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June 18, 2007

Abstract

The objective of this paper is to adopt a general equilibrium model and determine the socially efficient discount factor, risk free rate and discount rate when there are heterogeneous anticipations about the growth of the economy as well as heterogeneous time preference rates. Among others we tackle the following questions. Is the socially efficient discount factor an arithmetic average of the individual subjectively anticipated discount factors ? More generally, can the Arrow-Debreu prices, the risk free rates, the subjectively expected socially efficient discount factors and discount rates be obtained as an average of the individual subjectively anticipated ones ? Can beliefs dispersion be analyzed as a sort of additional risk or uncertainty leading to possibly lower discount rates ? Is it socially efficient, when diversity of opinion is taken into account, to reduce the discount rate per year for more distant horizons ? If so, what is the trajectory of the decline ?

1. Introduction

The concept of a discount rate is central to economic analysis, as it allows effects occurring at different future times to be compared by converting each future dollar amount into equivalent present dollars. The problem of the determination of a discount rate has acquired renewed relevance lately in order to analyze environmental projects or activities[1] the effects of which will be spread out over hundreds of years, and the evaluation of which, through Costs and Benefits Analysis, is very sensitive to the discount rate being used. For instance, concerning global climate change, it has been argued that the strong conclusions of the Stern Review were essentially driven by the low assumed discount rate (see, e.g. Nordhaus, 2006 or Weitzman, 2007).

In the short run, the use of the observed risk free rate to discount (public) investment projects leads to a socially efficient level of investment. The analysis is less easy to perform when benefits and costs of the set of current potential actions are expected to last in the long run. The carbon dioxide that one emits today will not be recycled for a couple of centuries, yielding long term costs like global warming. Some nuclear wastes like plutonium have half-life in the tens of thousands years. Financial markets are not very helpful to provide a guideline for investing in technologies that prevent this kind of long-lasting risks to occur. Liquid financial

[1] Prominent examples include: global climate change, radioactive waste disposal, loss of biodiversity, thinning of stratospheric ozone, groundwater pollution, minerals depletion, and many others.

instruments with such large durations do not exist. For the sake of comparison, US treasury bonds have time horizons that do not exceed 30 years. We must thus rely on the use of an economic model to value the distant future.

A critical feature that must be taken into account is divergence of opinion about the future of the economy. Estimating the growth rate for the coming year is already a difficult task. It is natural that the estimation of growth for the next century/millenium is subject to potentially enormous divergence. It is doubtful that agents or economists currently have a complete understanding of the determinants of long term consumption growth. The debate on the notion of sustainable growth is an illustration of the degree of possible divergence of opinion about future of society. Some will argue that the effects of improvement in information technology have yet to be realized and the world faces a period of more rapid growth. On the contrary, those who emphasize the effects of natural resource scarcity will see lower growth rates in the future. Some even suggest a negative growth of the GNP per head in the future, due to the deterioration of the environment, population growth and decreasing return to scales. Moreover, as underlined by Weitzman (2001), "these and many more are fundamentally matters of judgment or opinion, on which fully informed and fully rational individuals might be expected to differ". The objective of this paper is to determine the socially optimal discount rate, when explicitly taking into account possible disagreement among agents.

More precisely, we consider a pure endowment/exchange economy in which agents hold heterogeneous expectations about the growth of the economy. Agents may also differ in their rate of pure time preference. We want to analyse the properties of equilibrium risk free rates, discount factors and discount rates in such an economy; in particular, we want to determine the analog of Ramsey Equation[2] in a time-varying, stochastic setting (as in standard models of the term structure of interest rates[3]), in which agents are allowed to differ in their pure-time preference rate as well as in their subjective expectations of the growth rate. Among others, we wish to tackle the following questions. Is the socially efficient discount factor an arithmetic average of the individual subjectively anticipated discount factors ? More generally, can the Arrow-Debreu prices, the risk free rates, the subjectively expected socially efficient discount factors and discount rates be obtained as an average of the individual subjectively anticipated ones ? To what extent do equilibrium risk free rates and discount rates differ from the ones in an homogenous economy ? Can beliefs dispersion be analyzed as a sort of additional risk or uncertainty leading to possibly lower discount rates ? How do discount rates vary with the degree of divergence ? Is it socially efficient, when diversity of opinion is taken into account, to reduce the discount rate per year for more distant horizons ? If so, what is the trajectory of the decline ?

The last question about the shape of the yield curve is of particular interest. There is a wide agreement that discounting at a constant positive rate for long time horizons is problem-

[2] In a deterministic setting, the well-known Ramsey equation gives the following expression for the discount rate

$$R = \rho + \frac{1}{\eta} g,$$

where ρ denotes the rate of pure time preference, g is the per capita growth rate of consumption and $1/\eta$ is the elasticity of marginal utility, or equivalently the degree of relative risk aversion.

[3] See, e.g., Cox Ingersoll Ross, 1985, Ingersoll and Ross, 1992, Vasicek, 1977, Cochrane, 2001.

atic, irrespective of the particular discount rate employed. Indeed, with a constant rate, the costs and benefits accruing to generations in the distant future appear relatively unimportant in present value terms. Hence decisions made today on this basis, appear to tyrannise future generations and in extreme cases leave them exposed to catastrophic consequences. Weitzman (1998) summarises this succinctly when he states : « To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere ». A recently proposed solution to this problem is to use a discount rate which declines over time, according to some predetermined trajectory, thus raising the weight attached to the welfare of future generations. It is clear that using a declining discount rate could make an important contribution towards the goal of sustainable development. But what formal justifications exist for using a declining discount rate and what is the optimal trajectory of the decline[4] ? In a deterministic world, decreasing discount rates can arise as a result of known changes in the growth rate, changes in risk aversion, etc. Additional motivations emerge once uncertainty is considered. For example, Weitzman (1998, 2001) considers uncertainty on the discount rate itself. Starting from the fact that there is a huge divergence of expectations about future discount rates among economic experts, Weitzman (1998, 2001)[5] introduces a probability distribution for the future discount rate and its behaviour over time. The author adopts a certainty equivalent analysis in order to determine the discount rates for varying horizons and obtains decreasing discount rates. More generally, Gollier (2002a), Dybvig et al.(1996), Weitzman (1998, 2001) show in different contexts that from today's perspective, the only relevant limiting scenario is the one with the lowest interest rate. In the presence of uncertain growth, Gollier (2002a, 2002b) shows that shape of the yield curve depends upon preferences for risk and prudence, and higher order moments of the utility function. Regardless of whether it is the discount rate or the growth rate that is uncertain, the nature of the distribution of random growth is of particular importance; for instance, decreasing discount rates are obtained with Bayesian learning in Weitzman (2004) and Gollier (2007) shows that serial correlation in the growth rates leads to downward sloping yield curves when the representative agent is prudent. Decreasing discount rates also emerge from the specification of a sustainable welfare function à la Chichinisky (1996) and Li and Löfgren (2000). Lastly, there is considerable empirical and experimental evidence to show that individuals are frequently hyperbolic discounters (see, e.g. Loewenstein and Thaler, 1989). In this paper, we want to examine if divergence of beliefs and heterogeneity of time preference rates can be a justification for the use of declining discount rates.

We provide the following answers to the questions above. We first obtain that the equilibrium Arrow Debreu prices can be obtained as an average of the individually anticipated Arrow-Debreu prices[6]. However, the socially efficient discount factor is not necessarily an average of the individual subjectively anticipated ones. There is a bias, induced by beliefs and time preference rates heterogeneity. Depending on the level of elasticity of substitution, this bias

[4]See Groom et al. (2005) for a survey.

[5]Weitzman (2001) undertakes a survey of over 2 000 academic economists, and a so-called blue ribbon selection of 50, as to their opinion on the constant rate of discount to use for Cost Benefit Analysis. The responses were distributed with a gamma distribution with mean 4% and standard deviation of 3%.

[6]For a given agent i, the individually anticipated prices, discount rates, discount factors, risk-free rates are those that would prevail if the economy was made of agent i only.

is towards lower or higher discount factors. This result is consistent with the interpretation of beliefs and time preference heterogeneity as an additional source of risk or uncertainty in the future, leading agents to value more or less future consumption (with respect to present consumption) depending on their level of elasticity of substitution. Moreover, the right concept of average to consider is not an equally weighted arithmetic average of the individually expected discount factors but a weighted η-average, where η represents the intertemporal elasticity of substitution. Note however that two settings are specific. The first is the case of myopic utility functions (i.e. when $\eta = 1$); the equilibrium discount factors are then arithmetic averages of the individually anticipated ones, which means that the certainty equivalent approach of Weitzman (1998, 2001) is compatible with a general equilibrium approach with logarithmic utility functions. It suffices to interpret each discount rate/discount factor that is announced by an expert as her individually anticipated discount rate/discount factor. The second specific setting is when there are deterministic time preference rates and no beliefs dispersion. In this case, the socially efficient discount factor is an η-average of the individually anticipated ones (as shown in Gollier-Zeckhauser, 2005, in a fully deterministic setting).

We also examine the impact of beliefs and time preference heterogeneity on the expression of the risk free rate and the discount rate as well as on the relationship between the discount rate and the time horizon (the possible shapes of the yield curve). We first obtain that the equilibrium risk free rate is not an average of the individually anticipated ones. There is a bias related to beliefs dispersion. As in the discount factors case, this bias is towards higher or lower risk free rates depending on the position of η with respect to one. Note that for risk free rates, this bias is only induced by beliefs dispersion and not by time preference rates heterogeneity. Aggregate pessimism as well as aggregate patience reduce the rates. Since these aggregate levels are given by stochastic, time-varying (risk-tolerance) weighted averages of the individual levels of pessimism and patience, possible correlation effects are induced. Moreover, increased beliefs dispersion can lead to a decrease of risk free and discount rates or to an increase of the risk free and discount rates depending on the level of elasticity of substitution ($\eta < 1$ or $\eta > 1$). This leads in the medium term to different possible shapes of the yield curve (decreasing discount rates or increasing discount rates). Finally we show that the bias due to beliefs dispersion vanishes in the long run and that the asymptotic discount rate is essentially given by the lowest individually expected asymptotic discount rate among all agents, which is another element in favor of decreasing discount rates in the long run.

The paper is organized as follows. Section 2 presents the model. Section 3 deals with consensus Arrow-Debreu prices and consensus socially efficient discount factors and their link with individually expected subjective Arrow-Debreu prices and discount factors. In Section 4, we analyse the expression of both the consensus risk free rate and the socially efficient discount rate and how they are impacted by beliefs dispersion. Section 5 is devoted to long term and short term considerations. In Section 6, we consider specific settings. Appendix A-1 establishes aggregation results. All proofs are in Appendix A-2.

2. The model

We consider a continuous-time Arrow-Debreu economy, in which risk averse agents try to maximize the expected utility of future consumption. A time horizon T is fixed, and a filtered probability space $\left(\Omega, F, (F_t)_{t\in[0,T]}, P\right)$ is given. The model is adapted from Jouini-Napp (2007). Each agent indexed by $i = 1, ...N$, has a current endowment at time t denoted by e_t^{*i} and a Von Neuman-Morgenstern utility function for future consumption of the form $E^{Q^i}\left[\int_0^T \exp^{-\int_0^t \rho^i(s,\omega)ds} u\left(c_t\left(\omega\right)\right)dt\right]$, which means that agents may differ in their subjective beliefs (represented by the probability measure Q^i), in their pure time preference rate process ρ^i, and in their endowment process e^{*i}. We suppose that agents share the same CRRA utility function[7] for consumption, of the form $u'(x) = x^{-1/\eta}$. We let M^i denote the positive density process of Q^i with respect to P and we let $D_t^i \equiv \exp^{-\int_0^t \rho^i(s,\omega)ds}$ denote the individual pure time preference discount factor. With these notations, the utility function of agent i can equivalently be written in the form $E\left[\int_0^T M_t^i D_t^i u\left(c_t\left(\omega\right)\right)dt\right]$. We will consider arbitrarily large time horizons which implicitly means that each individual takes into account his own preferences as well as the preferences of his descendants as if they were his own preferences.

We let $e^* \equiv \sum_{i=1}^N e^{*i}$ denote the aggregate endowment process. We make the assumption that e^* and M^i satisfy the following stochastic differential equations[8]

$$\begin{cases} de_t^* = \mu_t e_t^* dt + \sigma_t e_t^* dW_t & e_0^* = 1 \\ dM_t^i = \delta_t^i M_t^i dW_t & \end{cases}$$

where W denotes a standard unidimensional $\left((F_t)_{t\in[0,T]}, P\right)$ −Brownian motion.

We have $M_t^i = \mathcal{E}_t\left(\delta^i\right) \equiv \exp\left(\int_0^t \delta_s^i dW_s - \int_0^t \frac{\left(\delta_s^i\right)^2}{2} ds\right)$ and $e_t^* = \exp\left(\int_0^t \mu_s ds\right)\mathcal{E}_t\left(\sigma\right)$ which means that e^* and M^i are stochastic processes that depend upon risk and time. The other parameters $(\rho^i, \delta^i, \mu, \sigma)$ themselves might depend upon risk and time. According to Girsanov Theorem, for agent i, aggregate endowment is an Itô process with diffusion parameter σ and with subjectively expected drift parameter $\mu^i \equiv \mu + \sigma\delta^i$, which means that in our model, agents differ in their expected instantaneous growth rate of aggregate endowment[9]. Note that individual beliefs M^i and μ^i might result from a Bayesian udating of the investors predictive instantaneous growth rate through the observation of the aggregate endowment process as in e.g. Detemple-Murthy (1994) and Zapatero (1998).

We recall that an Arrow-Debreu equilibrium relative to the pure time preference discount factors $\left(D^i\right)_{i=1,...,N}$, the beliefs $\left(M^i\right)_{i=1,...,N}$ and the endowment processes $\left(e^{*i}\right)_{i=1,...,N}$ is defined by a positive price process q^* and a family of optimal consumption plans $\left(y^{*i}\right)_{i=1,...,N}$

[7] Our approach can be extended to the case with heterogeneous utility functions of the form $u_i'(t,x) = (\theta_i + \eta x)^{-1/\eta}$.

[8] There is no Markovian assumption; the coefficients μ, σ and δ^i for $i = 1, ..., N$ may depend on the entire past history of the economy. We only assume that $\int_0^T |\mu_t| dt < \infty$, $\int_0^T \left|\sigma_t^2\right| dt < \infty$, and $\int_0^T \left|\left(\delta_t^i\right)^2\right| dt < \infty$.

[9] More precisely, letting $W_t^i \equiv W_t - \int_0^t \delta^i ds$, we obtain through Girsanov Theorem that W^i is a Brownian motion under Q^i and $de_t^* = \left(\mu_t + \sigma_t\delta^i\right)e_t^* dt + \sigma_t e_t^* dW_t^i$.

such that markets clear, i.e.

$$\left\{ \begin{array}{c} y^{*^i} = y^i\left(q^*, M^i, D^i, e^{*^i}\right) \\ \sum_{i=1}^N y^{*^i} = e^* \end{array} \right.$$

where $y^i(q, M, D, e) = \arg\max_{E\left[\int_0^T q_t\left(y_t^i - e_t\right)dt\right] \leq 0} E\left[\int_0^T M_t D_t u(c_t)\, dt\right]$.

