On Sanskrit commentaries dealing with mathematics (Vth-XIIth century)

Agathe Keller

On Sanskrit commentaries dealing with mathematics (fifth-twelfth century)*

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Introduction

A renewed interest for contextualization in indological studies[1], is but slowly affecting publications on Indian mathematics. The isolation of history of mathematics within the general field of indology derives partly from a lively historiographical trend of technical and patriotic history of mathematics which remains oblivious to social science. Preservation plays a role as well: precious little information exists on the context in which mathematics and astronomy were practiced in India in the past [2]. To overcome this problem some historians of science have turned to periods (XVIth-XIXth century) and places where institutions, libraries and many texts help us contextualize their mathematical and astronomical ideas[3].

A focus on the kind of texts produced by astronomers and mathematicians of the Indian subcontinent and the history of how they were transmitted to us aids the contextualization of the knowledge they contain. An explanation of the diversity of textual forms that were produced, examination of the functions these forms filled, consideration of their self-proclaimed purpose, and an elucidation of their own conceptions of mathematical practices and ideas can provide information about who produced mathematical texts, why they were produced and how they were used[4]. This contextualized approach may prove as fruitful for the history of mathematical practices and conceptions in India as it has elsewhere[5]. These questions of author, audience and use pertain even to texts of early periods for which very little background information is known.

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[1]For the fundamental project headed by Sheldon Pollock, "Sanskrit Knowledge Systems at the Eve of Colonialism" (SKEC), see http://www.columbia.edu/itc/mealac/pollock/sks, (Pollock, 2002). For the results of this approach in literature (Pollock, 2006).

[2](Plofker, 2009) struggles to link a technical history to social questions, but both fields remain estranged from one another today.

[3]Among the publications on history of science, produced within Pollock's SKEC, see the works of Christopher Minkowski and Dominik Wujastyk, listed at http://www.columbia.edu/itc/mealac/pollock/sks/papers/index.html. We may add to this the recently defended thesis of Toke Lindegaard Knudsen, (Knudsen, 2008).

[4]A list of various categories of astronomical/astrological texts is given in(Pingree, 1981), and reformulated in (Plofker, 2009). Different kinds of texts, categorized by the subjects or styles they are likely to adopt, have long been identified by Indologists. Kim Plofker (p. 105-108) briefly contrasts the kind of trigonometrical astronomy stated in *karaṇas* with the astronomy given in *siddhāntas*. Mathematical difference and textual variations are noted, but not closely investigated. Plofker remarks that both kind of texts received commentaries.

[5]K. Chemla has published extensively on this question. Her latest synthesis is (Chemla, 2004).

Such a focus on the context of these texts requires clarification about which manuscripts can be found in libraries today, because they are the sources of editions and studies. Here, many questions arise[6]: Who copied the texts now at our disposal? Who had them copied? How were these texts used and read? How where these texts collected into libraries? Why? C. Minkowski and D. Raina address these questions more directly in their contributions to this volume.

In this contribution, however, the spotlight falls on the specific case of Sanskrit commentaries on mathematical subjects written between the fifth and the twelfth century A. D.,[7] with the broader purpose of understanding the social function of commentaries on mathematics. Therefore, this study will focus on how commentaries related to the texts which they were explicating. The answer to this question depends on an understanding of the points of view of three different sets of actors: those who wrote commentaries, those who had them copied, and those who analyzed them as historians of mathematics. The question of whether these actors were distinct must be addressed. Were commentaries on mathematics considered an adjunct but independent text? Were they thought of as reusable but ultimately disposable explanations to be summoned when useful to understanding an algorithm or an idea? In order to answer these questions, an attempt must be made to comprehend the scope of readings generated in India over time from an unchanging commentary.

Before these considerations, though, an introduction to what is known today about commentaries on the mathematics of the fifth to twelfth century. This introduction must situate these commentaries in the larger context of the literature on astral science (*jyotiṣa*)[8]. Then, a description of the manuscripts at our disposal provides information about who copied these commentaries as well as how such texts were copied. A survey of selected histories of mathematics in India begins with the XIXth century and continues to the present. This survey gives special attention to the story of the rediscovery and edition of the works of Āryabhaṭa (ca. 499), Brahmagupta (ca. 628) and Śrīdhara (ca. 950) and will illustrate how commentaries on mathematics have been read in more recent times. Finally, a close look at algorithms for the extraction of square roots will serve as an illustration of an analysis of a mathematical text and focus on how commentaries relate to the text they explain, thereby providing insights on the ways mathematicians writing in sanskrit conceived of and used the decimal place value notation.

