



**Dispelling mathematical doubts. Assessing
the mathematical correctness of algorithms in Bhâskara's
commentary on the mathematical chapter of the
Âryabhatîya.**

Agathe Keller

► **To cite this version:**

Agathe Keller. Dispelling mathematical doubts. Assessing the mathematical correctness of algorithms in Bhâskara's commentary on the mathematical chapter of the Âryabhatîya.. 2007. halshs-00151013v1

HAL Id: halshs-00151013

<https://shs.hal.science/halshs-00151013v1>

Preprint submitted on 1 Jun 2007 (v1), last revised 21 Jan 2011 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dispelling mathematical doubts.

Assessing the mathematical correctness of algorithms in Bhāskara's commentary on the mathematical chapter of the *Āryabhaṭīya*.

This is a Draft

Agathe Keller

Abstract

This article will analyze the vocabulary and the reasonings that Bhāskara (629 A. D.) uses to ground the procedures exposed in the mathematical chapter of the *Āryabhaṭīya*, a Sanskrit astronomical treatise of the fifth century. We will see that explaining, showing and verifying procedures are part of the general endeavor of the commentary to defend his interpretation of the procedures and their correctness. These attempts are in no way systematic, and when they appear, they are often fragmentary as if they were carried out orally. We will nevertheless argue, that one of the modes of mathematically grounding a procedure was to reread it with the help of an alternative procedure. We will see that the Rule of Three and the Pythagoras Theorem have a special value in this context.

Introduction

Contrary to the perception that was prevalent at the beginning of the twentieth century, we now know that the mathematical tradition in Sanskrit showed a preoccupation for the mathematical correctness of algorithms. The system-

atic *upapattis* of Gaṇeśa's commentary on the *Bījagaṇita* and *Līlāvati* or the explorations of the Mādhava school have been the subject of some first preliminary publications on the subject¹. The variations of this tradition of justification and explanation still very much needs to be studied. In the following we will look at the reasonings and vocabularies used for this effect in the *Āryabhaṭīyabhāṣya* of Bhāskara (BAB). The *Āryabhaṭīya* (Ab) an astronomical *siddhānta* composed in verse at the end of the fifth century, has a second chapter, devoted to mathematics (*gaṇita*). Respecting the requirements of the genre, these aphoristic *āryas* more often than not provide the gist of a procedure, an essential relation or the main steps of an algorithm. The BAB is not only the earliest known commentary on this treatise but also the oldest known prose mathematical text in Sanskrit that has been handed down to us. It thus gives us a first glimpse into the reasonings that were used in the scholarly mathematical tradition in Sanskrit at the beginning of the seventh century². In the following we will first see how reflections on the mathematical correctness of the procedures are inserted in the larger endeavor of the commentator who attempts to defend Āryabhaṭa's treatise, and his own interpretations of the verses. We will then look at the reasonings associated with the mathematical justification of algorithms. We will first look at attempts to re-read procedures with the help of a Rule of Three and the Pythagoras Theorem³. Before attempting to characterize what Bhāskara

¹Srinivas 1990, ***.

²See Keller 2006, Introduction.

³Despite a growing habit of changing the name of this property of right-triangles, I've decided, for the sake of simplicity, to continue to call it the Pythagoras Theorem despite the fact that it appears here as a *procedure* found by mathematicians living in the Indian

calls explanations, proofs and verifications.

1 Defending the treatise

Bhāskara's commentary, as a prolix prose text, gives us a glimpse into the intellectual world of scholarly astronomers and mathematicians. It seems to strive with intellectual debates. Thus, justifying the fact that the author of the treatise mentions his name in the opening verse of the treatise, Bhāskara explains :

Or as a heroic man on battle fields, whose arms have been copiously lacerated by the strength of vile swords, having entered publicly a battle with enemies, who proclaims the following, as he kills: 'This Yajñadatta here ascended, a descendant of the Aditis, having undaunted courage in battle fields, ⟨now⟩ strikes. If someone has power, let him strike back!' In the same way, this master also, who has reached the other side of the ocean of excessive knowledge about Mathematics, Time-reckoning and the Sphere, having entered an assembly of wise men, has declared: 'Āryabhaṭa tells three: Mathematics, Time-reckoning, the Sphere'.⁴

subcontinent.