We start from an equilibrium $\left(q^*, \left(y^{*^i}\right)_{i=1,\ldots,N}\right)$ relative to the pure time preference discount factors $\left(D^i\right)$, the beliefs $\left(M^i\right)$ and the endowment processes $\left(e^{*^i}\right)$. Note that since our utility functions satisfy Inada conditions, all equilibria are interior, hence there exist Lagrange multipliers $(\lambda_i)_{i=1,\ldots,N}$ such that for all i, the equality $M_t^i D_t^i u'\left(y_t^{*^i}\right) = \lambda_i q_t^*$ holds for all t.

3. Consensus Arrow-Debreu prices and consensus socially efficient discount factors

The first question deals with the possible aggregation of individually anticipated Arrow-Debreu prices into the equilibrium Arrow-Debreu prices. We let q^{i^*} denote the Arrow-Debreu prices that would prevail if the economy was made of agent i only, i.e. $q_t^{i^*} = M_t^i D_t^i u'(e_t^*)$. We let the risk tolerance of agent i be denoted by $T_i = \eta y^{*^i}$ and we introduce the notation $\tau_i = \frac{T_i}{\sum_{i=1}^N T_i} = \frac{y_i^*}{e^*}$. Moreover, we let $\gamma_i \equiv \frac{(1/\lambda_i)^\eta}{\sum_{j=1}^N (1/\lambda_j)^\eta}$. Finally, we denote by N^i the individual characteristic $N^i = M^i D^i$. We obtain the following result.

Proposition 3.1. *The Arrow-Debreu prices can be written in the form*

$$q_t^* = \left[\sum_{i=1}^N \gamma_i \left(q_t^{i^*}\right)^\eta\right]^{1/\eta} = \left[\sum_{i=1}^N \tau_i \left(q_t^{i^*}\right)^{-\eta}\right]^{-1/\eta}.$$

This means that the Arrow Debreu prices q^* can be considered as consensus prices, since they can be written in the form of an average of the individual subjectively expected Arrow-Debreu prices. The Arrow-Debreu price q^* is an $\eta-$average of the individually anticipated Arrow-Debreu prices q^{i^*} with deterministic weights γ_i. This result stems from the fact that the Arrow-Debreu prices in our economy are the same as in an economy with a unique agent with a consensus characteristic N which in $\eta-$average of the individual characteristics N^i (see Appendix A-1). Note that the Arrow-Debreu price is also a $(-\eta)-$average of the individually anticipated Arrow-Debreu prices q^{i^*} with stochastic weights, these weights being given by the individual relative levels of risk-tolerance. As already seen, the individual relative level of risk-tolerance is equal to the individual relative level of consumption and measures then the relative "size" of the agent in the economy at a given date t and in a given state of the world ω.

We now consider to what extent individual subjectively expected socially efficient discount factors can be averaged into a consensus discount factor. We recall that the individually anticipated discount factor of agent i is the discount factor that would prevail if the economy was made of agent i only. It is the discount factor that agent i "expects", i.e. $A_t^i \equiv E\left[M_t^i D_t^i u'(e_t^*)\right] = E^{Q_i}\left[D_t^i u'(e_t^*)\right] = E\left[q_t^{i^*}\right]$. Our question can then be rephrased as follows: can the socially efficient discount factor $A_t \equiv E\left[q_t^*\right]$ be represented as an average of the individual A_t^i ?

Proposition 3.2. *1. If there is no beliefs heterogeneity and if time-preference rates are deterministic, i.e., if $\delta^i \equiv \delta$ and $\rho^i(s,\omega) \equiv \rho^i(s)$, then the socially efficient discount factor is an $\eta-$average of the individual subjectively expected socially efficient discount factors, more precisely*

$$A_t = \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^\eta\right]^{1/\eta}.$$

2. In the general setting,

- *If $\eta = 1$, then $A_t = \sum_{i=1}^{N} \gamma_i \left(A_t^i\right)$.*
- *Otherwise, we have*

$$A_t \leq \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^\eta\right]^{1/\eta} \quad \text{for } \eta < 1$$

and

$$A_t \geq \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^\eta\right]^{1/\eta} \quad \text{for } \eta > 1,$$

with equality holding only when the divergence in individual characteristics is deterministic, i.e. if N_i/N_j is deterministic for all (i,j).

This means that the equilibrium aggregate discount factor can be recovered as an average of the individual subjectively expected discount factors only in the setting with no beliefs dispersion and deterministic divergence in time preference rates[10], or for logarithmic utility functions. In particular, this implies that the certainty equivalent approach of Weitzman (1998, 2001), in which A_t is taken as the equally weighted arithmetic average of the A_t^i, each A_t^i corresponding to a possible scenario, would be consistent with a general equilibrium approach.It suffices to assume that the model that expert i consults and that leads to the recommendation A^i results from a general equilibrium model with a subjective belief μ^i, time preference rate ρ^i, logarithmic utility functions and equal Lagrange multipliers (see Section 6.1.2 for more details). Assuming that experts differ in their expectation about the growth rate is fairly natural. Indeed, the expected growth rate reflects the opinion about the future. It is also natural to assume that they differ in their pure time preference since it may reflect their point of view about intergenerational equity.

More generally, Proposition 3.2 means first that the right concept of average to consider for discount factors (in the case of power utility functions) is an $\eta-$average, which is an arithmetic average only in the case of logarithmic utility functions. Moreover, this $\eta-$average is a weighted average, the weights being given by the parameters γ_i. Finally, in a general setting it is not possible to recover the socially efficient discount factor as an average of the individual subjectively expected discount factors. Except in specific settings (in the deterministic setting as in Gollier-Zeckhauser, 2005, or, more generally, with deterministic heterogeneity in pure time preference rates and no beliefs dispersion or in the case of logarithmic utility functions), there

[10] Note that no beliefs dispersion does not mean that individuals are rational; they can all share the same subjective belief. Analogously, all time preference rates ρ^i need not be deterministic but they need to be written in the form $\rho^i(t,\omega) = \rho(t,\omega)a^i(t)$ where ρ is a common term and where a^i is a deterministic process.

is an aggregation bias. The price at date 0 of a zero-coupon bond maturing at date t is lower (resp. higher) than the (weighted $\eta-$) average of the subjectively expected prices for $\eta < 1$ (resp. $\eta > 1$). This means that with beliefs dispersion and/or stochastic time preference heterogeneity, individuals value less (resp. more) one unit of consumption at date t when $\eta < 1$ (resp. $\eta > 1$).

It is usually assumed that risk aversion is greater than 1, which would lead to consider the case $\eta \leq 1$ as the most likely. However, the parameter $1/\eta$ has other possible interpretations than just risk aversion. In particular, our model with CRRA utility functions does not permit to distinguish intertemporal elasticity of substitution from risk aversion, and setting $\eta \geq 1$ can also be interpreted as assuming that intertemporal elasticity of substitution is higher than 1[11]. The condition $\eta \geq 1$ is also equivalent in our setting to the condition that prudence $(1 + 1/\eta)$ is larger than twice absolute risk aversion $(1/\eta)$, a condition which has been thoroughly studied by Gollier (2000), among others. Determining whether prudence should be smaller or larger than twice risk aversion or equivalently $(1/u')$ concave has appeared in different contexts (Drèze and Modigliani, 1972, Caroll and Kimball, 1996, Sinclair-Desgagné and Gabel, 1997, Dionne and Fombaron, 1996). More directly related to our results, Gollier and Kimball (1996) show that, in a standard portfolio problem, the opportunity to invest in a risky asset raises (resp. reduces) the aggregate saving if and only if absolute prudence is larger (resp. smaller) than twice absolute risk aversion. Moreover, Gollier (2000) studies the problem of the optimal use of a good whose consumption can produce damages in the future and shows that scientific progress providing information on the distribution of the intensity of damages induces earlier prevention effort only if prudence is larger than twice risk aversion. As underlined by Gollier (2000), maybe one of the best arguments in favor of the restriction $\eta \geq 1$ was made by Debreu and Koopmans (1982). These authors argue that a good measure of risk aversion should be the following concavity index $-\frac{u''}{(u')^2}$. In our setting, it is given by $\frac{1}{\eta}x^{-1+\frac{1}{\eta}}$. Decreasing absolute risk aversion is then equivalent to $\eta \geq 1$. The Debreu and Koopmans index was used lately in political economy (see Alesina and Tabellini, 1990, Chandler, 1998).

Hence, a possible interpretation of our results is the following. Interpret beliefs heterogeneity and time preference heterogeneity in a stochastic setting as additional risk or as less information or more uncertainty about the future. According to Gollier and Kimball (1996) or Gollier (2000), this should lead agents to value more future consumption in the case $\eta > 1$ and less future consumption in the case $\eta < 1$, which is essentially the result of Proposition 3.2.

4. Consensus risk free rates and consensus socially efficient discount rates

We let $R_t = -\frac{1}{t}\log A_t$ denote the (average) socially efficient discount rate between date 0 and date t. We denote by $R_t^i = -\frac{1}{t}\log A_t^i$ the socially efficient discount rate that would prevail if the economy was made of agent i only. It is also the discount rate subjectively expected by agent i. The results of Proposition 3.2 imply, in particular, that the socially efficient discount rate R_t

[11] See Hansen-Singleton (1982) for reasonable values of the Intertemporal Elasticity of Substitution.

is lower when $\eta > 1$ (resp. higher when $\eta < 1$) than the quantity

$$-\frac{1}{t}\log\left[\sum_{i=1}^{N}\gamma_i\left(A_t^i\right)^{\eta}\right]^{1/\eta} = -\log\left[\sum_{i=1}^{N}\gamma_i\left(\exp^{-R_t^i}\right)^{\eta t}\right]^{1/\eta t}$$

which is the discount rate associated to the average of the individually anticipated discount factors. The bias is all the more important as dispersion of individual characteristics N^i is important.

In order to better identify the bias, its origin and its impact on the socially efficient discount rate, let us analyse more precisely the expressions of the socially efficient discount rate and of the risk free rate. Note that in the homogeneous setting with constant parameters $(\delta, \rho, \mu, \sigma)$, the yield curve is flat and the discount rate for all horizons is equal to the risk free rate and given by $R = r^f = \rho + \frac{\mu}{\eta} - \frac{1}{2}\left(\frac{1}{\eta}\right)\left(1+\frac{1}{\eta}\right)\sigma^2$. This is an easy extension of Ramsey Equation to a setting with uncertainty. In our heterogeneous setting with parameters $(\delta^i, \rho^i, \mu, \sigma)$ which may be time and state dependent, we obtain the following result.

Proposition 4.1. *We obtain the following expressions for the risk free rate and the socially efficient discount rate.*

1. *The risk free rate is given by*[12]

$$\begin{aligned} r^f &= \rho_D + \frac{1}{\eta}\left[\mu + \delta_M\sigma\right] - \frac{1}{2}\frac{1}{\eta}\left(1+\frac{1}{\eta}\right)\sigma^2 + \rho_B && (4.1)\\ &= r^f(stdard) + \rho_D + \frac{1}{\eta}\delta_M\sigma + \rho_B && (4.2)\end{aligned}$$

with $\delta_M = \sum_{i=1}^N \tau_i\delta^i$, $\rho_D = \sum_{i=1}^N \tau_i\rho^i$, $\rho_B = \frac{1}{2}(1-\eta)\left[\sum_{i=1}^N \tau_i\left(\delta^i\right)^2 - \delta_M^2\right] \equiv \frac{1}{2}(1-\eta)\, Var^{\tau}(\delta)$.

2. *If we let* $\left(r^i\right)^f$ *denote the risk free rate that would prevail if the economy was made of agent* i *only, then*

$$r^f = \sum_{i=1}^{N}\tau_i\left(r^i\right)^f + \rho_B. \quad (4.3)$$

3. *The socially efficient discount rate* R_t *is given by*

$$\begin{aligned} R_t &= -\frac{1}{t}\log E\left[M_tB_tD_tu'(e)\right] && (4.4)\\ &= -\frac{1}{t}\log E^{\overline{Q}}\left[\exp - \int_0^t r_s^f ds\right] && (4.5)\end{aligned}$$

with $B_t = \exp\left(-\int_0^t \rho_B(s)ds\right)$, $D_t = \exp\left(-\int_0^t \rho_D(s)ds\right)$, $M_t = \mathcal{E}_t(\delta_M)$ *and* $\frac{d\overline{Q}}{dP} = \mathcal{E}_T\left(\delta_M - \frac{\sigma}{\eta}\right)$.

Let us interpret the expression of the risk free rate obtained in Equation (4.1). The first term corresponds to the time preference component of the risk free rate. In the homogeneous

[12] We let r^f (stdard) denote the risk free rate that would prevail in an economy with rational beliefs and with zero time preference rates.

setting, it is simply given by the time preference parameter ρ, that is common to all agents. In our setting, it is a stochastic risk tolerance weighted average of the individual time preference parameters. The second term corresponds to the wealth effect component of the risk free rate. As in the homogeneous setting, it is given by the product of the representative agent constant relative risk aversion level $1/\eta$ by the expected rate of return of aggregate consumption. In the homogeneous setting, this expected rate of return is the same for all agents and given by μ. In our setting, the quantity $\mu + \delta_M \sigma = \mu + \left(\sum_{i=1}^N \tau_i \delta^i\right) \sigma = \sum_{i=1}^N \tau_i \mu^i$ is the risk tolerance weighted average of the individual subjectively expected rates of return and corresponds to a consensus expected rate of return. The third term is related to future risk and corresponds to the precautionary savings component of the risk free rate. It is the same as in the standard setting and involves the prudence $\left(1 + \frac{1}{\eta}\right)$ and the risk aversion $\frac{1}{\eta}$ of the representative agent as well as the future risk measured by σ^2. Finally, the last term reflects a beliefs dispersion effect. As shown in Equation (4.3), due to this effect, the equilibrium risk free rate with heterogeneous beliefs can not be written as an average of the "individual risk free rates $\left(r^i\right)^f$". Note that when there is no beliefs dispersion but possible dispersion in time preference rates, the risk free rate is an average of the individually anticipated risk free rates.

As enlightened by Equation (4.1) , the risk free rate in our setting is the same as in an homogeneous economy in which the representative agent pure time preference process is given by $\sum_{i=1}^N \tau_i \rho^i$ and the representative agent belief is given by $\delta_M = \sum_{i=1}^N \tau_i \delta^i$, except that there is the beliefs dispersion bias. Pessimism[13] at the aggregate level, i.e. $\delta_M \leq 0$, induces a decrease of the risk free rate. This is easily understandable since a pessimistic agent will attempt to reduce current consumption and increase precautionary savings, putting downwards pressure on the risk free rate. Note that since $\delta_M = \sum_{i=1}^N \tau_i \delta^i$, there is no need for all individuals to be pessimistic to induce a lower risk free rate. It suffices that the weighted average of the individual beliefs be pessimistic, which is the case if the agents are on average rational and if there is a positive correlation between pessimism and risk tolerance. Analogously, as far as the aggregate rate of impatience $\sum_{i=1}^N \tau_i \rho^i$ is concerned, it induces a lower risk free rate if there is a negative correlation between impatience and risk tolerance and/or if the agents are on average more patient than is usually assumed. Finally, as far as the beliefs dispersion bias is concerned, it induces a decrease of the risk free rate if and only if $\eta > 1$. Therefore, everything works as if agents would face an additional source of risk or uncertainty. As underlined in the previous section, in the case $\eta > 1$, agents would react to this additional source of risk by saving more, which would put downward pressure on the risk free rate. In particular, an increased dispersion represented by a higher $Var^\tau(\delta)$ induces a lower risk free rate.

