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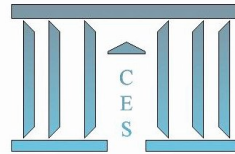
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**Natural rate of unemployment and efficiency : a dynamic
analysis with flexible prices and increasing returns**

Gaël GIRAUD, Céline ROCHON

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Natural rate of unemployment and efficiency: A dynamic analysis with flexible prices and increasing returns

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ABSTRACT.— We present a continuous time non tâtonnement process for frictionless and perfectly competitive markets with (possibly non convex) production, where the natural rate of unemployment (NRU) emerges as the asymptotic value of unemployment. Consumers and producers are myopic and repeatedly participate in Mertens’(2003) limit-price mechanism. We show that underemployment and unsold inventories can survive along the solution paths of our dynamics—the hallmark of the failure of Say’s law. The following paradox appears: A nonzero NRU is compatible with Pareto optimality; conversely, full employment is compatible with sub optimality. Nevertheless, each trade and production path of our price quantity dynamics converges to some infinitesimal Pareto optimal point as long as there are no unsold inventories at the limit.

KEYWORDS. Underemployment, Say’s law, inventories, non convex production.

JEL CLASSIFICATION. D11, D21, D51, E24.

1 Introduction

This paper arises from two motivations. First, the traditional conclusions of the “fixed price” equilibrium approach developed in the 1970s were largely correct,

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but they were not explored further because they failed to provide a theoretical explanation for price rigidities. Conversely, flexible-price equilibrium theory faces a basic stumbling block: How can underemployment emerge in a world where all markets are supposed to clear instantaneously ? Here we offer a paradigm where frictionless economies with perfectly flexible prices and wages exhibit Classical as well as Keynesian underemployment.

Second, standard theories of the natural rate of unemployment (NRU) are typically unable to draw any relationship between the equilibrium level of unemployment and the Pareto optimal level. Here, we provide a dynamic setup where the NRU can be thought of as the value to which the rate of unemployment converges as time goes to infinity and where efficiency properties of NRU can be tackled.

Let us look at these two issues in more detail. The fixed price equilibrium approach was based on the work by Barro and Grossman (1971), Bénassy (1973), Drèze (1975) and others¹; in essence, it consisted of allowing prices to differ from the flexible-price Walrasian equilibrium vector. Output in each market was then assumed equal to the minimum of supply and demand. According to this approach, if prices lead to excess supply in both goods and labor markets, as in Keynesian models, then firms are rationed in the goods markets so that the demand for labor is determined by output. The economy exhibits a *demand* multiplier: An increase in the demand leads to an increase in production, an increase in income, and further increases in the demand, and so forth. Excess demand in all markets yields a *supply* multiplier: Labor supply decreases and output decreases, which leads to further rationing in the goods market.

As a consequence, Blanchard (2000) notes that, without a theory of why prices differ from equilibrium prices, both outcomes (excess supply and excess demand on all markets) are equally probable. A theory of price setting is therefore required. Furthermore, as observed by Roberts (1987), in the early models of disequilibrium theory, at least some prices or wages are exogenously fixed at levels that are not market clearing, and the models offer no basis for analysis of the opportunities for changing prices or of the incentives to do so. Some of them even assume that there is some unmodeled mechanism at work that ensures the maximum level of transactions consistent with voluntary exchange at the fixed prices and feasibility. In this paper, prices are flexible and are determined instantaneously by a double auction, Mertens' (2003) limit-price mechanism. Out of equilibrium, this mechanism provides a natural rationing rule akin to the short-side rule but much closer to actual trading institutions used in contemporary markets. Yet we exhibit both Keynesian and Classical underemployment as well as contained inflation. Moreover, the micro-structure being analyzed as a strategic market game, we specifically address the issue of strategic incentives to change disequilibrium prices. Finally, the determination of quantities is made explicit and is also strategically chosen by players.

¹See Bénassy (1982) for references.

The main reason for the appearance of unemployment in our model is the postulated bounded rationality of economic agents. More specifically, we assume that all of them are myopic. On the consumption side, myopia is captured by assuming that workers instantaneously maximize the (linear) first-order approximation of their current utility. This conforms to a well-established way of modeling myopia in non tâtonnement models.² As a consequence, each worker behaves in the short run as if there were some reservation wage below which he refuses to work and at which his labor supply is vertical. On the production side, firms are myopic and instantaneously maximize their short run profit with respect to the (constant return) first-order approximation of their production function. Thus, they behave as if they had some reservation wage above which they refuse to hire and at which their labor demand is vertical. We show that both effects, when combined with a second ingredient detailed below, generate underemployment along trajectories that are solutions to our dynamics.

Our second ingredient is a reinterpretation of production sets in a dynamic perspective. To grasp the intuition behind this new point of view, consider the simplest possible example of an economy with one producer and one consumer. Suppose the producer's production set, Y_j , exhibits strictly decreasing returns so that, at a non trivial (standard) Walras equilibrium, this producer will have a positive profit, while the final production plan y_j will be located somewhere on the boundary ∂Y_j of the production set Y_j . Here, we question the logic behind this familiar picture, as the following remarks indicate:

1) First, actually, *production takes time*, so that, before reaching y_j , the producer must have produced say $\frac{1}{2}y_j, \frac{3}{4}y_j, \dots$. A little reflection suggests that, if each partial production plan is to be feasible and efficient, then the producer's plan should have evolved along the boundary ∂Y_j from 0 (no production) to y_j . In this paper, we provide a formalization of this intuitive dynamics which remains hidden in static equilibrium theory, and we show that it can be extended to non-convex production.

2) Second, in static equilibrium theory, the unique shareholder of the firm, the consumer, receives the whole profit as dividend. At a standard Arrow Debreu equilibrium, this additional income instantaneously enters into the consumer's budget and hence is used to finance the demand that is addressed to the unique firm. In this sense, Say's law is automatically satisfied but at the cost of a vicious circle: The redistributed profit is supposed to finance the demand that is responsible for this very profit. One interpretation of this paradox is to say that the consumer can take short positions during the production process: What actually happens is that he borrows (at zero cost) a certain amount of cash in order to finance his purchases; these purchases induce some profits that are afterwards redistributed to the same consumer at the end of the day; in equilibrium, nobody remains in the red because the dividends exactly compensate the consumer's loans.

²Smale (1976), Champsaur and Cornet (1990), Bottazi (1994), Bonnisseau and Florig (2003), Giraud (2007).

By contrast, in the situation we examine here, the output $y_j^+(t)$ produced at time t by means of input $y_j^-(t)$ *cannot* be put up for sale on commodity markets at time t , but only at $t + dt$. Because it is not available at t , $y_j^+(t)$ cannot be used to finance the input $y_j^-(t)$ from which it is supposed to be produced (again, “production takes time”). As a consequence, when buying consumption commodities, the consumer must be able to finance his purchases with the revenues from his current sales, not with the revenues of his future dividends.³ Since the mechanism used to implement trades is balanced, the value of the firm’s sales must be equal to that of the household’s wage. It should thus come as no surprise that, at market-clearing prices, firms’ profits are always zero. We view this property as a virtue because increasing returns are known to generate negative profits at static marginal cost equilibria (Cornet 1988).⁴ As a result, in this paper, Walras’ law is always satisfied, but Say’s law is not; underemployment can appear on the labor market as well as unsold inventories on the commodity market; firms’ profits are constantly equal to zero and, whether convex or not, none of them ever goes bankrupt.

Roberts (1987) also provides an equilibrium model with unemployment at flexible, competitive prices and wages. There, however, imperfect competition is assumed and permits to generate a continuum of equilibria; all but the Walrasian ones involve inefficiently low levels of activity. Here, by contrast, perfect competition is assumed throughout and captured by means of a continuum of negligible agents. In the present model, unemployment is not the consequence of informational asymmetries as in works based on efficiency wage models (Shapiro and Stiglitz 1984). Here, indeed, there are no problems of hidden knowledge or unobservable actions. Finally, unlike in the Clower-Leijonhufvud interpretations of Keynes’ chap. 2 of the *General Theory* (Ito 1979), unemployment is not the result of self-enforcing pessimism about demand, as firms do not make any conjectures about demand. The very simplistic way we capture myopia enables therefore to focus on one aspect, apparently neglected so far: the fact that markets populated by boundedly rational players are consistent both with full employment and with inefficient, “recessionary” unemployment.

Once it was realized that “fixed price” equilibrium theory, in its present form, leads to dead ends, many market imperfections have been extensively studied since the 1980s — for example, price staggering (Taylor 1988) and labor search and bargaining (Diamond 1982b; Mortensen 1982; Pissarides 1985, 1999, to name but a few seminal papers). All these approaches, in a sense, add some kind of “viscosity” to the basic equilibrating mechanism that leads to market clearing. This viscosity, in turn, is responsible for the appearance of unemployment. Such

³This is consistent with the fact that, in “real life”, even so-called short sales are accompanied by a private loan or repurchase contract. Following Mertens (2003, n. 3) we leave this aspect outside of the anonymous market by forbidding short sales.

⁴The introduction of *fiat* money (in the same vein as in Giraud and Tsomocos 2006) and of a banking system will allow firms’ to make positive profits along trade-and-production paths. This is left for further research.

imperfections are compatible with our paradigm, and their impact will be analyzed in subsequent work. The strength of our approach, however, is that we do not require them in order to exhibit underemployment, nor to state which type of underemployment will emerge. We show indeed that Classical underemployment and contained inflation should occur essentially when the short run (or “marginal”) rate of substitution (to be defined in subsection 2.2) of the representative consumer exceeds the short run (or “marginal”) productivity of labor (see also subsection 2.2). Conversely, Keynesian underemployment is a consequence of the fact that the marginal rate of substitution is *less* than the marginal productivity of labor.