⁴*athavā yaḥ tejasvī puruṣaḥ samareṣu nikṛṣṭāsītejovitānacchuritabāhuḥ śatruśaṅghātam prakāśam praviśya praharan evam āha 'ayam asau udiṭaḥ aditikulaprasūtaḥ samareṣu anivāritavīryaḥ yajñadattaḥ praharati — yadi kasyacit śaktiḥ pratipraharatu' iti — evam asau api ācāryaḥ gaṇitakālakriyāgolātīśayaajñānodadhipāraḡaḥ vitsabhām avagāhya 'āryabhaṭaḥ trīṇi gadati gaṇitam kālakriyām golam' iti uktavān* —. Shukla 1976:5 I would

In this atmosphere, one of the main aims of Bhāskara's commentary is thus to convince his reader of the coherence and validity of Āryabhaṭa's treatise. He does so by dispelling doubts that can arouse when explaining Āryabhaṭa's verses. One of the main ways to do so is by providing refutations (*parihāra*) to objections, and thus establishing (*sādhya*, *siddha*) his readings of Āryabhaṭa's verse. Discussions are in this text mainly syntactical and grammatical, debating the interpretation of a given word in the treatise. More often than not, the discussion of the meaning and use of a word is used to define and characterize the mathematical objects at stake (are squares all equal sided quadrilaterals ? do all triangles have equal halving heights ?, etc.)⁵. Bhāskara's technical acception of words, his specialized readings of the verses also show that it is his own interpretation of the verse that he gives. This is especially clear when he criticizes Prabhakara's interpretations⁶. For instance, when commenting the rule to compute sines, Bhāskara explains that with the expression *samavṛtta*, one should understand a circle, and not a disk, like Prabhakara did⁷. In a more crucial manner, by understanding the word *agra* as a synonym of *sankhyā*, Bhāskara can give a new interpretation of the rule given in BAB.2.32-33⁸. This is such a peculiar reading of the text, that it strikes as being an inventive commentarial device. It is in this context that sometimes leaving the syntactical discussion of the verse, Bhāskara considers directly the mathematical content of it. Defending Āryabhaṭa's approximation of π against those of competing schools, he thus

like to thank T. Kusuba, T. Hayashi and M. Yano for the translation.

⁵See Keller 2006, Introduction.

⁶See Keller 2006, BAB.2.11; BAB.2.12.

⁷Shukla 1976: 77; Keller 2006, volume 1: 57.

⁸Shukla 1976: 135; Keller 2006: 132-133.

launches into a refutation (*parihāra*) of the jaina value of $\sqrt{10}$ (*daśakaraṇī*).
Going so far as to claim that it rests only on a tradition and not on a proof.

In this case also, it is just a tradition (*āgama*) and not a proof
(*upapatti*) (...) But this also should be established (*sādhya*)⁹.

One should not however deduce from this a romantic vision of a revolutionary Bhāskara using reason to overthrow the prejudices transmitted through traditions. Criticizing here this reference to tradition, in other cases this very argument seems sufficient to accept the correctness of a mathematical statement¹⁰. The question nonetheless is raised : how does one ‘establish’ a rule, and what does Bhāskara call a ‘proof’ ? To attempt an answer, we will look at how he sometimes interprets mathematically a procedure given by Āryabhaṭa by re-reading it with the help of a Rule of Three or a Pythagoras Theorem.

2 Re-reading algorithms

In the following I will argue, that an explanation or a verification consisted either in providing an alternative procedure to arrive at a same result, or in rereading a given procedure with the help of a Rule of Three or a Pythagoras Theorem. In all cases the aim would have been to provide a mathematical grounding to what if not would appear as an arbitrary succession of operations. I will first present how a Rule of Three and the Pythagoras Theorem

⁹*atrāpi evāgamaḥ naivopapattiḥ . . . cetad api sādyam eva.* Shukla 1976: 72.

¹⁰See Keller 2006, Introduction.

are expressed in BAB, before examining occurrences of rereading of procedures in the commentary.

2.1 A Rule of Three

The Rule of Three (*trairāśika*¹¹) is given in verse Ab.2.26.

Now, when one has multiplied that fruit quantity of the Rule
of Three by the desire quantity|
The quotient of that divided by the measure should be this fruit
of the desire||¹²

Bhāskara understands this rule in two ways. He first underlines the classification of quantities this rule rests upon. Thus a measure (*pramāṇa*) quantity produces a fruit (*phala*). The desire (*iccha*) quantity is a kind of measure quantity. The fruit of the desire quantity (*iccha-phala*), is what is sought. The computation involves multiplication of the desire by the fruit, then its division by the measure. Note that this order makes the procedure appear as an arbitrary set of operations. If the division was made first (providing us with the ‘fruit’ of one measure), and the multiplication afterwards, the computation could have a meaning step by step. But this isn’t the case. When a situation involves proportional quantities, and thus a Rule of Three is applied, Bhāskara underlines this by using a verbal formulation (*vāco yukti*) of

¹¹For a general overview on the Rule of Three in India see Sarma 2002.