To sum up, we have exhibited three main effects of beliefs and time preference heterogeneity on the risk free rate. There is first a consensus belief effect. The consensus belief is a risk tolerance weighted average of individual beliefs. There is also a consensus pure time preference effect, the consensus pure time preference parameter being a risk tolerance weighted average of the individual pure time preference parameters. Aggregate pessimism as well as aggregate patience induce lower rates. Since the weights are not constant but stochastic, possible correlation effects are induced. Indeed, it is noteworthy that even in the case where all individual

[13] See Abel (2000) or Jouini-Napp (2006) for the impact of pessimism on the risk premium and risk free rates.

subjective beliefs δ^i and all individual time preference parameters are constant, the consensus belief $\delta_M = \sum_{i=1}^N \tau_i \delta^i$ as well as the consensus time preference rate $\rho_D = \sum_{i=1}^N \tau_i \rho^i$ are time-varying, stochastic weighted averages of the individual parameters. Finally, there is a beliefs dispersion bias which makes the heterogeneous beliefs setting fundamentally different from the homogeneous one. When $\eta > 1$, increased dispersion induces lower risk free rates.

As far as discount rates are concerned, notice first through Equation (4.5) that when r^f is deterministic (or when r^f and $\frac{d\overline{Q}}{dP}$ are independent), all that we have said about the impact of beliefs and time preference rates heterogeneity on r^f is true of R. In particular, aggregate pessimism, aggregate patience as well as increased beliefs dispersion when $\eta > 1$ induce a lower discount rate.

More precisely, we see on Equation (4.4) the impact of the consensus belief M, of the consensus time preference discount factor D and of the beliefs dispersion bias B on the socially efficient discount rate R. Intuitively, increased beliefs dispersion for $\eta > 1$ leads to a higher B hence, all other things being equal, to a lower discount rate R. Analogously, "aggregate patience" should lead, ceteris paribus, to a higher D hence to a lower R. As far as aggregate pessimism is concerned, intuitively, a pessimistic belief increases the expected value of a decreasing function of the total endowment e^*, hence should lead to a lower discount rate R.

The case with deterministic parameters μ_t and σ_t illustrates this intuition. Indeed, it is then easy to obtain (see Appendix A-2) that, when $\delta_M \leq 0$,

$$-\frac{1}{t} \log E\left[M_t u'\left(e_t^*\right)\right] \leq -\frac{1}{t} \log E\left[u'\left(e_t^*\right)\right],$$

hence the effect of pessimism only is toward a lower discount rate, which is intuitive. Moreover still in the same context of deterministic μ_t and σ_t we can show (see Appendix A-2) that if the consensus belief is neutral or pessimistic ($\delta_M \leq 0$) and when $\eta > 1$ then the impact of beliefs heterogeneity is towards a lower socially efficient discount rate, i.e.

$$-\frac{1}{t} \log E\left[B_t M_t u'\left(e_t^*\right)\right] \leq -\frac{1}{t} \log E\left[u'\left(e_t^*\right)\right].$$

We obtain analogously that if the consensus belief is neutral or optimistic (i.e. $\delta_M \geq 0$) and when $\eta < 1$, then the impact of beliefs heterogeneity is towards a higher socially efficient discount rate. This result is in the spririt of the findings of Dumas et al. (2005) that obtains in their specific sentiment framework that “whenever risk aversion is [an integer] greater than 1, an increase in the variance of sentiment reduces the expected values of all the future stochastic discount factors.”

5. Short term and long term considerations

We now turn to (very) short term and (very) long term considerations. In particular, we have seen in previous sections that there is an aggregation bias that makes the heterogeneous beliefs setting fundamentally different from an homogeneous setting with consensus characteristics. Is this bias present in the very short and very long run ? More generally, is it socially efficient, when diversity of opinion is taken into account, to reduce the discount rate per year for far distant horizons ?

In this section, since we let the time horizon T vary, we denote by λ_j^T the Lagrange multipliers, by $\gamma_j^T \equiv \frac{(1/\lambda_j^T)^\eta}{\sum_{j=1}^N (1/\lambda_j^T)^\eta}$ the normalized Lagrange multipliers and by Q^T the consensus probability measure. Similarly, we denote by A_t^T and R_t^T the discount factor and discount rate for date t amounts when the agents time horizon is T. Individual discount factors and rates will be denoted by $A_t^{T,i}$ and $R_t^{T,i}$.

For long term horizons, we obtain the following results.

Proposition 5.1. *We suppose that for all i, the individual asymptotic discount rate $R_\infty^i \equiv \lim_{t\leq T;t,T\to\infty} R_t^{T,i}$ exists*[14]*. Moreover, letting I denote an agent with the lowest subjectively expected asymptotic discount rate (i.e., $R_\infty^I = \inf\left\{R_\infty^i; i = 1, ...N\right\}$), we suppose that the weight of this agent does not vanish, i.e. $\gamma_I^T \geq \varepsilon > 0$.*

Under these assumptions, the asymptotic value for the socially efficient discount rate is given by the lowest subjectively expected discount rate i.e.

$$R_\infty \equiv \lim_{t\leq T;t,T\to\infty} R_t^T = \inf\left\{R_\infty^i, i = 1, ..., N\right\}.$$

In a model where each individual corresponds to an expert who consults a subjective model in order to recommend discount rates, the existence of R_∞^i for all i, means that each expert is able to propose an asymptotic discount rate. The condition $\gamma_I^T \geq \varepsilon > 0$ ensures that no agent vanishes when the economic horizon increases. We will see in the next section on specific examples that this condition is satisfied as long as the infinite horizon problem is well defined (i.e. the individual maximization problems are well defined for $T = \infty$).

Proposition 5.1 shows first that the bias, due to beliefs heterogeneity, that we have exhibited in previous sections vanishes in the long run. The socially efficient discount rate behaves asymptotically as the discount rate associated to an average of the individual subjectively anticipated discount factors. Indeed, under the conditions of the proposition, the socially efficient discount rate R_t^T converges to the lowest subjectively anticipated discount rate R_∞^I as does the quantity $-\log\left[\sum_{i=1}^N \gamma_i(T)\left(\exp^{-R_T^i}\right)^{\eta T}\right]^{1/\eta T}$. Nevertheless, we emphasize that the bias vanishes only asymptotically, and we shall see in the next section that we may have to consider very far horizons (hundreds of years) before observing this asymptotic behavior.

Proposition 5.1 proves foremost that, from today's perspective, among the possible subjectively expected asymptotic behaviors R_∞^i, the only relevant asymptotic behavior is the one with the lowest interest rate (as long as it exists in the long run). In other words, in a setting with heterogeneous agents, only the agent with the lowest expected discount rate matters in the long run. The interpretation is the following. From today's perspectives, all the other agents become negligible in the long run, because their "weight" has been reduced by the power of compound discounting at a higher expected discount rate. Asymptotically, the value of the socially efficient discount factor is then given by the discount factor that would prevail in an economy made of

[14] These limits can be replaced by limits along sequences, i.e. $R_\infty^i = \lim_{n\to\infty} R_{t_n}^{t'_n,i}$ for some sequence (t_n, t'_n) such that $t_n \leq t'_n$ and $\lim t_n = \lim t'_n = \infty$. In this case, the asymptotic socially efficient discount rate would be defined along the same sequences.

the agent with the lowest rate only. In the case of homogeneous beliefs and heterogeneous time preference rates, this implies that the asymptotic discount rate is given by the rate associated with the lowest rate of impatience. This means that the result of Gollier-Zeckhauser (2005) on the asymptotic discount rate with heterogeneous time preference rates[15] remains valid in a stochastic setting. In the case with homogeneous pure time-preference rates and heterogeneous beliefs, Proposition 5.1 implies that the asymptotic socially efficient discount rate is given by the most pessimistic rate.

Proposition 5.1 is another element in favor of declining discount rates in the long run. As underlined in differing contexts by Dybvig et al.(1996), Gollier (2002), Weitzman (1998, 2001), in the standard stochastic setting with a representative agent with rational beliefs and given time preference rate parameter, the long term discount rate is associated with the scenario with the lowest possible rate. In the case of heterogeneous agents, this implies that for each agent denoted by i, the asymptotic rate R^i_∞ corresponds to his worst expected scenario (over possible states of the world). Proposition 5.1 implies then that the asymptotic behavior of the discount rate is given by the lowest (among agents) of the lowest possible rate (among possible scenarios or states of the world). Consider for example the simple case in which the instantaneous rate of growth of aggregate endowment denoted by μ is a discrete random variable, independent of t and of W, taking values $\mu_1, ..., \mu_J$ with probability $p_1, ..., p_J$ and suppose that $\sigma_t, \delta^i, \rho^i$ are constants. We easily get that for all $i = 1, ...N$, the asymptotic subjectively expected discount rate is given by the rate associated to the worst possible scenario, i.e., $R^i_\infty = \rho^i + \frac{1}{\eta}\left(\inf_{j=1,...,J} \mu_j + \sigma\delta^i\right) - \left(\frac{1}{2\eta} + \frac{1}{2\eta^2}\right)\sigma^2$. According to Proposition 5.1, the socially efficient discount rate is then given by the lowest possible rate among the agents, i.e. $R_\infty = \inf_{i=1,...,N}\left(\rho^i + \sigma\delta^i\right) + \frac{\inf_{j=1,...,J}\mu_j}{\eta} - \left(\frac{1}{2\eta} + \frac{1}{2\eta^2}\right)\sigma^2$.

We now turn to short term horizons. We have essentially two ways of defining the limit of R^T_t at 0. We can consider the limit discount rate when the term goes to zero and the limit discount rate when the economic horizon goes to zero. The first one corresponds to the very short rate in a model where the agents have a given utility maximization horizon T. It informs us about the shape of the yield curve $\left(R^T_t\right)_{t\in[0,T]}$ for a given economic horizon T. It will be denoted by $R^T_0 \equiv \lim_{t\to 0} R^T_t$ and will be called the short term rate for the economic horizon T. The second one corresponds to the very short rate when the economic horizon goes to zero. It informs us about the shape of the curve $\left(R^T_T\right)_{T\geq 0}$ that corresponds to the discount rate at horizon T; agents are short-sighted for small values of T and far-sighted for large values of T[16]. It will be denoted by $R_0 \equiv \lim_{T\to 0} R^T_T$ and will be called the short-sighted rate[17].

Proposition 5.2. *For all T, the short term socially efficient discount R^T_0 is given by*

$$R^T_0 = \left(r^T_0\right)^f = \sum_{i=1}^N \gamma^T_i R^i_0 - \frac{1}{2}(\eta - 1)\sum_{i=1}^N \gamma^T_i \left(\delta^i_0 - \sum_{i=1}^N \gamma^T_i \delta^i_0\right)^2$$

[15] See also Blanchard and Fischer (1989) and Li and Löfgren (2000).

[16] The already introduced asymptotic discount rate R_∞ corresponds then to the long term rate for far-sighted agents.

[17] It corresponds then to the short term rate for an economy with short-sighted agents. Obviously, it would be possible to define the limit $\lim R^T_t$ along some sequences $0 < t_n \leq T_n \to 0$, as we discussed it for $\lim_\infty R^T_t$ in footnote 14.

and the short-sighted rate R_0 satisfies

$$R_0 = R_0^0 = \sum_{i=1}^{N} \gamma_i^0 R_0^i - \frac{1}{2}(\eta - 1) \sum_{i=1}^{N} \gamma_i^0 \left(\delta_0^i - \sum_{i=1}^{N} \gamma_i^0 \delta_0^i \right)^2$$

where $\gamma_i^0 = \lim_{T \to 0} \gamma_i^T = \frac{e_0^{*i}}{e_0^*}$ *and* $R_0^i \equiv \lim_{T \to 0} R_T^{T,i} = \lim_{t \to 0} R_t^{T,i} = \left(r_0^i\right)^f$.

The benchmark for the short rate is given by $\sum_{i=1}^{N} \gamma_i^T \left(r_0^i\right)^f$, the weighted average of the individually anticipated short rates. The difference between this benchmark and the actual short rate is proportional to beliefs dispersion and its sign depends upon the position of η with respect to 1. Recall that in the long term, the benchmark is given by the infimum of the individually anticipated long rates and there is no impact of beliefs dispersion. In particular, for $\eta < 1$ the short term rate is necessarily higher than the asymptotic rate.

In order to determine the whole shape of the yield curve, we must determine explicit formulas for the socially efficient discount rates. In particular, as highlighted by the previous analysis, we need to determine the coefficients γ_i^T. We need also to analyse how the individual risk tolerances evolve over time, since they represent the weights in the consensus characteristics. Besides, we have seen that the covariance between individual risk tolerance and individual beliefs and taste parameters play an important part. This means that it is useful to consider specific settings, which is the aim of the next section.

6. Specific settings

We consider the case in which the beliefs and taste parameters δ^i, ρ^i, as well as the aggregate endowment parameters μ and $\sigma > 0$ are constant[18].

For all i, the discount rate R_t^i is the same for all t and given by $R^i = \rho^i + \frac{\mu + \sigma \delta^i}{\eta} - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2$. We recall that even in this setting the consensus characteristics δ_M and ρ_D are time-varying, stochastic processes.

As an immediate consequence of Proposition 4.1 we obtain that, for all t, the risk free rate r_t^f and the socially efficient discount rate R_t^T are given by

$$\begin{aligned} r_t^f &= \sum_{i=1}^{2} \tau_i(t) \rho^i + \frac{\mu + \sum_{i=1}^{2} \tau_i(t) \delta^i \sigma}{\eta} - \frac{1}{2}\frac{1}{\eta}(1 + \frac{1}{\eta})\sigma^2 - \frac{1}{2}(\eta - 1) Var^\tau(\delta)(t) \quad (6.1) \\ R_t^T &= \frac{\mu}{\eta} - \frac{\sigma^2}{2\eta} + \frac{\delta^2}{2} - \frac{1}{t} \ln E\left[\left(\gamma_1^T \exp^{(\eta\delta - \sigma)W_t - \eta\rho^1 t} + \gamma_2^T \exp^{(-\eta\delta - \sigma)W_t - \eta\rho^2 t} \right)^{1/\eta} \right]. \end{aligned}$$

where $\tau_i = \frac{y^{*i}}{e^*}$ and $Var^\tau(\delta)(t) = \delta^2 \left[1 - \left(\frac{y_t^{*1} - y_t^{*2}}{e_t^*} \right)^2 \right] = 4\delta^2 \frac{y_t^{*1}}{e_t^*} \frac{y_t^{*2}}{e_t^*}$.