The second aspect of our contribution regards the relationships between underemployment and efficiency. The literature devoted to search and bargaining in a decentralized labor market usually leads to the conclusion that there will always be some unemployment and, moreover, that a very low rate of unemployment should be inefficient.⁵ However, as noted in Blanchard (2000), no such theory provides a clear-cut relationship between unemployment and efficiency.

Because, in this paper, the NRU does not depend upon issues of matching but only on the fundamental characteristics of the economy (agents’ short run preferences and firms’ short run productivity), efficiency issues can be formally explored. To be more precise, we prove that, provided there are no unsold inventories at the limit, every trade and production path converges towards some infinitesimal price equilibrium, hence to a locally optimal rest point. Moreover, we show by means of examples that a positive NRU is compatible with the resilience of unsold inventories, hence with optimality as well as with suboptimality. Perhaps even more surprisingly, the same holds for a zero NRU: A solution path may converge towards a price equilibrium with full employment, which remains nevertheless suboptimal (because of the persistence of unsold inventories).

There is a reason why this puzzling lack of correspondence between optimality and full employment has not been discovered earlier: macroeconomic markets of Keynesian inspiration (where unemployment emerges) are usually stated within a representative agent model, where welfare issues cannot be addressed. Conversely, general equilibrium models with heterogeneous agents do not permit one to analyze underemployment because, in such models, market clearing implies Say’s law. In this paper, we maintain market clearing throughout together with heterogeneous agents, but in a way that no longer automatically implies Say’s law.

The paper proceeds as follows. Section 2 sets-up our dynamics, emphasizing the reinterpretation of production sets—on which our whole approach rests. Trading rules are specified. At each instant, consumers and producers take part

⁵A similar conclusion is drawn, from different premisses, by Shapiro and Stiglitz (1984): a positive level of unemployment arising through wages that exceed the supply price of the amount of labor actually employed may be necessary to provide incentives not to shirk. Thus, equilibrium cannot involve full employment in such models. Moreover, the unemployment may not represent any inefficiency, given the informational constraints.

in a market by sending market and limit-price orders to a central clearing house. The resulting infinitesimal trades are dictated by Mertens' (2003) double auction.

Section 3 shows by means of various examples how underemployment can arise and survive along the solution paths of our dynamics. Remember that this holds even though all the short-run prices (including wages) are perfectly flexible. Even more strikingly, this section shows that a non-zero natural rate of unemployment is compatible with Pareto optimality and, conversely, that full employment is compatible with sub-optimality.

In section 4 we prove that, under weak conditions, our price-quantity dynamics with production admits solution paths, even when static Arrow-Debreu equilibria fail to exist. We then prove that each such trade and production path converges to (the analogue of) some infinitesimal price equilibrium provided there are no unsold inventories at the limit. Infinitesimal Pareto optimality can be turned into (standard) Pareto optimality whenever long-term returns to scale are strongly decreasing.

Section 5 concludes.

2 The model

2.1 The fundamentals

The long-run economy \mathcal{E} is defined by one consumption commodity, one type of labor, $N \geq 1$ households, $i = 1, \dots, N$, and $M \geq 0$ producers j .⁶

To each household i is associated a long-run trading set $X_i = (\{-a_i^L\} + \mathbb{R}_+) \times \mathbb{R}_+$, where $a_i^L \in \mathbb{R}_+$ represents some (institutional or physical) bound on labor for household i (e.g., 24 hours in a day). If $a_i^L \leq x_i^L < 0$, then household i provides a non trivial amount of work; if $x_i^L > 0$, then i is hiring the service of somebody else. The function $u_i : X_i \rightarrow \mathbb{R}$ is the (long-term) utility function of agent i . The vector $\omega_i = (\omega_i^L, \omega_i^C) \in X_i \setminus \{0\}$ is i 's initial endowment, and $\omega := \sum_i \omega_i$.

To each producer j is associated a (closed) production set $Y_j \subset \mathbb{R}^2$ and an initial state of production $\bar{y}_j \in \partial Y_j$, where ∂Y_j is the boundary of Y_j .

DEFINITION 2.1

- The set of feasible states is

$$\tau := \left\{ (x_i, y_j) \in \prod_i X_i \times \prod_j \partial Y_j \mid \sum_i x_i = \sum_j y_j + \omega \right\} \subset (\mathbb{R}^2)^{N+M}.$$

- For every i : $\hat{X}_i \subset X_i$ is the (canonical) projection of τ over X_i ; and $\hat{X}_i^* \subset \hat{X}_i$ is the subset of consumption plans that are individually rational with respect to i 's long-run preferences. Thus,

⁶Our results easily extend to the more general set up of economies with several commodities and various types of labor.

$$\hat{X}_i^* := \{x_i \in \hat{X}_i \mid u_i(x_i) \geq u_i(\omega_i)\}.$$

- τ^* denotes the subset of feasible and individually rational states:

$$\tau^* := \{(x, y) \in \tau \mid x_i \in \hat{X}_i^* \forall i\}.$$

- The configuration space of our dynamics is the Cartesian product $\tau^* \times \mathbb{R}_+^M$ of feasible and individually rational states $(x(t), y(t))$ and unsold inventories $s(t)$.

ASSUMPTION (C)

- (i) For every type i , the restriction $u_{i|\hat{X}_i^*}(\cdot)$ of $u_i(\cdot)$ to the subset \hat{X}_i^* is \mathcal{C}^1 and quasi-concave. Moreover, $\nabla u_{i|\hat{X}_i^*}(\cdot) > 0$, that is, u_i admits no critical point on \hat{X}_i^* .
- (ii) There exists at least one consumer i such that $\omega_i^C > 0$ and, for every $x \in \tau^*$, there exists an i' such that $\frac{\partial u_{i'}}{\partial x^c}(x) > 0$.
- (iii) Let \check{X}_i denote $\partial X_i \cap \hat{X}_i^*$. Then, for each $x_i \in \check{X}_i$, we have $\nabla u_i(x_i) \cdot x_i > 0$.

Regarding part (i), labor enters negatively in the utility function, whose monotonicity therefore corresponds to the utility of leisure. No condition is imposed on utility functions to keep our dynamics away from the boundary. Part (ii) imposes that no feasible and individually rational state be a collective satiation point. Part (iii) implies that no household admits satiation points on the boundary of its long-run consumption set intersected with its subset of feasible and individually rational bundles. All these assumptions are fairly standard and weak.

ASSUMPTION (P). For each j , the frontier ∂Y_j is a \mathcal{C}^1 submanifold of \mathbb{R}^2 that verifies the following claim: For each $y_j \in \partial Y_j$, the tangent space $T_{y_j} \partial Y_j$ is such that

$$T_{y_j} \partial Y_j \cap \left[\{y_j\} + \left(\mathbb{R}_+ \times (\mathbb{R}_+ \setminus \{0\}) \right) \right] = \{y_j\}. \quad (1)$$

LEMMA 2.1. Under (P), τ is a $2(N-1)$ -dimensional \mathcal{C}^1 -submanifold with corners of \mathbb{R}^{2N} .

Proof. It suffices to check that,⁷ in $X \times (\mathbb{R}^2)^M$, the submanifolds defined by the Y_j and the market-clearing condition $\sum_i x_i = \sum_j y_j + \omega$ all intersect transversally. So fix $x \in X$ and $y \in \prod_j \partial Y_j$ verifying the market-clearing equation. Let $\dot{y}_j \in T_{y_j} \partial Y_j$ ($j = 1, \dots, M$) be given.

If x belongs to the (relative) interior of X , then the $\dot{x}_i \in T_x X = \mathbb{R}^2$ satisfy the single equation $\sum_i \dot{x}_i = \sum_j \dot{y}_j$ with exactly $LM - L$ degrees of freedom.

⁷Here, $X := \prod_i X_i$.

This proves that the set of $(\dot{x}, \dot{y}) \in (\mathbb{R}^2)^N \times \prod_j T_{y_j} \partial Y_j$ with $\sum_i \dot{x}_i = \sum_j \dot{y}_j$ has dimension equal to $LM + \sum_j \dim \partial Y_j - L$.

If x belongs to a vertex of X , then the conclusion follows because such a vertex can be written as a submanifold with corners. \square

REMARK 2.1

(a) The restriction imposed on production sets guarantees that no (infinitesimal) free lunch is available. Observe nonetheless that the usual formulation of no free lunch in linear production economies requires, when put into the language of marginal economies,

$$T_{y_j} \partial Y_j \cap [\{y_j\} + \mathbb{R}_+^2] = \{y_j\}, \quad (2)$$

which is more demanding than (1). We use (1) because, unlike (2), it allows for “horizontal tangent planes” (i.e., faces of the form $\{y_j\} + \mathbb{R}_+^L \times \{0\}$). Indeed, when reinterpreted in terms of the long-run production set Y_j , such horizontal tangent planes correspond, for example, to fixed costs.

(b) The constant returns to scale technology of each firm j at $y_j \in \partial Y_j$ in the short term can be characterized by a $L \times C$ input matrix $A_j \in \mathcal{M}_{L \times C}(\mathbb{R})$. Then (P) amounts to the standard assumption that, for all $z \in \mathbb{R}^{L+C}$:⁸

$$A_j z \geq 0 \iff z = 0 \in \mathbb{R}.$$

(c) It is worth noticing that we allow the feasible set τ to have a zero Euler characteristic. In other words, some production sets may contain “holes”. Remember that the presence of “holes” is ruled out either by the assumption of convex production or, in the non-convex case, by the familiar free-disposal assumption. In any case, as can be expected from the role played by acyclicity in fixed-point theory, holes may be detrimental to the existence of marginal cost equilibria (cf. the torus example of Salchow 2003). As a consequence, our dynamic approach enables us to describe the behavior of the state of an economy that admits no competitive equilibrium.

