarrow C[0, 1]$  such that  $Tu = \mathbf{1}$  for some vector  $u \in \text{Int}(K)$ .*

An interesting consequence follows.

**Corollary 3.** *A nonempty subset  $C$  of  $\mathbb{R}^n$  is convex and compact if and only if there exist an  $(n + 1)$ -dimensional subspace  $E$  of  $C[0, 1]$  and a strictly positive linear functional  $f$  on  $E$  such that  $C$  and  $E_+ \cap \{x \in E: f(x) = 1\}$  are affinely homeomorphic.<sup>1</sup>*

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<sup>1</sup>Two nonempty convex sets  $A$  and  $B$  (in respectively two topological vector spaces) are affinely homeomorphic if there exists a surjective affine homeomorphism  $T: A \rightarrow B$ .

## 2 Background

### 2.1 Normal cones

Recall that a subset  $A$  of an ordered vector space  $E$  is said to be **full** if for each pair  $x, y \in A$  the order interval  $[x, y] := \{z \in E : x \leq z \leq y\}$  is contained in  $A$ .

**Definition 4.** A cone  $K$  of a topological vector space  $(E, \tau)$  is said to be **normal** whenever the topology  $\tau$  has a base at zero consisting of  $K$ -full sets (that is, of sets full for the order on  $E$  defined by  $K$ ).

The notion of normal cone is one of the most useful connections between topology and order of a vector space which implies several nice properties for topological vector spaces ordered by normal cones. In particular, order intervals are topologically bounded, and the existence of a normal cone implies that the topology  $\tau$  given on the vector space is Hausdorff. Also, if  $\tau$  is locally convex, then the dual wedge

$$K' =: \{f \in L' : f(x) \geq 0 \text{ for all } x \geq 0\}$$

is generating in  $L'$ . A useful characterization of normal cones is the following:

**Theorem 5.** For a cone  $K$  of a topological vector space  $(E, \tau)$  the following statements are equivalent:

1. The cone  $K$  is normal.
2. If two nets  $\{y_\alpha\}$  and  $\{x_\alpha\}$  of  $E$  (with the same index set) satisfy  $0 \leq_K y_\alpha \leq_K x_\alpha$  for each  $\alpha$  and  $x_\alpha \xrightarrow{\tau} 0$ , then  $y_\alpha \xrightarrow{\tau} 0$ .

Using this characterization, it is easy to prove the second of the following three basic properties of closed cones in finite dimensional vector spaces:

**Lemma 6.** If  $K$  is a closed cone of a finite dimensional space  $E$ , then:

1. The  $K$ -order intervals of  $E$  are compact.
2. The cone  $K$  is normal, and
3. The dual wedge  $K'$  is a cone if and only if  $K$  is generating.

*Proof.* We shall denote by  $\leq$  the vector ordering induced by the cone  $K$ , that is

$$x \leq y \text{ if and only if } y - x \in K.$$

(1) The proof is standard. Let  $[0, u]$  be a  $K$ -order interval. From  $[0, u] = K \cap (u - K)$ , we see that  $[0, u]$  is closed. To see that  $[0, u]$  is also norm bounded, assume that a sequence

$\{y_n\} \subset [0, u]$  satisfies  $\|y_n\| \rightarrow \infty$ . Passing to a subsequence, we can assume  $\frac{y_n}{\|y_n\|} \rightarrow y$ . Clearly,  $\|y\| = 1$  and so  $y \neq 0$ . Now from  $0 \leq y_n \leq u$ , it follows  $0 \leq \frac{y_n}{\|y_n\|} \leq \frac{u}{\|y_n\|}$ , and from the closedness of  $K$  and  $\frac{u}{\|y_n\|} \rightarrow 0$ , we see that  $0 \leq y \leq 0$  or  $y = 0$ , which is impossible. Hence  $[0, u]$  is also bounded and thus a compact set.