By Proposition 5.1, we know that $R_\infty = \inf_{i=1,\dots,N} \left(\rho^i + \frac{\sigma}{\eta}\delta^i \right) + \frac{\mu}{\eta} - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2$ if the agent with the lowest subjectively expected socially efficient discount rate "does not vanish."

[18] One could object that when μ is constant, agents will learn and then that the $\delta^{i\prime}s$ can not be taken as constant. However; the case with constant parameters without taking learning into account seems to be a good approximation of the situation where all the parameters are stochastic and where learning is regularly compensated by new shocks on μ.

6.1. Logarithmic utility functions

As we have underlined it in previous sections, the case of "myopic" utility functions is very specific. Indeed, in that case, the socially efficient discount factor can be expressed as a weighted arithmetic average of the individual subjectively expected ones[19], more precisely $A_t = \sum_i \gamma_i A_t^i$. There is no beliefs aggregation bias in the expression of the risk free rate and the socially efficient risk free rate can be written as an average of the individual subjectively expected risk free rates, i.e. $r^f = \sum_i \tau_i \left(r^i\right)^f$.

6.1.1. With two agents and rational average belief

We start by considering the case with two agents, $i = 1, 2$, differing in their beliefs δ^i. Moreover, we suppose that agent 1 is optimistic (i.e., $\delta^1 \geq 0$), that agent 2 is pessimistic (i.e., $\delta^2 \leq 0$), and that the agents are "on average" rational i.e. $\delta^1 + \delta^2 = 0$ (we shall consider below a more general case). We let w_i denote the relative wealth level of agent i, and we suppose that $e^{*^i} = w_i e^*$.

Proposition 6.1. *Consider the case of logarithmic utility functions. Suppose that $\delta^1 + \delta^2 = 0$. We obtain the following results.*

1. *For all time horizon T, $\gamma_i^T = \frac{w_i \rho^i \left(1 - \exp -\rho^i T\right)^{-1}}{\sum_{j=1}^2 w_j \rho^j (1 - \exp -\rho^j T)^{-1}}$.*
2. *The socially efficient discount rate satisfies*

$$\begin{aligned} R_t^T &= \mu - \sigma^2 - \frac{1}{t} \log \left[\gamma_1^T \exp^{-\left(\rho^1 + \delta\sigma\right)t} + \gamma_2^T \exp^{-\left(\rho^2 - \delta\sigma\right)t} \right] \\ R_0 &= \mu - \sigma^2 + w_1 \left(\rho^1 + \delta\sigma\right) + w_2 \left(\rho^2 - \delta\sigma\right) \\ R_\infty &= \mu - \sigma^2 + \inf \left(\rho^1 + \sigma\delta, \rho^2 - \sigma\delta\right). \end{aligned}$$

   The yield curve $\left(R_t^T\right)_t$ is downward sloping and $R_0 \geq R_\infty$.

We recall that in the standard setting with logarithmic utility functions, the yield curve is flat and for all wealth distribution, the rational discount rate R_t is given by $R_t = r_t^f = \mu - \sigma^2$ for all t. When beliefs and time preference rates are heterogeneous, the situation is more complex and we sum up in Table 1 the possible behaviors of the socially efficient discount rate as a function of time and of beliefs dispersion.

Insert Table 1

Consider first the particular setting with $w_1 = w_2$ and $\rho^1 = \rho^2 = 0$, as illustrated in Figure 1. In this case, the discount factor is an equally weighted arithmetic average of the individual subjectively expected discount factors, as in Weitzman (1998, 2001). The short term rate is the rational rate and the long term rate is the pessimistic rate. Moreover, for all t, the discount rate R_t decreases with beliefs dispersion (see Table 1).

[19] As shown in Appendix A-1, in that case, individual heterogeneous beliefs can be aggregated into a consensus belief and time preference rates can be aggregated into a consensus time preference rate and there is no aggregation bias.

Insert Figure 1

The setting with possibly different wealth levels is illustrated in Figure 2. The consensus discount factor is a wealth-weighted arithmetic average of the individual subjective discount factor. The yield curve is still downward sloping. The short term rate is a wealth weighted average of the individually expected short term rates, which differs from the rational rate if $w_1 \neq w_2$. If there is a positive correlation between optimism and initial wealth, then the short-term discount rate is higher than in the rational setting and an increase in beliefs heterogeneity increases the short-term rate. The long term rate is still given by the pessimistic rate and an increase in beliefs dispersion always decreases the long term rate. Finally, an increase in the initial relative wealth of the optimistic agent induces a higher short-term discount rate and a greater spread between the short-term discount rate R_0 and the long-term discount rate R_∞, which is always positive.

Insert Figure 2

Consider now the setting with possible divergence in the pure time preference rates. The yield curve $\left(R_t^T\right)_t$ is still decreasing. The asymptotic as well as the short term behaviors of the discount rate are similar to those observed in the setting with zero pure time discount rate, replacing $\delta^i\sigma$ by $\rho^i + \delta^i\sigma$. In particular, the asymptotic discount rate is the lowest subjectively expected asymptotic discount rate. However, as a consequence of the dependence in T of γ_i^T, R_T^T is not necessarily decreasing with T (see Table 1). Consider the case in which $\rho^1 + \sigma\delta < \rho^2 - \sigma\delta$, which is illustrated in Figure 3. In that case, the relative weight of agent 1 decreases with T: there is a bias in favor of agent 2. For small T, an increase in T induces a higher weight on the discount rate of agent 2, which is the highest discount rate, inducing an increase in R_T.

Insert Figure 3

6.1.2. More general settings (with logarithmic utility functions)

The results of Section 6.1.1 are easily generalized to the setting with N agents, with constant beliefs δ^i (not necessarily neutral on average), constant pure time preference rate ρ^i and wealth levels w_i. In such a setting, we obtain as an immediate extension of the results of the previous section that

$$\begin{aligned}
\gamma_i^T &= \frac{w_i \rho^i \left(1 - \exp -\rho^i T\right)^{-1}}{\sum_{j=1}^N w_j \rho^j \left(1 - \exp -\rho^j T\right)^{-1}} \\
R_t^T &= \mu - \sigma^2 - \frac{1}{t} \log \left[\sum_{i=1}^N \gamma_i^T \exp^{-\left(\rho^i + \delta^i \sigma\right) t} \right] \\
R_0 &= \mu - \sigma^2 + \sum_{i=1}^N w_i \left(\rho^i + \sigma \delta^i\right), \ R_\infty = \mu - \sigma^2 + \inf_{i=1,\dots N} \left(\rho^i + \sigma \delta^i\right).
\end{aligned}$$

Let us first analyse the impact of the average level of pessimism/optimism. In the previous sections, we have supposed $\sum_{i=1}^{N}\delta^i = 0$. If there is a (pessimistic or optimistic) bias on average with $\frac{1}{N}\sum_{i=1}^{N}\delta_i = \overline{\delta} \neq 0$, then there is an additional optimism (when $\overline{\delta}$ is positive) or pessimism (when $\overline{\delta}$ is negative) effect on the risk free rate and on the discount rate. Pessimism at the aggregate level reduces the risk free rate as well as the discount rate, which is intuitive.

This "average level of pessimism" effect disappears in the long run, since the long run discount rate is given by the lowest subjectively expected discount rate. This is not the case for the short-term discount rate which is given by an average of the individual subjectively expected (constant) discount rates. The long run discount rate is then necessarily lower than the short term discount rate. Moreover, the yield curve $\left(R_t^T\right)_{t\leq T}$ is still decreasing.

We can consider settings with a continuous distribution on the individual beliefs $\left(\delta^i\right)$. If we suppose that wealth is equally distributed[20], then we obtain as an easy extension of the discrete setting that the discount factor is an average of the individual subjectively expected discount factors, i.e. $A_t = E^i\left[A_t^i\right]$. For instance in the case of a normal distribution on individual beliefs, i.e. if $\left(\delta^i\right) \sim \mathcal{N}\left(0, \varpi^2\right)$, we obtain that $R_t = \mu - \sigma^2 - \frac{1}{2}\sigma^2\varpi^2 t$; the discount rate is clearly decreasing and converges asymptotically to the lowest possible rate (among agents) which is $-\infty$. Consider now the more realistic case of a Gamma distribution[21] on the individual subjectively expected discount rates, as in Weitzman (2001), i.e. if $\left(R^i\right) \sim \gamma\left(\alpha, \beta\right)$. Note that in our framework, $R^i = \mu - \sigma^2 + \delta^i\sigma$, and considering a given distribution on the $\left(R^i\right)$ is equivalent to considering the translated distribution on the $\left(\delta^i\right)$. As in Weitzman (2001), we obtain (see Appendix A-2) that $A_t = \left(1 + t\sigma^2/\mu\right)^{-\mu^2/\sigma^2}$, hence $-\frac{A_t'}{A_t} = \frac{\mu}{1+t\sigma^2/\mu}$, or $R_t = \frac{\mu^2}{\sigma^2 t}\log\left(1 + t\sigma^2/\mu\right)$ which is also clearly decreasing and converges to the lowest possible discount rate (which is zero) when t goes to infinity.

The main difference between the setting with two agents, who are on average rational, and the general setting with N agents who are not necessarily rational on average lies in the distribution among agents of individual optimal consumptions as well as of individual wealth levels at time t. Consider for instance the setting of Proposition 6.1 with $\rho^1 = \rho^2 = 0$ and $w_1 = w_2$. It is easy to obtain that for all t, the ratio of individual wealth levels at time t as well as the ratio of individual optimal consumptions at time t are given by $\frac{{y_t^*}^1}{{y_t^*}^2}$ and that this ratio has the same distribution as its inverse $\frac{{y_t^*}^2}{{y_t^*}^1}$, which means that none of the agents "wins". What happens now if, for instance, one agent is pessimistic (or optimistic) with $M^1 = \mathcal{E}\left(\delta_1\right)$ and the other is rational with $M^2 \equiv 1$. In this case, the weights γ_i^T are the same as in the previous setting since they do not depend upon individuel beliefs. However, agent 1 is "wrong" while agent 2 is "right" and the consumption level of agent 1 vanishes in the long run[22]. Indeed, the ratio of optimal consumptions at time t, $\frac{{y_t^*}^2}{{y_t^*}^1}$ has the same distribution as $\exp^{\delta^2 t}\frac{{y_t^*}^1}{{y_t^*}^2}$, which means that as t increases, ${y_t^*}^2$ becomes very important with respect to ${y_t^*}^1$. The same is true of the ratio of the individual wealth levels at time t. The equilibrium risk free rate converges then to the rational rate[23]. However, we still have that the asymptotic discount rate is given

[20] More precisely, we need that $E^i\left[\frac{M^i}{\lambda_i}\right] = E^i\left[M^i\right]$.

[21] We recall that the density of a Gamma distribution $\gamma\left(\alpha, \beta\right)$ is $f\left(x\right) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp^{-\beta x} 1_{x>0}$.

[22] The same would be valid as soon as one agent is "more wrong" than the other agent.

[23] Indeed, it can be shown that $\tau_1 \to 0$, a.s., and $\tau_2 \to 1$, a.s.

by the more pessimistic individually anticipated asymptotic rate. The asymptotic rate R_∞ is then equal to $\mu-\sigma^2$ if the irrational agent is optimistic, and to $\mu-\sigma^2+\sigma\delta_1<\mu-\sigma^2$ if the irrational agent is pessimistic.

6.2. Power utility functions

We still consider the setting with two agents, one optimistic ($\delta^1=+\delta\in\mathbb{R}_+^*$), one pessimistic $\left(\delta^2=-\delta\right)$ but we now assume that they are endowed with power utility functions. As we have underlined it previously, there are essentially two different settings, $\eta<1$ or $\eta>1$, for which the impact of beliefs divergence is opposite.

6.2.1. The case $\eta<1$

Let us start by considering the specific case $\eta=1/2$. In this case, we recall that in the standard setting the yield curve is flat and, for all t, $R_t=r_t^f=2\mu-3\sigma^2$.

Proposition 6.2. *Consider the case of power utility functions with $\eta=1/2$. We suppose*[24] *that $\mu-\sigma^2-\sigma\delta>0$. Suppose that $\delta^1+\delta^2=0$, $w_1=w_2$ and $\rho^1=\rho^2=0$.*

1. *We have $\left(\frac{\gamma_1^T}{\gamma_2^T}\right)^2=\frac{\left(-\mu+\sigma^2-\sigma\delta\right)}{\left(-\mu+\sigma^2+\sigma\delta\right)}\frac{\exp^{\left(-\mu+\sigma^2+\sigma\delta\right)T}-1}{\exp^{\left(-\mu+\sigma^2-\sigma\delta\right)T}-1}$, which is an increasing function of T, and*

$$\lim_{T\to0}\frac{\gamma_1^T}{\gamma_2^T}=1,\lim_{T\to\infty}\frac{\gamma_1^T}{\gamma_2^T}=\sqrt{\frac{\mu-\sigma^2+\sigma\delta}{\mu-\sigma^2-\sigma\delta}}$$

2. *For all t, the discount rate is given by*

$$\begin{aligned}
R_t^T &= 2\mu-3\sigma^2-\frac{1}{t}\ln\left(\left(\gamma_1^T\right)^2\exp^{-2\delta\sigma t}+\left(\gamma_2^T\right)^2\exp^{2\delta\sigma t}+2\gamma_1^T\gamma_2^T\exp^{-\frac{1}{2}\delta^2 t}\right)\\
R_0^T &= \left(r_0^T\right)^f=2\mu-3\sigma^2+2\delta\sigma\left(\gamma_1^T-\gamma_2^T\right)+\frac{1}{4}\delta^2\left[1-\left(\gamma_1^T-\gamma_2^T\right)^2\right]\\
R_\infty &= 2\mu-3\sigma^2-2\sigma\delta\leq R_0^T
\end{aligned}$$

The discount rate is a decreasing function of t.

According to the first point of the proposition, the relative weight of the optimistic agent is greater than the relative weight of the pessimistic agent, i.e., $\gamma_1^T>\gamma_2^T$: there is an optimistic bias at the aggregate level, and this bias increases with the time horizon, i.e. γ_1^T increases with T. This result is valid in the general setting with $\eta<1$ (see Appendix A-2). This means that an increase in beliefs dispersion leads to an increase of the short term rate. Notice that in the long term an increase in beliefs dispersion leads to a decrease of the discount rate since it corresponds to a more pessimistic belief for the more pessimistic agent.

