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Dispelling Mathematical Doubts.
Assessing Mathematical Correctness of Algorithms in
Bhāskara's Commentary on the Mathematical Chapter
of the *Āryabhaṭīya*.

Agathe Keller

Abstract

In his commentary on the mathematical chapter of the *Āryabhaṭīya*, a Sanskrit astronomical treatise of the fifth century, Bhāskara (629 CE) provides explanations, demonstrations and verifications to check the computational accuracy and logical consistency of some procedures. These unsystematic attempts at justification are often fragmentary and perhaps mere echoes of an activity carried out orally. Nonetheless, one can identify recurring modes of reasoning. To ground mathematically a procedure, a 're-interpretation' via the Rule of Three and the 'Pythagorean Theorem' could be provided. Another way of assessing the validity of a procedure was to establish an independent alternative procedure arriving at the same result.

Dans son commentaire sur le chapitre mathématique de l'*Āryabhaṭīya*, un traité d'astronomie en sanskrit du cinquième siècle, Bhāskara (629) fournit des explications, des démonstrations et des vérifications pour s'assurer de l'exactitude et de la consistance de quelques procédures. Ces tentatives non systématiques de justification sont souvent des échos fragmentaires d'explications orales. Néanmoins, on peut identifier différents types de raisonnements. Pour fonder mathématiquement une procédure une "ré-interprétation" par l'intermédiaire de la Règle de Trois et du "Théorème de Pythagore" était fournie. Une autre manière d'évaluer la validité d'une procédure consistait à fournir une procédure indépendante arrivant au même résultat.

Introduction

Contrary to the perception prevalent at the beginning of the twentieth century, a concern for the mathematical correctness of algorithms existed in the mathematical tradition in Sanskrit. The systematic *upapattis* of Kṛṣṇa's (fl. ca. 1600–1625) commentary on the *Bījagaṇita* and the explorations of the Mādhava school (XIVth-XVIth century) have already been published.¹ Still, the variations among this tradition of justification and explanation need to be studied.

In the following sections, the *Āryabhaṭṭyabhāṣya* of Bhāskara (BAB) is analyzed with regard to its reasoning and vocabulary. The second chapter of *Āryabhaṭṭya* (Ab)—an astronomical *siddhānta* composed in verse at the end of the fifth century—treats mathematics (*gaṇita*). Respecting the requirements of the genre, these aphoristic *āryas* usually provide the gist of a procedure, such as an essential relationship or the main steps of an algorithm. The BAB is not only the earliest known commentary on this treatise but also the oldest known text of mathematics in Sanskrit prose that has been handed down to us. The BAB thus gives us a glimpse into the reasonings used in the scholarly mathematical tradition in Sanskrit at the beginning of the seventh century². The text also provides clues on the context in which it was composed.

First, the commentator's defense of Āryabhaṭa's treatise (and the commentator's own interpretations of the verses) will provide a backdrop for reflections on the mathematical correctness of procedures. Next, the argu-

¹Srinivas 1990, but some of Kṛṣṇa's demonstrations are noted, among others, in the footnotes of Colebrooke 1817.

²Keller 2006, Introduction.

ments behind the algorithms of mathematical justification will be clarified. Specifically, these strategies consist in a re-interpretation³ of procedures via the Rule of Three and the Pythagorean Theorem⁴. Afterwards, Bhāskara’s vocabulary including explanations, proofs and verifications will be more precisely characterized.

1 Defending the Treatise

Bhāskara’s commentary, a prolix prose text, gives us a glimpse into the intellectual world of scholarly astronomers and mathematicians. The commentary records their intellectual debates. For the opening verse in which the author of the treatise mentions his name, Bhāskara’s commentary explains:

...as a heroic man on battle fields, whose arms have been copiously lacerated by the strength of vile swords, having entered publicly a battle with enemies, who proclaims the following, as he kills: ‘This Yajñadatta here ascended, a descendant of the Aditis, having undaunted courage in battle fields, ⟨now⟩ strikes. If someone has power, let him strike back!’ In the same way, this master also, who has reached the other side of the ocean of excessive knowledge about Mathematics, Time-reckoning and

³A “re-interpretation” provides a new mathematical context for the different steps of a procedure which is not modified. Another name for this commentarial technique could be “re-reading” a procedure.

⁴Despite the growing trend of changing the name of this property of right-triangles, for the sake of simplicity, the phrase “Pythagorean Theorem” appears in reference to a *procedure* found by mathematicians living in the Indian subcontinent.

the Sphere, having entered an assembly of wise men, has declared: ‘Āryabhaṭa tells three: Mathematics, Time-reckoning, the Sphere’.⁵

Within this hostile atmosphere, Bhāskara’s commentary attempts to convince the reader of the coherence and validity of Āryabhaṭa’s treatise. To this end, the commentary dispels “doubts” (*sandeha*) that arise in the explanations of Āryabhaṭa’s verses. Thus, the analysis provides refutations (*parihāra*) to objections and establishes (*sādhya*, *siddha*) Bhāskara’s readings of Āryabhaṭa’s verse. This text presents mainly syntactical and grammatical discussions which debate the interpretation of a given word in the treatise. More often than not, the discussion of the meaning and use of a word defines and characterizes the mathematical objects in question. (Are squares all equal sided quadrilaterals? Do all triangles have equally halving heights? etc.).⁶ Bhāskara’s commentary adopts technical words, and the specialized readings of the verses show that the Ab cannot be understood in a straight-

5

*...yaḥ tejasvī puruṣaḥ samareṣu nikṛṣṭāsītejovitānacchuritabāhuḥ
śatruśaṅghātam prakāśam praviśya praharan evam āha ‘ayam asau
uditaḥ aditikulaprasūtaḥ samareṣu anivāritavīryaḥ yajñadattaḥ praharati
/ yadi kasyacit śaktiḥ pratipraharatu’ iti / evam asau api ācāryaḥ
gaṇitakālakriyāgolātīśayajñānodadhipāragāḥ vitsabhām avagāhya ‘āryabhaṭaḥ
trīṇi gadati gaṇitam kālakriyām golam’ iti uktavān /.*

Shukla 1976:5. Unless otherwise specified, the text follows the critical edition published in Shukla 1976. I would like to thank T. Kusuba, T. Hayashi and M. Yano for the help they provided in translating this paragraph, during my stay in Kyoto in 1997.