¹²

trairāśikaphalarāśiṃ tam athecchārāśinā hataṃ kṛtvā|
labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idaṃ syāt||

(Shukla 1976: 112-223).

the Rule of Three. This verbal formulation is a syntactically rigid question, that goes as follows :

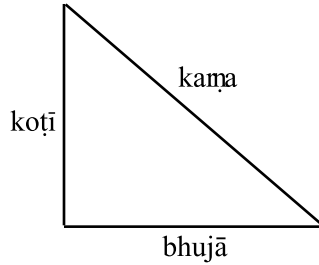
If the measure produces the fruit, then with the desire what is produced? The fruit of the desire is produced.

This question, when it appears, serves as a leitmotiv : when formulated, you know that a Rule of Three can be applied. This is probably the way proportionality was stated in the BAB.

2.2 The Pythagoras Theorem

Bhāskara's mathematics, as in all medieval Sanskrit texts, does not use the concept of angles. In trigonometry it uses lengths of arcs. Concerning right triangles, they are distinguished in the text from ordinary triangles by the fact that each of its sides has a specific name. Whereas scalene, isosceles and equilateral triangles have sides (*aśra*), flanks (*pārśva*) and sometimes earths (*bhū*), right angle triangles have a 'base' (*bhujā*), an 'up-right side' (*koṭi*) and a 'hypotenuse' (*kārṇa*), as shown in Figure 1.

Figure 1: Names of the sides of a right triangle



The Pythagoras Theorem is stated in the first half of Ab.2.17:

That which precisely is the square of the base and the square of the upright side is the square of the hypotenuse.¹³

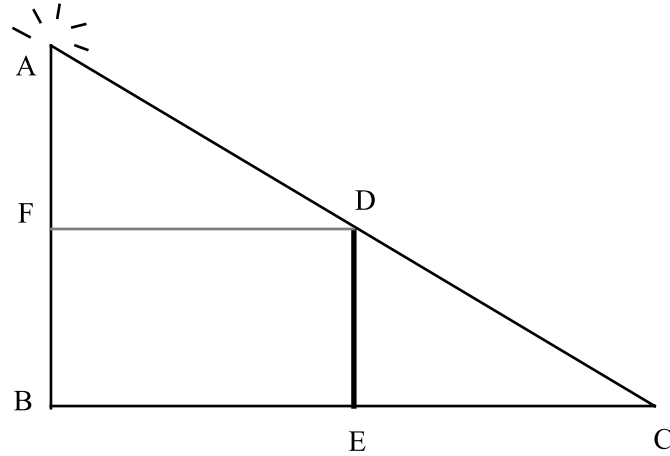
Thus when a situation involves a right-triangle, Bhāskara underlines this by renaming segments by the specific names of the sides of a right-triangle.

Let us now look at two examples of re-reading.

2.3 Rereading

The two cases we will look at are both in the section devoted to gnomons (*śaṅku*). The standard situation is the following: A gnomon (*śaṅku*, DE) casts a shadow (EC), produced by a source of light (AB). This is illustrated in Figure 2.

Figure 2: A schematized gnomon and a light



Let us first look at the procedure given in Ab.2.15:

¹³*yaś caiva bhujāvargaḥ koṭīvargaś ca karṇavargaḥ saḥ* Shukla 1976:96.

The distance between the gnomon and the base, multiplied by the ⟨height of⟩ the gnomon, is divided by the difference of the ⟨heights of the⟩ gnomon and the base.

What has been obtained should be known as that shadow of the gnomon ⟨measured⟩ indeed from its foot.||¹⁴

This procedure, involves a multiplication and a division. It appears here as an arbitrary list of operations. However, Bhāskara writes, after a solved illustration of this procedure:

This computation is a Rule of Three. How? If from the top of the base which is greater than the gnomon, the size of the space between the gnomon and the base, which is a shadow, has been obtained, then, what is ⟨obtained⟩ with the gnomon? the shadow is obtained¹⁵.

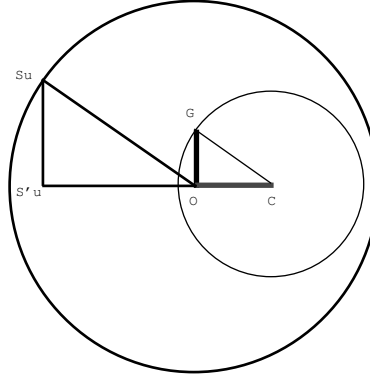
We can recognize here the standard formulation of a Rule of Three, applied to the similar triangles ABC and DEC. We can thus realize that the standard expression of the Rule of Three provides the proportional entities on which the computation is based. Here it explains that the ratio of AB to BC is equal to the ratio of DE to EC. It gives to the arbitrary set of rules a mathematical signification.

