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Change analysis of dynamic copula for measuring dependence in multivariate financial data

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Abstract. Most of the data both in finance and insurance almost always covers a reasonably long period relatively to the provided time horizon, then one may expect that economic factors induced changes in their dependence structures. Recently, along with the application of dynamic copulas, Dias, Embrechts (2004) proposed change-point techniques for specific parametric copula families while assuming that only the parameters change with the copula family unchanged. However, we are nearly sure that the copula family changes. For instance, other researchers determine the best copula on the different subsamples using moving-windows. Combining these two ideas, we propose a series of tests to inspect at first the variance of the copula's family before deciding how to deal with the changes. Undergoing the whole process of change analysis, we could point out exactly which aspect leads to the evolution of the variance in time. Then, we can distinguish two kinds of changes in the dependence: changes in the parameters of a copula and changes in copulas. We apply this procedure to investigate the dependence structure of multivariate data in finance with the empirical case of FX spot data for Asian market.

Keywords: GOF test; change-point; conditional pseudo copula; dynamic copula; U -statistics; foreign exchange data.

1 Introduction

Along the recent years, the notion of copula provides an ideal tool to construct models for extreme moves in correlation and more general measures of dependence both in finance and insurance. In the static case, we can choose the best copula adjusted on the whole period according to different criteria: for instance, the AIC criterion (Akaike, (1974)) or D_2 diagnostic (Caillault and Guégan, (2005)), or attaching importance in the forecasts

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such as VaR and ES measures. In the same time, we can consider both the empirical copula and the analytical copula specifying the kernels.

Most of data considered for research covers a reasonably long period relatively to the provided time horizon. Thus, one may expect that economic factors induced changes in dependence structures since the basic properties of financial assets are not the same in the stable periods and during the crisis. Therefore, when we inspect some changes in the dependence concepts, we should carry out a structure in a dynamic way.

The concept of dynamical copulas has been recently introduced in risk management by Dias and Embrechts (2003), Patton, Granger and Terasvirta (2004) and Jondeau and Rochinger (2004) for instance. In all these papers, the authors investigate the dynamical evolution of the copula's parameter, which means that they assume that the copula's family remains the same all along the period under study. Considering that the family of the copula may change during a considerable long time including perhaps the crisis, Caillault and Guégan (2005) propose to determine the best copula on the different subsamples using the moving windows.

Combining these two main ideas (changes of parameter and changes of copula), here we adopt a different way based on a series of tests for these two kinds of hypotheses. We would like to test at first the variance of the copula's family, then we decide how to deal with the change in copulas and in parameters. With the aid of goodness-of-fit (GOF) test statistics (see, for instance, Fermanian (2003), Fermanian and Wegkamp (2004)), we are able to ensure the variance in the dependence structure. Then, using the conditional pseudo copula, we decide whether the family of copulas changes or whether only the parameters change with a family type unchanged, in the similar way of GOF tests.

Through the nested tests, if we admit that the change only happens in the parameters, we can apply the change-points analysis (e.g. Csörgő and Horváth (1997), Gombay and Horváth (1999), Dias and Embrechts (2003)) to decide where is the change time. Considering that change-points tests have less power in case of "small" changes, we refined the dependence analysis assuming a time-varying behavior for the correlated parameter when we combine univariate dynamic models such as ARMA-GARCH with a time-varying copula whose parameter changes on the

basis of a varying equation.

If we could not deny the change of the copula's family, we use U -statistics for change-points theory (Horváth and Shao (1996) Gombay (2001), Gombay and Horváth (2002)) to detect the change time for the copula. Then we can decide which is the best copula on subsamples using the moving windows and different criteria (AIC, D_2).

The rest of this paper is organized as follows. In Section 2, static case for the copulas is considered. We try to choose the best copula for the whole considered period according to different criteria such as distance criterion and forecast criterion. In Section 3, a simple test is presented to test the time-dependence changes in the copula. In Section 4, the paper goes further to decide which type of change (change of parameter or change of family) happens in the copula. In section 5, we explore the approaches for the changes of parameters with copula's family remaining unchanged. In section 6, parallel analysis is rendered to the changes of copula's family. In section 7, we apply the nested test process and corresponding approaches above to the foreign exchange (FX) spot rates for Asia market. Section 8 concludes.

2 Static Approach: How to Choose the Best Copula?

Let $\mathbf{X} = (X_1, X_2)$ be a random vector with multivariate distribution function F and continuous univariate marginal distribution functions F_1 and F_2 . In order to investigate the dependence structure, we fit copula-based models of the type

$$F(x_1, x_2; \theta) = C(F_1(x_1), F_2(x_2); \theta),$$

where C is a copula function, which we know to exist by Sklar's Theorem (Sklar (1959)) and to be unique as we specify that the marginal distributions are continuous, parameterized by the vector $\theta \in \mathbb{R}^q$ with $q \in \mathbb{N}$. Denote by $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}} = \{(X_{i1}, X_{i2}) : i = 1, 2, \dots, n\}$ a random sample of n bivariate observations. The dependence parameter θ of C is estimated by the pseudo log-likelihood estimator introduced by Genest *et al.* (1995). The marginal distribution function $F_j, j = 1, 2$ are estimated by the rescaled empirical distribution functions,

$$F_{nj}(x) = \frac{1}{n+1} \sum_{i=1}^n \mathbf{1}(X_{ij} \leq x).$$

After the marginal transformations to so-called pseudo observations ($F_{n1}(X_{i1})$, $F_{n2}(X_{i2})$) for $i = 1, 2, \dots, n$, the copula family C is fitted. Suppose that its density exists, we then maximize the pseudo log-likelihood function

$$L(\theta; \mathbf{X}) = \sum_{i=1}^n \log c(F_{n1}(X_{i1}), F_{n2}(X_{i2}); \theta), \quad (2.1)$$

where c is the copula density given by

$$c(u_1, u_2; \theta) = \frac{\partial^2 C(u_1, u_2; \theta)}{\partial u_1 \partial u_2}, \quad (u_1, u_2) \in [0, 1]^2.$$

In that later case, the pseudo log-likelihood estimator $\hat{\theta}$ that maximizes (2.1) is known to be consistent and asymptotically normally distributed (Genest et al. (1995)).

Here, we estimate the analytical copula. We can also apply a non-parametric approach to estimate the analytical copula using a kernel function K as a symmetric probability density function (see Chen and Tang (2005)). This will be included in the further research work.

The question now is how to choose the best copula according to different criteria. On the one hand, we can minimize the distance between the empirical copula and the copula estimated using tail dependence approach. On the other hand, we can evaluate the copulas by maximizing the pseudo log-likelihood function. We will use this last approach in this paper and now we describe it quickly.