⁶Keller 2006, Introduction.

forward way. The verses need interpretation and the interpretation should be the correct one. for the proper interpretation thus defines the

The search for the proper interpretation thus defines the commentator’s task. The importance of interpretation becomes especially clear when Bhāskara criticizes Prabhākara’s exegesis of the Ab.⁷ For instance, in his comment on the rule for the computation of sines, Bhāskara explains that the expression *samavṛtta* refers to a circle, not a disk as Prabhakara understood it.⁸ More crucially, through his understanding of the word *agra* (remainder) as a synonym of *sarikhyā* (number), Bhāskara provides a new interpretation of the rule given in BAB.2.32-33⁹: the verse giving the rule for a “pulverizer with remainder” (*sāgrakūṭṭakāra*) can now be read as giving a rule for the “pulverizer without remainder” (*niragrakūṭṭakāra*)¹⁰. This peculiar reading of the word *agra* is an extreme example of the technical and inventive devices commentators use for their interpretations.

Outside the syntactical discussion of a verse, Bhāskara sometimes considers the mathematical content of the procedure directly. Defending Āryabhaṭa’s approximation of π against those of competing schools, he undertakes a refutation (*parihāra*) of the jaina value of $\sqrt{10}$ (*daśakaraṇī*), claiming that the value rests only on tradition and not on proof.

In this case also, it is just a tradition (*āgama*) and not a proof

⁷Keller 2006, BAB.2.11; BAB.2.12.

⁸Shukla 1976: 77; Keller 2006, volume 1: 57.

⁹Shukla 1976: 77; Keller 2006:132-133.

¹⁰Both rules are mathematically equivalent but do not follow the same pattern. Furthermore, the second reading also involves omitting the last quarter of verse 33. See Keller 2006: Volume II, Appendix on BAB.2.32-33.

(*upapatti*) (...) But this also should be established (*sādhya*).¹¹

The above statement should not induce a romantic vision of an enlightened Bhāskara using reason to overthrow prejudices transmitted through (religious) traditions. Although here he criticizes the reasoning which cites “tradition” to justify a rule, in other cases Bhāskara accepts this very argument as evidence of the correctness of a mathematical statement¹². The question nonetheless is raised: Bhāskara argues that the procedures of the *Āryabhaṭṭīya* are correct, but how does Bhāskara “establish” a rule? Moreover, what does Bhāskara consider a “proof” ? The answer to these questions could be in the use of a specific commentarial technique: the re-interpretation of procedures.

2 The Re-interpretation of Algorithms

Bhāskara, in an attempt to elucidate Āryabhaṭa’s rules, gives interpretations of Āryabhaṭa’s verses. He thus makes clear what are the different steps required to carry out a procedure, or the word used to define a mathematical object. In certain cases, having put forth such an interpretation, Bhāskara re-invests his understanding of the rule with an additional meaning. This is what I call a “re-interpretation”. A re-interpretation does not invalidate a previous interpretation. It is somehow like the poetic process of *śleṣa* which

¹¹

atrāpi evāgamaḥ naivopapattiḥ . . . cetad api sādyaṃ eva.

Shukla 1976: 72.

¹²Keller 2006, Introduction.

reads several meanings in a same compound, creating thus a poetic aura. A re-interpretation adds a layer of meaning, gives depth, to the interpretation of a rule.

As described below, an “explanation”, or a “verification,” consisted of either providing an alternative procedure to arrive at the same result or a re-interpretation of a given procedure via the Rule of Three or the Pythagorean Theorem. In both cases, an explanation would provide a mathematical justification for what alone could appear as an arbitrary succession of operations.

Before examining re-interpretations of procedures in Bhāskara’s commentary, the expression of the Rule of Three and the Pythagorean Theorem in BAB must be explained.

2.1 Rule of Three

The Rule of Three (*trairāsika*¹³) appears in verse Ab.2.26.

Now, when one has multiplied that fruit quantity of the Rule of
 Three by the desire quantity|
 The quotient of that divided by the measure should be this fruit
 of desire||¹⁴

In other words, if M (the measure) is the multiple of a fruit F_M , and D is a desire for which the fruit, F_D , is sought, the verse may be expressed in

¹³For a general overview on the Rule of Three in India see Sarma 2002.
¹⁴

trairāsīkaphalarāsīṃ tam athecchārāsīnā hataṃ kṛtvā|
labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idaṃ syāt|

(Shukla 1976: 112-223).

modern algebraic notation as:

$$F_D = \frac{F_M \times D}{M} \quad (1)$$

Obviously, this expression can also be understood as a statement that the ratios are equal:

$$\frac{F_D}{D} = \frac{F_M}{M} \quad (2)$$

The procedure given in the verse provides an order for the different operations to be carried out. First, the desire is multiplied by the fruit. Next, the result is divided by the measure. This order of operations causes the procedure to appear as an arbitrary set of operations. If the division was made first (resulting in the “fruit” of one measure) and then the multiplication, the computation would have had a step-by-step meaning, but this is not the order adopted in this case.

Bhāskara provides a standard expression to define the kind of problem which the Rule of Three solves. When the commentator thinks that a situation involves proportional quantities and thus the Rule of Three is (or can be) applied, he brings this fact to light by using a verbal formulation (*vāco yukti*) of the Rule of Three. This verbal formulation is a syntactically rigid question which reads as follows:

If the measure produces the fruit, then with the desire what
is produced? The fruit of desire is produced.

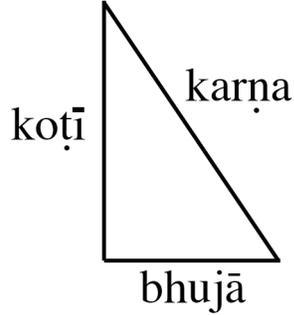
This question, when it appears, serves as a leitmotiv. When the formula appears, Bhāskara thinks that the Rule of Three can be applied. Reciprocally, when a Rule of Three can be applied, this standard expression can be given.