¹⁴*śaṅkuguṇaṃ śaṅkubhujāvivaraṃ śaṅkubhujayor viśeṣaḥṛtam yal labdaṃ sā chāyā jñeyā śaṅkoḥ svamūlāt hi//* Shukla 1976:90.

¹⁵*etat karma trairāśikam/ katham ? śaṅkuto 'dhikāyā uparibhujāyā yadi śaṅkubhujāntarālapramāṇaṃ chāyā labhyate tadā śaṅkunā keti chāyā labhyate* Shukla 1976:92.

In the previous verse commentary (BAB.2.14) Bhāskara exposes two procedures. Both rest on the proportionality of the right triangle formed by a gnomon and its midday shadow on one hand, and the right triangle composed by the Rsine of the altitude and the zenith distance on the other. One procedure uses only a Rule of Three, the other also uses a Pythagoras Theorem. They both compute the same entities. Look at Figure 3.

Figure 3: Proportional astronomical triangles



The situation here depends on a context in which a gnomon and its shadow are linked to the celestial sphere. The circle of radius OSu is the celestial meridian. Its radius is thus equal to the radius of the celestial sphere. The segment $SuS'u$ is what Bhāskara calls the ‘half-chord’ of the arc OSu . In other words it is the Rsine of the altitude. It is proportional to the length of the gnomon. The gnomon is represented by GO , and its midday shadow by OC . The segment $S'uO$ the Rsine of the zenithal distance. This is proportional to the shadow of the gnomon at noon.

Bhāskara writes¹⁶:

¹⁶*akṣajyā* “*nayane trairāśīkathāpanā- 13/ 5/ 3438/ labdham akṣajyā 1322/ eṣā bhujā, vyāsārdha karnaḥ, bhujākarnavargaviśeṣamūlam avalambakaḥ 3174* (Shukla 1976:90).

When computing the Rsine of latitude (*akṣajyā*) a Rule of Three is set down: 13, 5, 3438. What is obtained is the Rsine of latitude, 1322¹⁷. That is the base (*bhujā*), the semi-diameter is the hypotenuse (*karṇa*); the root of the difference of the squares of the base and the hypotenuse is the Rsine of colatitude (*avalambaka*), 3174¹⁸

In this case he uses thus both the fact that the triangles are right and similar.

But he then adds¹⁹:

With a Rule of Three also 13, 12, 3438; what has been obtained is the Rsine of the colatitude, 3174²⁰.

He has thus computed the same value twice, using two different methods. This gives the feeling that he is proceeding to verify one algorithm by the other. In both cases, the mathematical key is the existing relation between the gnomon and the celestial sphere. This relation is stated by establishing a syntactical connection between these two spaces. Indeed, the Rule of Three is invoked by a standard question. The right triangle is called in by naming two of its segments. This is highlighted here by a set of puns. Thus the gnomon and the Rsine of the altitude have the same name (*śaṅku*), the shadow of the gnomon and the Rsine of zenith distance also (*chāya*). This way of proceeding points not only to the mathematical property on which

¹⁷This is an approximate value. For more on this see Keller 2006, BAB.2.14.

¹⁸This value is also an approximation.

¹⁹*trairāśikenāpi 13/ 12/ 3438/ labdham avalambakaḥ 3174/* (Shukla 1976, 90).

²⁰This value is an approximation again.

the situation is based but also to a list of operations (a multiplication and a division, squaring and then summing or subtracting lengths). Rereading with the help of a Rule of Three or the ‘Pythagoras Theorem’ thus enables to transfer a localized astronomical problem to a more general and abstract mathematical situation (Rsines of altitudes and zenith distances becoming just a right-triangle). We thus see how re-reading the arbitrary steps of the procedure enables a mathematical grounding of it that may serve as a justification of the algorithm itself. We will thus turn to that part of the reasonings of the BAB which explicitly have a persuasive aim, that of showing that the algorithms of the Ab are correct.

3 Explanations, Verifications and Proofs

There is indeed a number of reasonings which bear a specific name: explanations, proofs (*upapatti*) and verifications (*pratyāyakaraṇa*). They do not appear systematically in each verse commentary, and are always fragmentary as we will see.