(2) Assume first that  $K$  is generating (that is,  $E = K - K$ ), thus has a nonempty interior. Assume that two sequences  $\{y_n\}$  and  $\{x_n\}$  of  $E$  satisfy  $0 \leq y_n \leq x_n$  for each  $n \in \mathbb{N}$  and  $x_n \rightarrow 0$ . Let  $u$  be an interior point of  $K$ . As  $0 \in \text{int}(u - K)$ , for large enough  $n$  we have  $0 \leq y_n \leq x_n \leq u$ , hence  $0 \leq (x_n - y_n) \leq x_n \leq u$ . In view of the compactness of  $[0, u]$ , passing to a subsequence we can assume that  $(x_n - y_n) \rightarrow z \in [0, u]$ , and so, using again the closedness of  $K$ , we see that  $-y_n \rightarrow 0$ .

If the cone  $K$  is non-generating, it is at least a generating cone of the vector subspace  $K - K$  of  $E$ . Since two sequences  $\{y_n\}$  and  $\{x_n\}$  of  $E$  satisfying  $0 \leq y_n \leq x_n$  for each  $n \in \mathbb{N}$  and  $x_n \rightarrow 0$  are actually sequences lying in the finite dimensional space  $K - K$ , the desired conclusion follows from the first part of the proof.

(3) Let  $x' \in K' \cap (-K')$ . If  $E = K - K$  then  $x' \cdot x = 0$  for all  $x \in E$ , which proves that  $x' = 0$ . Conversely, assume that some  $x \in E \setminus (K - K)$ . As a vector subspace of a finite dimensional vector space, the set  $K - K$  is closed. From the separation theorem between a closed convex set and the compact set  $x$ , we have  $x' \cdot (K - K) = 0$  and  $x' \cdot x > 0$  for some  $x' \in E'$ . If  $K' \cap (-K') = \{0\}$ , from  $x' \cdot (K - K) = 0$  we deduce  $x' = 0$ , which contradicts  $x' \cdot x > 0$ . ■

## 2.2 The Cantor set $\mathcal{C}$ and the space $C(\mathcal{C})$

The Cantor set can be defined as the countable product  $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ , where the two-point set has the discrete topology. As such, when equipped with the product topology, it is easily seen to be a compact metric space. It can also be thought of as a subset of the real interval  $[0, 1]$ , in an inductive construction where at each step one removes from each closed interval the open middle third-interval. Viewed as a subset of the unit interval, the Cantor set  $\mathcal{C}$  is a nowhere dense set of Lebesgue measure zero.

For details and more about the above assertions we refer the reader to [1, pp. 98–101] and to [2, pp. 41–42]. For the rest of our discussion, we need the following well known theorem which can also be found in [1, p. 100].

**Theorem 7.** *Every compact metric space is the image of the Cantor set under some continuous function.*

Let us recall some terminology. A mapping  $f: X \rightarrow Y$  between two topological spaces is a **topological embedding** if  $f: X \rightarrow f(X)$  is a homeomorphism. Likewise, a linear operator  $T: X \rightarrow Y$  between two ordered vector spaces is an **order-embedding** if  $T$  is one-to-one and if  $x \geq 0$  holds in  $X$  if and only if  $Tx \geq 0$  holds in  $Y$ . As usual,  $\mathbf{1}_\Omega$  denotes the constant function one on  $\Omega$ , i.e.  $\mathbf{1}(t) = 1$  for all  $t \in \Omega$ . If  $\Omega = [0, 1]$ , we will simply write  $\mathbf{1}$  for  $\mathbf{1}_{[0,1]}$ .

We shall use below the following easy observation.

**Lemma 8.** *If  $\phi: \Omega_1 \rightarrow \Omega_2$  is a continuous surjective function between two compact topological spaces, then the mapping  $x \mapsto x \circ \phi$  is a norm-preserving order-embedding of  $C(\Omega_2)$  into  $C(\Omega_1)$  satisfying  $\mathbf{1}_{\Omega_2} \circ \phi = \mathbf{1}_{\Omega_1}$ .*

*Moreover,  $x \mapsto x \circ \phi$  is a lattice isomorphism.*

*Proof.* The proof of the first part is straightforward. For the second part note that for each pair  $x, y \in C(\Omega_2)$  and each  $\omega \in \Omega_1$  we have

$$\begin{aligned} [(x \circ \phi) \vee (y \circ \phi)](\omega) &= \max\{(x \circ \phi)(\omega), (y \circ \phi)(\omega)\} \\ &= \max\{x(\phi(\omega)), y(\phi(\omega))\} = (x \vee y)(\phi(\omega)) \\ &= [(x \vee y) \circ \phi](\omega). \end{aligned}$$

Thus,  $(x \circ \phi) \vee (y \circ \phi) = (x \vee y) \circ \phi$  and hence  $x \mapsto x \circ \phi$  is a lattice isomorphism. ■

The next lemma is an immediate consequence of Theorem 7 and Lemma 8.