Furthermore, I believe that for Bhāskara this standard expression or the simple reference to a Rule of Three invokes proportionality.

2.2 The Pythagorean Theorem

Bhāskara, like other medieval Sanskrit mathematicians, does not use the concept of angles. In his trigonometry, Bhāskara uses lengths of arcs. As for right triangles, Bhāskara distinguishes them from ordinary triangles by giving to each side a specific name. Whereas scalene, isosceles and equilateral triangles have sides (*aśra*, for all sides), flanks (*pārśva*, a synonym) and sometimes earths (*bhū*, for the base), right angle triangles have a “base” (*bhujā*), an “up-right side” (*koṭī*) and a “hypotenuse” (*karṇa*), as shown in Figure 1. In the first half of Ab.2.17, Bhāskara states the Pythagorean

Figure 1: Names of the Sides of a Right Triangle



Theorem:

That which precisely is the square of the base and the square of

the upright side is the square of the hypotenuse.¹⁵

Therefore, in order to indicate that a situation involves a right-triangle, Bhāskara gives the names of the sides of a right-triangle to the segments concerned by his reasoning. Two examples of Bhāskara's re-interpretation will demonstrate how he employed this theorem, as well as show his terminology.

2.3 Re-interpretation

The section devoted to gnomons (*śāṅku*) contains two illuminating cases.

2.3.1 A Gnomon and a Source of Light

The standard situation is as follows: A gnomon (*śāṅku*, *DE*) casts a shadow (*EC*), produced by a source of light (*A*), as illustrated in Figure 2.

First, consider the procedure given in Ab.2.15:

The distance between the gnomon and the base, with the ⟨height of⟩ the gnomon for multiplier, divided by the difference of the ⟨heights of the⟩ gnomon and the base. |

Its computation should be known indeed as the shadow of the gnomon ⟨measured⟩ from its foot. ||¹⁶

15

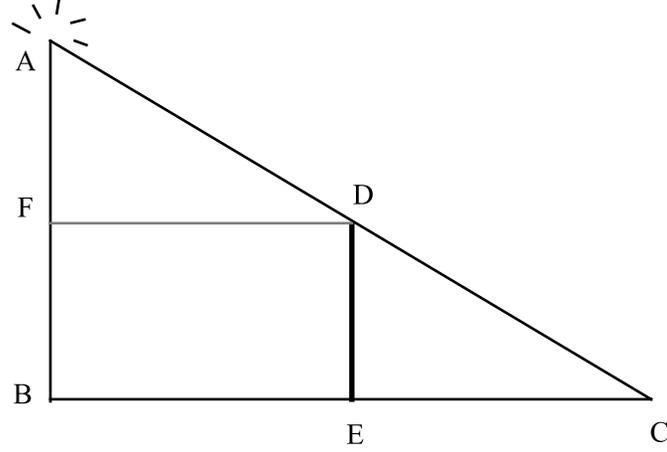
yaś caiva bhujāvargaḥ koṭivargaś ca karṇavargaḥ saḥ

Shukla 1976:96.

16

śāṅkuguṇaṃ śāṅkubhujāvivarāṃ śāṅkubhujayor viśeṣahṛtam |

Figure 2: A Schematized Gnomon and Light



This procedure involves a multiplication and a division. In modern algebraic notation:

$$EC = \frac{BE \times DE}{AF}$$

In this way, the procedure appears to be an arbitrary set of operations. Bhāskara begins with a general gloss. Then, as in all his verse commentaries, Bhāskara's commentary provides a list of solved illustrations. These examples have a standard structure: first comes a versified problem, then a 'setting down' (*nyāsaḥ*) section, and finally a resolution (*karaṇa*). Thus, in his re-interpretation of the above procedure after a solved illustration, Bhāskara writes:

This computation is a Rule of Three. How? If from the top of the base which is greater than the gnomon $[AF]$, the size of the space between the gnomon and the base, which is a shadow, $[FD = BE]$

yal labdaṃ sā chāyā jñeyā śaṅkoḥ svamūlāt hi

Shukla 1976:90.

is obtained, then, what is ⟨obtained⟩ with the gnomon $[DE]$? The shadow $[EC]$ is obtained¹⁷.

The standard formulation of a Rule of Three, applied to the similar triangles AFD and DEC, can be recognized here. The standard expression of a Rule of Three provides the proportional elements on which the computation is based. Here the rule indicates that the ratio of AF to FD is equal to the ratio of DE to EC. The application of the rule thus gives the arbitrary set of operations a mathematical significance. Rather than just a list of operations, the rule in Ab.2.15 becomes a Rule of Three.

2.3.2 A Gnomon in Relation to the Celestial Sphere

In the previous commented verse (BAB.2.14), Bhāskara sets out two procedures. Both rest on the proportionality of the right triangle formed by the gnomon and its midday shadow with the right triangle composed by the Rsine of the altitude and the zenith distance. In the present example, one procedure uses only the Rule of Three, while the other uses the Rule of Three with the Pythagoras Theorem. Both procedures compute the same results.

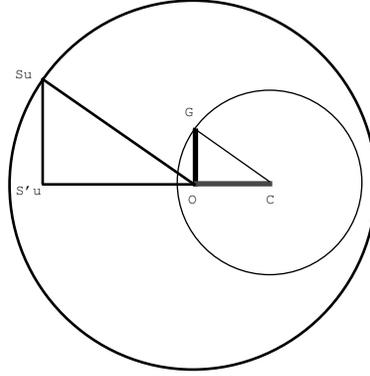
Consider Figure 3. Here, GO represents a gnomon and OC indicates its midday shadow. The circle of radius OSu (Su symbolising the sun) represents

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*etat karma trairāśikam/ katham ? saṅkuto 'dhikāyā uparibhujāyā yadi
śaṅkubhujāntarālapramāṇam chāyā labhyate tadā śaṅkunā keti chāyā labhy-
ate*

Shukla 1976:92.