3.1 Explaining

We will illustrate an example of explanation by using Bhāskara’s commentary on verse 8 of the mathematical chapter of the Ab. Verse 8 describes two computations within a trapezium. The first evaluates the length of two segments (*svapātalekha*) of a hight whose vertex is the point of intersection of the diagonals. The procedure is made of a multiplication followed by a

division²¹:

Ab.2.8. The two sides, multiplied by the height ⟨and⟩ divided by their sum are the ‘two lines on their own fallings’.[|]
When the height is multiplied by half the sum of both widths, one will know the area.||

Bhāskara comments²²:

The size of the ‘lines on their own fallings’ should be explained (*pratipādayitavya*) with the computation of a Rule of Three on a figure drawn by ⟨a person⟩ properly instructed. Then, by means of just a Rule of Three with both sides, the computation of ⟨the lines whose top is⟩ the intersection of the diagonals and a perpendicular ⟨is performed⟩.

The explanation consists in re-reading the procedure with a Rule of Three. It proceeds in two steps. It first looks at the proportionality in a diagram, then re-reads the list of operations of the algorithm investing it with the mathematical background of a Rule of Three, giving a clue that this re-reading invests a mathematical meaning to what before could have seemed an arbitrary list of information.

²¹

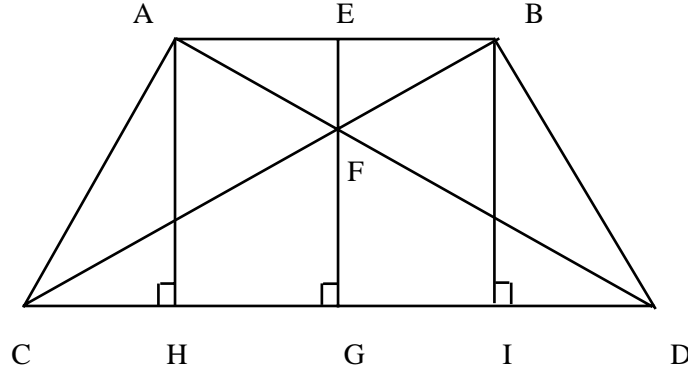
āyāmaguṇe pārsve tadyogahrte svapātalekhe te|
vistarayogārdhaguṇe jñeyam kṣetraphalam āyāme||

(Shukla 1976: 63).

²²*samyagādiṣṭena alikhite kṣetre svapātalekhāpramaṇam trairāśīkagaṇitena*
pratipādayitavyam/ tathā trairāśīkenaivobhaya pārsve karṇāvalambakasampātānayanam/
(Shukla 1976: 63).

The second computation of verse 8 provides the computation of the area of the trapezium. As shown in Figure 3.1, the area of the trapezium can be seen as the sum of the areas of several triangles, or of the area of a rectangle with two triangles.

Figure 4: Inner fields in a trapezium



Although not explicitly drawn, it seems that such a diagram is referred to when Bhāskara writes²³:

Here, with a previous rule (Ab.2.6.ab) the area of isosceles and uneven trilaterals should be shown/explained (*darśayitavya*). Or, with a rule which will be stated (Ab.2.9.) the computation of the area of the inner rectangular field (should be performed);

Even though it is not explicit, the drawing shows to us the sums possible. Here again, the computation is given a mathematical and immediately understandable meaning.

²³*pūrvāsūtreṇātra* *dvisamaviṣamatryaśrakṣetraphalaṃ* *darśayitavyam/*
vakṣyāṇasūtreṇāntarāyatacaturaśrakṣetraphalānayanam (...) *vā/* (Shukla 1976: 63).

The reasonings we have observed here seem to consist in a first step of an explanation in a diagram followed by the re-reading of a procedure, that providing the same result gives a mathematical meaning to the different steps to be carried out. This re-reading seems to confirm or verify the reasoning that was seen in the diagram.

Three words are used to refer to an explanation: *vyākhyāna*, *pradarśaṇa*, and *pratipādita*. The word *vyākhyāna* can be used to name the commentary as a place where explanations are given. It is also used for a reasoning that takes place in a diagram²⁴:

Or else, all the procedures ⟨used⟩ in the production of chords are in the realm of a diagram, and a diagram is intelligible ⟨only⟩ with an explanation (*vyākhyāna*). Therefore it has not²⁵ been put forth (*pratipādita*)⟨by Āryabhaṭa in the *Āryabhaṭīya*⟩.

Note that this emphasizes how explanations belong to the commentarial genre and should not, for Bhāskara, be exposed in a treatise.