The main result we obtain is the fact that the yield curve is decreasing. In fact, as shown on Equation (4.3), the risk free rate is equal to an average of the individually anticipated risk free rates plus a beliefs dispersion term $\frac{1}{2}\left(\eta-1\right)Var^\tau\left(\delta\right)$. For $\eta=1$, there is no beliefs

[24] This condition ensures that the individual optimization programs are well posed for both agents even for an infinite horizon.

dispersion term and, as already seen, the socially efficient discount rate R_t^T decreases with t and converges to the more pessimistic individually anticipated one. For general η, we have $\left(\frac{y_t^{*^1}/\gamma_1^T}{y_t^{*^2}/\gamma_2^T}\right) = \left(\frac{M_t^2}{M_t^1}\right)^{\eta} \sim \ln \mathcal{N}\left(0, 2\eta\delta\sqrt{t}\right)$. In particular, this implies that, for large t, "$y_t^{*^1}$ is large with respect to $y_t^{*^2}$ or $y_t^{*^2}$ is large with respect to $y_t^{*^1}$" with a probability near 1. Loosely speaking, there are two kinds of states of the world, those for which $y_t^{*^1}$ vanishes for large t and those for which $y_t^{*^2}$ vanishes for large t. The beliefs dispersion term $\frac{1}{2}(\eta-1)\,Var^{\tau}(\delta) = \frac{1}{4}\frac{y_t^{*^1}}{e_t^*}\frac{y_t^{*^2}}{e_t^*}$ vanishes then asymptotically. The discount rate curve is then globally decreasing and converges, as in the logarithmic case, to the most pessimistic rate. Everything works then as if we had two scenarios one with the optimistic rate and one with the pessimistic rate and the asymptotic socially efficient discount rate is associated to the worst scenario as in Gollier (2002) and Weitzman (1998, 2001).

In Figure 4, we represent[25] the discount rate curve R_T^T.

Insert Figure 4

In Figure 5 we represent the yield curve as well as the rates associated to the $\eta-$average and to the arithmetic average of the individual discount factors. All these curves converge asymptotically to the rate associated to the most pessimistic belief but the "η" one is a much better approximation of the yield curve than the "arithmetic" one. The distance between the yield curve and the "η" curve measures the impact of the bias due to beliefs dispersion. The variance term increases the short rate but its impact decreases with t and vanishes asymptotically. However this impact can remain non negligible for centuries.

Insert Figure 5

We have assumed so far that both agents are endowed with the same initial wealth. If we relax this assumption, we still obtain decreasing yield curves converging to the most pessimistic rate. However, when the more optimistic agent is wealthier, she has a greater weight in the average formula and the impact of her optimism lasts longer. The yield curve has then a higher starting point at $T = 0$ and its initial slope is smaller. When the more pessimistic agent is wealthier, she has a larger weight, the starting point of the yield curve is lower and the convergence to its asymptotic rate is more rapid.

6.2.2. The case $\eta > 1$

As can be seen e.g. in Equation (6.1) and as already underlined, the risk free rate exhibits both an average belief/time preference effect that is measured by $\sum \tau_i \delta_i$ and by $\sum \tau_i \rho_i$ and a beliefs dispersion effect that is measured by the variance term. As in the case $\eta < 1$, the average effect induces a decrease in the yield curve (since the associated rate converges asymptotically to the lowest rate) and the variance term decreases and vanishes asymptotically. However, we have now $\eta < 1$ and the sign of the beliefs dispersion term is then negative. This leads then to two

[25] In order to evaluate the integrals and expectations that are involved in the formulas we used the adaptive quadrature implemented in the R function INTEGRATE to approximate the target integral. This function is based on QUADPACK routines DQAGS and DQAGI (R. Piessens and E. deDoncker-Kapenga,1983) available from Netlib.

opposite effects, the average effect inducing a decrease of the yield curve and the dispersion effect inducing an increase in the yield curve. Depending on the relative size of these effects, we may obtain decreasing curves as in the case $\eta < 1$ as well as increasing then decreasing curves as in Figure 6. The case $\eta > 1$ leads then to a richer family of possible shapes and is compatible with the fact that long-term rates in bonds markets (i.e. $T = 30$) are usually higher than short-term rates. In the case of an initially increasing yield curve, this initial shape results from beliefs dispersion and is not retrieved when we approximate the yield curve by the rates associated to the $\eta-$average or the arithmetic average of the individual discount factors as can be seen in Figure 6.

Insert Figure 6

7. Conclusion

When public investment projects entail costs and benefits in the very long run, a question arises about the selection of the relevant discount rate to use for the Cost-Benefit Analysis. Indeed, financial markets do not provide a guideline in this case.

In this paper we provide an equilibrium analysis of the effect of expectations heterogeneity and pure time preference rates heterogeneity on the socially efficient discount factor in a (quite general) stochastic setting.

First, we show that the certainty equivalent approach of Weitzman (1998, 2001) that consists in taking the arithmetic average of the individually recommended discount factors is compatible with an equilibrium approach if we assume that all utility functions are logarithmic and that all agents have the same initial wealth. However, for more general utility functions and wealth distributions, the right concept of average to consider within an equilibrium approach is not the arithmetic average but an $\eta-$average. Moreover, this $\eta-$average is a weighted average. Finally, there is a bias induced by beliefs and pure time preference rates dispersion.

Increased beliefs and/or time preference rates dispersion leads to higher or lower discount rates depending on the position of the relative risk aversion parameter with respect to 1. In fact, beliefs and time preference rate heterogeneity impact the socially efficient discount factor as would an additional source of risk.

In the long run, we show that the asymptotic discount rate is the lowest individually anticipated discount rate, which is another element in favor of decreasing discount rates in the long run.

In the short and in the medium term, beliefs heterogeneity leads to a richer class of possible shapes for the yield curve. In particular, we may obtain increasing yield curves for the first 50 years as it is usually the case in financial markets.

Appendix

A-1 Aggregation of individual beliefs and time-preferences

The aim of this Appendix is to adapt the results of Jouini and Napp (2007) to our setting with heterogeneous time preference rates. We deal with aggregation issues in the spirit of Varian

(1985, 1989), Abel (1989), Calvet et al. (2002), Shefrin (2005). We obtain the following result.
Proposition A-1 *We let N^i denote the individual composite characteristic $M^i D^i$.*

1. *The individual characteristics N^i can be aggregated into a consensus characteristic N such that*
$$q_t^* = N_t u'(e_t^*)$$
with
$$N = \left[\sum_{i=1}^{N} \gamma_i \left(N_t^i\right)^{\eta}\right]^{1/\eta} = \left[\sum_{i=1}^{N} \tau_i \left(N_t^i\right)^{-\eta}\right]^{-1/\eta}.$$

2. *The consensus characteristic N can be written in the form*
$$N = BDM$$
where M is a consensus probability belief, D is a pure time consensus discount factor and B is an aggregation bias, related to beliefs dispersion. More precisely, the martingale process M and the finite variation processes D and B satisfy $dM_t = \delta_M M_t dW_t$, $dD_t = -\rho_D D_t dt$, $dB_t = -\rho_B B_t dt$ with $M_0 = D_0 = B_0 = 1$ and
$$\begin{aligned} \delta_M &= \sum_{i=1}^{N} \tau_i \delta^i, \qquad \rho_D = \sum_{i=1}^{N} \tau_i \rho^i \\ \rho_B &= \frac{1}{2}(1-\eta)\left[\sum_{i=1}^{N} \tau_i \left(\delta^i\right)^2 - \delta_M^2\right] = \frac{1}{2}(1-\eta)\, Var^{\tau}(\delta) \end{aligned}$$
We let Q denote the consensus probability measure aggregating the individual subjective probability measures Q^i, i.e. $\frac{dQ}{dP} = M_T$.

Proof
1. Since q^* is an interior equilibrium price process, we know that there exist Lagrange multipliers (λ_i) such that for all i and for all t,
$$\frac{1}{\lambda_i} N_t^i u'\left(y_t^{i^*}\right) = q_t^* \tag{7.1}$$
We easily obtain that
$$\left(\frac{1}{\lambda_i} N_t^i\right)^{\eta} = \left(q_t^*\right)^{\eta} \left(y_t^{i^*}\right).$$
Since $\sum_{i=1}^{N} y_t^{i^*} = e_t^*$, we get
$$q_t^* = N_t u'(e_t^*) \text{ with } N_t = \left[\sum_{i=1}^{N} \gamma_i \left(N_t^i\right)^{\eta}\right]^{1/\eta}. \tag{7.2}$$
The other formulation for N is obtained analogously; indeed, since
$$\frac{1}{\lambda_i} N_t^i u'\left(y_t^{i^*}\right) = N_t u'(e_t^*), \tag{7.3}$$

we have

$$\sum_{i=1}^{N}\left(N_t^i\right)^{-\eta} y_t^{i^*}=\sum_{i=1}^{N}\left(\frac{1}{\lambda_i}\right)^{\eta} u'\left(e_t^*\right)^{-\eta} N_t^{-\eta}$$

hence

$$N_t=\left[\sum_{i=1}^{N}\tau_i\left(N_t^i\right)^{-\eta}\right]^{-1/\eta}.$$

2. We can write that $dy_t^{i^*}=a_i(t)dt+b_i\left(t\right)dW_t$ for processes (a_i) and (b_i) such that $\sum_{i=1}^{N} a_i\left(t\right)=\mu_t e_t^*$ and $\sum_{i=1}^{N} b_i\left(t\right)=\sigma_t e_t^*$. Analogously, we introduce the processes μ_N and δ_M such that $dN_t=\mu_N\left(t\right)N_t dt+\delta_M\left(t\right)N_t dW_t$. We apply Itô's Lemma to both sides of Equation (7.3). Identifying the diffusion parts leads to

$$\frac{1}{\lambda_i}N^i u_i''\left(y^{*^i}\right)b_i+\frac{1}{\lambda_i}N^i u_i'\left(y^{*^i}\right)\delta^i=Nu'\left(e^*\right)\delta_M+Nu''\left(e^*\right)\sigma e^*$$

hence, using Equation (7.3) ,

$$\delta_M=\frac{u_i''\left(y^{*^i}\right)}{u_i'\left(y^{*^i}\right)}b_i+\delta^i-\frac{u''\left(e^*\right)}{u'\left(e^*\right)}\sigma e^* \tag{7.4}$$

and

$$\delta_M=\sum_{i=1}^{N}\tau_i\delta^i.$$

Identifying the drift parts leads to

$$\mu_N=-\frac{1}{T_i}a_i-\frac{u''\left(e^*\right)}{u'\left(e^*\right)}\mu e^*-\rho_i+\frac{1}{2}\left(\eta+1\right)\frac{b_i^2}{T_i^2}-\frac{\delta^i b_i}{T_i}-\frac{1}{2}\frac{u'''\left(e^*\right)}{u'\left(e^*\right)}\sigma^2\left(e^*\right)^2-\delta_M\frac{u''\left(e^*\right)}{u'\left(e^*\right)}\sigma e^*.$$

Replacing b_i by its expression in Equation (7.4) and simple computation leads to

$$\mu_N=-\frac{1}{T_i}a_i-\frac{u''\left(e^*\right)}{u'\left(e^*\right)}\mu e^*-\rho_i+\frac{1}{2}\left(\eta+1\right)\delta_M^2+\frac{1}{2}\left(\eta-1\right)\delta_i^2-\eta\delta_M\delta^i+\sigma\delta^i-\sigma\delta_M.$$

Simple computation leads to

$$\mu_N=\frac{1}{2}\left(\eta-1\right)\left[\sum_{i=1}^{N}\tau_i\left(\delta^i\right)^2-\delta_M^2\right]-\sum_{i=1}^{N}\tau_i\rho^i.$$

Since $dN_t=\mu_N\left(t\right)N_t dt+\delta_M\left(t\right)N_t dW_t$ with $N_0=1$, we have for all $t, N_t=\exp\int_0^t \mu_N-\frac{\left(\delta_M\right)^2}{2}ds+\int_0^t\delta_M dW_s$, or equivalently

$$N_t=\exp^{\int_0^t\frac{1}{2}(\eta-1)\left[\sum_{i=1}^{N}\tau_i\left(\delta^i\right)^2-\delta_M^2\right]ds}\exp^{-\int_0^t\sum_{i=1}^{N}\tau_i\rho^i ds}\exp^{\int_0^t\delta_M dW_s-\int_0^t\frac{\left(\delta_M\right)^2}{2}ds}$$

which is of the form $N=BDM$. ■

Proposition A-1 means that the Arrow-Debreu prices are the same as in an economy in which the representative agent, endowed with the same utility function as the agents in the economy, has a consensus characteristic N which is an $\eta-$average of the individual initial characteristics

$N^i \equiv M^i D^i$. The process N can be decomposed into a consensus belief M and a consensus discount factor D modulo the introduction of a finite variation process B. The consensus belief parameter δ_M is the risk tolerance weighted arithmetic average of the individual beliefs (diffusion) parameters δ^i. Notice that under the consensus probability Q, the dynamics of aggregate endowment is given by $de_t^* = (\mu + \delta_M \sigma) e_t^* dt + \sigma_t e_t^* dW_t^Q$, where by Girsanov Theorem W^Q is a Brownian motion under Q, which means that the instantaneous expected rate of return at the aggregate level can be written as a risk tolerance weighted average of the individual subjectively expected rates of return, i.e., $\mu^Q \equiv (\mu + \delta_M \sigma) = \sum_{i=1}^N \tau_i \mu^i$. The consensus pure time discount rate ρ_D is also a risk tolerance weighted arithmetic average of the individual pure time discount rates ρ^i. The presence of the factor B makes the setting with heterogeneous beliefs fundamentally different from the setting with homogeneous beliefs. This factor represents an aggregation bias and is directly related to beliefs dispersion (and not to time preference rates dispersion). Notice that since the factor B is of finite variation, its effect can be interpreted as a modification of the rate of time preference. Indeed, since B is of the form $B_t = \exp^{-\int_0^t \frac{1}{2}(1-\eta) Var^\tau(\delta) ds}$, the presence of this aggregation bias is analogous to a bias on the consensus time preference rate. Since $Var^\tau(\delta)$ is always nonnegative, it induces a lower time-preference rate for $\eta > 1$ and a higher time-preference rate for $\eta < 1$. This means that beliefs heterogeneity induces a bias on the trade off between present and future consumption.

Let us underline two specific settings. The myopic logarithmic setting (with $\eta = 1$) appears as very specific; the consensus characteristic is then an arithmetic average of the individual ones, and there is no aggregation bias.

In the particular setting with rational beliefs, i.e. if for all i, $M^i \equiv 1$, we obtain that $N = D = \left[\sum_{i=1}^N \tau_i \left(D^i\right)^{-\eta}\right]^{-1/\eta}$. There is then no aggregation bias. The consensus characteristic N is also the consensus discount factor and is given by the risk tolerance weighted average of the individual discount factors. This is the analog of the result of Gollier and Zeckauser (2005), obtained in a deterministic setting. Analogously, if there is no divergence of opinion, i.e. for all i, $M^i \equiv M$ (beliefs are then subjective but not heterogeneous) then $N = MD$ and there is no aggregation bias.

A-2 Proofs

Proof of Proposition 3.1

By Proposition A-1, we have

$$\begin{aligned} q_t^* &= \left[\sum_{i=1}^N \gamma_i \left(N_t^i\right)^\eta\right]^{1/\eta} \left(e_t^*\right)^{-\frac{1}{\eta}} = \left[\sum_{i=1}^N \gamma_i \left(N_t^i \left(e_t^*\right)^{-\frac{1}{\eta}}\right)^\eta\right]^{1/\eta} \\ &= \left[\sum_{i=1}^N \gamma_i \left(q_t^{i*}\right)^\eta\right]^{1/\eta}. \quad \blacksquare \end{aligned}$$

Analogously, using Proposition A-1, we obtain that $q^* = \left[\sum_{i=1}^N \tau_i \left(q_t^{i*}\right)^{-\eta}\right]^{-1/\eta}$.