Assume that we estimate the copulas' parameters maximizing the pseudo log-likelihood function, then θ is equal to:

$$\hat{\theta} = \arg \max_{\theta} L(\theta; \mathbf{X}) = \arg \max_{\theta} \sum_{i=1}^n \log c(F_{n1}(X_{i1}), F_{n2}(X_{i2}); \theta). \quad (2.2)$$

Now we fit all the possibly suited copulas to the two dimensional random variables to obtain the dependence parameter estimates $\hat{\theta}$. The choice of the best fit is based on the AIC values of the models. From the maximized log-likelihood function we compute for each family AIC:

$$AIC = -2L(\hat{\theta}; \mathbf{X}) + 2q,$$

q is the number of parameters of the family fitted. The smaller the AIC value, the better the model fits to the data.

The limitations of this approach are:

- **an *a priori* choice:** We have to estimate the parameter θ maximizing the pseudo log-likelihood function defined by (2.2) after an *a priori* choice of copulas.
- **a difficult calculation:** Maximization of the equation (2.2) is not always easy to obtain.
- **a requirement for the convergence of algorithms:** In order to obtain the convergence of the algorithms, we need to initialize them graphically.
- **a strong assumption:** We also assume that the observations are independent which is a strong assumption. We bypass this problem using a first modelisation step: an ARMA process or a GARCH modelling.

As we said previously, we can choose the best copula according to other criteria. Then, we can use a L^2 distance that allows to measure how far the Deheuvels (1979) empirical copula \hat{C} and the copula $C_{\hat{\theta}}$ estimated from the estimators of tail behavior is (for instance). Moreover, we can consider the method of choosing the copula in the view of the market by computing VaR and ES using copulas, in this way, we can consider the best copula as in Caillault and Guégan (2004)

3 Time-dependent Test for the Change of Copulas

In practice, conditional distributions with respect to past observations are crucial to specify the underlying model. One key issue is to state if we have a static stationarity on the whole sample? This will say that the copula will be invariant on all the period under study. This assumption is made most of the time in practice (see among others, Rosenberg (2001) and Cherubine and Luciano (2000)). Few papers try to model time dependent conditional copulas, Exceptions are Fermanian and Wegkamp (2004) who introduced pseudo-copulas, see also Patton (2001a), Rockinger and Jondeau (2001) and Genest et al. (2003).

In this section we apply the concept of conditional pseudo copulas to specification tests in which we detect whether the copula remains unchanged during a specific period.

3.1 Conditional pseudo copulas

Patton (2001a, 2001b) has introduced so-called conditional copulas, which are associated with conditional laws in a particular way. We extend his definition to cover a much larger scope of situations. Firstly, we introduce the concept of pseudo-copula.

Definition 1. A 2-dimensional pseudo-copula is a function $C: [0, 1]^2 \rightarrow [0, 1]$ such that

- (a) For every $\mathbf{u} \in [0, 1]^2$, $C(\mathbf{u}) = 0$ when at least one coordinate of \mathbf{u} is zero,
- (b) $C(1, \dots, 1) = 1$,
- (c) For every \mathbf{u} and \mathbf{v} in $[0, 1]^2$ such that $\mathbf{u} \leq \mathbf{v}$, the C -volume of $[\mathbf{u}, \mathbf{v}]$ is positive.

Invoking the same type of arguments as in Sklar's theorem (see Nelsen (1999)), we get the following result:

Theorem 1. Assume that for every $\mathbf{x} = (x_1, x_2)$, $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$,

$$F_j(x_j) = F_j(\tilde{x}_j) \quad \text{for all } j = 1, 2 \Rightarrow F(\mathbf{x}) = F(\tilde{\mathbf{x}}).$$

Then there exists a pseudo-copula C as defined in definition 3, such that

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2)), \quad (3.1)$$

for every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. C is uniquely defined on $\text{Ran}F_1 \times \text{Ran}F_2$, the product of the values taken by the F_j . Conversely, if C is a pseudo-copula and if F_1, F_2 are some univariate cumulative distribution functions (cdf), then the function F defined by (3.1) is a 2-dimensional cdf.

For stationary multivariate process $(\mathbf{X}_n)_{n \in \mathbb{Z}}$, we use the short-hand notation \mathbf{X}_m^n for the vector $(\mathbf{X}_m, \mathbf{X}_{m+1}, \dots, \mathbf{X}_n)$. Similarly we write $\mathbf{X}_{m,j}^n = (X_{m,j}, X_{m+1,j}, \dots, X_{n,j})$, $j = 1, 2$. Let \mathbf{X} defined on the probability space $(\Omega, \mathcal{A}_0, \mathbb{P})$, the sub- σ -algebras $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{B} are conditioning sets containing the past values of $\mathbf{X}_{n-p,j}^{n-1}$ ($p \in \mathbb{N}$) such that $\mathcal{B}_n = \mathcal{A}_n$. Specifically we set $\mathcal{B}_n = (\mathbf{X}_{n-p}^{n-1} = \mathbf{y}^*)$ with $\mathbf{y}^* = (y_1, \dots, y_p)$ and $\mathcal{A}_{n,j} = (\mathbf{X}_{n-p,j}^{n-1} = \mathbf{y}_j^*)$ with $\mathbf{y}_j^* = (y_{1j}, \dots, y_{pj})$.

Now we want to estimate conditional pseudo copulas to test its constancy with respect to n , on specific period of observations.

3.2 Time-varying test

Using the conditional pseudo copula introduced in definition 3, we test the null hypothesis

$$\mathcal{H}_0^{(1)} : \text{For every } \mathbf{y}, C(\cdot|\mathcal{A}, \mathcal{B}) = C_0(\cdot),$$

against

$$\mathcal{H}_a^{(1)} : \text{For some } \mathbf{y}, C(\cdot|\mathcal{A}, \mathcal{B}) \neq C_0(\cdot),$$

where C_0 denotes a fixed pseudo-copula function.

We propose a nonparametric estimator of the conditional pseudo-copulas, and derive its (normal) limiting distribution which allows us to build some GOF test statistics to derive a statistic which permits to decide between $\mathcal{H}_0^{(1)}$ and $\mathcal{H}_0^{(2)}$.

