



Chaos in economics and finance

Dominique Guegan

► To cite this version:

| Dominique Guegan. Chaos in economics and finance. 2007. halshs-00187885v1

HAL Id: halshs-00187885

<https://shs.hal.science/halshs-00187885v1>

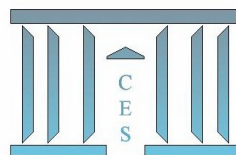
Submitted on 15 Nov 2007 (v1), last revised 16 Dec 2009 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License



Chaos in Economics and Finance

Dominique GUEGAN

2007.54



CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

Chaos in Economics and Finance

Guégan D. *

June 30, 2007

Abstract: In this article, we specify the different approaches followed by the economists and the financial economists in order to use chaos theory. We explain the main difference using this theory with other research domains like the mathematics and the physics. Finally, we present tools necessary for the economists and financial economists to explore this domain empirically.

Keywords: Chaos theory - Attractor - Economy - Finance - Estimation theory - Forecasting.

1 Introduction

The chaotic systems belong to complex systems which are important in a lot of fields to model data sets in order to do control, forecasts and also to understand the evolution of an experiment. These systems are characterized by a strong non linearity. They permit to take into account non-periodical fluctuations, mixing cycles, switches inside data sets. These systems are interesting in the sense that they are characterized by an invariant distribution function and their orbits evolve inside an attractor in which it is possible to do forecasts. Working inside an attractor permits also to control the system and to be able to avoid explosions and strong volatility.

One of the main property of a chaotic system is its randomness although such a system is deterministic. It is this kind of "‘contradiction'" which makes these systems so attractive, flexible and interesting in the applications. This

*Centre d'Economie de la Sorbonne, Paris 1, Panthéon-Sorbonne, UMR 7184, e-mail : guegan@ecogest.ens-cachan.fr, talk prepared for the 11Chaos'06 Conference in Reims, June 28th - 30th June, 2006

explains why so many research domains use these models.

Historically, the mathematicians were the first to be interested by this theory. Since 1800, works concerning deterministic dynamical systems which generate random behaviors have been investigated and the Poincaré's works (Poincaré, 1908) permit to those studies to observe a growing interest and to be developed in a lot of communities. In biology, people who study the population's evolution use chaotic deterministic systems since the years 1970, May (1976). In physics it is a long tradition for the researchers, working mainly with an empirical approach, to use these models. The works of Ruelle and Takens (1971) propose a new approach lying on non-linear modelling with few parameters and also give the opportunity to extend the works in this community. In economy, people doing works concerning stability, instability flirts with the bifurcation theory since the 1980. This approach is really new and unexpected coming from this community. Indeed, working with chaotic systems is in opposition with most of the different concepts developed by macro-economists: we cite for instance, the neo-classic theory with the works of Lucas, Sargent, Prescott, etc, the work concerning the 'rational' behavior based only on a linear approach or the Keynes' theory which is not concerned with the complex systems, Medio (1992). Between the years 1986 to 1998, a lot of studies, following mainly the idea of Grandmont (1988), guide a part of the research in economy towards chaotic systems. In particular, it permits to explain randomness through endogenous explanation. New econometric models are developed and irreversibility time is introduced. In finance the interest for the chaotic theory is more recent and sparsed. The craze for this theory from financial people date from the 1990 years. People wait for to obtain better and robust forecasts using chaos approach. Finally, chaos has recently developed as an area of increasing interest and importance for statisticians. The main reason, for this last community, is that a chaotic system is deterministic but appears to behave randomly, and determining the properties of a chaotic system from observations involves statistical estimation. In practice, there will be measurement error in observing the process, which may also be subject to intrinsic system noise. Even when noise free, the process can be interpreted as stochastic with respect to an invariant measure. Using all these classical properties of chaos permit to the statisticians to develop a new domain of research, since the 1992's.