Proof of Proposition 3.2

1. In the deterministic setting, we have $A_t^i = q_t^{i^*}$ and $A_t = q_t^*$, hence by Proposition 3.1,

$$A_t = \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^{\eta}\right]^{1/\eta}.$$

2. Since $A_t = E\left[q_t^*\right]$, we get from Proposition 3.1 that

$$A_t = E\left\{\left[\sum_{i=1}^{N} \gamma_i \left(q_t^{i^*}\right)^{\eta}\right]^{1/\eta}\right\}.$$

In the case $\eta = 1$, we have $A_t = E\left[\sum_{i=1}^{N} \gamma_i q_t^{i^*}\right] = \sum_{i=1}^{N} \gamma_i E\left[q_t^{i^*}\right] = \sum_{i=1}^{N} \gamma_i A_t^i$.
In the case $\eta < 1$, we have

$$\begin{aligned} A_t^{\eta} &= \left\|\sum_{i=1}^{N} \gamma_i \left(q_t^{i^*}\right)^{\eta}\right\|_{1/\eta} \\ &\leq \sum_{i=1}^{N} \gamma_i \left\|\left(q_t^{i^*}\right)^{\eta}\right\|_{1/\eta} = \sum_{i=1}^{N} \gamma_i \left\|q_t^{i^*}\right\|_1^{\eta} = \sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^{\eta} \end{aligned}$$

hence $A_t \leq \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^{\eta}\right]^{1/\eta}$.

In the case $\eta > 1$, using Minkovski's Lemma, we get analogously that

$$E\left\{\left[\sum_{i=1}^{N} \gamma_i \left(q_t^{i^*}\right)^{\eta}\right]^{1/\eta}\right\}^{\eta} \geq \sum_{i=1}^{N} \gamma_i E\left[q_t^i\right]^{\eta},$$

hence $A_t \geq \left[\sum_{i=1}^{N} \gamma_i \left(A_t^i\right)^{\eta}\right]^{1/\eta}$. ■

Proof of Proposition 4.1

We adopt the notations of Appendix A-1. We let μ_{q^*} (resp. σ_{q^*}) denote the drift (resp. diffusion) parameter of the process q^*, i.e. $dq_t^* = \mu_{q^*} q_t^* dt + \sigma_{q^*} q_t^* dW_t$. Since, by Proposition A-1, $q^* = Nu'(e^*)$, we easily get through Itô's Lemma that

$$\mu_{q^*} = \mu_{u'(e^*)} + \mu_N + \delta_M \left(\frac{u''(e^*)}{u'(e^*)} \sigma e^*\right)$$

where $\mu_{u'(e^*)}$ denotes the drift parameter of the process $u'(e^*)$, and

$$\sigma_{q^*} = \delta_M + \frac{u''(e^*)}{u'(e^*)} \sigma e^*$$

1. Since q^* is a state price density, we obtain as in the standard setting that $r^f = -\mu_{q^*}$. We know from the proof of Proposition A-1 that $\delta_M = \sum_{i=1}^{N} \tau_i \delta^i$ and that $\mu_N = \frac{1}{2}(\eta - 1)\left[\sum_{i=1}^{N} \tau_i \left(\delta^i\right)^2 - \delta_M^2\right] - \sum_{i=1}^{N} \tau_i \rho^i$. Letting r^f(stdard) denote $-\mu_{u'(e^*)}$, i.e. the risk free rate that would prevail if agents

held rational beliefs ($Q^i = P$), and had a zero rate of time preference, we easily deduce that

$$\begin{aligned} r^f &= r^f(\text{standard}) - \frac{1}{2}(\eta-1)\left[\sum_{i=1}^{N}\tau_i\left(\delta^i\right)^2 - \delta_M^2\right] + \sum_{i=1}^{N}\tau_i\rho^i + \frac{1}{\eta}\left(\sum_{i=1}^{N}\tau_i\delta^i\right)\sigma \\ &= \sum_{i=1}^{N}\tau_i\rho_i + \frac{1}{\eta}\left[\mu + \left(\sum_{i=1}^{N}\tau_i\delta_i\right)\sigma\right] - \frac{1}{2}\frac{1}{\eta}\left(1+\frac{1}{\eta}\right)\sigma^2 - \frac{1}{2}(\eta-1)\,Var^{\tau}(\delta)\,. \end{aligned}$$

2. Immediate using 1.
3. The first equation results from the definition of $R_t = -\frac{1}{t}\log E\left[q_t^*\right]$ and from Proposition A-1. For the second equation, we have $R_t = -\frac{1}{t}\log E\left[q_t^*\right]$ hence

$$\begin{aligned} R_t &= -\frac{1}{t}\log E\left[\exp\int_0^t\left(\mu_{q^*} - \frac{\left(\sigma_{q^*}\right)^2}{2}\right)ds + \int_0^t\sigma_{q^*}dW_s\right] \\ &= -\frac{1}{t}\log E\left[\exp^{\int_0^t\mu_{q^*}ds}\exp^{\int_0^t\sigma_{q^*}dW_s - \int_0^t\frac{\left(\sigma_{q^*}\right)^2}{2}ds}\right]. \end{aligned}$$

Since $\mu_{q^*} = -r^f$ and $\sigma_{q^*} = \delta_M - \frac{\sigma}{\eta}$, we obtain

$$R_t = -\frac{1}{t}\log E^{\overline{Q}_T}\left[\exp^{-\int_0^t r_s^f ds}\right].$$

■

Proof of $-\frac{1}{t}\log E\left[M_t u'\left(e_t^*\right)\right] \leq -\frac{1}{t}\log E\left[u'\left(e_t^*\right)\right]$ when $\delta_M \leq 0$ and $-\frac{1}{t}\log E\left[M_t B_t u'\left(e_t^*\right)\right] \leq -\frac{1}{t}\log E\left[u'\left(e_t^*\right)\right]$ when $\delta_M \leq 0$, $\eta > 1$ (μ and σ are deterministic).

Since $de_t^* = e_t^*\left(\mu_t dt + \sigma_t dW_t\right)$ with $e_0^* = 1$, we have

$$e_t^* = \exp\left[\int_0^t\left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^t\sigma_u dW_u\right].$$

Letting $W_t^Q \equiv W_t - \int_0^t\delta^M(u)\,du$, we know, by Girsanov's Lemma, that W^Q is a Q-Brownian motion where Q is defined by $\frac{dQ}{dP} = M$ and, for $s \geq t$, we have

$$\left(\frac{e_s^*}{e_t^*}\right)^{-1/\eta} = \exp\left(-1/\eta\right)\left[\int_t^s\left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_t^s\sigma_u dW_u^Q\right]\exp\left(-1/\eta\right)\int_t^s\sigma_u\delta^M(u)\,du.$$

If $\delta^M \leq 0$, then $\exp\left(-1/\eta\right)\int_t^s\sigma_u\delta^M(u)\,du \geq 1$, hence,

$$\begin{aligned} E_t\left[M_s\left(e_s^*\right)^{-1/\eta}\right] &= \left(e_t^*\right)^{-1/\eta}E_t\left[M_s\left(\frac{e_s^*}{e_t^*}\right)^{-1/\eta}\right] \\ &\geq \left(e_t^*\right)^{-1/\eta}E_t^Q\left[\exp\left(-1/\eta\right)\left[\int_t^s\left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_t^s\sigma_u dW_u^Q\right]\right] \\ &\geq \left(e_t^*\right)^{-1/\eta}E_t\left[\exp\left(-1/\eta\right)\left[\int_t^s\left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_t^s\sigma_u dW_u\right]\right] \\ &\geq E_t\left[\left(e_s^*\right)^{-1/\eta}\right]. \end{aligned}$$

It suffices now to remark that $B_t \geq 1$, a.s. for $\eta \geq 1$ to obtain $E_t\left[M_s B_s \left(e_s^*\right)^{1-1/\eta}\right] \geq E_t\left[\left(e_s^*\right)^{1-1/\eta}\right]$. ■

Proof of Proposition 5.1

We let q_t^T (resp. $q_t^{T,i}$) denote the Arrow-Debreu prices (resp. the individual subjectively anticipated Arrow-Debreu prices) when the agents horizon is T. We start by comparing in 1. the socially efficient discount factor A_t^T with the individually anticipated ones.

1. For $\eta = 1$, we know by Proposition 3.2 that $A_t^T = \sum_{i=1}^N \gamma_i^T A_t^{T,i}$, hence

$$\sup_i \left(\gamma_i^T A_t^{T,i}\right) \leq A_t^T \leq \sup_i A_t^{T,i}.$$

For $\eta < 1$, we have seen in Proposition 3.2 that $A_t^T \leq \left[\sum_{i=1}^N \gamma_i^T \left(A_t^{T,i}\right)^\eta\right]^{1/\eta}$. Since $\sum_{i=1}^N \gamma_i^T = 1$, we get that $A_t^T \leq \sup_i A_t^{T,i}$. Moreover, since $A_t^T = E\left[q_t^T\right]$, we get from Proposition 3.1 that

$$\begin{aligned} A_t^T &= E\left\{\left[\sum_{i=1}^N \gamma_i^T \left(q_t^{T,i}\right)^\eta\right]^{1/\eta}\right\} \\ &\geq E\left\{\left[\gamma_i^T \left(q_t^{T,i}\right)^\eta\right]^{1/\eta}\right\} = E\left[\left(\gamma_i^T\right)^{1/\eta} q_t^{T,i}\right] \end{aligned}$$

hence $A_t^T \geq \sup_i \left(\left(\gamma_i^T\right)^{1/\eta} A_t^{T,i}\right)$.

For $\eta > 1$, we get analogously that $A_t^T \geq \sup_i \left(\left(\gamma_i^T\right)^{1/\eta} A_t^{T,i}\right)$. Moreover, since for nonnegative real numbers

$$\left(\sum_{i=1}^N a_i\right)^{1/\eta} \leq \sum_{i=1}^N a_i^{1/\eta} \text{ for } \eta > 1,$$

we get that

$$\begin{aligned} A_t^T &= E\left\{\left[\sum_{i=1}^N \gamma_i^T \left(q_t^{T,i}\right)^\eta\right]^{1/\eta}\right\} \\ &\leq E\left[\sum_{i=1}^N \left(\gamma_i^T\right)^{1/\eta} q_t^{T,i}\right] = \sum_{i=1}^N \left(\gamma_i^T\right)^{1/\eta} A_t^{T,i}. \end{aligned}$$

2. Consider first the case $\eta = 1$. Since $A_t^{T,i} = \exp^{-R_t^{T,i} t}$, we have $\sup_i A_t^{T,i} = \exp^{-\inf_i R_t^{T,i} t}$. For all (t, T), let us denote by $i(t,T)$ an agent such that $\inf_i R_t^{T,i} = R_t^{T, i(t,T)}$. For t and T large enough, it is easy to check that $\exp^{-\left(\inf_i R_T^i\right) T} = \exp^{-R_T^I T}$. Since for all i, $\gamma_i^T \leq 1$, and since $\gamma_i^T \geq \varepsilon$, we have $\sup_i \left(\gamma_i^T A_T^i\right) \geq \gamma_i^T \exp^{-R_T^I T}$ for T large enough. We then get, for T large enough,

$$-\frac{1}{T} \log\left(\exp^{-R_T^I T}\right) \leq R_T \leq -\frac{1}{T} \log\left(\gamma_i^T \exp^{-R_T^I T}\right)$$

or equivalently

$$R_T^I \leq R_T \leq -\frac{\log\left(\gamma_i^T\right)}{T} + R_T^I$$

hence $\lim_{T\to\infty} R_T = R_\infty^I$.

Consider the case $\eta < 1$. As above, it is easy to verify that for T large enough $\sup_i \left(\left(\gamma_i^T\right)^{1/\eta} A_T^i \right) \geq \left(\gamma_i^T\right)^{1/\eta} \exp^{-R_T^I T}$ and that $\sup_i A_T^i = \exp^{-R_T^I T}$, hence that $R_T \rightarrow_{T\rightarrow\infty} R_\infty^I$.

Consider now the case $\eta > 1$. We have

$$\sum_{i=1}^N \left(\gamma_i^T\right)^{1/\eta} A_t^{T,i} \leq \left(\sum_{i=1}^N \left(\gamma_i^T\right)^{1/\eta} \right) \sup_i A_t^{T,i} \leq N \sup_i A_t^{T,i}.$$

For t and T large enough, we then have

$$R_t^{T,i(t,T)} - \frac{1}{t} \log N \leq R_t^T \leq R_t^{T,I} - \frac{1/\eta}{t} \log \varepsilon$$

and $R_t^T \rightarrow_{t,T\rightarrow\infty} R_\infty^I$. ■

Proof of Proposition 5.2

We know from Proposition 4.1 that for all (t,T),

$$R_t^T = -\frac{1}{t} \log E^{\overline{Q^T}} \left[\exp - \int_0^t r_s^f ds \right]$$

with $\frac{d\overline{Q^T}}{dP} = \mathcal{E}_t \exp \left(\sum_{i=1}^N (\tau_i \delta_i)(t) - \frac{\sigma(t)}{\eta} \right)$.

Let us denote by $S_t = \exp - \int_0^t r_s^f ds$, we have $dS_t = -S_t r_t^f dt$ and $E^{\overline{Q^T}} \left[\exp - \int_0^t r_s^f ds \right] = E^{\overline{Q^T}} [S_t] = 1 - E^{\overline{Q^T}} \left[\int_0^t S_s r_s^f ds \right]$. We have then $R_t^T = -\frac{1}{t} \log \left(1 - E^{\overline{Q^T}} \left[\int_0^t S_s r_s^f ds \right] \right)$ which is equivalent in a neighborhood of 0 to $\frac{1}{t} E^{\overline{Q^T}} \left[\int_0^t S_s r_s^f ds \right]$. We have then $\lim_{t\rightarrow 0} R_t^T = S_0 r_0^f = r_0^f$.

We also have $R_T^T = -\frac{1}{T} \log \left(1 - E^P \left[\int_0^T E_t^P \left(\frac{d\overline{Q^T}}{dP} \right) S_t r_t^f dt \right] \right)$ which is equivalent in a neighborhood of 0 to $\frac{1}{t} E^P \left[\int_0^t E_t^P \left(\frac{d\overline{Q^T}}{dP} \right) S_t r_t^f dt \right]$. We have then $\lim_{T\rightarrow 0} R_T^T = E^P \left(\frac{d\overline{Q^T}}{dP} \right) S_0 r_0^f = r_0^f$.

Applied to the case $N = 1$, this leads to $R_0^i = \lim_{t\rightarrow 0} R_t^{T,i} = \lim_{T\rightarrow 0} R_T^{T,i} = r_0^i$.