1 Commentaries on mathematics from the fifth to the twelfth century

Today, Indologists intent on making new editions may easily be overwhelmed by the number of manuscripts at their disposal. The case of Sanskrit astronomy and mathematics is quite exceptional in this respect,

[6] A similar set of questions about another kind of text, the colonial archives, can be found in (Stoler, 2002).

[7] Stated more precisely, these commentaries were written between the seventh and twelfth century about texts which had been composed between the fifth and eighth century.

[8] *Jyotiṣa* (lit. "the ⟨sky's⟩ luminaries"), which is commonly translated as "astronomy," more exactly means "astral science" because the field includes horoscopy and mathematics together with observational and computational astronomy. See (Pingree, 1981, Introduction and Table of contents).

since a census has been undertaken which enables us to evaluate the number of manuscripts and published editions for the specific field of *jyotiṣa*. Indeed, David Pingree's *Census of the Exact Sciences in Sanskrit* (CESS)[9] lists most of the manuscripts on astronomy and mathematics in Sanskrit existing today.

The near exhaustivity of the CESS enables quantitative reflection on the collected and preserved manuscripts for astronomical, astrological or mathematical texts. Since the census not only gives manuscript references but tracks published editions, it can help us evaluate how many of these texts buried in manuscripts have been edited and studied in the last two hundred years. A close look at the census[10] shows that mathematical texts in Sanskrit have been significantly better-studied than texts on *jyotiṣa*. Indeed, texts on mathematics are comparatively rare in comparison to the other texts included in the census[11], but those in Sanskrit have been much more actively edited, studied and translated than the texts concerned solely with astronomy or astrology[12].

In this context, commentaries on mathematical texts have been closely studied, even though they do not represent much of the transmitted textual tradition[13]. However, as we will see, studies have often treated the commentaries in bits and pieces, or considered at them independently from the texts they explicate. Furthermore, even when commentaries have been edited, they have often not always been completely translated. When compared with the sea of all astral texts known in Sanskrit, this specialized treatment of commentaries dealing with mathematics can be considered within the sub-set of known texts on mathematics for the period ranging from the fifth to twelfth century of our common era.

[9](Pingree, 1970-1995). In 1955, D. Pingree started (see CESS I, preface.) a survey of manuscripts on *jyotiṣa* that was remained unfinished when he died fifty years later in November, 2005. The CESS spans 5 volumes, and Pingree's death interrupted the completion of volume 6. For authors who have not yet been treated in the published volumes of CESS, one can refer to (Sen, Bag, & Sarma, 1966) and to individual library catalogs. Despite its name, the CESS lists texts in many languages of the Indian subcontinent. Pingree cast a large net when undertaking his census and included texts that may refer to some part of *jyotiṣa* only in passing. While some manuscripts doubtless exist in private collections, other have escaped classification by libraries, and still others have been misclassified, most known manuscripts are probably included in his census.

[10]Because no electronic version of the text was available, the entries were counted manually: the evaluation may be subject to human error. Nonetheless, the investigation give a general idea of the proportions involved. Since this investigation, the CESS has been partly digitalized (volumes 1, 2 and 4) on http://books.google.com/. I have counted only manuscripts and have thus excluded references to authors for which there is no remaining text, as well as XXth century publications by modern authors for which no manuscript remains.

[11]Indeed, of the 3,686 texts I have recorded in the first five volumes of the CESS, only 102 (2.7 %) are clearly devoted at least partly to mathematics (*gaṇita*).

[12]This list does not include texts which were edited and translated. However, among the 102 texts listed, a significant number are in vernacular languages (particularly, oriya and tamil) and have seldom been edited or even translated. A majority of the parts of the Sanskrit texts concerning *gaṇita* have been edited and translated.

[13]Of the 3,686 texts and 2,972 authors devoted to *jyotiṣa*, 816 (22%) are commentaries and 646 authors (21.7 %) are commentators. Before starting this article, I believed that commentaries on mathematics had been largely neglected in the historiography of Indian science, and that they were an important part of the past tradition. Indeed, I mentioned this in the introduction to my book, (Keller, 2006). This error was noted and rightly criticized by S. R. Sharma. (S. R. Sarma, 2006, p. 144).

1.1 A limited number of known texts on mathematics.

All the presently identified texts on mathematics of the fifth to seventh century for which we have manuscripts are enumerated in Table 1.1. Some mathematical commentaries from this period are now lost to us, such as Prabhākara's commentary on the *Āryabhaṭīya* (ca.sixth century) (CESS 4 227 a), and Balabadhadra's (fl. eighth century) commentary to the *Brahmasphuṭasiddhānta* (CESS 4 255 a). These texts are not taken into account here.