We can summarize this introduction saying that each community has a different strategy, does not use the same models, nor the same information set. Most of the mathematicians work with analytical expressions and characterize their models under specific assumptions to decide if they can exhibit

theoretically chaotic behavior. It is important to note that inside this community there exist a lot of notions of "chaos". The economists use generally analytical systems corresponding to a specific problem that they want to model. These systems depend on few parameters and they often study the range of evolution of these parameters to understand and control the passage of stable behavior to the unstable behavior. The bifurcation theory appears important in all these developments. An important feature is that the economists cannot follow the roads of the physicists. Indeed, physicists are interested by questions for universal laws and in economy the trend is to understand and document differences. In finance, practitioners do not use analytical systems and want to use the chaotic theory to robustify their forecasts, most of the works are empirical.

Now we introduce briefly the notion of chaos and give specific examples taken in the different communities.

2 Definition of chaos: Examples and Strategies

Let $(\underline{X}_t)_t$ be a stochastic vector process characterized by the following equation $\underline{X}_t = f(\underline{X}_{t-1})$, whose \underline{X}_0 is the initial condition. Such a process is considered as a dynamical deterministic system, defined on a metric space $A \in \mathbb{R}^d$, $d \in \mathbb{N}$ and $f : A \rightarrow A$. As a working definition for our purpose, we will say that such a system has a chaotic behavior if it is non linear, if it is characterized by the existence of an attractor inside A and if it is sensitive to initial conditions (which means that there exists almost one positive Lyapunov exponent). Such a dynamical system gives raise to observations that have the characteristics of random data, and yet nevertheless are deterministically generated. All along this paper we will limit to this definition which is sufficient to explain the attempts of the different communities with respect to the chaotic theory. For more details on the notion of chaos, attractor, and sensibility to initial conditions, we refer to Devaney (1989), Katok and Hasselblatt (1995) and Guégan (2003) among others.

It appears very interesting to use chaos in terms of model. Indeed, it is a non explosive system (no trend); it is an aperiodic system (no seasonality); it is a stationary system (invariant distribution - ergodicity). Thus, if we are able to know where is the attractor (determining the phase space and knowing the embedding dimension) or if we are able to re-build the orbits, then we can make predictions.

There exists a class of chaotic systems which has been developed, a lot mainly by the mathematicians, and which gives simple examples (or toys) permitting to illustrate this theory. We think to the maps developed on $[0, 1]$ (including the logistic map, the tent map and its generalization, the binary maps, etc.), see Guégan and Ladoucette (2002). But if the logistic map found applications in biology and ecology, these maps are not very interesting in terms of applications. Now, chaotic dynamics have been observed in a wide variety of experimental contexts, examples can be found inside the chemist, physicist, meteorologist, hydrologist, medical and biologist literature, Lorenz (1963), Schaffer (1985), Dvorak and Holden (1991) and Greenfeld (1992). Some chaos are also famous and permit to give nice examples to illustrate the theory (existence of attractor, sensibility to initial conditions, bifurcation theory with the exact computation of Lyapunov exponents). We think for instance to Hénon, Lorenz, Rossler, Chua and Makey Glass systems. These models permit to explain, for some, the behavior of a resistance in physics, the water's level river in hydrology, the population's growth in ecology, the fishes' flux in fishing research or the temperature's evolution, Bergé, Pomeau and Vidal (1984).

The analysis of the behavior of individuals (auto-organization - reproduction) depend also on complex models, "closed" models, and researchers are interested to know their asymptotic behavior. For instance, if we consider the organization of a market looking at the social behaviors, it is assumed, in the equilibrium theory, that the agents have a complete knowledge of the market. But in fact, any agent has an in-complete knowledge of the market. This knowledge lies on empirical observations, from which the agents learn a certain knowledge, and then take some decisions. It appears that the information setting is not complete. Thus, the auto-regulation can be interpreted in terms of attractor, but not known chaotic analytical models correspond a priori to this situation. Nevertheless these thinkings, in the social sciences permit to some population of researchers to be caught by this theory, and this approach will be mainly empirical.