ASSUMPTION (B). τ^* is compact.

Hurwicz and Reiter (1973) provide sufficient conditions on the (possibly non convex) production sets Y_j that ensure boundedness of τ . Observe that Assumption (B) is not necessary for proving existence of a static Walras equilibrium in convex economies, as shown by Debreu (1962).⁹ It is the unique restrictive assumption imposed by our dynamic framework. However, it is widely used for proving existence in non-convex production economies¹⁰ and has strong economic

⁸Where z is the projection of $z \in \mathbb{R}^{L+C}$ to \mathbb{R} .

⁹Hurwicz and Reiter (1971) provide an example of an economy verifying Debreu’s (1962) sufficient conditions for existence and where τ is not compact.

¹⁰Cf., e.g., Bonnisseau and Cornet (1988) and Giraud (2001).

appeal as a mathematical translation of the scarcity of economic resources.¹¹

REMARK 2.2. What is a “production set”? The classical study of static economies with production has mainly restricted attention to a static object encompassing all the production plans that are instantaneously technologically feasible, called the “production set”, Y_j (Debreu 1951; Arrow & Debreu 1974). Even in intertemporal economies, where commodities are indexed by their date of delivery, a producer contemplating its production set has in view all the production plans that will be feasible in the future. This way of modeling production without reference either to the past history of production or to (unforeseen) technological progress needs to be modified in order to fit with our dynamic viewpoint. Here, we propose a new mathematical object as a stylized representation of technological possibilities in a dynamic environment. This object is the tangent bundle $T\partial Y_j$ of (the boundary of) each “production set”. This fiber bundle should be understood as a description of all the possible *histories of production* that are technologically feasible, taking into account (at each point in time) the past history and the possible innovation due to technological progress.

We therefore modify the commonly accepted interpretation of the set Y_j in order to deal with such issues. Its boundary should no longer be understood as the set of static efficient production plans that can be faced a priori from a timeless viewpoint but rather as the space of vector fields where the marginal productivity at each instant t depends upon the current point $y_j(t) \in \partial Y_j$, that is, upon the quantity of inputs $y_j^-(t)$ already destroyed in past production processes.

Of course, when (part) of this input was devoted to R&D, an increase in the marginal productivity at t could be seen as resulting from technological progress. On the other hand, in this context the non convexities in ∂Y_j can also be interpreted as *learning effects*: as time goes by, more know-how is accumulated and so marginal productivity increases. This new interpretation implies, naturally, that a production process (i.e., a path along ∂Y_j) *must* be irreversible: a destruction of output does not amount to reversing the arrow of time.

We shall use simple examples to illustrate how technical progress can be incorporated as an endogenous variable in our approach. Consider first the textbook case of a CRS production function per capita with exogenous technical progress:

$$f(k, t) = e^{gt} Ak,$$

where $A, g > 0$. Here, the marginal productivity of the capital per capita at time t , being simply equal to $e^{gt} A$, is evidently independent of the cumulative amount of capital invested earlier. Next, consider the following production function:

$$f(k(t)) = e^{k(t)} k(t).$$

¹¹In Giraud and Rochon (2007), however, this restriction is dropped. Allowing for non-compact feasible set is then shown to open the door for endogenous growth within a general equilibrium framework.

In this case, capital's marginal productivity per capita at time t is $(1 + k(t))e^{k(t)}$, and thus depends upon the cumulative capital $k(t)$ invested at time t , while returns to scale are strictly increasing. Now, a more interesting example follows by considering an analog of the celebrated Cobb Douglas example:

$$f(k(t)) = e^{k(t)}k^\alpha(t),$$

with $\alpha \in (0, 1)$. Here, returns to scale vary qualitatively over time, depending on the investment policy of the firm. If the firm does not invest enough capital so as to reach the level $\sqrt{\alpha} - \alpha$ in finite time, then it will never experience an increase in marginal productivity. The speed of investment (i.e., the magnitude of infinitesimal investments $\dot{y}^-(t)$ in each marginal economy to be defined) will be responsible for the time at which the aforementioned threshold will be attained.

Underlying our approach is the (Classical) distinction between long-term and short-term decisions. *Long-term* decisions, such as capital budgeting or the choice of capital structure, usually involve long-lived assets and liabilities and are not easily reversed, so they commit the firm to a particular course of action for several years. We regard the shape of the production set Y_j of firm j as the result of a long-term investment decision, which is not modeled in this paper, but will be taken as exogenously given. *Short-term* decisions, by contrast, amount to moving along the boundary ∂Y_j . They will be the central focus of this paper.

2.2 Infinitesimal trades and production plans

Households and firms are assumed to act as independent players (dropping this assumption would force us to address the problem of the firms' objectives, which is left for further research).

Long-term production possibilities are allowed to exhibit increasing returns to scale as well as more general non convexities. However, each first-order approximation will, by definition, involve constant returns to scale; hence short-run profit maximization is well-defined. As a result, and at variance with current practice in static general equilibrium theory (GET), there is no need for defining any ad hoc behavior of regulated non-convex firms.¹²

At each instant t , consumers and firms meet on marginal markets. Infinitesimal trades $\dot{x}(t)$ and production plans $\dot{y}(t)$ proceed in the direction indicated by Mertens' (2003) limit price mechanism applied to the marginal economy (to be defined shortly). In this way, a cone field is defined that almost everywhere reduces to a vector field and thus to an ordinary differential equation (ODE) whose (maximal) solutions will be the trade and production paths of our dynamics. By construction, every solution path is such that producer j moves along the boundary ∂Y_j of his production set while every consumer moves within her consumption set.

¹²See Cornet (1988) and the references therein for a survey of the various pricing rules that can be attributed to non convex firms in GET.

Consumers. At date t , if the state of the economy is $(x(t), y(t), s(t)) \in \tau^* \times \mathbb{R}_+^M$ (where $s(t)$ represents unsold inventories, to be defined below), household $i = 1, \dots, N$ is characterized in the marginal economy $T_{x(t), y(t), s(t)}\mathcal{E}$ by its infinitesimal trade space, $(\{-a_i^L - x_i^L(t)\} + \mathbb{R}_+) \times (-x_i^C(t) + \mathbb{R}_+)$ and its (infinitesimal) initial endowment of zero. A consumer is allowed to go short in each marginal economy, but her short-sales are bounded by her current endowment both in consumption, $x_i^C(t)$, and in labor, $a_i^L + x_i^L(t)$. The consumer is characterized by a marginal utility that is given by¹³

$$b_i(t) := \frac{\nabla_x u_i(x_i(t))}{\|\nabla_x u_i(x_i(t))\|_{\ell_1}}. \quad (3)$$

At time t , the infinitesimal trade of agent i is $\dot{x}_i(t)$. In $T_{x(t), y(t), s(t)}\mathcal{E}$, household i 's (myopic) behavior consists of solving

$$\max b_i(t) \cdot \dot{x}_i(t)$$

$$\text{s.t. } \dot{x}_i(t) \geq (-a_i^L - x_i^L(t), -x_i^C(t)), \quad p(t) \cdot \dot{x}_i(t) \leq 0.$$

Producers. Given the state $y_j(t) \in \partial Y_j$ (induced by the history of past production), each firm $j = 1, \dots, M$ is characterized by the constant return marginal (i.e., short-run) production set $T_{y_j(t)}\partial Y_j$. The infinitesimal production plan of firm j will be denoted $\dot{y}_j(t) = (\dot{y}_j^-(t), \dot{y}_j^+(t)) \in T_{y_j(t)}\partial Y_j$, where $\dot{y}_j^-(t)$ is the increment of investment at date t and $\dot{y}_j^+(t)$ the (infinitesimal) quantity of output produced out of $\dot{y}_j^-(t)$. In order to capture the feature that “production takes time”, we assume that $\dot{y}_j^+(t)$ is not instantaneously available on the market at time t but rather will be *put on the market at time $t + dt$* . At time t , j 's unsold inventories are denoted $s_j(t)$ with $s_j(0) = \bar{y}_j$. Firm j solves the following profit maximization program with respect to its marginal production set:

$$\max p(t) \cdot \dot{y}_j(t)$$

$$\text{s.t. } \dot{y}_j(t) \in T_{y_j(t)}Y_j, \quad p(t) \cdot \dot{y}_j^-(t) \leq p(t) \cdot s_j(t). \quad (4)$$

Such behavior requires that j be myopic in two senses: (i) j takes only into account $T_{y_j(t)}Y_j$, the first-order approximation of Y_j at $y_j(t)$ (instead of optimizing over the whole set Y_j as in classical GET) and (ii) j assumes constant prices p in t and $t + dt$. Indeed, recall that the output $\dot{y}_j^+(t)$ will be effectively sold at the price $p(t + dt)$. Here, bounded rationality thus takes the form of trivial price expectations (“between” t and $t + dt$!) on the part of producers. Moreover, for positive prices $p(t) \gg 0$, the constraint $p(t) \cdot \dot{y}_j^-(t) \leq p(t) \cdot s_j(t)$ in (4) provides a bound on the subset of attainable infinitesimal production plans: it guarantees that the program just described has a solution even when $p(t)$ is not normal to Y_j at y_j . The constraint itself can be best understood when cast as a rule of the limit-price mechanism. We shall therefore return to it when discussing the mechanism itself.

¹³ $\|x\|_{\ell_1} := \sum_k |x_k|$ is the ℓ_1 -norm.