The word *pradarśaṇa* is derived from the verbal root *dṛṣ-*, ‘to see’. Its meaning and ambiguity is similar to the English ‘to show’. It is often hard to distinguish if it refers to the visual part of an explanation or to the whole of the explanation itself. For instance, in BAB.2.11, Bhāskara uses a diagram and writes²⁶:

²⁴*athavā jyotpattaṃ yatkaraṇaṃ tatsarvaṃ chedyakaviṣayaṃ, chedyakaṃ ca vyākhyānagamyamiti na pratipāditam/* (Shukla 1976: 79).

²⁵*na* has been added by the editor, K. S. Shukla and is not found in the manuscripts. An other possible interpretation of the sentence is: ‘Therefore it has been put forth ⟨by Bhāskara in his commentary⟩

²⁶*evam ālikhite kṣetre sarvaṃ pradarśayitavyam* (Shukla 1976: 79).

In the field drawn in this way all is to be shown/explained
(*pradarśayitavya*).

Finally the word *pratipādita* is more technical and straightfoward. It essentially appears in the list of solved examples that can be found in most commented verses in the mathematical part of BAB.

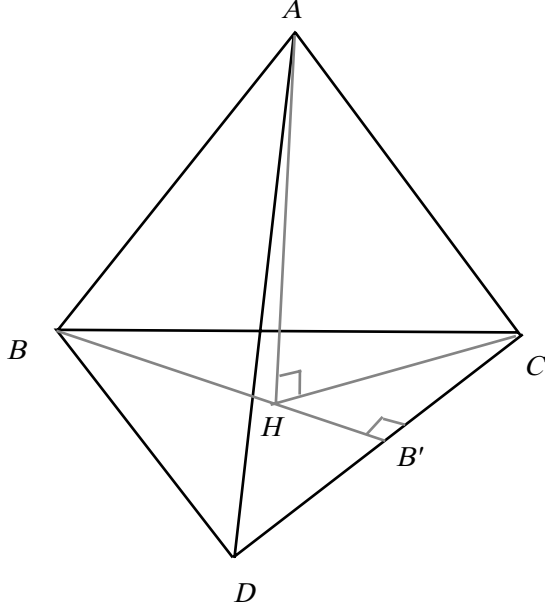
In the example of explanations that we have looked at the commentator re-reads geometrical procedures with a Rule of Three or the ‘Pythagoras Theorem’. Geometrical procedures are the only ones for which we have such kind of reasonings. But each time the reasoning stops short of a diagram in which it seems to be orally held. The explanations are ‘seen’ in them. This is specifically what happens in the only occurrence of the word proof that can be found in the BAB.

3.2 The one and only proof

The Sanskrit word *upapatti* directly referring to a reasoning is used twice in Bhāskara’s commentary, as noticed by Takao Hayashi²⁷. It is a feminin word derived from the verbal root *upa-PAD-*, whose colloquial meaning is ‘to reach’. Litterally then, an *upapatti*, is ‘what is reached’, and has thus been translated by ‘proof’. In the both cases where the word appears in BAB, we do not know exactly what is the reasoning called in this was. One occurrence, has been quoted above, when proofs (*upapatti*) are opposed to tradition. The other is used to name the reasoning by which from the sides of a regular tetrahedron the height is obtained. The rule given by Āryabhaṭa, in the second half of verse 6 of the mathematical chapter is understood by

²⁷Hayashi 1985: 75-76.

Figure 5: An equilateral pyramid with a triangular base

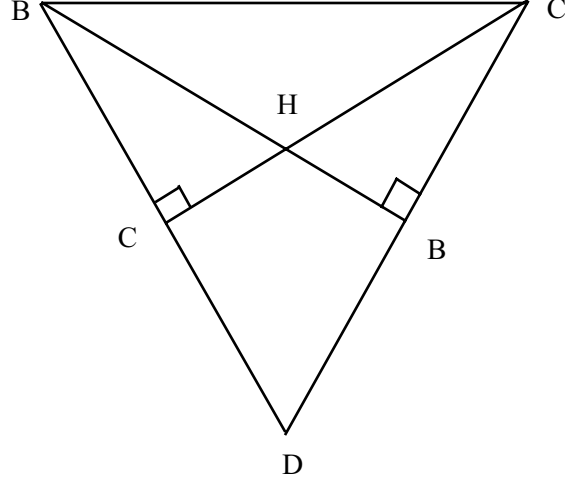


Bhāskara as computing the volume of a regular tetrahedron. The situation involved is described in Figure 5.