Macroeconomists have long realized that a certain class of deterministic non-linear systems was capable of producing a self-sustained fluctuation without any shock from outside of the models. A group of researchers including Benhabib and Nishimura (1979), Day (1982), Grandmont (1988) developed many examples of deterministic economic models that could generate non periodic fluctuations. Recent studies, Brock and Hommes (1998), have shown that routes to chaos can arise in the traditional expectation models such as the cobweb model and the asset pricing model by the introduction of het-

erogenous beliefs. In finance, some models try to explain the behavior of the chartists and the fundamentalists on a market. In that case, the model concerns the evolution for change rates and one of its expression can be summarized as follows $S_t = X_t S_{t-1}^\alpha S_{t-2}^\beta$, where S_t represents the exchange rate at time t and X_t some behavioral variable, α and β being the parameters of the system. For a specific range of these parameters, we can get an attractor whose an illustration is given in Figure 1, de Grauwe, Dewatcher and Embrechts (1993).

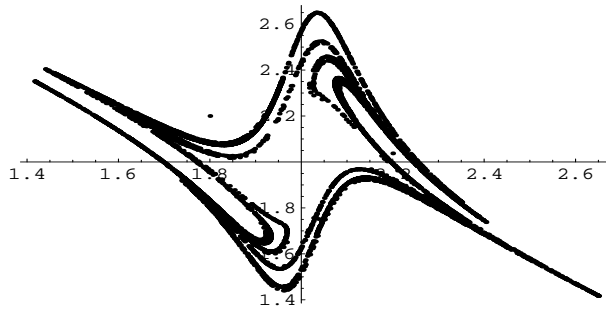


Figure 1: $S_t = X_t S_{t-1}^\alpha S_{t-2}^\beta$

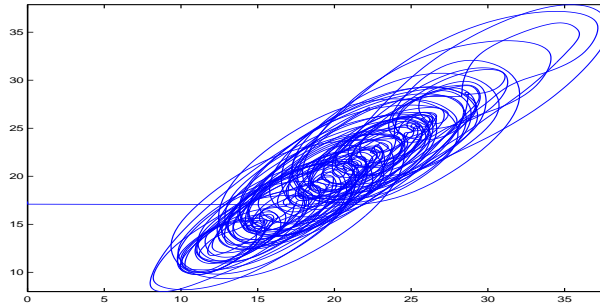


Figure 2: Attractor for electricity spot prices: Hourly prices from 16th of June, 2000 to the 16th of December, 2004, German market.

Thus, theoretical developments revealed that, depending on the choice of parameters values, most economic models can produce either stable solutions or complex solutions including chaos. For this reason, the theoretical analysis have been followed by an active research program to find empirical evidence of chaos as the source of the business cycles, Shintani and Linton (2003). These works have been developed also to understand the dynamics

of the labour market, or the actions of the agents and their interaction on markets. Works based on stock prices and currency exchanges rates were extensively developed. The detection of attractors empirically appears difficult because the presence of measurement noise. Thus, the detection and the use of attractors to make predictions from empirical (economic or financial) data necessitate to develop specific tools. On the figure 2, we provide the attractor corresponding to the evolution of hourly electricity spot prices, after deconvolution using wavelets. This example illustrates the fact that on the outside of estimation and prediction theory, deconvolution theory appears important.

Indeed, in practice, we observe only a trajectory X_1, \dots, X_n corresponding to the process $(X_t)_t$ that we want to explain. This evolution is not sufficient to observe the auto-regulation or the reproduction of the behaviors which can exist inside the data. The existence of an attractor will permit to bypass this problem. Practically, how do we know that some chaos exists? We can look after the divergence inside the trajectories from two initial conditions computing a positive Lyapunov exponent λ ; we can exhibit a bifurcation diagram; we can make evidence of the attractor in the phase space of dimension d ; we can show that the attractor has a fractal dimension d_H , $d \geq 2d_H + 1$. But, with real data sets corresponding to a unique trajectory (that is the case in economics and finance), we cannot use the bifurcation theory because we just have only an experience. The computation of d_H is generally very complicated. Thus, specific tools need to be developed in order to detect existence of an attractor from time series. We can distinguish two kinds of approaches.