Figure 3: Proportional Astronomical Triangles



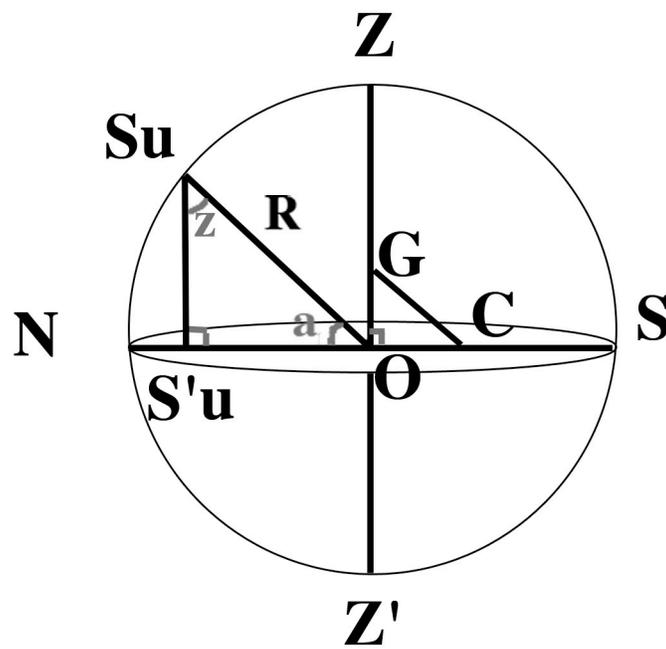
the celestial meridian. The radius OSu is thus equal to the radius of the celestial sphere. $S'u$ designates the projection of the Sun onto the horizon. The segment $SuS'u$ illustrates the Rsine of altitude. Bhāskara I notes that the triangle $SuS'uO$ is similar to GOC . Therefore the segment $S'uO$ (that is, the Rsine of the zenithal distance) is proportional to the shadow of the gnomon at noon and the Rsine of the altitude is proportional to the length of the gnomon. This proportionality is further illustrated in Figure 4.

In modern algebraic notation,

$$\frac{SuS'u}{GO} = \frac{S'uO}{OC} = \frac{SuO}{GC}$$

The mathematical key to this situation is the relationship between the celestial sphere and the plane occupied by the gnomon, which Bhāskara and Āryabhaṭa call ‘one’s own circle’ (*svavṛtta*). This relationship is highlighted here by a set of puns. Thus, the gnomon and the Rsine of the altitude have the same name (*śaṅku*), as do the shadow of the gnomon and the Rsine of zenith distance (*chāya*). GC is the ‘half-diameter of one’s own circle’.

Figure 4: Altitude and Zenith



z is the zenith distance
a is the altitude

Bhāskara states this relationship by considering the Rule of Three¹⁸:

In order to establish a Rule of Three -‘ If for the half-diameter of one’s own circle both the gnomon and the shadow ⟨are obtained⟩, then for the half-diameter of the ⟨celestial⟩ sphere, what are the two ⟨quantities obtained⟩?’ In that way are obtained the Rsine of altitude and the Rsine of the zenith distance.

He also adds¹⁹:

Precisely these two [i.e. the Rsine of the sun’s altitude and the Rsine of the sun’s zenith distance] on an equinoctial day are said to be the Rsine of colatitude (*avalambaka*) and the Rsine of the latitude (*akṣajyā*)

Indeed, as illustrated in Figure 5, on the equinoxes the sun is on the celestial equator. At noon, the sun occupies the intersection of the celestial equator and the celestial meridian. At that moment, the zenithal distance z equals the latitude of the gnomon (ϕ) and the altitude (a) becomes the co-latitude ($90 - \phi$). As in the previous case, the Rsine of latitude ($SuSu'$) is called “perpendicular” (*avalambaka*).

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*trairāśika-prasiddhy-artham- yady asya sva-vṛtta-viṣkambha-ardhasya ete
śaṅkuc chāye tadā gola-viṣkambha-ardhasya ke iti śaṅkuc chāye labhyete*

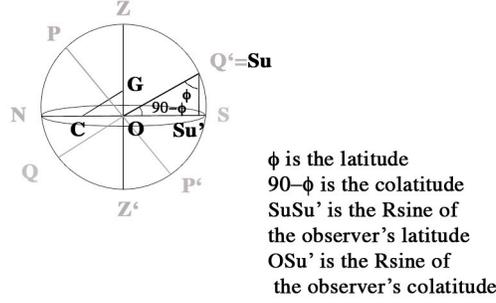
Shukla 1976: 89.

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tāv eva viṣuvati avalambakākṣajye ity ucyete/

Shukla 1976: 89.

Figure 5: Latitude and colatitude on an equinoctial day



Now, Bhāskara considers an example for an equinox in which $OG = 13$, $OC = 5$, and the radius of the celestial sphere (SuO) is the customary 1348. Bhāskara writes²⁰:

When computing the Rsine of latitude (*akṣajyā*) a Rule of Three is set down: 13, 5, 3438. What is obtained is the Rsine of latitude, 1322²¹. That is the base (*bhujā*) the half-diameter is the hypotenuse (*karṇa*); the root of the difference of the squares of the base and the hypotenuse is the Rsine of co-latitude (*avalambaka*), 3174²²

In this case, Bhāskara uses the fact that the triangles are both right and

²⁰

akṣajyā “*nayane trairāsīkasthāpanā- 13/ 5/ 3438/ labdham akṣajyā 1322/ eṣā bhujā, vyāsārdha karṇaḥ, bhujākarṇavargaviśeṣamūlam avalambakaḥ 3174*

Shukla 1976:90.

²¹This is an approximate value. For more on this value, see Keller 2006, BAB.2.14.

²²This value is also an approximation.

similar. Bhāskara then uses this similarity to compute $SuS'u$. Bhāskara employs the ‘Pythagorean Theorem’ to compute $OS'u$. In order to identify the right triangle, Bhāskara renames the Rsine of latitude (*aksajyā*, $SuS'u$) as the base of a right triangle (*bhujā*) and he identifies the radius of the celestial sphere as the hypotenuse. Thus, the Rsine is identified with the up-right side of a right-triangle. This identification implicitly explains how the computation is carried out. However, Bhāskara immediately adds²³:

With a Rule of Three also 13, 12, 3438; what has been obtained
is the Rsine of the colatitude, 3174²⁴.