Let $\beta_j(t)$ be the quantity of output effectively sold at time t according to the limit-price mechanism (which will be defined later). Clearly,

$$0 \leq \beta_j(t) \leq s_j(t).$$

At this stage it suffices to say that, when $\beta_j(t) < s_j(t)$, the firm is rationed (more on this to follow). The current production plan $y_j(t) \in \partial Y_j$ at time $t \geq 0$ is given by:

$$y_j(t) := \left(\left(\int_0^t \dot{y}_j^-(s) ds, s_j(0) + \int_0^t y_j^+(s) ds \right) \right),$$

and the dynamics of unsold inventories is given by the ODE

$$\dot{s}_j(t) = \dot{y}_j^+(t) - \beta_j(t), \quad s_j(0) = \bar{y}_j.$$

In our dynamic setting, the marginal (i.e., short-term) profit of a firm is given by the scalar product $\dot{y}_j(t) \cdot p(t)$ of its current production increment $\dot{y}_j(t)$ by current prices $p(t)$. In particular, the product $y_j(t) \cdot p(t)$ no longer has any economic meaning.

REMARK 2.2. In the set up of Remark 2.1, the infinitesimal production plan $\dot{y}_j(t)$ can be expressed as

$$\dot{y}_j(t) := (\dot{y}_j^-(t), A_j \dot{y}_j^-(t)).$$

REMARK 2.3. Let us briefly recall the constraint (4):

$$p(t) \cdot \dot{y}_j^-(t) \leq p(t) \cdot s_j(t).$$

As we shall see in the following examples, this constraint is responsible for most of the peculiarities of our dynamics; (4) is a *feasibility* constraint on the consumption good market. Recall that, in a standard Shapley Shubik strategic market game (or Shapley windows model),¹⁴ no player is allowed to put on a trading post (resp. send to the clearing house) an amount of commodity exceeding her endowment. Indeed, when offering something for sale, a player must be able to physically show that she indeed possesses what she offers. Similarly, in Mertens's mechanism, no player is allowed to send a sell order whose associated quantity exceeds her current endowment. When the player entering such a (stylized) market is a firm offering for sale an output of its production technology, there is no reason to allow this firm to sell a product that it does not already physically possess.¹⁵

¹⁴See, e.g., Giraud (2003).

¹⁵Companies selling houses before they are built are also obliged to collateralize their offers.

2.3 The marginal economy

The marginal economy $T_{x(t),y(t),s(t)}\mathcal{E}$ is defined by

$$((b_i(t), x_i(t), a_i^L)_{i=1,\dots,N}, (T_{y_j(t)}\partial Y_j, s_j(t))_{j=1,\dots,M}).$$

To each $T_{x(t),y(t),s(t)}\mathcal{E}$ is associated a unique order book \mathcal{O} corresponding to the market and limit-price orders sent by the agents of $T_{x(t),y(t),s(t)}\mathcal{E}$ to the (marginal) market. According to Mertens's (2003) path breaking viewpoint, this order book \mathcal{O} can be interpreted as a linear economy where "agents" are orders, "utilities" are limit prices, and "endowments" are quantities put up for sale. (All the technical details can be found in the Appendix). To be more precise, \mathcal{O} is populated by $N + M$ "agents" of whom the N first "agents" correspond to orders sent by consumers and the M last to orders sent by producers. In \mathcal{O} , "agent" $i = 1, \dots, N$ is characterized by the "utility" given by $b^i(t)$, the "short-sale bounds" $x_i(t) + (a_i^L(t), 0)$, and 0 as "initial endowments". "Agent" $j = N + 1, \dots, M$ is characterized by $N_{y_j(t)}\partial Y_j$ (the unit vector normal to ∂Y_j at $y_j(t)$) as linear "utility", $s_j(t)$ as "short-term bound", and 0 as "initial endowments". In the sequel, "agents" in \mathcal{O} will be denoted h , with "utility" b^h and "short-sale bound" e^h .

To help the reader distinguish between real players and fictitious "agents" (representing orders sent by the former), we shall describe the latter with quotation marks. Hence, an "agent" means an *order*, a "utility" is a *limit price* (denoted by a vector b^h), and a "short-sale bound" is a *quantity put up for sale* (denoted by e^h). Superscripts are reserved for (fictitious) "agents" and subscripts for real households.

For simplicity, we assume that an order disappears immediately after it has been sent, whether or not it could be executed. This is innocuous because, whenever the corresponding player still wants to send the same order at time $t + dt$, she need only resend it.

We now define the marginal (short-term) *outcome* that will be induced by a collection of order books sent by players in the marginal economy $T_{x,y,s}\mathcal{E}$. This outcome will provide the direction in which the state of the underlying economy \mathcal{E} , starting from $(x(t), y(t), s(t))$, will move at time t in the configuration space.

2.4 The marginal outcome

Infinitesimal trades cannot be formally analyzed without referring to a specific micro-structure. The trading paradigm used here is Mertens's (2003) limit-price mechanism. Suppose, to fix ideas, that our unique firm has constant returns to scale equal to 2 and that the unique consumer has linear preferences with a marginal rate of substitution equal to 1. Both players enter the market for labor and consumption by playing a strategy in Mertens's strategic market game. As it

is traditional in this literature,¹⁶ no player can offer more than what she actually possesses. As a consequence, at time t , the firm can put up for sale not the output of the current, period t -investment, but rather the output induced by the investment made at time $t - dt$.¹⁷ Therefore, the producer will act, within our strategic market game, as a trader willing to buy labor against consumption and whose initial endowment is simply the output of period $t - dt$. Hence, unless the initial endowment of each firm is exactly equal to the output prescribed by a Walras equilibrium, there is no reason for the outcome of the limit-price mechanism to coincide with such a competitive equilibrium.

Take an order book \mathcal{O} as given. The set of feasible trades in the corresponding fictitious linear economy is

$$\tau_{\mathcal{O}} := \left\{ (\dot{x}, \dot{y}) \in \mathbb{R}^{2(N+M)} \mid \sum_{i=1}^N \dot{x}^i + \sum_{j=1}^M \dot{y}^j = 0, \dot{x}^i \geq -x_i(t) + (-a_i^L, 0), \dot{y}^j \geq (0, s_j(t)) \right\}.$$

DEFINITION 2.2. A *pseudo-outcome* of \mathcal{O} is a price system $p \in \mathbb{R}_+^2 \setminus \{0\}$ and a trade $\dot{x} \in \tau_{\mathcal{O}}$ for which the following statements hold.

- (i) For every “agent” h , $p \cdot b^h = 0$ implies $\dot{x}^h = 0$.
- (ii) For every h , $\dot{x}^h \in \mathbb{R}_+^2$ maximizes $b^h \cdot \dot{x}$ subject to the (infinitesimal) budget constraints:

$$p \cdot \dot{x}^h \leq 0 \text{ and } (p_c = 0 \Rightarrow \dot{x}_c^h = 0). \quad (5)$$

- (iii) For commodity c , if $p_c = 0$ then $p \cdot e^h > 0$ implies $b_c^h = 0$.

DEFINITION 2.3. A pseudo-outcome is said to be *proportional* if there is a pair of weights μ_{CL}, μ_{LC} with $\mu_{CL} + \mu_{LC} > 0$ such that all “agents” whose demand set at p includes both commodity C and labor L receive them in the same proportions μ_{CL} and μ_{LC} . The *demand set* of i at price p is

$$\delta_p^i := \left\{ \ell \in \{C, L\} \mid p_\ell \leq r(b_i, \ell, k)p_k \quad \forall k \in \{C, L\} \right\},$$

with $r(b_i, \ell, k) := b_\ell^i / b_k^i$ denoting the marginal rate of substitution between ℓ and k .¹⁸

The trading procedure in the limit-price mechanism is then defined as follows: For an order book \mathcal{O} , choose a proportional pseudo-outcome and settle the corresponding infinitesimal trades. Mertens (2003) shows that there always exists a unique proportional pseudo-outcome allocation and that the corresponding price

¹⁶Cf. the 2003 special issue of *Journal of Mathematical Economics*.

¹⁷This is consistent with Definition 2.2 (ii).

¹⁸With the convention $b/0 := 0$.

is also unique provided the final allocation induces non-trivial trades. Let us call this final allocation, together with some of its corresponding positive prices, a *marginal outcome* of \mathcal{O} .

In sum: To each state (x, y, s) in \mathcal{E} corresponds a linear marginal economy $T_{x,y,s}\mathcal{E}$, and to each such $T_{x,y,s}\mathcal{E}$ there corresponds an order book \mathcal{O} . The direction $(\dot{x}, \dot{y}, \dot{s})$ in which the state of \mathcal{E} evolves is given by the marginal outcome of \mathcal{O} . For $(x, y, s) \in \tau^* \times \mathbb{R}_+^M$, we denote by $\text{MO}(x, y, s)$ the subset of marginal outcomes of the economy \mathcal{E} in state (x, y, s) .

REMARK 2.4. One might be tempted to define a short-term outcome $(\dot{x}(t), \dot{y}(t))$ of the short-term economy $T_{x(t),y(t),s(t)}\mathcal{E}$ simply (and more directly) as a Walras equilibrium of the linear economy $T_{x(t),y(t),s(t)}\mathcal{E}$. However, doing so would be problematical for at least two reasons: (i) it would leave unmodeled all the situations of interest where the marginal economy fails to admit any competitive equilibrium; and (ii) it would lead to indeterminacy in every circumstance where there are multiple short-run Walras equilibria.

REMARK 2.5. In the simplistic situation considered in this paper, firms reinvest (in each period t) the total profit gained from selling $\beta_j(t)$, after buying input $y_j^-(t)$. Hence, a firm's instantaneous profit is always zero. On the other hand, we show by means of examples that, in the short run, a firm may sell outputs at a higher price than its current marginal cost of production. This means that the zero profit property of production trajectories cannot be attributed to the usual Bertrandian argument, according to which the selling prices of the perfectly competitive firms should equal marginal cost of production.