**Lemma 9.** *If  $\Omega$  is a compact metrizable topological space, then there exists a norm-preserving order-embedding of  $C(\Omega)$  into  $C(\mathcal{C})$  that carries  $\mathbf{1}_\Omega$  to  $\mathbf{1}_\mathcal{C}$ .*

Our major intermediate result is the following:

**Lemma 10.** *There is a norm-preserving order-embedding of  $C(\mathcal{C})$  into  $C[0, 1]$  that maps  $\mathbf{1}_\mathcal{C}$  to  $\mathbf{1}$ .*

*In particular, if  $\Omega$  is any compact metrizable topological space, then there exists a norm-preserving order-embedding of  $C(\Omega)$  into  $C[0, 1]$  in such a way that  $\mathbf{1}_\Omega$  is mapped to  $\mathbf{1}$ .*

*Proof.* Recall that the complement of the Cantor set  $\mathcal{C}$  can be written as a countable union of pairwise disjoint open intervals. That is, we can write  $[0, 1] \setminus \mathcal{C} = \bigcup_{n=1}^{\infty} (a_n, b_n)$ , where  $(a_n, b_n) \cap (a_m, b_m) = \emptyset$  for  $n \neq m$ . Now each  $x \in C(\mathcal{C})$  can be extended to a function  $\hat{x} \in C[0, 1]$  by extending the graph of  $x$  on each open interval  $(a_n, b_n)$  to coincide with the graph of the line segment joining the points  $(a_n, x(a_n))$  and  $(b_n, x(b_n))$ . That is, for each  $a_n < t < b_n$  we let

$$\hat{x}(t) = \frac{x(b_n) - x(a_n)}{b_n - a_n}(t - a_n) + x(a_n).$$

Some easy verifications show that:

- (a)  $\hat{x}$  is a continuous function.
- (b) If  $x = \mathbf{c}$ , the constant function  $c$ , then  $\hat{x} = \mathbf{c}$ . In particular,  $\widehat{\mathbf{1}_\mathcal{C}} = \mathbf{1}$ .
- (c)  $\hat{x} \geq 0$  holds in  $C[0, 1]$  if and only if  $x \geq 0$  holds in  $C(\mathcal{C})$ .
- (d) If  $x, y \in C(\mathcal{C})$  and  $\lambda \in \mathbb{R}$ , then  $\widehat{x + y} = \hat{x} + \hat{y}$  and  $\widehat{\lambda x} = \lambda \hat{x}$ .

$$(e) \max_{t \in \mathcal{C}} |x(t)| = \max_{t \in [0,1]} |\widehat{x}(t)|.$$

The above properties show that  $x \mapsto \widehat{x}$  is a norm-preserving order-embedding of  $C(\mathcal{C})$  into  $C[0, 1]$  satisfying  $\widehat{\mathbf{1}}_{\mathcal{C}} = \mathbf{1}$ .

The last part follows easily from the above conclusion and Lemmas 8 and 9. ■

### 3 The Proof of Theorem 2

We shall actually prove a more general result from which Theorem 2 is a simple consequence.

**Theorem 11.** *For a separable ordered Banach space  $E$  with a closed normal positive cone  $K$  we have:*

- (a) *There is a one-to-one, order-preserving, linear operator  $T : E \rightarrow C[0, 1]$ .*
- (b) *If, in addition,  $K$  satisfies  $\overline{K - K} = E$ , then the operator  $T$  [from  $E$  onto  $T(E)$ ] is also a homeomorphism.*

*Proof.* (a) Let  $\Omega := \{x' \in K' : \|x'\| \leq 1\}$ . From the separability of  $E$  and the Alaoglu–Bourbaki Theorem, it follows that  $\Omega$  equipped with its  $w^*$ -topology is a compact metrizable topological space (see [1, Theorem 6.30, p. 239]).