We introduce some assumptions:

H₁: For $p \in \mathbb{N}$, we choose K (resp. \bar{K}), a probability kernel function on \mathbb{R}^{2p} (\mathbb{R}^p , resp.), twice continuously differentiable, vanishing outside a compact interval and satisfying $\int u_j K(\mathbf{u}) d\mathbf{u} = 0$ for $1 \leq j \leq 2p$ ($\int u_j \bar{K}(\mathbf{v}) d\mathbf{v} = 0$ for $1 \leq j \leq p$, resp.). $h = h_n$ ($\bar{h} = \bar{h}_n$, resp.) is the sequence of bandwidths satisfying $0 < h_n \rightarrow 0$ ($0 < \bar{h}_n \rightarrow 0$, resp.), $nh_n^{2p+2+\delta} \rightarrow \infty$ ($n\bar{h}_n^{p+1+\delta} \rightarrow \infty$, resp.) and $nh_n^{2p+4} \rightarrow 0$ ($n\bar{h}_n^{p+2} \rightarrow 0$, resp.) as $n \rightarrow \infty$, where δ is as given as:

H₂: For $p \in \mathbb{N}$ and $k \in \mathbb{N}$, the sequence $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is stationary and strongly mixing, i.e., there exists a function $\alpha(n)$ defined on \mathbb{N} with $\alpha(n) \downarrow 0$ as $n \rightarrow \infty$ and

$$\sup_k |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \alpha(n)$$

for all $A \in \sigma(\mathbf{X}_1, \dots, \mathbf{X}_k)$ and $B \in \sigma(\mathbf{X}_{k+n}, \mathbf{X}_{k+n+1}, \dots)$. In addition, we assume that for some $0 < \delta < 1$,

$$\sum_{j=1}^{\infty} j^{4(p+1)} \alpha^\delta(j) < +\infty.$$

We set

$$K_h(\mathbf{x}) = h^{-2p} K\left(\frac{x_1}{h}, \dots, \frac{x_{2p}}{h}\right),$$

and

$$\overline{K}_h(\mathbf{x}) = \overline{h}^{-p} \overline{K}\left(\frac{x_1}{\overline{h}}, \dots, \frac{x_p}{\overline{h}}\right).$$

For every $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{y}^* \in \mathbb{R}^{2p}$, we can estimate the conditional distribution

$$m(\mathbf{x}|\mathbf{y}^*) = \mathbb{P}\{\mathbf{X}_p \leq \mathbf{x} | \mathbf{X}_0^{p-1} = \mathbf{y}^*\}$$

by

$$m_n(\mathbf{x}|\mathbf{y}^*) = \frac{1}{n-p} \sum_{l=0}^{n-p} K_n(\mathbf{X}_l^{l+p-1}) \mathbf{1}(\mathbf{X}_{l+p} \leq \mathbf{x}),$$

where

$$K_n(X_l^{l+p-1}) = K_h(F_{n1}(X_{l1}) - F_{n1}(y_{11}), F_{n2}(X_{l2}) - F_{n2}(y_{12}), \dots, \\ \dots, F_{n1}(X_{(l+p-1),1}) - F_{n1}(y_{p1}), F_{n2}(X_{(l+p-1),2}) - F_{n2}(y_{p2})).$$

Similarly, for all $x_j \in \mathbb{R}$ and $\mathbf{y}_j^* \in \mathbb{R}^p$, the conditional marginal cdf's

$$m_j(x_j|\mathbf{y}_j^*) = \mathbb{P}\{X_{pj} \leq x_j | \mathbf{X}_{0,j}^{p-1} = \mathbf{y}_j^*\}$$

can be estimated in a nonparametric way by

$$m_{n,j}(x_j|\mathbf{y}_j^*) = \frac{1}{n-p} \sum_{l=0}^{n-p} \overline{K}_h(F_{nj}(X_{l,j}) - F_{nj}(y_{1j}), \dots, \\ F_{nj}(X_{(l+p-1),j}) - F_{nj}(y_{pj})) \mathbf{1}(X_{l+p,j} \leq x_j),$$

for every $j = 1, 2$.

We propose now an estimate of the conditional pseudo-copula, using the previous notations as:

$$\hat{C}(\mathbf{u}|\mathbf{X}_{n-1}^{n-p} = \mathbf{y}^*) = m_n(m_{n,1}^{-1}(u_1|\mathbf{y}_1^*), m_{n,2}^{-1}(u_2|\mathbf{y}_2^*)|\mathbf{y}^*).$$

And a simple test procedure may be based on the statistics

$$\mathcal{T}_n(\mathbf{u}, \mathbf{y}_1^*, \dots, \mathbf{y}_q^*) = (nh_n^{2p}) \sum_{k=1}^q \frac{\{\hat{C}(\mathbf{u}|\mathbf{X}_{n-1}^{n-p} = \mathbf{y}_k^*) - C_0(\mathbf{u})\}^2}{\sigma^2(\mathbf{u})} \quad (3.2)$$

for different choices of \mathbf{u} and conditioning values \mathbf{y}_k^* , where q is the number with which all prerequisite assumptions hold, and

$$\sigma^2(\mathbf{u}) = C_0(\mathbf{u}) \cdot (1 - C_0(\mathbf{u})) \int K^2.$$

Under the previous assumptions \mathbf{H}_1 and \mathbf{H}_2 , the statistic \mathcal{T}_n defined in (3.2) tends to a Chi-square distribution with q -degrees of freedom under

the null hypothesis, (Fermanian and Wegkamp (2004)).

As in Fermanian (2003), we can choose

$$\hat{h} = \frac{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}{n^{1/6}},$$

where σ_j^2 is the empirical variance of $F_{n,j}$ ($j = 1, 2$). We also select a kernel, for example, we can take:

$$K(\mathbf{u}) = \left(\frac{15}{16}\right)^2 \prod_{j=1}^2 (1 - u_j^2) \mathbf{1}(u_j \in [0, 1]).$$

4 Type Test for the Origin of Change

If we reject $\mathcal{H}_0^{(1)}$ (which is almost surely over a relatively long period possibly consisting some crisis), then we would study the dependence structure in a dynamical way.

The concept of dynamical copulas has been recently introduced in risk management. In all the research papers, the authors investigate the dynamical evolution of the copula's parameter assuming the copula's family is the same all along the period under study (Patton (2003), Dias and Embrechts (2003), Patton, Granger and Terasvirta (2004) and Jondeau and Rochinger (2004) for instance). Other authors determine the best copula on the different subsamples using moving-windows (Caillault and Guégan(2003)). Combining these two main ideas, we propose to test at first the variance of the copula's family, then decide how to deal with the change.

Similarly to the GOF test using conditional pseudo-copula, we consider the null hypothesis

$$\begin{aligned} \mathcal{H}_0^{(2)} : \text{For some function: } \theta(\mathbf{y}) = \theta(\mathcal{A}, \mathcal{B}), \\ C(\cdot|\mathcal{A}, \mathcal{B}) = C_{\theta(\mathbf{y})} \in \mathcal{C}, \forall \mathbf{y}, \end{aligned}$$

against the alternative:

$$\mathcal{H}_a^{(2)} : \text{For some } \mathbf{y}, C(\cdot|\mathcal{A}, \mathcal{B}) \notin \mathcal{C},$$

where $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$ denotes a family of pseudo-copulas and we denote $\theta(\mathbf{y})$ the parameter family because we may allow the parameter θ to

depend on past values of the process on the basis of the rejection of the hypothesis $\mathcal{H}_0^{(1)}$.