Furthermore, for a given horizon T, we know that $r_0^f = \sum_{i=1}^N \tau_i^T r_0^i - \frac{1}{2}(\eta - 1) \sum_{i=1}^N \tau_i^T \left(\delta_0^i - \sum_{i=1}^N \tau_i^T \delta_0^i \right)^2$ with $\tau_i^T = \frac{y_0^{T,*^i}}{e_0^*}$. It is easy to check that $\lim_{T\rightarrow 0} \gamma_i^T = \lim_{T\rightarrow 0} \tau_i^T = \frac{e_0^{*^i}}{e_0^*}$.

The comparison with $\sum_{i=1}^N \gamma_i^0 r_0^i$ is immediate.

Proof of Proposition 6.1

1. The relative wealth level w_i of agent i when time horizon is T must satisfy

$$w_i E \left[\int_0^T q_t^* e_t^* dt \right] = E \left[\int_0^T q_t^* y_t^{*^i} dt \right] = E \left[\int_0^T \frac{1}{\lambda_i} \exp^{-\rho^i t} M_t^i dt \right] = \frac{1}{\lambda_i} \int_0^T \exp^{-\rho^i t} dt.$$

If $\rho^i = 0$, then $\gamma_i^T = w_i$. Otherwise, $w_i E \left[\int_0^T q_t^* e_t^* dt \right] = \frac{1 - \exp - \rho^i T}{\lambda_i \rho^i}$, hence $\gamma_i^T = \frac{w_i \rho^i \left(\frac{1}{1 - \exp - \rho^i T} \right)}{\sum w_i \rho^i \left(\frac{1}{1 - \exp - \rho^i T} \right)}$.

2. The expression for R_t^T is a direct consequence of the fact that $R_t^T = -\frac{1}{t} \log \left[\sum_{i=1}^2 \gamma_i^T \exp^{-R^i t} \right]$. The expression for R_∞ results from Proposition 5.1 and from the fact that, according to 1., none of the agents vanishes. The expression for R_0 results from Proposition 5.2. ■

Proof of the result with a Gamma distribution

We have $A_T = E \left[E^i \left[\frac{1}{e^*} M^i \right] \right] = E^i \left[\exp^{-R^i T} \right]$. Since $\left(R^i\right) \sim \gamma(\alpha, \beta)$, we have $A_T =$

$\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\exp^{-xT}f(x)\,dx=\left(\frac{\beta}{\beta+t}\right)^{\alpha}=\frac{1}{(1+T\sigma^2/\mu)^{\mu^2/\sigma^2}}$. ■

Proof of Proposition 6.2

1. Since $w_1=w_2$, the relative weights γ_i^T must solve

$$E\left[\int_0^T (e_t^*)^{-1}\left[\gamma_1^T\left(M_t^1\right)^{1/2}-\gamma_2^T\left(M_t^2\right)^{1/2}\right]\left[\gamma_1^T\left(M_t^1\right)^{1/2}+\gamma_2^T\left(M_t^2\right)^{1/2}\right]dt\right]=0.$$

This is equivalent to

$$\int_0^T \exp^{\left(-\mu+\sigma^2/2-\delta^2/2\right)t}E\left[\left(\gamma_1^T\right)^2\exp^{(\delta-\sigma)W_t}-\left(\gamma_2^T\right)^2\exp^{(-\delta-\sigma)W_t}\right]dt=0$$

or

$$\int_0^T \exp^{\left(-\mu+\sigma^2/2-\delta^2/2\right)t}\left(\left(\gamma_1^T\right)^2\exp^{\frac{(\delta-\sigma)^2}{2}t}-\left(\gamma_2^T\right)^2\exp^{\frac{(\delta+\sigma)^2}{2}t}\right)dt=0.$$

This implies that

$$\frac{\left(\gamma_1^T\right)^2}{\left(\gamma_2^T\right)^2}=\frac{\exp^{\left(-\mu+\sigma^2+\sigma\delta\right)T}-1}{\exp^{\left(-\mu+\sigma^2-\sigma\delta\right)T}-1}\frac{\left(-\mu+\sigma^2-\sigma\delta\right)}{\left(-\mu+\sigma^2+\sigma\delta\right)}.$$

This function is of the form $\frac{\exp^{aT}-1}{\exp^{bT}-1}$ with $a,b<0$ and $a>b$, it is then easy to verify that this function is increasing in T.

2. We know that $R_T=-\frac{1}{T}\log\left[\left\{\sum_{i=1}^2\gamma_i^T\left(M_T^i\right)^{\eta}\right\}^{1/\eta}\left(e_T^*\right)^{-1/\eta}\right]$. It is then easy to obtain that $R_T=\frac{\mu}{\eta}-\frac{\sigma^2}{2\eta}+\frac{\delta^2}{2}-\frac{1}{T}\ln E\left[\left(\gamma_1^T\exp^{(\eta\delta-\sigma)W_T}+\gamma_2^T\exp^{(-\eta\delta-\sigma)W_T}\right)^{1/\eta}\right]$. For $\eta=1/2$, we have $R_T=2\mu-\sigma^2+\frac{\delta^2}{2}-\frac{1}{T}\ln V_T$ with

$$V_T=E\left[\left(\gamma_1^T\right)^2\exp^{2(\delta/2-\sigma)W_T}+\left(\gamma_2^T\right)^2\exp^{-2(\delta/2+\sigma)W_T}+2\gamma_1^T\gamma_2^T\exp^{-2\sigma W_T}\right].$$

We have then

$$\begin{aligned}V_T&=\left(\gamma_1^T\right)^2\exp^{2(\delta/2-\sigma)^2T}+\left(\gamma_2^T\right)^2\exp^{2(\delta/2+\sigma)^2T}+2\gamma_1^T\gamma_2^T\exp^{2\sigma^2T}\\&=\exp^{2\sigma^2T}\left(\left(\gamma_1^T\right)^2\exp^{\left(\delta^2/2-2\sigma\delta\right)T}+\left(\gamma_2^T\right)^2\exp^{\left(\delta^2/2+2\sigma\delta\right)T}+2\gamma_1^T\gamma_2^T\right),\end{aligned}$$

hence

$$R_T=2\mu-3\sigma^2-\frac{1}{T}\ln\left(\left(\gamma_1^T\right)^2\exp^{-2\delta\sigma T}+\left(\gamma_2^T\right)^2\exp^{2\delta\sigma T}+2\gamma_1^T\gamma_2^T\exp^{-\frac{1}{2}\delta^2T}\right).$$

The result on R_∞ comes from the expression for γ_i^T obtained in 1., which implies that none of the agents vanishes and from Proposition 5.1. The result on R_0 comes from Proposition 5.2. ■

Proof of the optimistic bias $\gamma_1^T>\gamma_2^T$ for $\eta<1$ and of the pessimistic bias $\gamma_1^T<\gamma_2^T$ for $\eta>1$.

The proof is taken from Jouini-Napp (2007). As in Proposition 6.2, we have

$$A\equiv E\left[\int_0^T\left(e_t^*\right)^{1-1/\eta}\frac{\left(M^1/\lambda_1\right)^{\eta}-\left(M^2/\lambda_2\right)^{\eta}}{\left\{\left(M^1/\lambda_1\right)^{\eta}+\left(M^2/\lambda_2\right)^{\eta}\right\}^{1-1/\eta}}dt\right]=0.$$

It is immediate that A can be written in the form $A = \frac{1}{\lambda_1} x^{-1/\eta} g(x)$ with $x^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^\eta$ and

$$g(x) = E\left[\int_0^T (e_t^*)^{1-1/\eta} \frac{x\left(M_t^1\right)^\eta - \frac{1}{x}\left(M_t^2\right)^\eta}{\left\{x\left(M_t^1\right)^\eta + \frac{1}{x}\left(M_t^2\right)^\eta\right\}^{1-1/\eta}} dt\right].$$

For $\eta < 1$, we show that $\lambda_1 \leq \lambda_2$. We prove 1) that $g(1) \leq 0$, and 2) that g is increasing with x, which implies that for $\lambda_1 > \lambda_2$ we would have $A < 0$, which is impossible. We have

$$g'(x) = E\left[\int_0^T \frac{x\left(\left(M_t^1\right)^\eta + \left(M_t^2\right)^\eta / x^2\right)^2 - x\left(\left(M_t^1\right)^\eta - \left(M_t^2\right)^\eta / x^2\right)^2 (1 - 1/\eta)}{\left(x\left(M_t^1\right)^\eta + \left(M_t^2\right)^\eta / x\right)^{2-1/\eta}} (e_t^*)^{1-1/\eta} dt\right],$$

which is positive for $\eta < 1$ and proves 2). Now, $g(1) = \int_0^T E\left[(e_t^*)^{1-1/\eta} \frac{\left(M^1\right)^\eta - \left(M^2\right)^\eta}{\left\{\left(M^1\right)^\eta + \left(M^2\right)^\eta\right\}^{1-1/\eta}}\right] dt$. With deterministic coefficients, $\left(M^1\right)^\eta$ and $\left(M^2\right)^\eta$ can be written in the form $\left(M^1\right)^\eta(t) = Z(t)\left(e^*(t)\right)^{\frac{\eta\delta}{\sigma}}$ and $\left(M^2\right)^\eta(t) = Z(t)\left(e^*(t)\right)^{-\frac{\eta\delta}{\sigma}}$ for some deterministic process Z. It is easy to see that $\frac{\left(M^1\right)^\eta - \left(M^2\right)^\eta}{\left\{\left(M^1\right)^\eta + \left(M^2\right)^\eta\right\}^{1-1/\eta}}$ is increasing in e^*, hence decreasing in $(e^*)^{1-1/\eta}$, leading to

$$E\left[(e_t^*)^{1-1/\eta} \frac{\left(M_t^1\right)^\eta - \left(M_t^2\right)^\eta}{\left\{\left(M_t^1\right)^\eta + \left(M_t^2\right)^\eta\right\}^{1-1/\eta}}\right] \leq E\left[(e_t^*)^{1-1/\eta}\right] E\left[\frac{\left(M_t^1\right)^\eta - \left(M_t^2\right)^\eta}{\left\{\left(M_t^1\right)^\eta + \left(M_t^2\right)^\eta\right\}^{1-1/\eta}}\right].$$

Since $E\left[\frac{\left(M_t^1\right)^\eta - \left(M_t^2\right)^\eta}{\left\{\left(M_t^1\right)^\eta + \left(M_t^2\right)^\eta\right\}^{1-1/\eta}}\right] = 0$, we obtain that $g(1) \leq 0$. ■

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	γ_i^T	A^T	R_t^T as a function of t	R_T^T as a function of T	R_0	R_∞	$R_t^T(\delta)$ as a function of δ
$w_1 = w_2$ and $\rho_1 = \rho_2 = 0$	$\gamma_i^T = \frac{1}{2}$	$\frac{1}{2}A^{T,1} + \frac{1}{2}A^{T,2}$	Decreasing	Decreasing	$\mu - \sigma^2$ rational rate	$\mu - \sigma^2 - \delta\sigma < R_0$ pessimistic rate	Decreasing
$\rho_1 = \rho_2 = 0$	$\gamma_i^T = w_i$	$w_1 A^{T,1} + w_2 A^{T,2}$	Decreasing	Decreasing	$\mu - \sigma^2 + (w_1 - w_2)\delta\sigma$	$\mu - \sigma^2 - \delta\sigma \leq R_0$ pessimistic rate	If $w_1 \leq w_2$, decreasing. If $w_1 > w_2$, increasing for small t and decreasing for large t.
General case	$\gamma_i^T = \frac{\frac{w_i \rho^i}{1-\exp -\rho^i T}}{\sum_j \frac{w_j \rho^j}{1-\exp -\rho^j T}}$ $\gamma_i^0 = w_i$ $\gamma_i^\infty = \frac{w_i \rho^i}{\sum_j w_j \rho^j}$	$\gamma_1^T A^{T,1} + \gamma_2^T A^{T,2}$	Decreasing	If $\rho^1 > \rho^2$ or if $\rho^1 < \rho^2$ and $\rho^1 + \delta\sigma < \rho^2 - \delta\sigma$ increasing then decreasing. Decreasing otherwise.	$\mu - \sigma^2 + w_1(\rho^1 + \delta\sigma)$ $+ w_2(\rho^2 - \delta\sigma)$	$\mu - \sigma^2$ $+ \inf(\rho^i + \delta^i\sigma)$ lowest rate	

Table 1. Yield curve properties in the case of logarithmic functions when agents are on average rational i.e. $\delta_1 = \delta > 0$ and $\delta_2 = -\delta$

The yield curve is always downward sloping. The behavior of R_T^T (the discount rate at the economic horizon T) as a function of T and the behavior of $R_t^T(\delta)$ (the discount rate for a given maturity t and a given economic horizon T) as a function of beliefs dispersion δ depend upon individual wealth and time preference rates distribution.

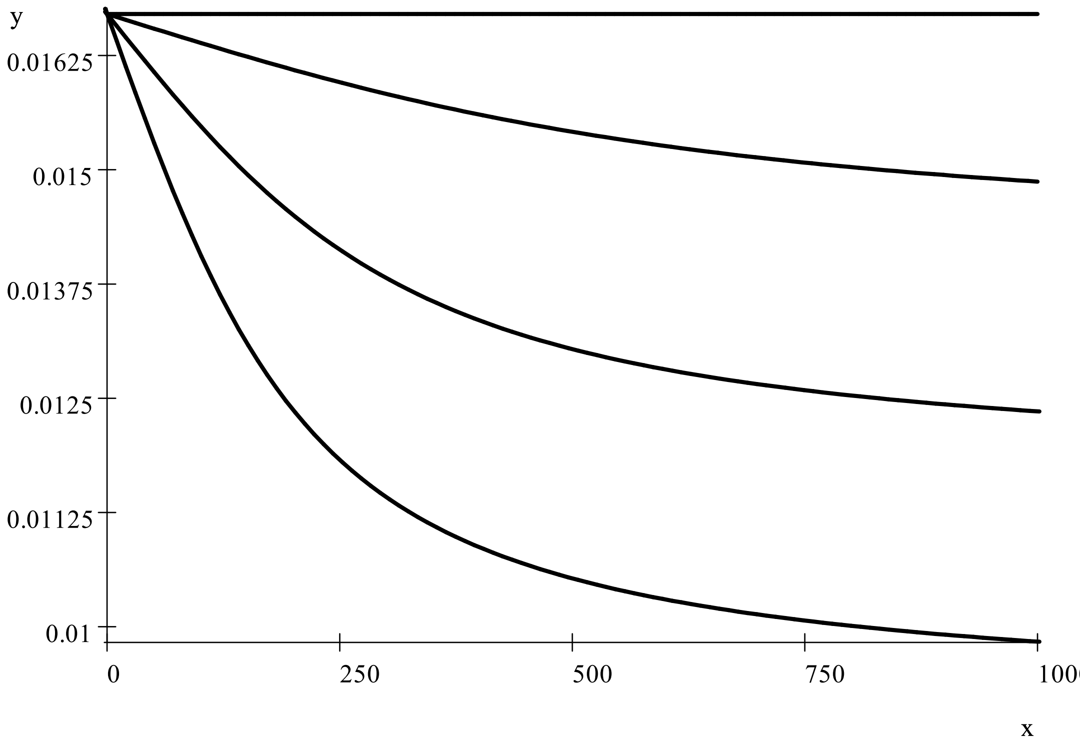

Figure 7.1: This figure represents R_t as a function of t in the case of logarithmic utility functions, zero time-preference discount rates $\left(\rho^1 = \rho^2 = 0\right)$, same relative initial wealth levels $(w_1 = w_2)$, and for three different levels of beliefs dispersion δ_j, $j = 1, 2, 3$. The straight line represents the rational rate. We take $\mu = 1.8\%$, $\sigma = 3.6\%$. We take $\delta_1 = 0.07$, $\delta_2 = 0.14$, $\delta_3 = 0.21$. The three curves are decreasing (see Proposition 6.1). For all t, the discount rate R_t decreases with the level of beliefs dispersion (see Table 1), hence the three curves do not cross and the lowest curve corresponds to the highest level of beliefs dispersion. The three curves start from the rational rate. Each curve $\mathcal{C}_j$ converges asymptotically to the pessimistic discount rate $\mu - \sigma^2 - \sigma\delta_j$.