Now define the mapping  $R: E \rightarrow C(\Omega)$  by letting  $(Rx)(\omega) = \omega(x)$  for all  $x \in E$  and all  $\omega \in \Omega$ . It should be clear that  $R$  is a linear operator. The normality of the cone  $K$  implies that the wedge  $K'$  is generating in  $E'$ . This guarantees that a linear functional on  $E'$  is the zero functional if and only if it vanishes on  $\Omega$ . Consequently, from

$$Rx = 0 \iff \omega(x) = 0 \text{ for all } \omega \in \Omega \iff x = 0,$$

it follows that  $R$  is one-to-one. Moreover, using that  $K$  is closed, we see that

$$\begin{aligned} Rx \geq 0 &\iff \omega(x) \geq 0 \text{ for all } \omega \in \Omega \\ &\iff x'(x) \geq 0 \text{ for all } x' \in K' \\ &\iff x \in K'' = K, \end{aligned}$$

where  $K''$  is the dual cone in  $E$  of  $K'$  with respect to the dual system  $\langle E, E' \rangle$  (that  $K'' = K$  follows from the bipolar theorem). This implies that  $R: E \rightarrow C(\Omega)$  is an order-embedding. Now apply Lemma 10.

(b) Notice first that for each  $x \in E$  we have  $\|Rx\|_{\infty} = \sup_{\omega \in \Omega} |\omega(x)| \leq \|x\|$ . Now assume  $\overline{K - K} = E$ . As in the finite dimensional case, we can easily see that  $K'$  is a closed cone, generating since  $K$  is normal and  $E$  locally convex. It then follows from a theorem of Andô [9] (see also [8]) that  $\Omega - \Omega$  is a 0-neighborhood for the norm topology

of  $E'$ . This implies that there exists some  $\rho > 0$  such that for each  $x'$  such that  $\|x'\| \leq 1$  there exist  $y', z' \in \Omega$  satisfying  $\|y'\| \leq \rho$ ,  $\|z'\| \leq \rho$ , and  $x' = y' - z'$ . In particular, for each  $x'$  in the unit ball  $U'$  of  $E'$  and each  $x \in E$  we have

$$|x'(x)| \leq \rho \left| \frac{y'}{\rho}(x) \right| + \rho \left| \frac{z'}{\rho}(x) \right| \leq 2\rho \|Rx\|_\infty.$$

We have also  $\|x\| = \sup_{x' \in U'} |x'(x)| \leq 2\rho \|Rx\|_\infty$ . Therefore, for each  $x \in E$  we have

$$\frac{1}{2\rho} \|x\| \leq \|Rx\|_\infty \leq \|x\|$$

so that (in this case)  $R$  is also a topological order-embedding. To complete the proof now note that (according to Lemma 10)  $C(\Omega)$  is topologically order-embeddable in  $C[0, 1]$ . ■

To complete the section, we show how Theorem 2 can be deduced from the previous one.

**Corollary 12.** *Every closed cone  $K$  of a finite dimensional vector space  $E$  is order-embeddable in  $C[0, 1]$ . If, moreover,  $K$  is generating (that is, if  $E = K - K$ ), then  $T$ , the linear operator which topologically order-embeds  $E$  into  $C[0, 1]$ , can be chosen so as  $T(u) = \mathbf{1}$  for some  $u \in \text{int } K$ .*

*Proof.* A finite dimensional (real) vector space is obviously a separable Banach space. Assume now that  $E = K - K = \overline{K - K}$ . The function  $\mathbf{1}$  is an order-unit thus an interior point of  $C_+[0, 1]$ . Thus  $T^{-1}(\mathbf{1})$  is an interior point of  $K$ . ■

## 4 The wedge of arbitrage free prices

The present work originated from questions in financial economics. It is motivated by the counter example in [7] and the resolution of the economic problem highlighted by the example in [4, 5, 6]. We briefly illustrate this connection below.

We consider the standard two-period securities model. That is, we suppose that there are two periods 0 and 1 (“today” and “tomorrow”). In period 0 everything is known while in period 1 there is uncertainty. The uncertainty is described by a probability space  $(\Omega, \mathcal{B}, P)$ . We view the vector space  $L_0(\Omega, \mathcal{B}, \pi)$  of all equivalence classes of measurable real functions on  $\Omega$  as the asset space. The elements of  $L_0(\Omega, \mathcal{B}, \pi)$  are called **assets**.