A positive by-product of our approach is that it circumvents the difficult issue of how to finance (non convex) firms' deficits at equilibrium. Indeed, as shown by Guesnerie (1975), non convex firms following, say, the marginal pricing rule (captured e.g. through Clarke's normal cone) may incur a deficit at equilibrium. As long as the equity market is not open (so that portfolios of equities are not endogenously defined, enabling shareholders to refuse to finance a firm's deficit by selling its stocks), it hardly makes sense to consider privately held firms incurring a deficit at equilibrium. One is then usually led to consider public or at least mixed ownership structures.¹⁹ Here, the issue vanishes because the instantaneous profit of firm j at time t is zero even when long-run returns to scale are increasing.

2.5 Some special cases

We now examine two specific cases of linear fictitious economies associated to an order book that will play a major role in our dynamic theory.

CASE 1. Consider a 2×2 fictitious economy represented by the Edgeworth box of Figure 2.5.1. Since the "agents" of a fictitious economy exhibit linear preferences, their indifference curves are straight lines. Here, the indifference

¹⁹See e.g. Dall'olio and Vohra (1999) on this topic.

curves of the “agents” coincide and the initial endowment w is Pareto optimal. However, the outcome E of the limit-price mechanism does not correspond to w and thus involves trades.

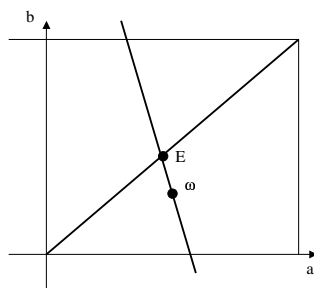


Fig. 2.5.1

This property, which is specific to the proportional rule, will be responsible for several features of our dynamics:

1. The vector field will be *discontinuous*,²⁰ so we may rely on Filippov’s (1988) solutions.²¹
2. Even if the true economy crosses an infinitesimally Pareto optimal point (in which case the marginal economy looks like the one depicted in Figure 2.5.1), non trivial infinitesimal trades will occur.
3. This property of the proportional rule will lead to short-run underemployment, as shown in Example 1 of Section 3.

Infinitesimally Pareto optimal points turn out to be compatible with underemployment. In fact, Example 1 of Section 3 shows that it is possible for an economy to converge to full employment even though its starting point was already infinitesimally Pareto optimal. With this in mind, feature 2. should rather be interpreted as a positive property. On the other hand, Examples 2 and 3 of Section 3 show that underemployment arises in our dynamics even when the proportional rule is not at work. Thus, the proportional rule is not solely responsible for involuntary unemployment; it is but one of the various causes of the failures of labor markets.

CASE 2. Consider a fictitious linear economy such as that depicted in Figure 2.5.2, with no Walrasian equilibrium and for which the initial endowment ω is not Pareto optimal. In order to improve the situation of “agent” 1, one must take from “agent” 2 some good that has zero utility for him and then give it to “agent” 1. The limit-price mechanism selects the no-trade solution, so the final outcome is not Pareto optimal. As a consequence, solution paths of our dynamics can be trapped at points that are not Pareto-optimal (this was the qualification alluded to at the end of Section 1). However, the no-trade solution is Pareto

²⁰This was already the case in the pure-exchange situation; see Giraud (2007).

²¹Filippov’s definition is recalled in the Appendix, Section 6.3.

optimal when computed in strict trades (as defined in Section 4.1). This will enable us to prove convergence of our solution paths to a weaker notion than mere Pareto optimality.

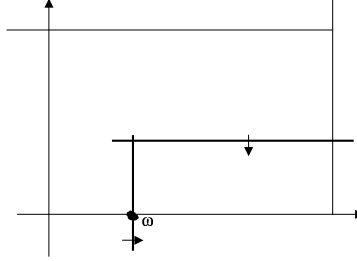


Fig. 2.5.2

2.6 The dynamics

Wrapping up everything, we define a *marginal equilibrium* $(\dot{x}(t), \dot{y}(t), \dot{s}(t), p(t))$ of the economy $T_{x(t), y(t), s(t)} \mathcal{E}$ at time t as being the marginal outcome of its associated order book $\mathcal{O}(t)$.

Recall that for $(x, y, s) \in \tau^* \times \mathbb{R}_+^M$ we use $\text{MO}(x, y, s)$ to denote the subset of marginal outcomes of the economy \mathcal{E} in state (x, y, s) . The sets $M(x, y, s)$ and $P(x, y, s)$ will denote the corresponding marginal quantity outcomes and marginal prices respectively:

$$M(x, y, s) := \left\{ (\dot{x}_1, \dots, \dot{x}_N, \dot{y}_1, \dots, \dot{y}_M) \in \mathbb{R}^{2(N+M)} \mid \exists p \in \mathbb{R}_+^2 : \right. \\ \left. (\dot{x}_1, \dots, \dot{x}_N, \dot{y}_1, \dots, \dot{y}_M, p) \in \text{MO}(x, y, s) \right\},$$

$$P(x, y, s) := \left\{ p \in S_+^2 \mid \exists (\dot{x}, \dot{y}) \in \mathbb{R}^{2(N+M)} : (\dot{x}, \dot{y}, p) \in \text{MO}(x, y, s) \right\}.$$

A *trade and production path* $(x(\cdot), y(\cdot), p(\cdot))$, is a maximal Filippov solution²² of the differential inclusion

$$(\dot{x}(t), \dot{y}(t)) = M(x(t), y(t)) \quad \text{and} \quad p(t) \in P(x(t), y(t)) \quad \text{a.e. } t \geq 0, \quad (6)$$

with $x(0) = \omega$, and $y(0) = \bar{y}$.

²²The definition of such a solution is presented in the Appendix.

3 Examples of underemployment

Let us first propose various understandings of underemployment. We start by distinguishing three types of demand: (α) *market demand* for good c , denoted $MD_c(p)$, which is the aggregate demand observed on the market (through the aggregate limit orders) at every vector p ; (β) *maximal Walrasian demand*, $WD_c(p^*)$, which is the maximal (in terms of volume) demand satisfied at some Walrasian equilibrium price p^* (when it exists) of the fictitious linear economy induced by some order book; and (γ) *realized demand*, $RD_c(\bar{p})$, which is the demand satisfied at a marginal outcome induced by the proportional rule applied to some order book; here \bar{p} is the marginal (short-term) price selected by the limit-price mechanism. Obviously, the three definitions do not coincide in general: As soon as some demand orders are not fully executed (or, equivalently, as soon as some player is rationed), $MD_c(p^*)$ will be larger than $WD_c(p^*)$ (provided this latter object is defined). As soon as the proportional rule comes into play, $WD_c(\bar{p})$ will itself be larger than $RD_c(\bar{p})$. Note that $MD_c(\cdot)$ and $RD_c(\bar{p})$ are always defined, whereas $WD_c(p^*)$ may fail to exist.

Given these definitions, we can similarly define three different kinds of labor supplies (for each type of labor ℓ): the market supply $MS_\ell(p)$, the maximal Walrasian supply $WS_\ell(p)$, and the realized supply $RS_\ell(p)$. These lead to the following definitions of underemployment:

(α) *Walrasian underemployment*, $WU(p)$, is $MS_\ell(p^*) - WS_\ell(p^*) \geq 0$ at some Walrasian equilibrium price p^* of the linear fictitious economy induced by some order book;

(β) *realized underemployment*, $RU(\bar{p})$, is the difference $WS_\ell(\bar{p}) - RS_\ell(\bar{p}) \geq 0$ at the marginal price \bar{p} (supposed to be also a Walrasian equilibrium price);

(γ) *market underemployment*, $MU(p)$, is the sum $WU(p^*) + RU(\bar{p})$.

We shall see that all three types of underemployment emerge along trade and production paths of our dynamics. Clearly, $RU(\bar{p})$ is entirely due to the proportional rule. Given a trade and production path, we define the natural rate of unemployment (NRU) as the limit of market underemployment $MU(p(t))$ as $t \rightarrow \infty$. Being the limit of the observed long term underemployment, the NRU is affected by the history of observed underemployment. In this sense, hysteresis is accounted for.

3.1 Example 1: Realized “Classical” underemployment.

We assume that there is a single producer j facing a single consumer i . The production function of j is given by $f_j(y_L) = -y_L + 1$ with marginal productivity of labor $MPL_j = 1$. Hence $Y_j = \{(y_L, y_C) \in \mathbb{R}_- \times \mathbb{R} : y_C \leq 1 - y_L\}$. The starting

inventory is $s_j(0) = \bar{y}_j = (0, 1)$. The consumer's utility is linear and equal to $u(x_L, x_C) = x_C + x_L$; also, $(\omega_L, \omega_C) = 0$, and $a_i^L = 10$.

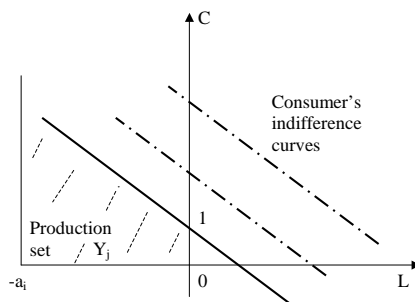


Fig. 3.1.1. Long-run economy \mathcal{E}

• **At $t = 0$:** At time $t = 0$, associated to $T_{0,0,(0,1)}\mathcal{E}$ is the order book \mathcal{O} , which is generated by the limit-price orders of the limit-price mechanism, as described in Section 2. The underlying 2×2 Edgeworth box is represented in the next figure. The “endowment” of the (fictitious agent representing the limit-order sent by the) producer is $e^1 = (0, 1)$, and the “endowment” of the consumer is $e^2 = (10, 0)$. The consumer (resp. producer) sends exactly one limit-price order associated to the limit price equal to her (resp. its) marginal rate of substitution $MRS = 1$ (resp. marginal productivity of labor $MPL = 1$).