In this way, Bhāskara again computes $OS'u$ by using the similarity of $OSuSu'$ and OGC . Bhāskara thus computes the same value twice, using two different methods. The most likely explanation is that he verifies the results obtained with one algorithm by using another independent process.

The mathematical key to both these computations is the prior relationship between the gnomon and the celestial sphere. Establishing a syntactical connection between these two spaces makes this relationship explicit. Indeed, the invocation of the Rule of Three begins with a standard question. Through the naming of two of its segments, the example specifies the figure as a right triangle and its associated properties may be assumed. This identification as a right triangle indicates not only the mathematical property underpinning the procedure but also a set of operations (first multiplication and division,

²³

trairāśikenāpi 13/ 12/ 3438/ labdham avalambakaḥ 3174/

Shukla 1976, 90.

²⁴This value is an approximation again.

then squaring the lengths with subsequent additions or subtractions of the results). Consequently, the re-interpretation according to the Rule of Three or the ‘Pythagorean Theorem’ maps the specific astronomical problem onto a more general and abstract mathematical situation. (That is, Rsines of altitudes and zenithal distances become the legs of a simple right-triangle.) The re-interpretation of the unexplained steps of the procedure thus permits a mathematical grounding that may serve as a justification of the algorithm itself.

This analysis thus brings to light two kinds of reasoning: the verification of a result by using two independent procedures and the mathematical grounding of a set of operations via their re-interpretation according to the Rule of Three and/or the ‘Pythagoras Theorem’. However, the topic of mathematical reasoning is also addressed by the parts of BAB which explicitly have a persuasive aim, attempting to convince the reader that the algorithms of the Ab are correct.

3 Explanations, Verifications and Proofs

Bhāpefic names when referring to a number of arguments: ‘explanations’, ‘proofs’ (*upapatti*) and ‘verifications’ (*pratyāyakaṛaṇa*). These arguments do not appear systematically in each verse commentary and – as will be seen below – are always fragmentary. In order to understand when and why these reasonings appear, an understanding of how they explain, prove or verify is necessary.

3.1 Explanations

Bhāskara’s commentary on verse 8 of the mathematical chapter of the Ab presents an example of explanation. Verse 8 describes two computations concerning a trapezoid. (See Figure 3.1.) The first calculation evaluates the length of two segments (*svapātalekha*, EF and FG) of the height of a trapezoid. In this case, the height is bisected at the point of intersection for the diagonals. The procedure is made of a multiplication followed by a division²⁵:

Ab.2.8. The two sides, multiplied by the height ⟨and⟩ divided by their sum are the ‘two lines on their own fallings’.
When the height is multiplied by half the sum of both widths, one will know the area.||

In other words, with the labels used in Figure 3.1, we have:

$$EF = \frac{AB \times EG}{AB + CD};$$
$$FG = \frac{CD \times EG}{AB + CD}.$$

Likewise, the area \mathcal{A} is:

$$\mathcal{A} = EG \times \frac{(AB + CD)}{2}.$$

On the first part of the verse, Bhāskara comments²⁶:

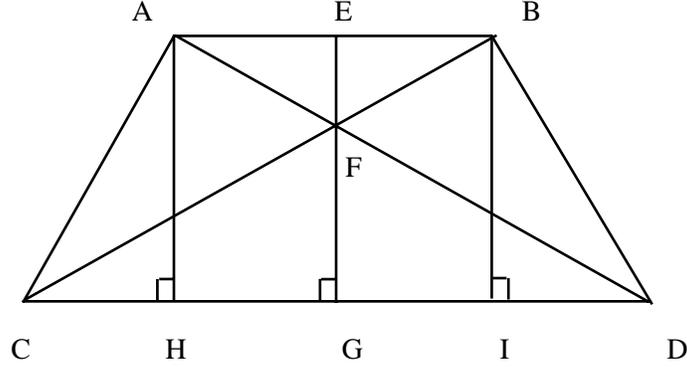
²⁵

āyāmaguṇe pārśve tadyogahr̥te svapātalekhe te|
vistarayogārdhaguṇe jñeyam̐ kṣetraphalam āyāme||

(Shukla 1976: 63).

²⁶

Figure 6: Inner segments and fields in a trapezoid



The size of the ‘lines on their own fallings’ should be explained (*pratipādayitavya*) with the computation of a Rule of Three on a figure drawn by ⟨a person⟩ properly instructed. Then, by means of just a Rule of Three with both sides, the computation of ⟨the lines whose top is⟩ the intersection of the diagonals and a perpendicular ⟨is performed⟩.

This explanation consists of re-interpreting the procedure—which is a multiplication followed by a division—according to the Rule of Three. The explanation contains two steps. The first step considers the proportionality in a diagram, then re-interprets the set of operations of the algorithm as the application of the Rule of Three. A re-interpretation of the procedure in this way endows the seemingly arbitrary set of operations with a mathematical meaning.

samyagādiṣṭena ālikhite kṣetre svapātalekhāpramaṇaṃ
trairāśikagaṇitena pratipādayitavyam/ tathā trairāśikenaivobhaya pārśve
karṇāvalambakasampātānayanam/

Shukla 1976: 63.

The second computation in verse 8 determines the area of the trapezoid. As shown in Figure 3.1, the area of the trapezoid can be broken into the summation of the areas of several triangles. Alternately, the trapezoid can be decomposed into a rectangle and two triangles.

Although no figure is explicitly drawn, Bhāskara seems to have such a diagram in mind. Moreover, he seems to refer to such a drawing when he writes²⁷:

Here, with a previous rule [Ab.2.6.ab] the area of isosceles and uneven trilaterals should be shown/explained (*darśayitavya*). Or, with a rule which will be stated [Ab.2.9.] the computation of the area of the inner rectangular field (should be performed);

Even though it has not survived, such a figure shows how areas can be added to give the area of the trapezoid. Here again then, an enigmatic computation is given a convincing meaning by re-interpreting it. This time the procedure is re-interpreted as a collection of already known procedures, those computing the area of triangles and rectangles. Bhāskara alludes to another verse, in which another interpretation of the area of the trapezoid can be understood in a different way. However, the procedures of Ab.2.9 will be analyzed below.