In an $ABCD$ regular tetrahedron, AH is the perpendicular issued from A onto the triangle BDC . AH is what is called the upward-side (*ūrdhvabhujā*). AC , the ear (*karṇa*), is the side of the tetrahedron and also the hypotenuse of AHC . Bhāskara explains how to compute the upright side by using the Pythagorean theorem, and a rule of Three. The computation of CH , from which the upright side AH may be computed rests upon the proportional properties of similar triangles, illustrated in Figure 6. The triangles $BB'C$ and $B'CH$ are similar:

$$BB' : CB = CB' : CH.$$

Figure 6:



So that in other words:

$$CH = \frac{CB \times CB'}{BB'}.$$

This is expressed by Bhāskara with a Rule of Three. The precise reasoning is not given in the text, but alluded to as being shown in a diagram. This is when the word *upapatti* is used²⁸:

In order to show the proof (*upapatti*) of ⟨that⟩ Rule of Three, a field is set-down.

What reasoning is understood with this word here is silenced by a diagram: as in explanations, we can infer that the proof was given orally. We can note here a difference with the acts of ‘re-reading’ seen previously. In this case, a reasoning is created. Its foundations are proved in a diagram. There is no previous algorithm, re-read in this case. The link with the kind

²⁸*trairāśīkopapattipradarśanārthaṃ kṣetranyāsaḥ*-(Shukla 1976: 59).

of explanations we have seen before is not only that there is an oral part that escapes the written text, but the procedure used, which is a Rule of Three.

We will now turn to another type of reasoning concerning the correctness of an algorithm: verification.

3.3 Verification

Verifications are distinguished from explanations and proofs, by their name: *pratyayakarāṇa*. Indeed, *pratyaya*'s etymological meaning is 'to come back', which is also a way of evoking conviction. *pratyayakarāṇa* is 'enabling to come back' or 'producing conviction'. Historians of mathematics in India usually understand it as a verification and translate it this way²⁹.

A verification looks like an explanation: it rereads a procedure by another one, giving to it a mathematical grounding. One of the two reasonings that our commentator provides for verifications is to a certain extent puzzling, and we often have the feeling we do not quite understand it. In the following, I will thus expose both and give a certain number of hypothesis concerning how they can be understood.

3.3.1 Verifying an arithmetical computation

Bhāskara states a verification of the rule given in Ab.2.25, using a Rule of Five. The rule stated in Ab.2.25 goes as follows³⁰:

²⁹See Hayashi 1995:73-74.

³⁰

mūlaphalaṃ saphalaṃ kālamūlaguṇam ardhamūlakṛtiyuktam|
tanmūlaṃ mūlārdhonaṃ kālahṛtaṃ svamūlaphalam||

The interest on the capital, together with the interest (on the interest), with the time and capital for multiplier, increased by the square of half the capital|

The square root of that, decreased by half the capital and divided by the time, is the interest on one's own capital||

It can be formalized as follows. Let m (*mūla*) be a capital; p_1 (*phala*) the interest on m during a unit of time, usually a month, $k_1 = 1$ (*kāla*) ; p_2 , the interest on p_1 , at the same rate, for a period of time k_2 . When $p_1 + p_2$, m , and k_2 are known, in a modern mathematical notation; the rule can be understood as:

$$p_1 = \frac{\sqrt{mk_2(p_1 + p_2) + (\frac{m}{2})^2} - \frac{m}{2}}{k_2}.$$

This rule, derives from a constant ratio:

$$\frac{m}{p_1} = \frac{p_1}{p_2} k_2.$$

The Rule of Five, as described in BAB.2.26-27.ab typically concerns such commercial problems, where k_1 - here always equal to one- may be variable, and where a different value than the initial interest p_1 may be considered as lent at the same rate for a time k_2 . The Rule of Five computes a value for p_2 :

$$p_2 = \frac{p_1^2 k_2}{m k_1}$$

A reversed Rule of Five would therefore give a value for p_1 .

(Shukla 1976: 114).

The Rule of Five, in fact, rests upon the same ratio as the rule given in Ab.2.25, only k_1 may be different from 1:

$$\frac{m}{p_1}k_1 = \frac{p_1}{p_2}k_2.$$

In this case an independent procedure is produced that provides the same result. The use of the Rule of Five, which is according to Bhāskara a combination of two Rules of Three, also invests the computation with a mathematical grounding in terms of proportionality.

3.3.2 Verifying the area of plane figures

The first half of Ab.2.9 is understood by Bhāskara as a way to verify the procedures for areas that Āryabhaṭa has given in the previous verses.