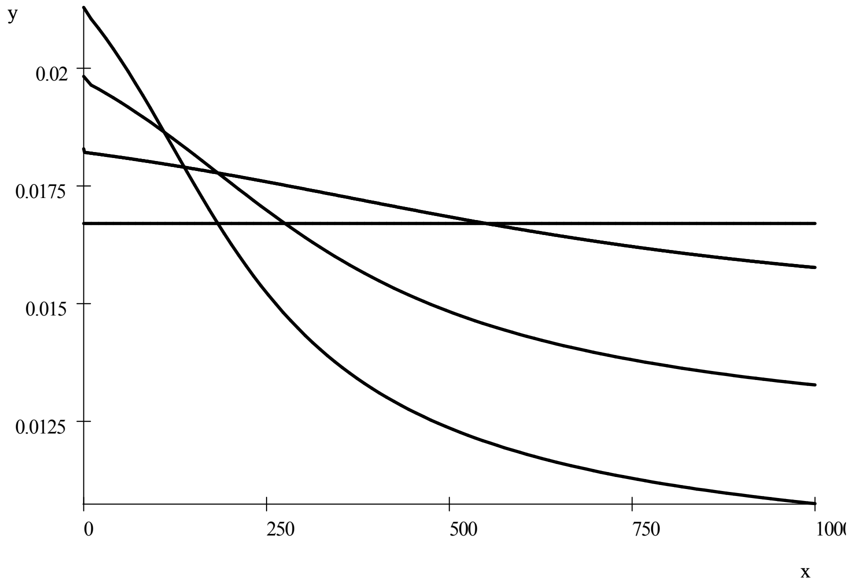

Figure 7.2: This figure represents R_t as a function of t for three different levels of beliefs dispersion δ_j, $j = 1, 2, 3$ in the case of logarithmic utility functions, zero time-preference discount rates $\left(\rho^1 = \rho^2 = 0\right)$ and a positive correlation between wealth and optimism. The straight line represents the rational rate. We take $\mu = 1.8\%$, $\sigma = 3.6\%$; $\delta_1 = 0.07$, $\delta_2 = 0.14$, $\delta_3 = 0.21; w_1 = 0.8, w_2 = 0.2$. The three curves are decreasing, each curve $\mathcal{C}_j$ converging asymptotically to the pessimistic discount rate $\mu - \sigma^2 - \sigma\delta_j$ (see Proposition 6.1). Since $w_1 > w_2$, for small t the discount rate R_t increases with the level of beliefs dispersion, and for larger t, it decreases with the level of beliefs dispersion, hence the three curves cross. Since $w_1 \neq w_2$, the three curves start from different initial points given by $\mu - \sigma^2 + (w_1 - w_2)\sigma\delta_j$. The lowest curve at $t = 1000$ corresponds to the highest level of beliefs dispersion.

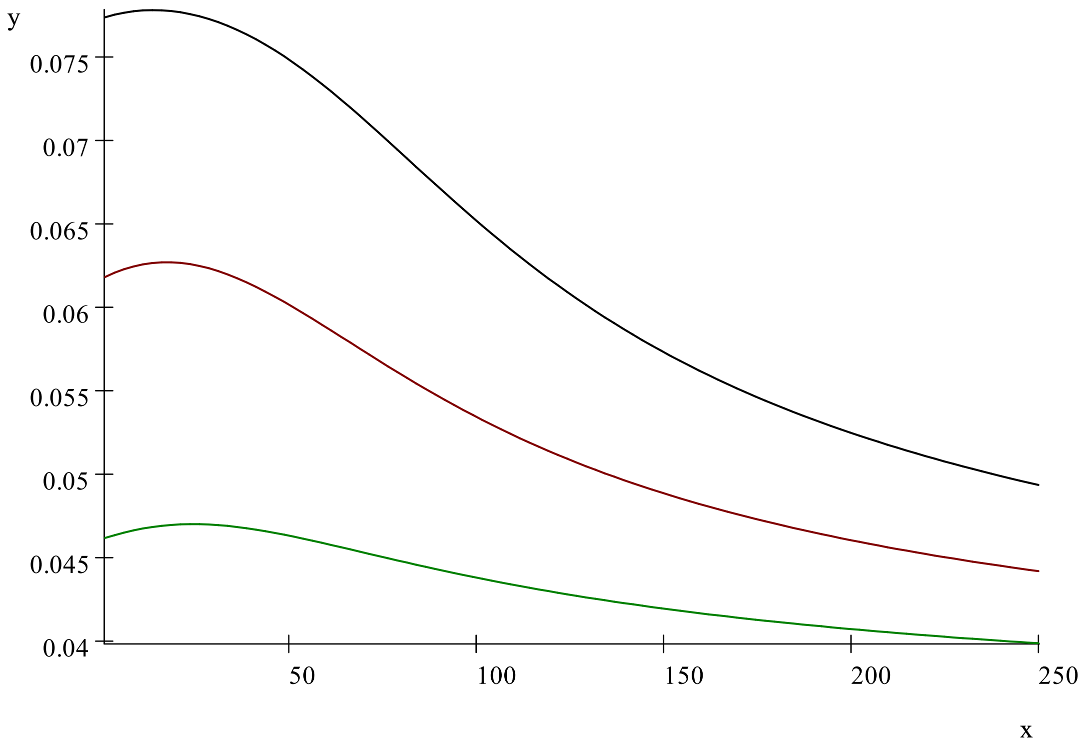

Figure 7.3: This figure represents R_T as a function of T for three different levels of relative initial wealth w_j, $j = 1, 2, 3$ in the case of logarithmic utility functions when the discount rate of the optimistic agent is lower than the discount rate of the pessimistic agent. The three curves are first increasing, then decreasing (see Table 1), each curve $\mathcal{C}_j$ converging asymptotically to the pessimistic discount rate $\mu - \sigma^2 + \rho^2 - \sigma\delta$ (see Proposition 6.1). The three curves start from different points, which are given by $\mu - \sigma^2 + w\left(\rho^1 + \delta\sigma\right) + (1-w)\left(\rho^2 - \delta\sigma\right)$. We take $\mu = 1.8\%$, $\sigma = 3.6\%$; $w_1 = 0.2$, $w_2 = 0.5$, $w_3 = 0.8$; $\rho^1 = 1\%, \rho^2 = 8\%; \delta = 0.25$. The highest curve corresponds to the lowest w.

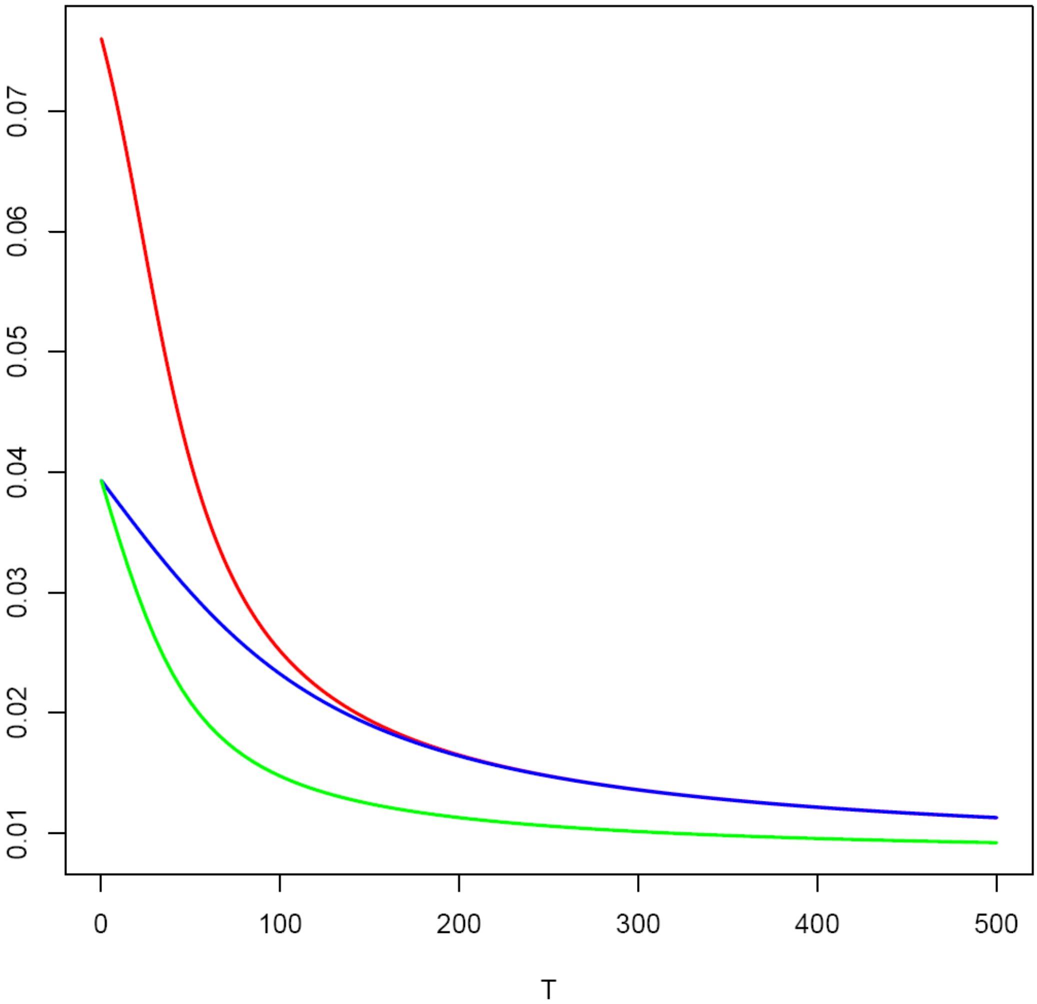

Figure 7.4: This figure represents the socially efficient discount rate for $\gamma_1 = \gamma_2, \eta = 0.4$, $\mu = 0.018$, $\sigma = 0.036$ and $\delta = 0.35$ as well as the curves associated to the Hölder-η (resp. arithmetic) average of the individual discount factors. The discount rate curve dominates the $\eta-$average that dominates the arithmetic average. With these parameters values, the rational rate is equal to 3.2%.

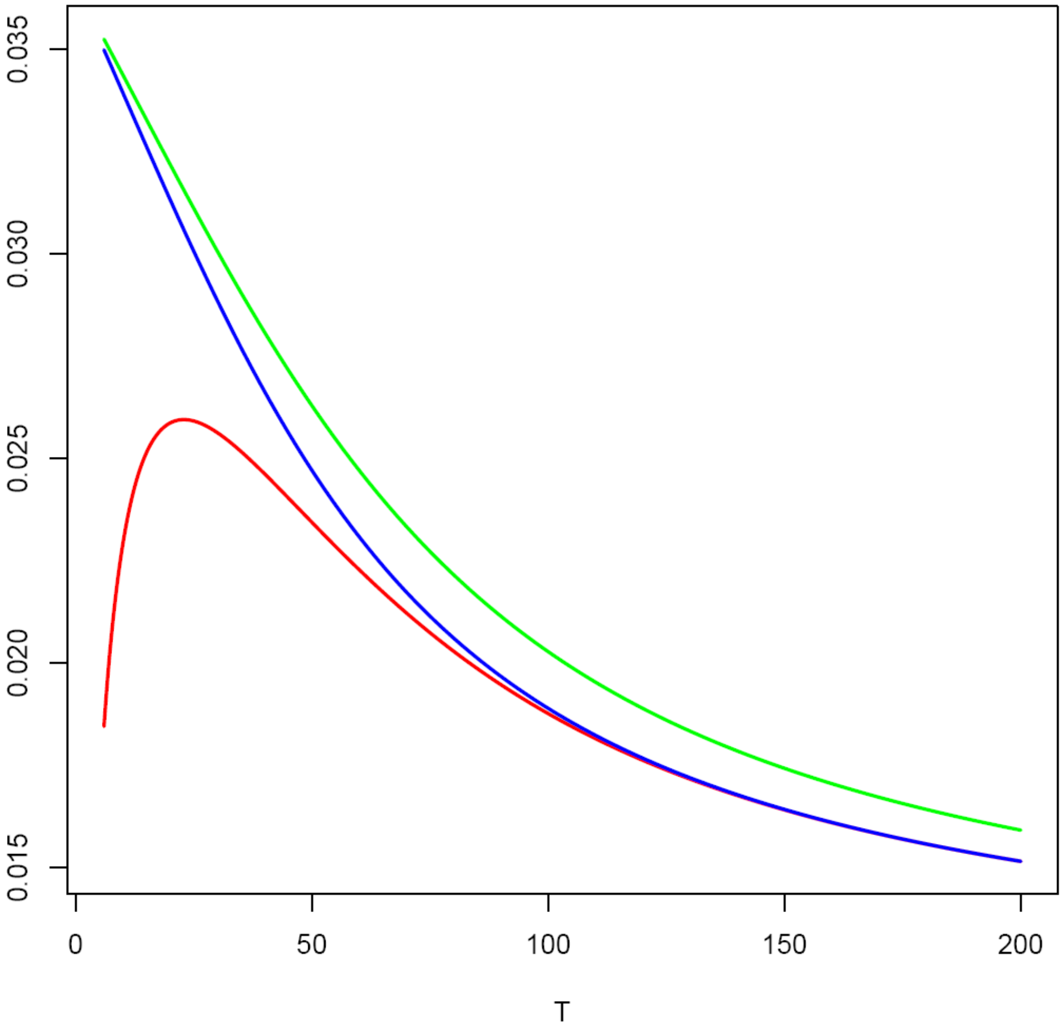

Figure 7.5: This figure represents the socially efficient discount rate for $\gamma_1 = 0.6, \eta = 1.2$, $\mu = 0.04$, $\sigma = 0.036, \delta = 0.7$ as well as the curves associated to the Hölder-η (resp. arithmetic) average of the individual discount factors. The arithmetic average dominates the $\eta-$average that dominates the discount rate curve.