In the same way previously shown, using the same notation as in section 3, we denote again the statistics

$$\mathcal{R}_n(\mathbf{u}, \mathbf{y}_1^*, \dots, \mathbf{y}_q^*) = (nh_n^{2p}) \sum_{k=1}^q \frac{\{\hat{C}(\mathbf{u} | \mathbf{X}_{n-1}^{n-p} = \mathbf{y}_k^*) - C_{\hat{\theta}_k}(\mathbf{u})\}^2}{\hat{\sigma}_k^2(\mathbf{u})} \quad (4.1)$$

for different choices of \mathbf{u} and conditioning values \mathbf{y}_k^* , where $\hat{\sigma}_k$ is some consistent estimators of $\theta(\mathbf{y}_k^*)$, $k = 1, \dots, q$, and

$$\hat{\sigma}_k^2(\mathbf{u}) = C_{\hat{\theta}_k}(\mathbf{u}) \cdot (1 - C_{\hat{\theta}_k}(\mathbf{u})) \int K^2.$$

On the previous assumption, the statistic \mathcal{R}_n in (4.1) tends to a Chi-square distribution with q -degrees of freedom under the null hypothesis $\mathcal{H}_0^{(2)}$.

According to both marginal time series which are normally regarded serial independence, we know that

$$\mathbb{P}\{X_j \leq x_j | \mathcal{B}\} = \mathbb{P}\{X_j \leq x_j | \mathcal{A}_j\}, a.e.$$

for all $j = 1, 2$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. This means that \mathcal{B} cannot provide more information about X_j than \mathcal{A}_j , for every j . Then C is a true copula (conditional copula), (Fermanian and Wegkamp (2004)).

5 Change of Parameters with Copula Family Remaining Invariant

If we could not reject $\mathcal{H}_0^{(2)}$, that is, while the parameters of copula change, the copula family remains invariant, then we can deal with analysis for the dynamic copula as in Dias and Embrechts (2004) with some technique disposing of the parameters.

5.1 Change-point analysis for the parameter

In this part, we apply the change-point theory to the changing parameters and estimate the size of those changes and the corresponding time of occurrence (change-points).

According to the static case of copulas, we choose the best copula over all the period considered in the research. If this copula has more than one parameter, we consider the one which is regarded as more important to be variant and the others invariant. For example, if the copula is Student t copula, we can think that the correlation parameter changes while the degree of freedom remains unchanged; for the convex combination of the Archimedean copulas, we decide the variant parameter on the basis of the weights in the combination.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be a sequence of independent random vectors in $[0, 1]^2$ with univariate uniformly distributed margins and copulas $C(\mathbf{u}; \theta_1, \eta_1)$, $C(\mathbf{u}; \theta_2, \eta_2), \dots, C(\mathbf{u}; \theta_n, \eta_n)$ respectively, where θ_i and η_i are the copula parameters such that $\theta_i \in \Theta^{(1)} \subseteq \mathbb{R}^p$ and $\eta_i \in \Theta^{(2)} \subseteq \mathbb{R}^q$. We will be primarily interested in one single change-point analysis for the θ_i 's, whereas the η_i 's will be nuisance parameters. As a consequence, we test the null hypothesis

$$\mathcal{H}_0^{(3)} : \theta_1 = \theta_2 = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n$$

versus the alternative

$$\mathcal{H}_a^{(3)} : \theta_1 = \dots = \theta_{k^*} \neq \theta_{k^*+1} = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n.$$

Here k^* is the location or time of the single change-point if we reject the null hypothesis. All the parameters $(\theta, \eta) \in \Theta^{(1)} \times \Theta^{(2)}$ are supposed to be unknown under both hypotheses. If we assume that $k^* = k$ is known, the question consists of testing if these two subsamples come from the same population and can be done through the generalized likelihood ratio test. The null hypothesis would be rejected for small values of the likelihood ratio

$$\Lambda_k = \frac{\sup_{(\theta, \eta) \in \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq n} c(\mathbf{u}_i; \theta, \eta)}{\sup_{(\theta, \theta', \eta) \in \Theta^{(1)} \times \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq k} c(\mathbf{u}_i; \theta, \eta) \prod_{k < i \leq n} c(\mathbf{u}_i; \theta', \eta)}.$$

where c is the density of C given by

$$c(\mathbf{u}_i; \theta, \eta) = \frac{\partial^2 C(u_{i,1}, u_{i,2}; \theta, \eta)}{\partial u_{i,1} \partial u_{i,2}},$$

with $(u_{i,1}, u_{i,2}) \in [0, 1]^2$. As the estimation of Λ_k is carried out through maximum likelihood, all the necessary conditions of regularity and efficiency have to be assumed, (Lehmann and Casella (1998)).

If we denote

$$L_k(\theta, \eta) = \sum_{1 \leq i \leq k} \log c(\mathbf{u}_i; \theta, \eta)$$

and

$$L_k^*(\theta, \eta) = \sum_{k < i \leq n} \log c(\mathbf{u}_i; \theta, \eta),$$

then the likelihood ratio equation can be written as

$$-2 \log(\Lambda_k) = 2(L_k(\hat{\theta}_k, \hat{\eta}_k) + L_k^*(\theta_k^*, \hat{\eta}_k) - L_n(\hat{\theta}_n, \hat{\eta}_n)).$$

While k is unknown, $\mathcal{H}_0^{(3)}$ will be rejected for large values of

$$Z_n = \max_{1 \leq k < n} (-2 \log(\Lambda_k)).$$

Taking $h(n) = l(n) = (\log n)^{3/2}/n$ as in Gombay and Horváth (1996), the following approximation holds:

$$\begin{aligned} \mathbb{P}(Z_n^{1/2} \geq x) &\approx \frac{x^p \exp(-x^2/2)}{2^{p/2} \Gamma(p/2)} \cdot \left(\log \frac{(1-h)(1-l)}{hl} \right. \\ &\quad \left. - \frac{p}{x^2} \log \frac{(1-h)(1-l)}{hl} + \frac{4}{x^2} + O\left(\frac{1}{x^4}\right) \right), \end{aligned}$$

as $x \rightarrow \infty$ (see Dias and Embrechts (2003)).

If we assume that there is exactly one change-point, then the maximum likelihood estimation for the time of change is given by

$$\hat{k}_n = \min\{1 \leq k < n : Z_n = -2 \log(\Lambda_k)\}.$$

In the case that there is no change, \hat{k}_n will take a value near the limits of the sample. This holds because under the null hypothesis

$$\hat{k}_n/n \xrightarrow{\mathcal{D}} \xi, \text{ as } n \rightarrow \infty,$$

under regularity conditions, where $\mathbb{P}(\xi = 0) = \mathbb{P}(\xi = 1) = 1/2$, (see Csörgö and Horváth (1997) and verified in Dias and Embrechts (2002)).