- We use an analytical system (approach developed mainly by dynamists and some economists). In that case, we need to estimate the parameters and to study a bifurcation proposal. This approach is mainly used in control theory.
- We do not use an analytical system, only time series. It is the favourite approach for statisticians, practitioners in finance, insurance and economic researchers. We need to embed the time series to detect the attractor - if it exists - that characterizes the process $(X_t)_t$. Then, from observations, for certain $d \in N$ and $\tau \in R$, we build the orbits: $(X_t, X_{t-\tau}, X_{t-2\tau}, \dots, X_{t-d\tau})$, see Takens (1996).

In both cases an estimation theory is necessary specifying the assumptions and the study's framework. The embedding technique provides an estimate for d . Then, in the phase space, we estimate the Lyapunov exponent, λ , in order to test its positivity. In all procedures, we need to reconstruct the

chaotic function f using non parametric methods. Finally, we do forecasts.

In another hand, we know that a random behavior does not create chaos: this is the case for noisy economic data sets or noisy financial data sets (no attractor appears in the embedding). But, an ordered behavior does not fulfill the space providing an attractor. It is this last property that is useful in time series modelling. Now, be free of the presence of noise is fundamental in applications. Recent work use wavelets theory to suppress the measurement noise inside chaotic data and they have given very interesting results, see Guégan and Hoummiya (2005).

3 Chaos in Economics and Finance: A Statistical Approach

In this Section, we specify now the statistical tools which permit to answer to the attempts of the practitioners in economics and finance. We do not enter in details and refer to basic works.

The observation of a trajectory is not sufficient to detect the existence of an attractor which is obtained using embedding. Thus, we have to estimate the embedding dimension \hat{d} . Most of the methods are non parametric: The delay method (Takens, 1981), the Grassberger and Procaccia method (1986) or the Zero-one explosive method, see Guégan (2003) for a review. As soon as we have found the smallest embedding dimension, we can work with the orbits which characterize the attractor. To be sure of the existence of the attractor, we can also estimate the Lyapunov exponents and test their positivity. In that case, we use the following relationship:

$$\hat{\lambda} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^T \ln |\hat{f}'(\underline{X}_i)|,$$

when this limit exists and if the chaotic map f is differentiable. The \underline{X}_i are the points of the trajectory coming from an initial condition \underline{X}_0 . To get $\hat{\lambda}$, we need to have an estimate of the chaotic function \hat{f} and of its derivative. To solve this problem a lot of non parametric methods have been developed using non parametric methods, nearest neighbors, radial basis functions, neural networks or kernels, we refer to Pesaran and Potter (1993) and references therein. Then, we can test the positivity of the Lyapunov exponents, see

Shintani and Linton (2004). Another approach consists in studying the evolution of the Lyapunov exponents to try to detect some changes which can permit to determine when the observations are explained by a chaotic behavior or by a stable behavior, Bayley (1998). Obtaining this kind of results necessitates a lot of data. It is generally easier working with financial data, mainly if we have intra day data.

In economy, we will be mainly interested in detection and control: this can be done playing with the values of the parameters which characterize the systems. In that case, classical estimation theory is used but the assumptions are generally strong.

In finance the objective concerns predictions. Different kinds of predictions can be discussed. As soon as the function f is well re-built, short term prediction inside the attractor are easy to obtain. Now, the question of mean term or long term prediction is opened. Traditionally, if we observe a chaos, people decides that it is impossible to do forecasts. We know, using "stochastic" processes that it is possible to do it. If the model is short memory, we will do short term predictions. If the model has a long memory behavior we will do long term predictions. Recall that we say that a process has a short memory behavior if its autocorrelation function decreases with an exponential rate towards zero. It has a long memory behavior if its autocorrelation function decreases with an hyperbolic rate towards zero, Guégan (2005).

Here the underlying model is deterministic and the question is to decide if we can do forecasts with chaotic systems. Some chaos have a similar behavior to the one of a white noise like the logistic map and then we cannot do more than short term predictions with these models. We will use the estimates of the chaotic map to get them, computing iteratively $\hat{f}_n(X_n)$, $\hat{f}_n(X_{n+1})$, \dots , $\hat{f}_n(X_{n+h})$, Examples of short term predictions in finance is provided in Guégan and Mercier (2005).