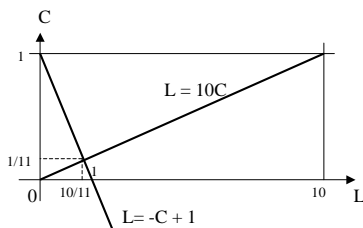
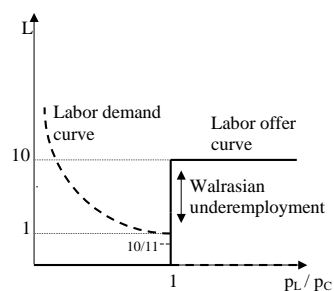
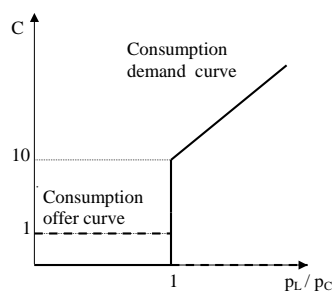


Fig. 3.1.2. Order book $t = 0$

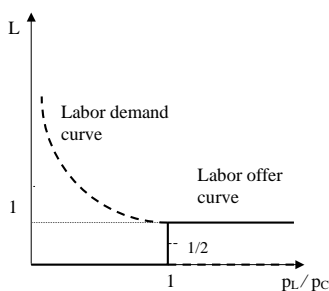
The line $L = -C + 1$ passing through the “endowment point” $(0, 1)$ corresponds to the common indifference curve of the “agents” in the fictitious marginal economy, and the proportionality rule selects the unique point at the intersection of the diagonal $L = 10C$ and the common indifference line; $(\bar{L}, \bar{C}) = (10/11, 1/11)$. The resulting price coincides with the unique (normalized) Walras equilibrium price of \mathcal{O} : $p^* = (1/2, 1/2)$. The unique Walrasian allocation selected by the limit-price mechanism among the continuum of Walrasian allocations is therefore $(10/11, 10/11)$. In other words, the consumer has sold $10/11$ of the ten units of labor put on the market. On the other hand, the producer has sold $10/11$ unit of (unsold) consumption commodity put on the market. The realized demand of labor of producer j is $10/11$ and his realized offer of goods is $10/11$. Thus, at date $t = 0$, we have $\dot{y}_j(0) = (-10/11, 10/11)$, and $\dot{x}_i(0) = (-10/11, 10/11)$. Market demand for C , MD_C , is here equal to 10, maximal Walrasian demand $WD_C = 1$, and realized demand $RD_C = 10/11$. Hence, both Walrasian and realized underemployment exist.

Fig. 3.1.3. Labor market $t = 0$ Fig. 3.1.4. Consumption market $t = 0$

As shown in the figures above, there is excess labor supply and excess consumption demand in the marginal economy $T_{0,0,(0,1)}\mathcal{E}$. The economy starts at $t = 0$ at a global and strict Pareto optimum; however, there are (infinitesimal) trades induced by the proportional rule of the limit-price mechanism. In other words, $(0, 1)$ is not a rest point of our dynamics.

• **At $t > 0$:** At time $t = dt$, the market labor offer is $10 - (10/11)dt$, with $p(dt) = p(0) = (1, 1)$. The indifference curve of the agents in the fictitious linear economy associated to \mathcal{O} at time dt is $L = -C + 1$, and the proportional rule defines the curve $L = \frac{1}{10 - (10/11)dt}C$, from which the marginal outcome can be computed. It is equal to $C = 1 - \frac{1}{11 - (10/11)dt}$. Hence, realized underemployment *increases* and is equal to $\left(1 - \frac{1}{11 - (10/11)dt}\right)^{-1}$.

Over time, the state of the economy evolves in a similar fashion along the boundary of the production set (or, equivalently, along the consumer's indifference line). The consumer's long-run utility remains constant. Let $t = T$ denote the time at which the "agent" of the corresponding marginal economy offers exactly one unit of labor. For any $t < T$, there is labor supply rationing as well as short-run Classical realized underemployment. At $t = T$ (see figure 3.1.5) there is no Walrasian underemployment any more but there is still realized underemployment by the proportional rule, according to which only half of the short-run labor supply is executed.

Fig. 3.1.5. Labor market $t = T$

At date $T + dt$, the firm is rationed. There is excess market labor demand and,

moreover,

$$\dot{y}_j(T + dt) = \left(\frac{1/2dt - 1}{2 - 1/2dt}, \frac{1 - 1/2dt}{2 - 1/2dt} \right).$$

We observe Walrasian excess consumption demand (in a broad sense). The economy therefore displays *contained inflation* with *realized underemployment*.

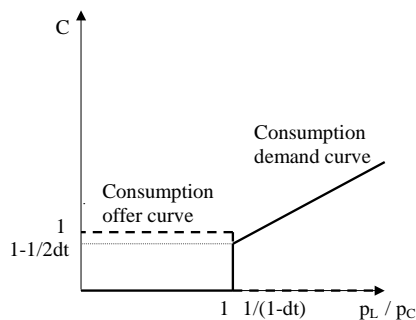
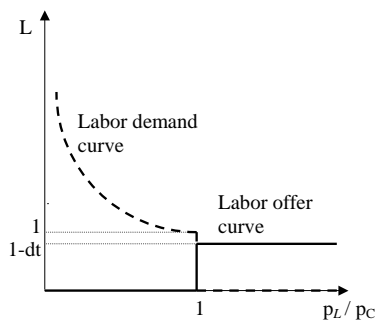


Fig. 3.1.6. Labor market $t = T + dt$ Fig. 3.1.7. Consumption market $t = T + dt$

At date $t > T$ (Figures 3.1.6 and 3.1.7), the economy moves slowly to $(-10, 10)$, that is, towards full capacity utilization of labor, reached at some date $t = \hat{T}$, where trades cease. As a consequence, $NRU = 0$. In this simplistic example, the unique solution path of our dynamics is therefore a straight line connecting $(0, 1)$ to $(-10, 11)$. Every point on the trajectory is a global strict Pareto optimum (and also, of course, a strict infinitesimal Pareto point), both prices and the consumer's long run utility remain constant. Hence, the producers' myopic price expectations turn out to be rational. This shows there may exist Pareto optima that are not steady states of our dynamics, and short-run underemployment is compatible with global Pareto optimality.

REMARK 3.1 If we assume in this example that there are no initial inventories (i.e., $s_j(0) = 0$), then the starting point becomes a steady state of our dynamics. It is a strict Pareto optimum and nevertheless $NRU > 0$.

3.2 Example 2: “Classical” underemployment

Take the same economy as before, except that now

$$\begin{aligned} Y_j &= \{(y_L, y_C) \in \mathbb{R}_- \times \mathbb{R} : y_C \leq 1 - \frac{1}{2}y_L\} \\ &\cup \{(y_L, y_C) \in [0, \epsilon) \times \mathbb{R} : y_C \leq -y_L^2 - \frac{1}{2}y_L + 1\} \\ &\cup \{(y_L, \epsilon) : y_L \leq -\epsilon^2 - \frac{1}{2}\epsilon + 1\} \end{aligned}$$

and $\bar{y}_j = (0, 1)$ (Figure 3.2.1).

At time $t = 0$, both the marginal productivity of labor at $(0, 1)$ ($1/2 = MPL < MRS_{CL}$) and the amount of inventories $s_j(0) = 1$ put up for sale are too weak: There is short-run underemployment and no trade takes place for all relative prices $p_L/p_C \in [MPL, MRS]$. There is a continuum $[MPL, MRS]$ of marginal (short-run) price vectors at $t = 0$. Thus, the example illustrates that our price dynamics may induce a continuum of final prices (although there is uniqueness along non degenerate trajectories, except at the limit; here, the problem clearly arises because the path is degenerate).

If $p_L/p_C = MRS$ emerges then there is market underemployment, which can be interpreted as Classical underemployment. The starting point is a steady state. Note that, because of the unsold inventories, it is not an infinitesimal Pareto optimum; $NRU > 0$. Here, evidently, a policy promoting the supply side should be implemented to increase the marginal productivity of labor MPL .

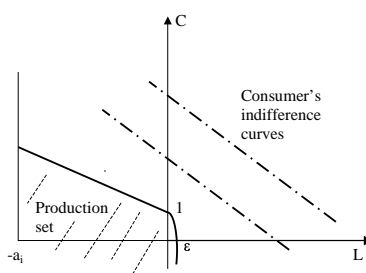


Fig. 3.2.1. Long-run economy \mathcal{E}

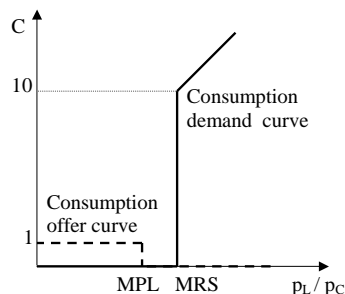


Fig. 3.2.2. Labor market $t = 0$

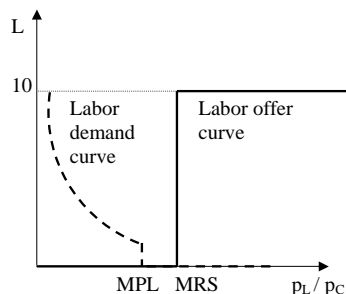


Fig. 3.2.3. Consumption market

$t = 0$

3.3 Example 3: “Keynesian” underemployment.

Now, let $f_j(y_L) = 2y_L + 1$, $s_j(0) = 1$, and $u(x_L, x_C) = x_C + x_L$ with $x(0) = 0$. Hence $Y_j = \{(y_L, y_C) \in \mathbb{R}_- \times \mathbb{R} : y_C \leq -2y_L + 1\}$. In this case, the marginal productivity of labor is too strong: $MPL > MRS_{CL}$. Consequently, there is underemployment at the marginal outcome of the marginal economy at time $t = 0$.