For all fields, when one has acquired the two sides, the area is their product |³¹

Two moments can be distinguished in the verifications carried out in this verse commentary, each rythmed by a diagram. In the first step a diagram of the figure for which the area should be verified is drawn. The length and width of a rectangle having the same area as the figure are identified. These ‘length’ and ‘width’ are usually values that were used in Āryabhaṭa’s procedure one seeks to verify. For instance, to verify the area of a triangle, the length identified is the height of the triangle, the width is half the base. In the first half of verse 6 the area of a triangle is given as the product of half the base and the height of a triangle. The second step of the reasoning draws a diagram of the rectangle thus identified and computes the multiplication.

³¹ *sarveṣāṃ kṣetrāṇāṃ prasādhya pārśve phalaṃ tadabhyāsaḥ* | Shukla 1976: 66.

How should one understand this reasoning?

A first way of understanding it would be to see in it a re-interpretation of the procedures to be verified. This reinterpretation would consider formally the procedure to be verified as a multiplication of two quantities. The procedure would then be read geometrically, each quantity becoming the length and width of a rectangle having the same area as the figure whose area is being verified. This is how Bhāskara would find, as the verse 9 requires, the two lengths by which such a rectangle is produced.

A second way to understand the reasoning would be to consider that a verification, for a given figure, *produces* a rectangle with the same area. It would then be grounded in an implicit conception of plane geometry that all figures should have a rectangle with the same area. This is how T. Hayashi has understood this reasoning³², which would then consist in producing a rectangle and verifying that its area is equal to the area of the figure.

A third way to understand the reasoning, suggested by the ‘setting down’ parts which contain diagrams, would be to verify if from a given figure, one can *construct* a rectangle with the same area. Thus in the second moment of the verification of the area of a triangle, Bhāskara specifies that when the sides of such a triangle when rearranged (*vyasta*) produce the rectangle which is then drawn. The construction of a rectangle from the original figure is not described in Bhāskara’s commentary. However, such constructions could have been known, as shown by the methods exposed in BAB.2.13. Furthermore, this process recalls the type of algorithms produced in the *sulbasūtras*, the earliest known texts of Sanskrit geometry. The aim being

³²Hayashi 1995: ***.

a construction which is not described in the text, the reasoning would just verify arithmetically that the construction is correct.

These three interpretations can be united by understanding that a verification would be simultaneously in this case geometrical and arithmetical. Geometrically Bhāskara would produce a rectangle with the same area as the figure whose area is verified, and one should know how to construct it from the initial figure. The construction, obvious, would not be specified, but the lengths of the rectangle would be given. Arithmetically, this rereading would reinterpret the rule given by Āryabhaṭa. By ‘verifying’ arithmetically, Bhāskara would also explain the geometrical verification, because it explicits the link in between the sides of the initial figure and the lengths and widths of the rectangle with the same area.

In all cases, the verification re-reads a first algorithm and produces thus a new understanding of it, or a new procedure that gives the same result.

In both cases then the reasoning called ‘verification’ would consist in a rereading of a mathematical procedure by another one that provides a mathematical meaning to it. The independance of the two procedures producing the same result would also be considered as convincing the reader, one of the procedures however providing a ‘grounding’ that the other one does not have, appearing as a set of arbitrary steps.

Conclusion

How should one qualify the Rule of Three and the so-called ‘Pythagoras Theorem’? One can notice that they are technically mathematical, enabling

one to take concrete like situations or astronomical situations and rereading them in the more abstract and general case of right triangles and proportional entities. The reasonings we have observed are hard to understand and pin down. This difficulty could spring from the oral nature of explanations that thus remain fragmentary in Bhāskara's written text. This could also be due to the fact that we can observe here the very beginning of a tradition of written justifications which seems to have flourished 1000 years later. Further detailed explorations of the explaining, proving and verifying mathematical algorithms in Sanskrit texts will slowly enable to better map such reasonings.

Bibliography

Hayashi, T. (1995) *The Bhakhshālī Manuscript, An ancient Indian mathematical treatise*. Groningen: Egbert Forsten.

Keller, A. (2006) *Expounding the mathematical seed, Bhāskara and the mathematical chapter of the Āryabhaṭīya*. 2 volumes. Basel: Birkhäuser.

Sarma, S. R. (2002) 'Rule of Three and its variation in India'. In: Yvonne Dold-Samplonius, Joseph W. Dauben, Menso Folkerts and Benno Van Dalen (eds.), *From China to Paris: 2000 years transmission of mathematical ideas*. Stuttgart: Franz Steiner Verlag.

Shukla, K. S. (1976) *Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara*. Delhi: Indian National Science Academy.

Srinivas, M. D. (1990)'The Methodology of Indian Mathematics and its contemporary relevance'. In: Sundeep Prakashan (ed.), *History of Science and Technology in India; Mathematics and Astronomy*; pp. 1990.