Others have a similar behavior to the one of a long memory process thus we can make long term predictions. This means that, inside the estimated attractor, we can explore the behavior of the sample autocorrelation function and of the periodogram of the orbits. If we suspect existence of long memory behavior, we can test it using some well known long memory processes like the Gegenbauer process whose a representation is

$$(I - 2\nu B + B^2)^\omega \underline{X}_t = \varepsilon_t, \quad (1)$$

where B is the backshift operator, ν the frequency where the periodogram

explodes and ω the long memory parameter, Gray et al (1989). This model permits to take into account the slow decay of the autocorrelation function. In that case, we compute $E[X_{n+h}|I_n]$ using the model (1) as soon as we have estimated ω and ν , using a sample size n . Long term prediction is discussed in Guégan (2003). To illustrate our purpose we exhibit evidence of long memory behavior inside the well known Henon system. Using a simulated n -data set from this system and after examination of the sample ACF and the periodogram (see Figure below), we decide to adjust a 2 factors Gegenbauer process. The model has the following expression :

$$(I - \nu_1 B + B^2)^{\omega_1} (I - \nu_2 B + B^2)^{\omega_2} \underline{X}_t = \varepsilon_t, \quad (2)$$

where $(\varepsilon_t)_t$ is a Gaussian white noise $N(0, 1)$. The estimates for $(\nu_i, \omega_i), i = 1, 2$ are given in the table below with respect to n . Whatever the values of the sample size, we observe some stability in the results. After an analysis of the residuals, the results confirm presence of long memory behavior. Thus, the model (2) can be used to do long term prediction from this data set, inside the attractor.

Another way to improve the forecasts inside the attractor consists in using the sign of the Lyapunov exponent on the region where we are on the attractor. This work is in progress. Making predictions inside the attractor is interesting because when we embed the data, all the topological properties of the underlying system are conserved. We work in a close space which is fundamental for the predictions. There is no noise or very small noise. Thus, the randomness of the data set is explain by the complexity of the function f , independently of any noise.

n	$(\hat{\nu}_1, \hat{\omega}_1)$	$(\hat{\nu}_2, \hat{\omega}_2)$
10000	0.6504; 0.2543	-0.9695 ; 0.2851
5000	0.6340; 0.2573	-0.9695; 0.2814
2500	0.6948; 0.2335	-0.9667; 0.2685
1250	0.7185; 0.2211	-0.9646; 0.2596
500	0.7203; 0.2013	-0.9548; 0.2274

4 Opened problems

This paper has discussed the approaches followed by the economists and financial economists community when they use the chaos theory. The main

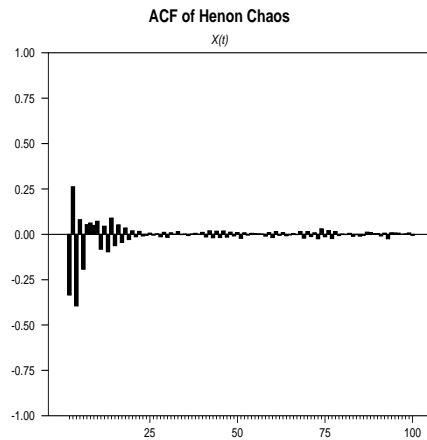


Figure 3: Sample ACF for Henon system.

difference in their approach with the one of physicists for instance come from the existence of an unique experience to treat the data sets and also of the presence of measurement noise in the time series. Thus one of the main problem for applications is to get very robust and simple techniques to suppress the noise. In another hand, working with nonlinear deterministic models is always a choice and a complementary approach in the sense that until now no test permitting to distinguish between non linear stochastic and non linear deterministic models exists. The estimation theory is very often used under strong assumption like independence of the data sets or Gaussianity of the observations : assumptions which are not verified in practice. Thus, new developments need to be develop for data characterized by skewness or kurtosis. Another problem which is important is the existence of an invariant measure for chaotic system. We can question if this assumption is realistic and we refer to Guégan (2007) for discussion on this subject.