Both of Bhāskara's explanations in BAB.2.8 consist of

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*pūrvasūtreṇātra dvisamaviśamatryaśrakṣetrāphalaṃ darśayitavyam/
vaksyāṇasūtreṇāntarāyatacaturaśrakṣetrāphalānayanam (...) vā/*

Shukla 1976: 63.

1. an explanation of a diagram, and
2. a re-interpretation of a procedure. The different steps to be carried out are mathematically grounded. This re-interpretation seems to confirm or verify the reasoning by looking at a diagram.

Three words refer to an explanation: *vyākhyāna*, *pradarśana*, and *pratipādita*. The word *vyākhyāna* indicates that the commentary gives an explanation, but it is also used for an argument connected with a diagram²⁸:

Or else, all the procedures ⟨used⟩ in the production of chords are in the realm of a diagram, and a diagram is intelligible ⟨only⟩ with an explanation (*vyākhyāna*). Therefore it has not²⁹ been put forth (*pratipādita*)⟨by Āryabhaṭa in the *Āryabhaṭīya*⟩.

Note that this passage emphasizes that explanations belong to the genre of commentary and, at least according to Bhāskara, should not be exposed in a treatise.

The word *pradarśana* is derived from the verbal root *dṛṣ-*, ‘to see’. It has a similar range of meaning as the English verb ‘to show’. It is often hard to distinguish if the word refers to the visual part of an explanation or to the entirety of the explanation. For instance, in BAB.2.11, Bhāskara uses a diagram and writes³⁰:

²⁸*athavā jyotpattau yatkarāṇaṃ tatsarvaṃ chedyakaviśayaṃ, chedyakaṃ ca vyākhyānagamyamiti na pratipāditam/* (Shukla 1976: 79).

²⁹*na* has been added by the editor, K. S. Shukla and is not found in the manuscripts. Another possible interpretation of the sentence reads: ‘Therefore it has been put forth ⟨by Bhāskara in his commentary⟩

³⁰*evam ālikhite kṣetre sarvaṃ pradarśayitavyam* (Shukla 1976: 79).

In the field drawn in this way all is to be shown/explained
(*pradarśayitavya*).

*2: "a word's ambiguity" is a strange formulation. I don't like the unref-
erenced "it" that begins a cleft sentence, but I'll allow it. What is hard to
distinguish?*

Finally, the word *pratipādita* is more technical and straightforward. It
commonly appears in lists of solved examples found in most of the commented
verses in the mathematical part of BAB.

*1: A comma is needed. 2: "essentially" has a different sense in English.
"commonly" is only a suggestion. Rephrasing to avoid an ambiguity wherein
it is unclear what "most" modifies...*

In the example of explanations presented above, the commentator re-
interprets geometrical procedures according to the Rule of Three or the
'Pythagorean Theorem'. Only geometrical procedures receive such argu-
ments. Each time, the argument omits a diagram to which the text seemingly
refers. Among the geometrical processes, explanations are 'seen', as will be
seen in the only example from the BAB in which the word 'proof' occurs.

*1: More of an oral composition. "With" is wrong preposition, again. 2:
Trying to make a more direct presentation. Standard vocabulary. 3: Syntax
is off – I've tried to fix it, but I'm unsure what "in which it seems orally to be
held" could mean. I think I have it from context. I'm not sure how to explain
how it missed the mark. 4: "them" needed a referent, so I reworked it. Kill
the first person. Adding a little joke with the repetition of "see" emphasizing
the ambiguity noted before.*

3.2 The Only Two Occurrences of the Word “Proof”

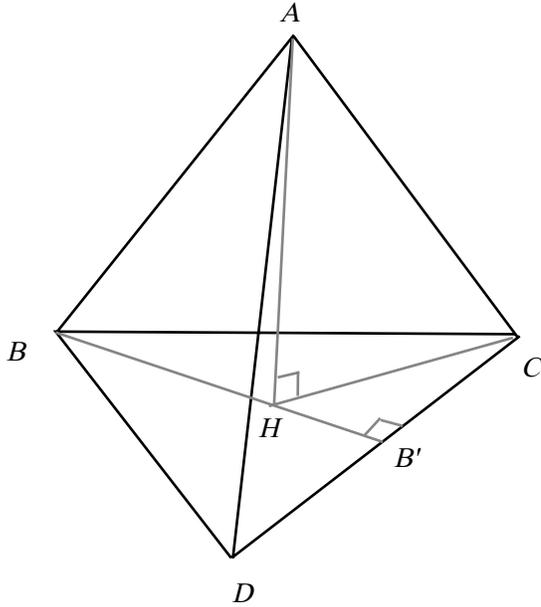
The Sanskrit word *upapatti* refers directly to a logical argument. This word is used twice in Bhāskara’s commentary as noted by Takao Hayashi³¹. The gender of this word is feminine and it is derived from the verbal root *upa-PAD-*, meaning ‘to reach’. Thus, an *upapatti*, is literally ‘what is reached’ and has consequently been translated as ‘proof’. In both instances, some ambiguity surrounds this word, and the meaning of the word is not certain. One occurrence has been quoted above, wherein proofs (*upapatti*) are described as opposed to tradition. The other instance refers to the reasoning whereby the height of a regular tetrahedron is determined from its sides. In this case, Bhāskara understands Āryabhaṭa’s rule in the second half of verse 6 of the mathematical chapter as the computation of the volume of a regular tetrahedron. Such a situation is described in Figure 7.

1: More direct statement, but the ideas should be split apart. 3: Because English has very little gender, the grammatical terms ought to be invoked. Stated more directly. 4: Syntax and repetition. 5: Killing first person. No terminating prepositions. 6: "when" implies a TIME. 6: "other" needs a noun. Trying to mirror the syntax to the standard translations of Euclid (which will be familiar to anglophones.) 7: More direct statement of A.'s authority.