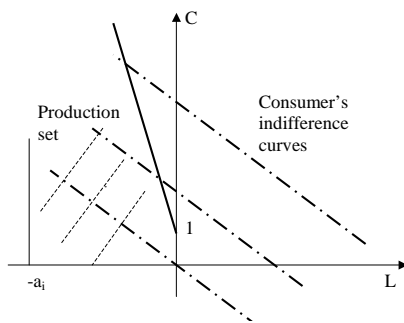


Fig. 3.3.1. Long-run economy \mathcal{E}

At time $t = 0$, the fictitious linear economy associated to the order book sent by the players in $T_{x(0),y(0),s(0)}\mathcal{E}$ admits a unique Walrasian equilibrium price $p_L^*/p_C^* = 1$; Figure 3.3.2 shows the Edgeworth box associated to this order book at time $t = 0$:

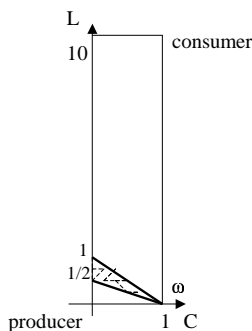


Fig. 3.3.2. Edgeworth box at $t = 0$

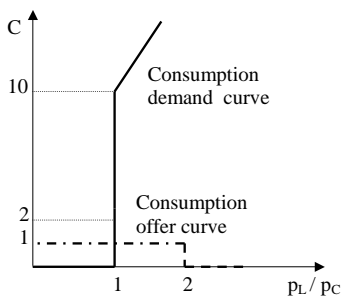


Fig. 3.3.3. Consumption market $t = 0$

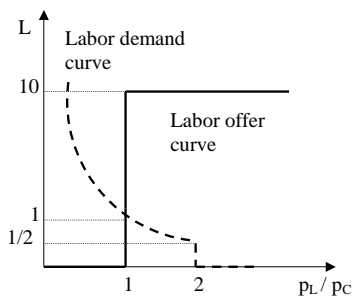


Fig. 3.3.4. Labor market $t = 0$

At time $t = 0$, the limit-price mechanism selects the price $(1, 1)$ together with $\dot{y}_j(0) = (-1, 2)$, $\beta_j(0) = (-1, 1)$, $\dot{x}_i(0) = (-1, 1)$, and $s_j(0) = dt$ ($\dot{s}_j(0) = 1$). Note that the proportional rule does not apply (see Figures 3.3.3 and 3.3.4). At time $t = 0$ there is market underemployment but no realized underemployment. At $t = dt$, we have $\dot{y}_j(dt) = (-1 - dt, 2 + 2dt)$, $\dot{x}_i(dt) = (-1 - dt, 1 + dt)$, $p_L(t)/p_C(t) = 1$, and $s_j(dt) = dt$.

There is market underemployment until some finite T ; at $t = T$, there is full market employment and $NRU = 0$. For $t > T$, firms are rationed on the labor market. Prices $p(t)$ increase from $p_L/p_C = MRS$ to $p_L/p_C = MPL$ and then stay equal to MPL . Let T^* denote the first time t for which $p(t) = MPL$. The trajectory of the dynamics is linear for $t \leq T$ and for all $t \geq T^*$. For $MRS < p_L/p_C < MPL$ (i.e. $T \leq t \leq T^*$), the path is nonlinear. The economy converges to a point in $\partial\tau$ with full employment, but it is neither a Pareto optimum nor an infinitesimal Pareto optimum because there remain unsold inventories. This example shows that full employment is compatible with Pareto suboptimality.

In our opinion, this example displays “Keynesian” underemployment (except at the limit) in the following sense. The effective consumption demand that is addressed to the firm on the consumption market is equal to 1 for $t = 0$ to $t = T$, whereas the consumption offer is at least $1 + dt$ for each $t > 0$. Hence there is a lack of effective demand, which prevents the firm from hiring more employees.

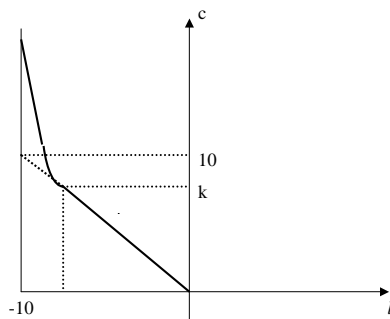


Fig. 3.3.5. The path

One may wonder why the (Walras) equilibrium price $(p_L, p_C) = (2, 1)$ is not selected by our mechanism as a marginal outcome of the economy at time $t < T$. The reason, as already suggested in section 1, is that the output produced at time $t - dt$ is too low for $(p_L, p_C) = (2, 1)$ to emerge. Were the producer able to put up for sale 20 units of consumption good rather than 1, then $(p_L, p_C) = (2, 1)$ would be the unique market-clearing price (see Figures 3.3.3 and 3.3.4).

4 Results

4.1 Preliminaries

In the next definition we consider an order book \mathcal{O} .

DEFINITION 4.1. In \mathcal{O} viewed as a linear fictitious economy, a *strict trade* is a feasible trade $\dot{x} \in \tau_{\mathcal{O}}$ such that, for all h , either $\dot{x}^h \geq 0$ or $b^h \cdot (\dot{x}^h + e^h) > 0$. Here b^h is the “utility vector” of “agent” h .

Extend functions $u_i : \tau \rightarrow \mathbb{R}$ ($i = 1, \dots, N$) by $u_i(x, y) := u_i(x_i)$.

DEFINITION 4.2

(1) An *admissible curve* is a smooth curve $\phi : [0, \varepsilon) \rightarrow \tau$ with $\phi(0) = (x, y)$ and $\frac{d}{dt}(u_i(\phi(t))) \geq 0$ for all $i, t \in [0, \varepsilon)$, with a strict inequality for at least one individual i , at every t .

(2) A *strict admissible curve* at x is an admissible curve that passes through x at some time $t \in (0, \varepsilon)$ and such that $\phi'(t)$ is a strict trade in the order book associated to $T_{x,y,s}\mathcal{E}$.

(3) $(x, y) \in \tau$ belongs to θ (resp. Θ) if and only if there is no admissible curve that passes through (x, y) (resp. no strict admissible curve at (x, y)).

(4) For $K \in \{\theta, \Theta\}$, K^* is the subset of points in K that are individually rational with respect to long-run utilities.

REMARK 4.1 Observe that, when $(x, y) \in \Theta$, the short-term equilibrium price fails to be unique (even after normalization). This is one reason why we end up with an inclusion instead of an equality in (8). Another reason is that, at infinitesimal Pareto allocations, the short-run outcome may induce price vectors that do not sustain this very Pareto point as a price quasi-equilibrium.²³ Infinitesimal Pareto points (computed with strict trades) are also the unique points where indeterminacy of prices occurs.

LEMMA 4.1. *Under (C)-(i)-(ii) and (P), a point (x, y) of τ belongs to θ if and only if there exist vectors $p \in \mathbb{R}_+^2$ and $\lambda \in \mathbb{R}_+^N$ such that $p, \lambda \neq 0$ and:*

- (a) $\lambda_i b_i \leq p$, $(p - \lambda_i b_i) \cdot (x_i + (a_i, 0_{\mathbb{R}})) = 0$ for all i ;
- (b) $p \in N_{y_j}(\partial Y_j)$, the normal half-line to ∂Y_j at y_j , for all j .

Proof. This follows from first-order conditions. (See e.g. Mas-Colell (1985, Theorem 4.3.1)), and take $(-a_i^L + \mathbb{R}_+) \times \mathbb{R}_+$ as the consumption set for household i .)

□

ASSUMPTION (C*). Assumptions (C) (i) and (ii) are in force; in addition,

$$\nabla u_{i|\hat{X}_i^*}(\cdot) \gg 0 \quad \text{for every } i.$$

ASSUMPTION (D). For each j and each $p \in S_+^2$, the map $f_p : Y_j \rightarrow \mathbb{R}$ (which sends $y_j \in Y_j$ to $p \cdot y_j$) has exactly one critical point, and that critical point is a non degenerate maximum.

This restriction is borrowed from Smale (1976), and captures the strict convexity of each production set Y_j . It will enable us to deduce that every *infinitesimal* Pareto optimum is, in fact, *globally* Pareto optimal, as shown in the next lemma.

LEMMA 4.2. (1) *Under (C*) and (I)²⁴, and for fixed $x \in \tau^*$, the following three statements are equivalent:*

²³For a definition, see Example 2 (cf. Mas-Colell (1985, Def. 4.2.12)).

²⁴This is an irreducibility assumption, the technical details of which are found in Giraud (2007).

- (a) $(x(t), y(t)) \in \Theta^*$;
 - (b) $0 \in F_\varphi(x(t), y(t))$ and $s_j(t) = 0$ for all j ;
 - (c) $b_i(t) \cdot \dot{x}_i = 0$, for all $i, \dot{x} \in F_\varphi(x(t))$.
- (2) If also (C)(iii) and (D) hold, then θ coincides with the subset of strict (global) Pareto optimal points in τ .

Proof. Part (1) follows Giraud (2007), while Part (2) follows from Smale (1976), and Giraud (2007). □

4.2 Existence and convergence

Given a trade and production path $\gamma := (x(\cdot), y(\cdot))$, it follows that $(x^*, y^*) \in \tau^*$ is a *limit point* if there exists a sequence $(t_n)_n \in \mathbb{R}^{\mathbb{N}}$ converging to $+\infty$ such that $(x(t_n), y(t_n)) \rightarrow_{n \rightarrow \infty} (x^*, y^*)$. We denote by $\Omega(\gamma)$ the set of limit points of γ . A *stationary point* (x, y) of our dynamics is such that $0 \in F(x, y)$, where F is defined as in (8) (Filippov (1988, p. 124)). A trajectory $(x(\cdot), y(\cdot))$ is said to be *degenerate* if there exists some time T for which $(x(t), y(t))$ is independent of t for all $t \geq T$.