References

Bayley B.A. (1998) Local Lyapunov exponents: predictability depends on where you are, in "Nonlinear Dynamics and Economics", Proceedings of the Tenth International Symposium in Economic Theory and Econometrics, W. Barnett, A.P. Kirman, M. Salmon, Wiley, NY. Benhabib J., Nishimura K. (1979) The Hopf bifurcation and the existence and stability of closed orbits in multi sector models of optimal economic growth, *Journal of Economic Theory*, 21, 421 - 444. Bergé P., Pomeau Y., Vidal C. (1984) *L'ordre dans le chaos*, Hermann, Paris.

- Brock W.A., Hommes C.H. (1998) Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *J. of Economic Dynamics and Control*, 22, 1235 - 1274.
- Day R.H. (1992) Complex Economic Dynamics: Obvious in History, Generic in Theory, Elusive in Data, *J. of Applied Econometrics*, 7, S9-S23.
- Devaney R.L. (1989) An introduction to chaotic dynamical systems, Addison Wesley.
- Dvorak I., Holden A.V. eds. (1991) Mathematical approaches to brain functioning diagnostics, Manchester University Press, Manchester.
- Gray H.L., Zhang N., Woodward A. (1989) On generalized fractional models, *J.T.S.A.*, 10, 233 - 257.
- Grandmont J.M. (1988) Nonlinear economic dynamics, Academic Press, NY.
- Grassberger P, Procaccia I. (1983) Measuring the strangeness of strange attractors, *Physica D*, 9, 189 - 208.
- Greenfeld B.T.(1992) Chaos and Chance in Measles dynamics, *J.R.S.S. B*, 54, 383 - 398.
- Guégan D., Ladoucette S. (2002) Extreme values of particular nonlinear processes, *C.R.A.S.*, t. 335, 73 - 78.
- Guégan D. (2003) Les chaos en finance : Approche statistique, *Economica*, Série Mathématique et Probabilité, Paris.
- Guégan D. (2005) How can we define the concept of long memory ? An econometric survey, *Econometric Reviews*, 24.
- Guégan D., Houmniya K. (2005) Denoising with wavelets method in chaotic time series: application in climatology, energy and finance, in *Noise and Fluctuations in econophysics and finance*, ed. By D. Abbott, J.P. Bouchaud, X. Gabaix, J.L. McCauley, *Proc. SPIE*, vol 5848, 174-185.
- Guégan D., Mercier L. (2005) Prediction in Chaotic Time Series : Methods and Comparisons with an Application to Financial Intra-day Data, *The European Journal of Finance*, 11, 137-150.
- Guégan D., Rioublanc S. (2005) Regime switching models: real or spurious long memory, *Note de recherche MORA - IDHE - n° 02-2005*, Février 2005, Cachan, France.
- Guégan D. (2007) Global and Local Stationarity Modelling in Finance: Theory and Empirical Evidence, Submitted to *Econometrics Review*.
- de Grauwe P., Dewachter H, Embrechts M. (1993) Exchange rate theory: chaotic models of foreign exchange markets, Blackwell Publishers.
- Katok A., Hasselblatt B. (1995) Introduction to the modern theory of dynamical systems, Cambridge University Press.
- Lorenz E. (1963) Deterministic nonperiodic flow, *J. Atmosp. Sci.*, 20, 130 - 141.
- May R.M. (1976) Simple mathematical models with complicated dynamics,

Nature, 261, 459 - 467.

Pesaran M.H., Potter S.M. (1993) Nonlinear dynamics, chaos and econometrics, John Wiley, N.Y.

Poincaré H. (1908) Science et Méthode, Flammarion, Paris.

Ruelle D., Takens F. (1971) On the nature of turbulence, Communications in Mathematical Physics, 20, 167 - 192.

Schaffer W.M. (1985) Can nonlinear dynamics elucidate mechanisms in ecology and epidemiology? IMA J. Math. Appl. Med. Bio., 2, 221 - 252.

Shintani N., Linton O. (2004) Non parametric neutral network estimation of Lyapunov exponent and a direct test for chaos, Journal of Econometrics, 120, 1 - 33.

Takens F. (1981) Detecting strange attractors in turbulence, Lectures Notes in Mathematics, 898, Springer Verlag.

Takens F. (1996) Estimation of dimension and order of time series, Progress in Nonlinear Differential equations and their Applications, 19, 405 - 422.