The detection of several change-points in multidimensional processes with unknown parameters can be done using so called binary segmentation procedure proposed by Vostrikova (1981) which enables to simultaneously detect the number and the location of the change-points. The method consists of first applying the likelihood ratio test for one change. If $\mathcal{H}_0^{(3)}$ is

rejected, then we have the estimate of the time of the change \hat{k}_n . Secondly, we divide the sample in two subsamples $\{\mathbf{u}_i : 1 \leq i \leq \hat{k}_n\}$ and $\{\mathbf{u}_i : \hat{k}_n < i \leq n\}$ and test $\mathcal{H}_0^{(3)}$ for each one of them. If we find a change in any of the subsets, we continue this segmentation procedure until we could not reject $\mathcal{H}_0^{(3)}$ in any of the subsamples.

5.2 Refine the variance equation for the parameter

As change-points tests have less power in the case of small changes, then we refined the dependence analysis assuming a time-varying behavior for the correspondent parameter.

If we deal with the dynamic t-copula, we assume that the degree of freedom ν is invariant, while the dynamical correlation is defined by:

$$\rho_\tau = h^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 h(\rho_{t-1})),$$

where r_0, r_1, s_1 are parameters and $h(\cdot)$ is Fisher's transformation for the correlation

$$h(\rho) = \log\left(\frac{1+\rho}{1-\rho}\right),$$

(Dias and Embrechts(2004)).

For the Archimedean copula, we decide which parameter changes according to the weight in the convex combination. Moreover, when we can also consider in the way of the tail dependence concept defined as:

Definition 2. *If a bivariate copula C is such that*

$$\lim_{u \uparrow 1} \bar{C}(u, u)/(1 - u) = \lambda_U$$

exists, with $\bar{C}(u, u) = 1 - 2u + C(u, u)$, then the copula C has an upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. Moreover if a bivariate copula C is such that

$$\lim_{u \downarrow 0} \bar{C}(u, u)/u = \lambda_L$$

exists, we will say that the copula C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$.

If the lower tail dependence λ_L represents more significant than the upper tail dependence λ_U , we can choose the symmetrized Joe-Clayton (SJC) copula in Patton (2003) in the form of

$$C_{SJC}(u, v | \lambda_U, \lambda_L) = \frac{1}{2}(C_{JC}(u, v | \lambda_U, \lambda_L) + C_{JC}(1-u, 1-v | \lambda_U, \lambda_L) + u + v - 1)$$

where C_{JC} is the Joe-Clayton copula defined as:

$$C_{JC}(u, v | \lambda_U, \lambda_L) = 1 - (1 - \{[1 - (1-u)^\kappa]^{-\gamma} + [1 - (1-v)^\kappa]^{-\gamma} - 1\}^{-1/\gamma})^{1/\kappa}$$

where

$$\kappa = 1/\log_2(2 - \lambda_U)$$

and

$$\gamma = -1/\log_2(\lambda_L)$$

with $\lambda_U \in (0, 1)$, $\lambda_L \in (0, 1)$.

We assume that λ_U remains unchanged and

$$\lambda_{L,t} = A(\omega_L + \beta_L \lambda_{L,t-1} + \alpha_L \cdot \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}|)$$

where

$$A(x) \equiv \frac{1}{1 + e^{-x}}$$

is the logistic transformation and ω, β, α are its parameters.

Then we combine two marginal time series with one dynamic copula as in Dias and Embrechts (2004).

6 Change of Copula's Family

If we do reject $\mathcal{H}_0^{(2)}$ which means that the family of the copula may change, we have to consider the change of the multivariate distribution. Extending the change-points theory in Section 5.1, generalized analysis is rendered to the changes of copula's family.

6.1 Change-point analysis for the copula

We can use U -statistic to detect the possible change-points of the multivariate distribution as in Horváth and Shao (1996) and Gombay (2001).

For the reason of serial independence, we regard $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}} = \{(X_{i1}, X_{i2}) : i = 1, 2, \dots, n\}$ as independent variables. We want to test the null hypothesis

$$\mathcal{H}_0^{(4)} : C_1(u, v) = C_2(u, v) = \dots = C_n(u, v) \quad \text{for all } u, v \in [0, 1]$$

against

$$\begin{aligned} \mathcal{H}_a^{(4)} : C_1(u, v) = C_2(u, v) = \dots = C_{k^*}(u, v) \quad \text{and} \\ C_{k^*+1}(u, v) = C_{k^*+2}(u, v) = \dots = C_n(u, v) \quad \text{for all } u, v \in [0, 1], \\ C_{k^*}(u_0, v_0) \neq C_{k^*+1}(u_0, v_0) \quad \text{for some } u_0, v_0 \in [0, 1]. \end{aligned}$$

If we reject the null hypothesis, k^* is the time of the change.

The change-point problem has been studied extensively in the literature. Csörgö and Horváth (1988) used U -statistics which are generalizations of Wilcoxon-Mann-Whitney statistics to detect a possible change-point. For surveys on U -statistics we refer to Serfling (1980), Lee (1990) and Koroljuk and Borovskich (1994).

Let $l(x, y)$ be a symmetric function and denote

$$l(\mathbf{X}, \mathbf{Y}) \equiv (l(X_1, X_2), l(Y_1, Y_2)),$$

such that

$$l(\mathbf{X}, \mathbf{Y}) = l(\mathbf{Y}, \mathbf{X}) \quad \text{for all } \mathbf{X}, \mathbf{Y} \in \mathbb{R}^2.$$

We define the U -statistics $\hat{U}(k)$ and $\tilde{U}(k)$ by

$$\hat{U}(k) = \frac{1}{\binom{k}{2}} \sum_{1 \leq i < j \leq k} l(\mathbf{X}_i, \mathbf{X}_j), \quad 1 < k \leq n$$

and

$$\tilde{U}(k) = \frac{1}{\binom{n-k}{2}} \sum_{k < i < j \leq n} l(\mathbf{X}_i, \mathbf{X}_j), \quad 1 \leq k < n - 1.$$

If $\hat{U}(k)$ significantly differs from $\tilde{U}(k)$ for at least one k , then we reject $\mathcal{H}_0^{(4)}$ in favor of $\mathcal{H}_a^{(4)}$. Our procedures will be based on functionals of

$$U_{k,n} = \begin{cases} n^{\frac{1}{2}} \frac{k}{n+1} (1 - \frac{k}{n+1}) (\hat{U}(k) - \tilde{U}(k)) & \text{for } 2 \leq k \leq n-2 \\ 0 & \text{for } k = 1, n-1, n \end{cases},$$

and

$$T_n = \max_{1 \leq k < n} \frac{|U_{k,n}|}{(\text{Var} U_{k,n})^{1/2}}.$$

For each k , $U_{k,n}$ compares the first k observations to the last $n-k$ using the kernel l , and T_n selects the maximum of the standardized U -statistics. According to Csörgö and Horváth (1988), we should reject $\mathcal{H}_0^{(4)}$ for large values of T_n . The limit distribution of T_n under $\mathcal{H}_0^{(4)}$ in case of nondegenerate kernels was obtained by Csörgö and Horváth (1988) and Gombay and Horváth (2002), we use the following result:

Theorem 2. *Let $\tilde{l}(\mathbf{t}) = E\{l(\mathbf{X}_1, \mathbf{t}) - E_{\mathcal{H}_0^{(4)}} l(\mathbf{X}_1, \mathbf{X}_2)\}$. Assume that $\mathcal{H}_0^{(4)}$ holds, $E|l(\mathbf{X}_1, \mathbf{X}_2)|^\nu < \infty$ with some $\nu > 2$ and $\tau^2 = E\tilde{l}^2(\mathbf{X}_1) > 0$. Then*

$$\lim_{n \rightarrow \infty} \mathbb{P}\{a(\log n) \frac{1}{2\tau} \max_{1 \leq k < n} \frac{|U_{k,n}|}{(\frac{k}{n+1}(1 - \frac{k}{n+1}))^{1/2}} - b(\log n) \leq \mathbf{t}\} = \exp(-2e^{-\mathbf{t}})$$

for all $\mathbf{t} \in \mathbb{R}^2$, where $a(x) = (2 \log x)^{1/2}$ and $b(x) = 2 \log x + \frac{1}{2} \log \log x - \frac{1}{2} \log \pi$.

We denote

$$\eta_n = a(\log n) \frac{1}{2\tau} \max_{1 \leq k < n} \frac{|U_{k,n}|}{(\frac{k}{n+1}(1 - \frac{k}{n+1}))^{1/2}} - b(\log n),$$

η has the distribution functions $\exp(-2e^{-\mathbf{t}})$, then $\eta_n \xrightarrow{\mathcal{D}} \eta$.

Further results on the applications of U -statistics to change-point analysis can be found in Ferger and Stute (1992) and Ferger (1994a-c).

If we assume that there is exactly one change-point, then the estimator for the time of the change is given by

$$\hat{k}_n = \min\{1 \leq k < n : T_n = \frac{|U_{k,n}|}{(\text{Var} U_{k,n})^{1/2}}\}.$$

In the case that there is no change, \hat{k}_n will take a value near the limits of the sample as described in Section 5.1, and also similarly to what is

introduced in Section 5.1, the detection of several change-points in multi-dimensional processes can be done using the so called binary segmentation procedure.

6.2 Subsamples analysis using moving windows

More tractably, we can use moving windows to decide the best copula on subsamples as in Caillault and Guégan (2004). For the whole period, we divide it into several subintervals, which seems like a moving window for us to study the observations. On each subinterval, we apply the methodology for the static case as presented in Section 2 to choose the best copula according to the different criterions.

7 Change Analysis of Dynamic Copula for FX Data in Asian Market

In this section, we apply the above change analysis of dynamic copula to the foreign exchange markets in Asia. We are interested in the FX spot rates for USD Dollar quoted against Thai Baht (USD/Baht) and FX spot rates for USD Dollar quoted against Malaysian Dollar (USD/MYR) in which 3726 daily observations from November 6, 1991 to February 16, 2006 — including the Asian crisis in 1997 to 1998 — are used. As usual, we analyze the log-difference of each exchange rate.

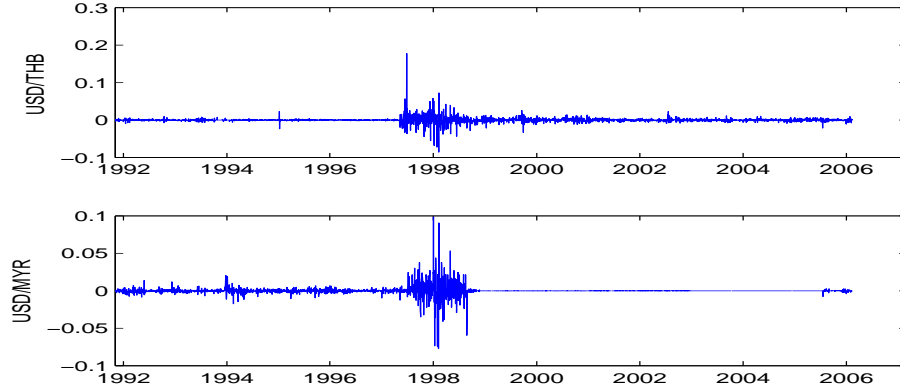


Fig. 1. Daily FX logreturn of USD/THB and USD/MYR from November 6, 1991 to February 16, 2006

Among the empirical stylized facts for univariate financial returns are weak linear dependence in time, heteroscedasticity and non-normal innovations. The FX observations of USD/THB and USD/MYR are no exception. We ran the Jarque-Bera test, and normality is rejected at the usual probability levels for all considered time series. Moreover, the test for the absence of ARCH effects is also rejected. In our discrete-time setting, we model stochastic volatility effects by GARCH type models. In particular, we fit univariate GARCH models to each of the marginal series with innovations assuming come from a t distribution. We fit the models, perform the tests and obtain the standardized residuals.

After plotting the sample autocorrelograms and cross-correlograms for the absolute values of the USD/THB and USD/MYR residual vectors resulting from the above marginal GARCH fitting, it seems to be no evidence against serial independence of the absolute residual values. Only the contemporaneous cross-dependence remains and that is exactly where our interest lies.

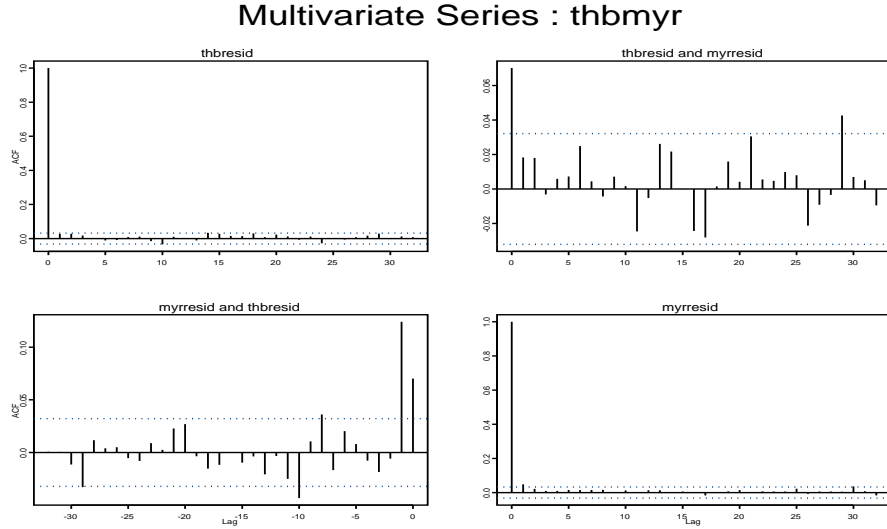


Fig. 2. Sample autocorrelograms for USD/THB and USD/MYR residuals, respectively top left and bottom right, and cross-correlograms for USD/THB on past USD/MYR (top right) and USD/MYR on past USD/THB (bottom left).