Table 1: A list of known texts on mathematics, in Sanskrit, fifth-twelfth century.

Dates (A.D.)	Author	Title	Abbreviation
499	Āryabhaṭa	*Āryabhaṭīya* (Chapter 2)	Ab
628	Brahmagupta	*Brahmaspuṭasiddhānta* (Chapter XII)	BSS
629	Bhāskara	*Āryabhaṭīyabhāṣya* (Chapter 2)	BAB
VIIth-Xth century	unknown	*Bhakshālī* manuscript	BM
ca 864	Pṛthudakśvamin	*Vāsanābhāṣya* on the BSS (verses of Chapter XII)	PBSS
850-950	Śrīdhara	*Pāṭīgaṇita*	PG
idem	idem	*Triśatika*[a]	T
tenth century	Mahāvīra	*Gaṇitasārasaṃgraha*	GSS
c. 1039 A.D.	Śrīpati	*Gaṇitatilaka*	GT
ca. 1040	Someśvara	on the *Āryabhaṭīya* (chapter 2)	SAB
ca. 1150	Bhāskara II	*Līlāvatī*	L
idem	idem	*Bijagaṇita*	BG
[IXth-twelfth century	Āryabhaṭa II	*Mahāsiddhānta* (Chapters XV and XVIII)	MS][b]
ca. 1200	Sūryadeva Yajvan	on the *Āryabhaṭīya* (Chapter 2)	SYAB
[Unknown[c]	Unknown	commentary on the *Pāṭīgaṇita*	APG]

[a] As [Hayashi 1995] notes, we do not have any extant manuscript of his *Gaṇitapañcaviṃśi*.

[b] According to André Billard ((Billard, 1971, 161)), the *Mahāsiddhānta* contains observational data that was made in the first half of the sixteenth century. Generally, however,Āryabhaṭa II is considered to have lived between Śrīdhara and Bhāskara II (CESS I-II53). Thus, the text is provisionally added to this list.

[c] Shukla, who edited the text, considers that the commentary shares features with texts in the time span we have delineated, such as the Bhakshālī Manuscript and the BSS. (K. S. Shukla, 1959, xxviii-xxxiv).

The seventh to twelfth century is the beginning of an expanding mathematical and astronomical tradition. This tradition will permeate not only the Indian subcontinent, but extend in the East to

China and in the West to the Arabic peninsula. We have chosen the time before the works of Bhāskara II (ca.1114-1183) started to have an impact as an upper boundary. This decision enables us to include Sūryadeva Yājvan's commentary, which dates from some time after Bhāskara II (but is ignorant of his work) and after the Vedic period, which had its own specific mathematical tradition or style. The period we are considering thus ranges thus from 499 AD to 1200 precisely.

Although the number of mathematical texts transmitted is limited, their diversity is striking. This variety may be due in part to the limited autonomy of mathematics (*gaṇita*, "computation") with respect to astronomy. Indeed, a number of preserved texts on mathematics belong to astronomical treatises. To some extent, particularly for certain authors, Mathematics seems to have been a sub-discipline of astronomy. A commonly accepted use of *gaṇita* in this context is "computational astronomy." However, other Sanskrit authors of astronomical texts insisted that mathematics had an existence outside of astronomy[14]. This dubious division explains why a certain number of procedures with no application in astronomy were stated in astronomical chapters. Additionally, from the Vedic time onwards, independent texts on mathematics have been preserved. Accordingly, mathematical texts in an astronomical context differed from autonomous mathematical texts. In the first and second part of this study, these disparate texts will be collected together in order to focus on the fact that they concentrate on a same subject matter, which has a specific name, *gaṇita*. In the third part of this study, the difference will emerge again.

Table 1.1 lists 15 texts. It includes two sets of texts. "Primary texts" contain treatises and all other texts that stand alone; "secondary texts" refer to those texts that depend on another composition. Thus, "secondary texts" may be termed commentaries. All the texts on mathematics known for this period are summarized graphically in Figure 1, which delineates between primary and secondary texts.