ASSUMPTION (S). There exists at least one household i such that u_i is strictly quasi-concave on Θ^* .

PROPOSITION 4.1.

(i) Under (C), (P) and (I), there exists a trade and production path. Every such path satisfies:

(ii) for every $t \geq 0$, $(x(t), y(t)) \in \tau$ and $x_i(t) + (a_i^L, 0_{\mathbb{R}_+}) > 0$ for every i ;

(iii) the function $t \mapsto u_i(x_i(t))$ is non-decreasing for every i ;

(iv) every limit-point with no unsold inventories is in Θ^* .

(v) If, in addition, (B) and (S) hold, then every trade and production path admits limit-points and every limit-point is in θ^* .

Proof. (i) Observe first that the graphs of $M(\cdot)$ and $P(\cdot)$ can be expressed by means of a first-order formula over the field \mathbb{R} of real numbers. Thus, according to Tarski-Seidenberg's theorem,²⁵ $M(\cdot)$ and $P(\cdot)$ are semi-algebraic mappings and hence they are Borel-measurable. As a result, the set-valued map obtained from (8) by replacing f with $M(\cdot)$ and $P(\cdot)$ respectively, is upper semi-continuous, non-empty, convex valued, and compact valued. Consequently, existence of local absolutely continuous solutions $(x(\cdot), y(\cdot), p(\cdot)) : [0, T] \rightarrow \tau \times S_+^{L+C-1}$ for some $T > 0$ follows from standard existence results for differential inclusions (see e.g. Aubin and Cellina (1984)).

In order to show that this solution is a trade and production path, it suffices to check that every local solution remains in a compact subset. Indeed, as already observed in Champsaur and Cornet (1990), the standard proof that compactness

²⁵Cf. Bochnak, Coste & Roy (1987).

of the domain implies maximality of the solution carries over from the single-valued case to the multi-valued case. But compactness of the domain follows from the feasibility of every marginal equilibrium in every marginal economy,

$$\sum_i \dot{x}_i(t) = \sum_j \dot{y}_j(t),$$

and the fact that for every $t \geq 0$, one has $(x(t), y(t)) \in \tau^*$ — which is compact according to assumption (B).

(ii) That every trade and production path stays within τ has just been proved. That $x_i(t) + (a_i^L, 0_{\mathbb{R}_+}) > 0$ for every i and t is a consequence of the monotonicity of preferences, the individual rationality of each short-term outcome (Giraud (2007)) and the condition $\omega_i + (a_i^L, 0_{\mathbb{R}_+}) > 0$ for every i (Assumption (C)).

(iii) This, too, is a consequence of the individual rationality of trade and production paths; for a proof, see Giraud (2007).

(iv) The claim follows because (b) \Rightarrow (a) in Lemma 4.2.

(v) This follows from the standard argument; see Giraud (2007). \square

5 Conclusion

To summarize, we briefly review some properties of the approach proposed here in comparison with the standard treatment of production in static general equilibrium theory.

1. Existence of a trajectory obtains in our setting in more general circumstances than that of static competitive equilibria in Arrow-Debreu economies. In particular, we do not need the strong survival assumption that was shown by Kamiya (1988) to be a necessary condition for the existence of marginal cost pricing equilibria.

2. Whenever production sets exhibit increasing returns to scale, the behavior of a non-regulated firm can still be defined as short-run profit maximization, whereas static GET needs to rely on some regulated pricing behavior (such as marginal cost pricing, etc.). Even in this case, such a regulated firm may incur losses at a static competitive equilibrium, which makes sense only if it is publicly owned. There is room in our model for a theoretical debate about the need for publicly regulating firms with increasing returns. The reasoning runs as follows. Since no firm incurs losses along a solution trajectory, it makes sense to study the efficiency properties of a privately owned firm whose production set exhibits non convexities and to compare them with what could be obtained in a publicly regulated fashion.

3. When production is strongly convex (in the sense of Assumption (D) *supra*), steady states of our dynamics are always first-best optimal as soon as there are no unsold inventories. When production exhibits non convexities, steady states of

our dynamics may fail to be Pareto optimal. This is also the case in static GET: examples by Brown & Heal (1979) show that the community indifference curve defined by a marginal cost pricing equilibrium may cut inside the production possibility set. Nevertheless, we have shown that steady states of our dynamics (with no inventories) are always infinitesimally optimal. Quinzii (1992) (among others) gave sufficient conditions for a static marginal cost pricing equilibrium to be Pareto optimal. We leave to further research the task of formulating analogous conditions for the steady states of our dynamics to be also Pareto optimal even when returns to scale are increasing.

4. Finally, trajectories that are solutions to our dynamics may exhibit various kinds of underemployment. Actually, a large part of the lessons to be drawn from macroeconomic theories of market disequilibrium can be recovered within our flexible price setup.

We conclude with a few remarks of a more technical nature.

If we were to add some boundary condition on preferences together with the interiority of initial endowments, we might obtain sharper results. However, neither additional assumption is easy to justify from an economic viewpoint, especially in the presence of production. On the other hand, keeping the attainable set away from the boundary makes little sense when one of the goods to be traded is labor. Indeed, this would imply that households ask for an infinite amount of consumption good in exchange for the full utilization of labor capacity. Every feasible path would then exhibit (voluntary) underutilization of labor.

When (S) is not satisfied while τ is compact, a trade and production path still admits limit-points in τ but does not necessarily converge. In the latter case it must therefore exhibit *cycles*. This provides a new area where *real business cycles* can be studied (within a dynamic general equilibrium setting) as a consequence of the myopia of boundedly rational economic agents. We did not pursue this issue here.

Readers interested in the game-theoretic foundations of our approach will find in Giraud (2007) the underlying game played by agents in a marginal economy. There it is shown that, when each type of agent is represented by a continuum, there exists a unique dominant strategy equilibrium whose output coincides with the order book associated to this marginal economy. The same holds in the framework of this paper. As a consequence, the marginal outcome of a marginal production economy can be understood as the outcome induced by the dominant strategy equilibrium of some underlying market game.

6 Appendix

6.1 Limit orders

DEFINITION 6.1. A *limit order* to sell good ℓ in exchange for commodity c gives a quantity $0 \leq q_\ell \leq x_i^\ell$ to be sold and a relative price p_ℓ^+/p_c^+ . The order is to sell up to q_ℓ units of commodity ℓ in exchange for good c if the relative price p_ℓ/p_c is greater than, or equal to, p_ℓ^+/p_c^+ .

DEFINITION 6.2. A *market order* to sell good ℓ in exchange for commodity c gives quantity $0 \leq q_\ell$ of good ℓ to be sold at any positive value of the relative price p_ℓ/p_c .

REMARK 6.1. A limit order to “sell” ℓ against c at relative prices $p_c^+ = 0$, and $p_\ell^+ > 0$ is, in fact, an order *not* to buy c . (By means of comparison, in a standard Shapley and Shubik (1977) game, only market orders are allowed and only against the numéraire; whereas in a Shapley windows model (1976), market orders can be sent for any pair of commodities.)

The limit-price mechanism allows only sell orders. Agents can offer to sell a fixed quantity of one good in exchange for another good, as a market or limit order, but they cannot offer to buy a fixed quantity of one good in exchange for another. This, however, is not a restriction: in order to buy commodity ℓ against good c , it suffices to sell c against ℓ .

The limit-price mechanism is defined by reinterpreting market and limit orders, whether sent by a consumer or a producer in $T_{x(t),y(t),s(t)}\mathcal{E}$ as the “agents” of a fictitious pure-exchange linear economy \mathcal{L} as follows:

- I is the set of *orders*;
- For each fictitious “agent” i (i.e., for each order), its linear “utility” is given by $b^i := (p_1^+, p_2^+)$;
- The “initial endowment” of each such “agent” is defined as $e^i := (q_1, q_2)$.

The limit-price mechanism finds a final allocation for the “agents” of this fictitious economy in which the “agents” bring their entire endowment to the market and maximize their utility for up to a continuum of “agents”. A natural candidate for playing the role of a final outcome of this mechanism is the Walras equilibrium. Unfortunately, when restricted to linear pure-exchange economies, Walras equilibria do not necessarily exist and, when they do exist, may fail to provide a unique equilibrium allocation. The limit-price mechanism of Mertens generalizes the competitive equilibrium mechanism so as to always yield a *unique* final outcome, even when there are no Walrasian equilibria or when there is a multiplicity of equilibrium allocations. This explains why we call our dynamics “Walrasian”: it can roughly be seen as a non-tâtonnement dynamics where each

infinitesimal increment is (a generalized version of) a short-run Walras equilibrium.

6.2 Filippov's solutions

Consider a differential inclusion

$$\dot{x}(t) \in f(x(t)), \quad (7)$$

where $f : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$ is a (possibly discontinuous) generalized vector field.

DEFINITION (Filippov 1988) A *Filippov solution* of (7), is an absolutely continuous trajectory $\phi : [a, b] \rightarrow \mathbb{R}^m$ such that, for a.e. $t \in [a, b]$,

$$\dot{\phi}(t) \in F_f(\phi(t)) := \bigcap_{\varepsilon > 0} \bigcap_{A \in \mathcal{N}} \overline{\text{co}}\{y \mid d(y, f(\phi(t))) < \varepsilon, y \notin A\}, \quad (8)$$

where \mathcal{N} denotes the family of sets $A \subset \mathbb{R}^m$ of (Lebesgue) measure 0.

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