We show the scatter-plots of the residual-pairs. In order to investigate the residual dependence, we fit copula-based models as introduced in Section 2. The copula families fitted to the USD/THB and USD/MYR spot rate residuals are: t, Frank, Plackett, Gaussian, Gumbel, Clayton and the mixtures Gumbel with survival Gumbel, Clayton with survival Clayton, Gumbel with Clayton and survival Gumbel with survival Clayton (for details on these classes see Embrechts *et al.* (2002), Joe (1997) and Nelsen (1999)).

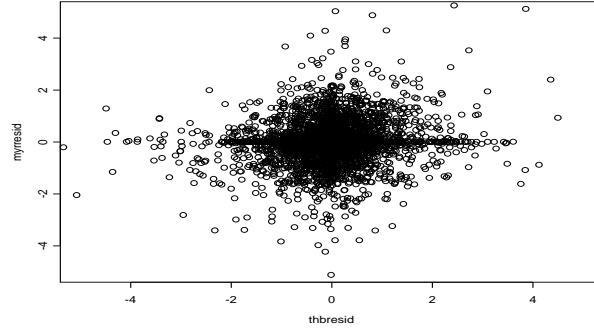


Fig. 3. scatter plot for USD/THB and USD/MYR residuals using ARMA-GARCH process filtering

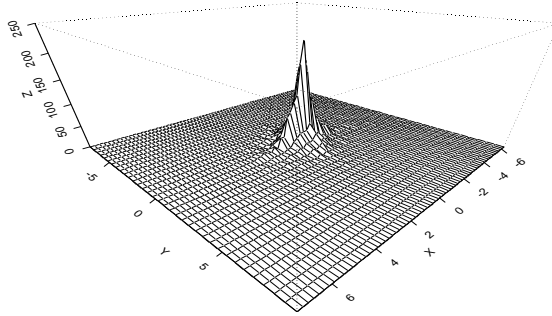


Fig. 4. 3D scatter plot for USD/THB and USD/MYR residuals using ARMA-GARCH process filtering

We fitted all the listed copula models to the residual-pairs to obtain the dependence parameter estimates. As shown in Section 2.1, we use the AIC diagnostic criterion to choose the best copula.

Copula	Parameters	Converge	AIC
Student t	0.120256, 36.27431	T	-53.6716
Gaussian	0.1213863	T	-52.91376
Gumbel	0.9684141 (invalid)	T	-26.9599
Clayton	1.110223×10^{-16}	F	-8313.906
Frank	-0.6680762 (invalid)	T	-43.8967
Plackett	4791.683	F	-279636.6
Surv Clay + Clay	2.75588, 0.165659 0.1401002	T	384.0014
Gum + Surv Gum	1.179328, 0.1903219 0.5	F	NA
Gum + Clay	1.196078, 1.814006 0.9808032	T	180.9153
Surv Clay + Surv Gum	1.179328, 0.1903219 0.5	F	NA

Table 1: AIC analysis for the static case for choosing the best copula. For the Student t copula, the first parameter is the correlation and the second is the degree of freedom. In case of the mixture models, the first two parameters are the dependence parameters respectively for the first and second terms of the mixture, the third is the mixture parameter which gives the proportion of the first term.

From the models fitted to the residuals, the one which has the best AIC is the Student t or the Gaussian for USD/THB and USD/MYR. Considering that the Student t copula tends to a Gaussian copula when its degree of freedom is large, and for that the AIC value for the Student t copula and the Gaussian copula is almost the same, we regard the Gaussian copula as the best copula for the residuals pair of USD/THB and USD/MYR.

Combining the GARCH marginal fits with the copula modelled residuals we end up with the dynamic, bivariate models.

As the data covers a reasonably long period of time, we may expect that economic factors induced changes. We apply the test in Section 3.2 to the conditional pseudo copulas in the aim of detecting the constancy with

respect to the time.

We restrict to $p = 1$, i.e.: only the last observed value is assumed to influence the dynamics of the dependence of the two FXs. We choose two sets of past observed values s.t. $q = 2$. Then we can get the test result for the null hypothesis $\mathcal{H}_0^{(1)}$ which is almost equal to 0 and we cannot expect a constant type conditional dependence structure between all the return series. That means that we should use the models of dependence in a dynamical way. The problem now is to decide whether the changes come from the parameters of copulas with invariant families or from the copulas' families. In order to specify the changes, we use the test introduced in Section 4.

We apply the binary segmentation method to detect the changes. Considering that the empirical results of testing $\mathcal{H}_0^{(2)}$ might not be very exact,³ we can determine the copula change type in a more immediate and obvious reasonable way. For every moving window, after the considered possible change point divides it into two parts, we compare all the copula fitting results of the whole part and the two sub-parts. If the best copulas are not identical for these three parts, we say that the copula family changes. Otherwise, we then compare the parameters for the two sub-parts to see whether the parameters change in the case that the copula family remain unchanged. If the change ratio exceeds 17%, we think that the parameter changes with invariant copula family, and when the change ratio shows below 17%, we say that the copula does not change.

In the case where only the parameters change while copulas' families unchanged, we can at first perform the change-points analysis to find the change times, and then refine the dynamic model by assigning the variance equations to the changing parameters.

On the other hand, if the copula's family changes, we can also carry out the change-points analysis using the U -statistics as presented in Section 6.1 which tells us the change times.

³ The estimated copula $\hat{C}(\mathbf{u}|\mathbf{X}_{n-1}^{n-p} = \mathbf{y}^*) = m_n(m_{n,1}^{-1}(u_1|\mathbf{y}_1^*), m_{n,2}^{-1}(u_2|\mathbf{y}_2^*)|\mathbf{y}^*)$ involves the inverse kernel probability function that we usually handle as the quantile function. For this reason, in such specification test process, the empirical results may appear not very precise.