Texts so far identified as commentaries on the subject of *gaṇita*, written during our chosen period[15] and for which we have extant manuscripts are thus, in a chronological order:

- Bhāskara's commentary on the second chapter of the *Āryabhaṭīya* (629 A. D.; hereafter the treatise is abbreviated as Ab and the commentary as BAB, with implicit reference to chapter 2 when cited this way),

- Pṛthudakśvamin's mathematical commentary on the twelfth chapter of the *Brahmasphuṭasiddhānta* of Brahmagupta (the treatise dates to 628 A. D. and the commentary dates to ca. 864 A. D. ; hereafter the treatise is abbreviated as BSS and the commentary as PBSS, with implicit reference to chapter XII when cited this way),

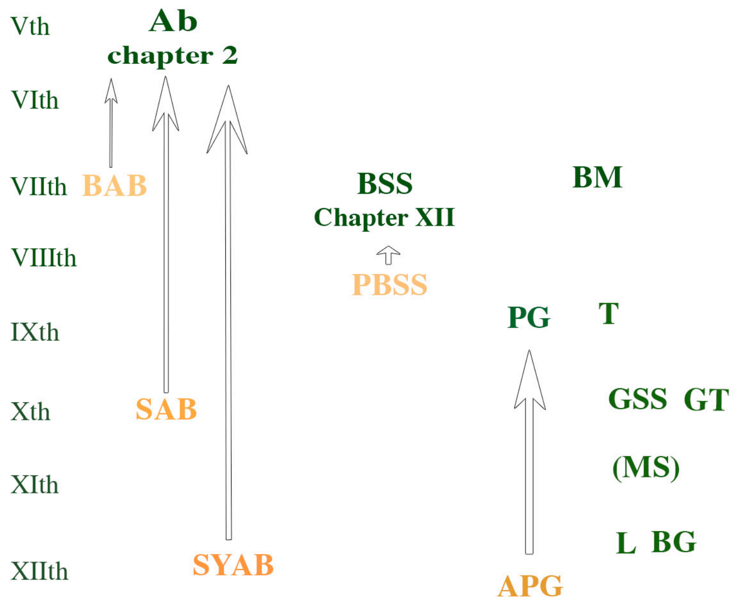
- Someśvara's commentary on the second chapter of the *Āryabhaṭīya* (ca. 1040, hereafter abbreviated as SAB, with implicit reference to chapter 2),

- Sūryadeva Yajvan's mathematical commentary on the *Āryabhaṭīya* (Sūryadeva Yajvan is believed

[14]See for instance (Keller, 2007), and (Plofker, 2009).

[15]Commentaries on the treatises enumerated here have been written after our period but these commentaries are not listed here. We will return to this situation below.

Figure 1: Texts on mathematics in Sanskrit, from the seventh to twelfth century



to have been born in 1191 A. D., his commentary is hereafter abbreviated as SYAB, with implicit reference to chapter 2).

To these we might also add:

- the anonymous and undated commentary on the *Pāṭīgaṇita* of Śrīdhara (fl. 850-950 A.D , date unknown for the commentary[16]; hereafter the treatise is abbreviated as PG and the commentary as APG).

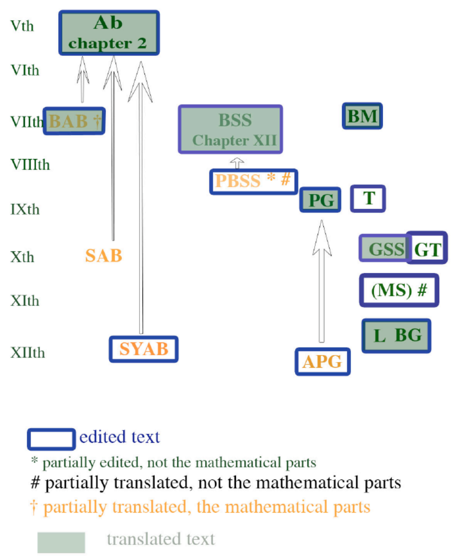
The following discussion concentrates especially on the edited commentaries of this list: BAB, SYAB and APG. First, let us return to the texts which these commentaries gloss. The *Āryabhaṭīya* has received extensive commentary. K. S. Shukla and K. V. Sarma count 19 commentators on the *Āryabhaṭīya*[17], 12 of which are in Sanskrit. Half of these are from after the Fifth century. In the case of Brahmagupta's BSS, on the other hand, only two commentators are known. Furthermore, only seven manuscripts of the original composition have been preserved. BSS, however, considers only one commentary: PBSS. Only two manuscripts of PBSS contain commentaries of Chapter XII. Four manuscripts of the 34 remaining manuscripts of the BSS, however, also provide an anonymous commentary on BSS[18]. Finally, as far as I know, PG is known through a single manuscript which contains a similarly unique anonymous commentary on the text. Briefly stated, early treatises dealing with mathematics have not always come down to us with a great number of commentaries.

[16]K. S. Shukla, who edited the text, believes that the commentary shares features with texts from the chosen time span. He especially draws similarities with the Bhakshālī Manuscript and the BSS. (K. S. Shukla, 1959, pp. xxviii-xxxiv).