We assume that in our market today there is a finite number of non-redundant (i.e., linearly independent) assets  $f_1, f_2, \dots, f_n$  that can be purchased by the consumers. A portfolio is a vector  $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$ . With each portfolio  $\theta$  we consider the asset  $T\theta$ , defined for each  $s \in \Omega$  by

$$[T\theta](s) = \sum_{i=1}^n \theta_i f_i(s). \quad (\star)$$

The interpretation of  $[T\theta](s)$  is the following: If a consumer holds the portfolio  $\theta$  and the materialized state of the world tomorrow is  $s$ , then the value (payoff) of the portfolio  $\theta$  is precisely  $[T\theta](s)$ .

It is not difficult to see that  $(\star)$  defines a one-to-one linear operator  $T: \mathbb{R}^n \rightarrow L_0(\Omega, \mathcal{B}, \pi)$ . This operator is called the *payoff operator* and its range is precisely the subspace  $M$  of  $L_0(\Omega, \mathcal{B}, \pi)$  spanned by the available assets  $f_1, f_2, \dots, f_n$ .

An asset price is also a vector  $q \in \mathbb{R}^n$ . It is called **arbitrage free** if for each portfolio  $\theta \in \mathbb{R}^n$  satisfying  $[T\theta](s) \geq 0$  for almost all  $s \in \Omega$  and  $P(\{s \in \Omega: [T\theta](s) > 0\}) > 0$  we have  $q \cdot \theta > 0$ . Let  $\mathcal{A}$  be the set of arbitrage free prices. Notice that  $\mathcal{A}$  is an **open wedge** i.e., it is an open convex set that satisfies  $\alpha q \in \mathcal{A}$  for all  $\alpha > 0$  and  $q \in \mathcal{A}$ . In the special case where  $\mathcal{A}$  satisfies  $\mathcal{A} \cap (-\mathcal{A}) = \emptyset$  we say that  $\mathcal{A}$  is an **open cone**. The notion of arbitrage free prices is of enormous importance in financial economics.

The set of arbitrage free prices  $\mathcal{A}$  is never empty because the set

$$K = \{\theta \in \mathbb{R}^n: [T\theta](s) \geq 0 \text{ a.e.}\} = T^{-1}(L_0^+)$$

is always a closed cone. The cone  $K$  is called the **portfolio cone** of the assets  $f_1, f_2, \dots, f_n$ . It induces a vector ordering on  $E$  called **portfolio dominance**; see [3]. The set of arbitrage free prices  $\mathcal{A}$  is the interior of the dual

$$K' = \{q \in \mathbb{R}^n: q \cdot \theta \geq 0 \text{ for all } \theta \in K\}.$$

Now we consider the space  $C[0, 1]$  as canonically embedded in  $L_0$  with the Lebesgue measure. Theorem 2 can easily be re-stated as follows.

**Theorem 13.** *If  $\mathcal{A}$  is a non-empty open wedge in  $E = \mathbb{R}^n$ , then there exist non-redundant assets  $f_1, f_2, \dots, f_n$  in  $C[0, 1]$  such that the set of arbitrage free prices is  $\mathcal{A}$ .*

*If  $\mathcal{A}$  is an open cone, then  $f_1$  can be chosen to be the constant function (bond)  $\mathbf{1}$  satisfying  $f_1(s) = 1$  for all  $s \in [0, 1]$ .*

*Proof.* Since  $\mathcal{A}$  is an open wedge, its dual is a closed cone  $K$  to which we can apply Theorem 2. Let  $T: \mathbb{R}^n \rightarrow C[0, 1]$  be a one-to-one operator such that  $K = T^{-1}(C_+[0, 1])$ . Take for assets  $f_1, f_2, \dots, f_n$  any basis of  $T(K)$ . The set of arbitrage free prices is the interior of  $K'$ , i.e., the set  $\mathcal{A}$ . To see the equivalence between the respective conditions in Theorem 13 and Theorem 2 that  $\mathcal{A}$  is an open cone and that  $K$  is generating, apply Lemma 6. ■

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