Therefore, we can divide the whole sample into subsamples separated by the change times which seems like the moving windows. So we are able to decide the best copula on each subsample.

windows on 500 observations	copula	parameters	change analysis from one window to the next one
1	Gaussian	0.09123	
2	Gaussian	0.15605	parameter changes
3	Gaussian	0.11489	parameter changes
4	Student t	0.13398,12.38166	copula family changes
5	Student t	0.23045,11.00495	parameter changes
6	Gaussian	0.24196	copula family changes
7	Gaussian	0.19999	parameter changes
8	Student t	0.13465,342.2022	copula family changes
9	Student t	0.29947,7.35822	parameter changes
10	Student t	0.38835,6.18583	parameter changes
11	Student t	0.33863,7.61399	copula dose not change
12	Student t	0.13212,10.11369	parameter changes
13	Student t	0.04966,341.3317	parameter changes
14	Student t	0.04194,342.5209	copula dose not change
15	Student t	0.02442,342.8223	parameter changes
16	Student t	0.02031,341.809	copula dose not change
17	Student t	0.03102,341.3082	parameter changes
18	Student t	0.0526,341.3494	parameter changes
19	Student t	0.04475,341.943	copula dose not change
20	Gaussian	0.04488	copula family changes
21	Clayton	0.07593	copula family changes
22	Student t	0.01586,342.3481	copula family changes
23	Student t	0.06559,342.2361	parameter changes

Table 2: Dynamic copula analysis according to the moving windows. We use 23 moving windows in which each consists 500 observations and we move the windows by 150 observations. We can choose the best copula on each subsample and observe the changes. For the change analysis from one window to the next one, we detect the change by 17% for the parameter change. For the Student t copula, the first parameter is the correlation that may change and the second is the degree of freedom that we suppose as invariant.

We detect several change-points in the process using the so called binary segmentation procedure as expressed in section 5.1 and section 6.1.

change type	$z_n^{1/2}$	n	$P(Z_n^{1/2} > z_n^{1/2})$	$\mathcal{H}_0^{(3)}(0.95)$	Time of change
parameter changes	Inf	3726	0	reject	28 Nov. 1998
parameter changes	3.291633	1863	0.04971634	reject	1 June 1995
parameter changes	0	1863	0.1994711	not reject	-
family changes	-	931	-	-	18 Aug. 1993
parameter changes	2.572383	932	0.27421	not reject	-
family changes	-	465	-	-	25 Sep. 1992
parameter changes	1.725376	466	0.9253652	not reject	-
family changes	-	232	-	-	16 April 1992
parameter changes	1.85954	233	0.6969473	not reject	-
parameter changes	0	116	0.1994711	not reject	-

Table 3: Change-point analysis for USD/THB and USD/MYR spot rate residuals. $z_n^{1/2}$ is the correspondent observation values for the statistic $Z_n^{1/2}$.

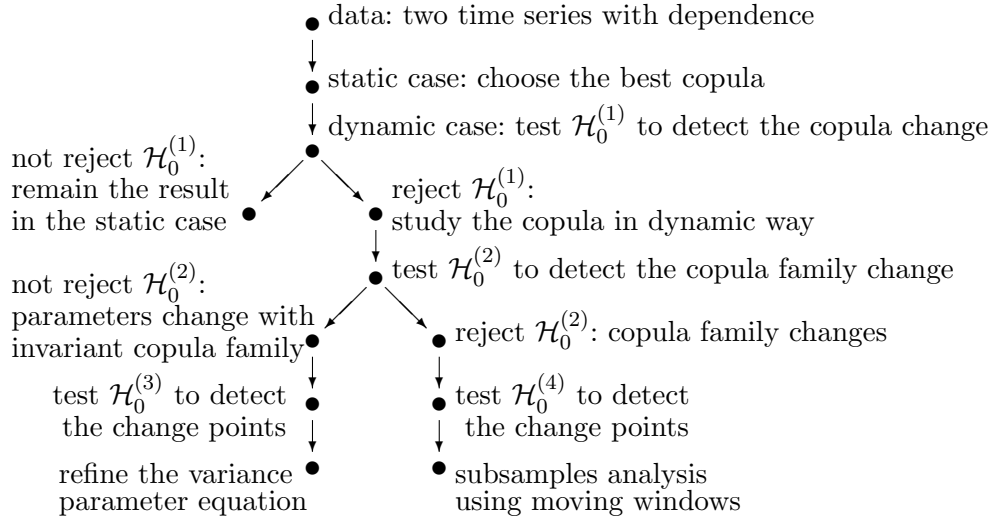
From the graph of the daily FX log-return, we can see that the data fluctuates very violently in 1998 when the Asian crisis broke out. So the result of the test for the change on 28 Nov. 1998 is extremely undoubted. In 1995, the daily log-return of FX USD/MYR changes normally, but for FX USD/THB, it changes little except for the middle of the year where there exists a large fluctuation which corresponds to the change point of 1 June 1995. From 1992 to 1994, FX USD/MYR changes a lot while FX USD/THB has almost no unusual cases, which brings about the three family change points on 16 April 1992, 25 Sep. 1992 and 18 Aug. 1993. After 1998, FX USD/THB changes normally while FX USD/MYR has nearly no changes, so we may think that the dependence structure does not change. Fortunately, the results using our approach shows that there is no change point in the last part from 1999, and this is what we expect.

8 Conclusion

In this paper we investigated a nested series of tests to analyze the change in the copula models. As the data considered for research covers a long period, one may not deny the possibility of change in the concept of copula. As concerned as the change in the copula structure, there exists two recent methodologies: one assumes the dynamical evolution of the cop-

ula's parameter with the invariant copula's family; the other one is more tractable but less accurate by using the moving windows to decide the best copula on the subsamples. Our goal in this study is to adopt a different way to deal with the dynamic copula. Through a chain of tests, we can decide the origin of the change and in the same time we can work with the dynamic copula. The rejection of the hypothesis $\mathcal{H}_0^{(1)}$ determines the time-dependent effect which allows us to study the structure of dependence in a dynamic way; the test result of hypothesis $\mathcal{H}_0^{(2)}$ points out how we should deal with the change. That is, the change originates from just the parameter or from the copula's family. Understanding which factor changes exactly, we could analyze the change-points and further to handle dynamic model.

The main idea of this paper can be expressed as following:



We apply the change analysis of dynamic copula to the foreign exchange spot rates in Asia markets, which comes into being several different results. During the test procedure, we estimate the copula by applying the kernel probability conditioned on the past observation values, and by using also the inverse kernel conditional probability. For some simple copula, such as the independent copula in some specified process in Fermanian (2004), the test works though the results may be not very accurate because that the results depend largely on the bandwidth choices in such specification test. In our empirical study, the copula is more complex

with a large set of data fluctuating violently, the inverse function in the form of quantile function might cause lots of deviations. Therefore, from the empirical study, we realize that the test using the conditional copula needs further research to raise the precision, especially when the inverse conditional kernel probability function with which we deal usually as the quantile function is applied. This point will become the emphasis of the future study in the aspect of dynamic copula.

Other aspects about modelling the structure of dependence are also concerned, such as the criterion to choose the best copula, conditional pseudo copulas, U -statistics used to detect the possible change-points of the multivariate distribution, which may still need further research.

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