Given a regular tetrahedron $ABCD$, AH is the line through A perpendicular to the plane defined by the triangle BDC . AH is called the ‘upward-side’ (*ūrdhvabhujā*). AC is called *karṇa* (lit. ear) because it is the hypotenuse of AHC . Bhāskara explains how to compute the upright side by using the

³¹Hayashi 1985: 75-76.

Figure 7: An Equilateral Pyramid with a Triangular Base



Pythagorean Theorem and the Rule of Three. The determination of CH , from which the upright side AH may be computed, rests upon the proportional properties of similar triangles, illustrated in Figure 8. The triangles $BB'C$ and $B'CH$ are similar:

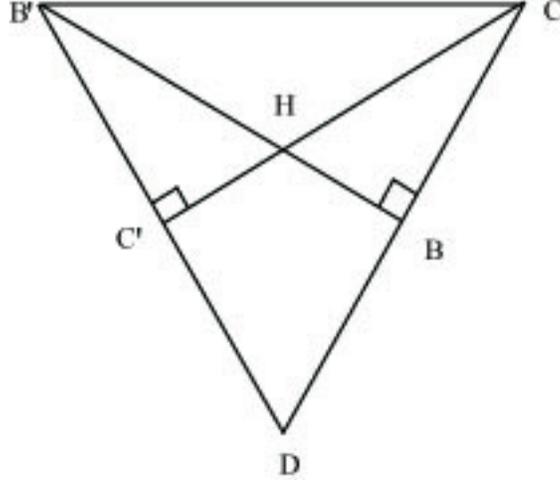
$$BB' : CB = CB' : CH.$$

From this relationship it is known that:

$$CH = \frac{CB \times CB'}{BB'}.$$

1: In general, I'm trying to follow Euclidean compositions. 2: Need quotation marks and not "as" but "because". 6: These aren't words, but equations. Bhāskara expressed this relationship as the Rule of Three. The text does not

Figure 8:



give a precise argument, but it alludes to the properties as being clear from a diagram. It is in this context that the word *upapatti* appears³²:

In order to show the proof (*upapatti*) of ⟨that⟩ Rule of Three, a field is set-down.

1: Shifting the emphasis to B. and his text. 3: "This" needs antecedent and when is temporal only.

The argument implied by this word depends on the diagram. As in the case of the explanations, the proof must have been presented orally. This situation differs from the acts of ‘re-interpretation’ seen above. In the present case, an argument is created, and there is no pre-existing algorithm to re-interpret. The foundations of this new argument are set out in a diagram. The procedure used is the Rule of Three, as in the “explanations” seen above. However, another type of argument concerns the correctness of algorithms:

³²*trairāśīkopapattipradarśanārtham kṣetranyāsaḥ*-(Shukla 1976: 59).

verification.

1: More concise. Not certain what "is silenced" would mean. Diacritical marks suggest that this word was a cut-n-paste suggestion from an online translator? I've tried to rephrase. Breaking ideas into 2 sentences. 2: Killing first person. 3: Killing first person. Standardizing vocab. 5: Trying to simplify so the idea is more clear. show/find is an odd juxtaposition. 7: "However" is a better coordination of ideas. Changing tense and making the statement more abstract.

3.3 Verification

Verifications are distinguished from explanations and proofs by their name, *pratyayakarāṇa*. Indeed, *pratyaya* has an etymological root in a verb meaning 'to come back', which has connotations of conviction. *Pratyayakarāṇa* thus means 'enabling to come back' or 'producing conviction'. Historians of Indian mathematics usually understand this word as a type of verification and translate it accordingly.³³

A verification resembles an explanation in that a verification re-interprets a given procedure according to another rule and establishes a mathematical grounding. The arguments that the commentator labels "verifications" sometimes present difficulties, and currently our understanding of them is not at all certain. Below are set out several hypotheses about how these verifications can be understood.

³³Hayashi 1995:73-74.

3.3.1 Verification of an Arithmetical Computation

Bhāskara states a verification by the Rule of Five for the rule given in Ab.2.25. Bhāskara states the rule in Ab.2.25 as follows³⁴:

The interest on the capital, together with the interest ⟨on the interest⟩, with the time and capital for multiplier, increased by the square of half the capital|
The square root of that, decreased by half the capital and divided by the time, is the interest on one's own capital||

This passage can be formalized as follows: Let m (*mūla*) be capital; let p_1 (*phala*) be the interest on m during a unit of time, $k_1 = 1$ (*kāla*), usually a month. Let p_2 be the interest on p_1 at the same rate for a period of time k_2 . If $p_1 + p_2$, m , and k_2 are known, the rule can be expressed in a modern mathematical notation as:

$$p_1 = \frac{\sqrt{mk_2(p_1 + p_2) + \left(\frac{m}{2}\right)^2} - \frac{m}{2}}{k_2}.$$

This rule is derived from a constant ratio:

$$\frac{m}{p_1} = \frac{p_1}{p_2} k_2.$$

The Rule of Five, described in BAB.2.26-27.ab, rests on the same ratio as the rule given in Ab.2.25. In the former instance though, only k_1 may

³⁴

*mūlaphalaṃ saphalaṃ kālamūlaguṇam ardhāmūlakṛtiyuktam|
tanmūlaṃ mūlārdhonaṃ kālahṛtaṃ svamūlaphalam||*

(Shukla 1976: 114).

numbers other than 1:

$$\frac{m}{p_1}k_1 = \frac{p_1}{p_2}k_2.$$

The Rule of Five indicates an expression equal in value to p_2 :

$$p_2 = \frac{p_1^2 k_2}{m k_1}$$

The Rule of Five may therefore be used in the opposite direction to find a value for p_1 .

In BAB.2.25 Bhāskara gives an example³⁵:

1. I do not know the ⟨monthly-⟩interest on a hundred. However, the ⟨monthly-⟩interest on a hundred increased by the interest ⟨on the interest⟩ | Obtained in four months is six. State the interest of a hundred produced within a month||

This example states a case in which:

$$\begin{aligned} m &= 100 \\ k_2 &= 4 \\ p_1 + p_2 &= 6 \end{aligned}$$

By the procedure given in Ab.2.25, the value of p_1 is 5.

Bhāskara then adds³⁶:

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*jānāmi śatasya phalam na ca kintu śatasya yatphalam saphalam | māsaḥ
caturbhiḥ āptam ṣaḍ vada vṛddhim śatasya māsothām||*

36

Verification (*pratyayakaraṇa*) with a rule of Five: “If the monthly interest (*vṛddhi*³⁷) on a hundred is five, then what is the interest of the interest [of value (*dhana*)-five] on a hundred, in four months?”

1 4

Setting down: 100 5 The result is one. This increased by the

5 0

⟨monthly⟩ interest on the capital is six *rūpas*, 6.

Simply stated, the verification consists of knowing m , p_1 and k_2 , finding p_2 and confirming that its value increased by p_1 will give the same value for $p_1 + p_2$ as stated in the problem.

The Rule of Five, as seen above, returns the value of p_2 . In this case, an independent procedure is established. This procedure does not deliver the same result but gives a method of inverting the procedure to check independently that the result makes sense. The use of the Rule of Five, which Bhāskara describes as a combination of two Rules of Three, also imbues the computation with mathematical basis in proportionality.

3.3.2 Verification of the Area of Plane Figures

Bhāskara interprets the first half of Ab.2.9 as a way to verify procedures for areas given by Āryabhaṭa in the previous verses.

*atyayakaraṇam pañcarāśīkena yadi śatasya māsikī vṛddhiḥ pañca tadā
caturbhiḥ māsaḥ śatavṛddheḥ [pañcadhanasya] kā vṛddhiḥ iti/ nyāsaḥ -*

1 4

100 5 labdham 1 / etatsahitā śatavṛddhiḥ ṣaḍ rūpāṇi 6/

5 0

³⁷From now on, unless otherwise stated this is the word translated as “interest”.

For all fields, when one has acquired the two sides, the area is their product |³⁸

Bhāskara endows the treatise with the goal of “verification”—a goal nowhere explicitly appearing in the treatise itself. Two steps can be distinguished in the verifications of this verse commentary, each corresponding to a diagram. The first step constructs a diagram of the figure for which area is verified. The length and width of a rectangle with the same area as the figure are identified. This ‘length’ and ‘width’ are usually values from Āryabhaṭa’s procedure for which verification is sought. For instance, to verify the area of a triangle, the length of the corresponding rectangle is identified as the height of the triangle, while the width of the rectangle is half the base of the triangle. In the first half of verse 6, the area of a triangle is given as the product of half the base by the height of a triangle. The second step of the argument presents a diagram of the rectangle and computes the multiplication.

How should this argument be understood? According to one means of understanding, this argument represents a re-interpretation of the procedures given for the areas to be verified. This re-interpretation formally considers the procedure to be a multiplication of two quantities. Each quantity is then interpreted geometrically as either the length or width of a rectangle with the same area as the initial figure. In this way, Bhāskara calculates the length and height of the rectangle, as required by verse 9.

Another way of understanding the argument begins with the fact that the verification for a given figure *produces* a rectangle of the same area as the given figure. The fact that all figures have a rectangle with the same

³⁸*sarveṣāṃ kṣetrāṇāṃ prasādhyā pārśve phalaṃ tadabhyāsaḥ* | Shukla 1976: 66.

area is in an implicit assumption of Sanskrit plane geometry. T. Hayashi has interpreted this argument in exactly such a manner.³⁹ This interpretation consists of producing a rectangle and verifying that its area is equal to the area of the figure.

A third approach relies on the ‘setting down’ parts which contain diagrams. This verification consists of *constructing* a rectangle with the same area from a given figure. For instance, in the second step of the verification of the area of a triangle, Bhāskara specifies that when the parts of the area of such a triangle are rearranged (*vyasta*), they produce the rectangle which is drawn. The construction of a rectangle from the original figure is not described in Bhāskara’s commentary. However, such constructions could have been known, as shown by the methods exposed in BAB.2.13. Furthermore, this process recalls the algorithms from the *sulbasūtras*, the earliest known texts of Sanskrit geometry. These algorithms produce a construction which, although not described in the text, corresponds with the discussion contained in the text. With just such a diagram, the argument in the text would arithmetically verify that the construction is correct.

These three interpretations can be combined if a verification is allowed to be simultaneously geometrical and arithmetical. Bhāskara relies on a geometrical strategy to produce a rectangle with the appropriate area, showing that he knows how to construct the corresponding rectangle from the initial figure. Because the construction is obvious, it would not be detailed, and only the lengths of the rectangle would be given. From an arithmetic perspective, this re-interpretation provides a new understanding of the rule

³⁹Hayashi 1995: 73.

given by Āryabhaṭa. Through his arithmetical ‘verification,’ Bhāskara explains the geometrical verification. Bhāskara explains the link between the sides of the initial figure and the lengths and widths of the rectangle with the same area as the initial figure.

Regardless of which interpretation is accepted, the verification either re-interprets a first algorithm (BAB.2.9) and produces a new understanding of the procedure, or it produces a new procedure that gives the same result (BAB.2.25). In either case, the so-called ‘verification’ confirms the numerical results and places the procedure in a secure mathematical context. Thus, after verification, the calculations do not appear to be a set of arbitrary steps.

Conclusion

How should the Rule of Three and the so-called ‘Pythagorean Theorem’ be described? First, they are mathematical tools which enable concrete examples and astronomical situations to be re-interpreted as abstract and general cases, involving right triangles and proportionalities. Nonetheless, the methods of reasoning are hard to understand and pin down. This difficulty may arise from their oral nature, of which Bhāskara’s written text preserves only a portion. These fragments provide us with a glimpse into the tradition of written justifications which flourished a thousand years later. Further detailed explorations of how Sanskrit texts explain, prove and verify mathematical algorithms will advance understanding about how the mathematical correctness of algorithms was conceptualized by mathematicians in the Indian

subcontinent.

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