[17]See (K. V. Shukla & Sharma, 1976, p. xxv-lviii). We have included Prabhākara in this account, although no extant commentary is known. Nonetheless, he is quoted by Bhāskara.

[18]CESS IV 255 b; V 239 b.

Figure 2: Edited and translated texts on mathematics from the fifth to the twelfth century



All of the primary texts enumerated here have been entirely edited and translated into English. The texts of Ab, BSS, BM, GSS, PG, L and BG have also been edited and translated into English[19]. All other texts have only been edited[20]. The BSS is, for this period, the only non-extensively edited treatise containing a mathematical part. Thus, BSS is an exception: only bits and pieces have been translated in English. This special situation probably derives from the fact that no extant ancient commentaries in surviving manuscripts are known for this text, part of which thus remains hard to understand.

Concerning commentaries, two (PBSS and BAB) have been partially translated into English[21], but only one (the BAB) was translated for its mathematical part. A portion of SAB was edited together with BAB.[22]. The situation concerning the editions and translations of these texts is given in Table 1.1 and Figure 2. Setting aside the special cases of BSS and PBSS, note that the commentaries have been generally edited, but usually not translated: a striking artefact of modern scholarship.

[19]See (K. V. Shukla & Sharma, 1976), (M. S. Dvivedin, 1902), (Hayashi, 1995), (K. S. Shukla, 1959), and (Rangacarya, 1912). There have been numerous printed editions of L and BG, two texts which are noted in CESS 4 308 a and 311 b. See the translation given in (Brahmagupta; Bhāskarācārya; Colebrooke, 1817).

[20]The text of T was edited in (S. E. Dvivedin, 1899). The text of GT was edited with a modern commentary (Kāpadīā, 1937). The text of MS was edited by Sudhākara Dvivedin in 1910 (S. Dvivedin, 1910), and partially translated into English by S. R. Sarma, (S. R. Sarma, 1966).

[21]Part of PBSS's astronomical commentary has been studied, edited and translated by Setsuro Ikeyama (Ikeyama, 2003). Pṛthudakṣvamin's commentary on the twelfth chapter of the BSS, is found in a manuscript at the Indian Office and in what appears as a copy of this manuscript used by S. Dvivedi in Benares (CESS A. IV. 221 b), (Ikeyama, 2003, S5-S7). This last commentary has not been edited entirely, probably because the only recension is at times quite difficult to understand.

[22](K. S. Shukla, 1976).

Table 2: Editorial situation of texts on Sanskrit mathematics from the fifth to twelfth century

Abbreviation	Treatise, Commentary, Others	Edited	Translated
Ab	Versified treatise	Yes	Yes
BSS	Versified treatise	Partly	Partly
BAB	Prose commentary	Yes	Partly
BM	Fragmentary text (versified rules and examples; prose resolutions)	Yes	Yes
PBSS	commentary	Partly	Partly
PG	Versified treatise	Yes	Yes
T	Treatise	Yes	No
GSS	Treatise	Yes	Yes
GT	Treatise	Yes	Partly (not the mathematical parts)
L	Treatise	Yes	Yes
BG	Treatise	Yes	Yes
MS	Treatise	Yes	Partly
SYAB	Commentary	Yes	No
APG	Commentary	Yes	No

This editorial situation can be taken as a symptom of the special treatment of commentaries as independent texts. The situation raises many questions: Were commentaries less frequently translated because historians of science were not interested in them? Or was their neglect a consequence of the state of manuscripts? These questions lead us to consider the way ancient collectors of texts, and maybe even authors of commentaries themselves, regarded the texts they were composing.

1.2 Reading and collecting commentaries on mathematical subjects

Who collected manuscripts? Who had them made? Who copied them? As C. Minkowski and D. Raina's articles in this volume emphasize, to date we have only sporadic information, which enables only partial answers, varying from library to library, region to region, collection to collection. Because we have restricted ourselves to a specific corpus, information is especially sparse. Usually, we do not even know the dates of manuscripts or their accession histories, but it is reasonable to believe that they are mostly products of the copying frenzy during the late modern period described by C. Minkowski in his article of this volume.

In certain cases, a great number of ancient hand made copies of a given text have been preserved. Thus, for Ab[23], the CESS counts 149 manuscripts of which 47 (more or less 1/3rd) include commentaries. These manuscripts can be found in 28 different libraries. These texts are mostly of unknown origin and are written on either paper or palm leaf.

[23]CESS 1 51a-52b; 2 15b; 3 16a; 4 27b.

The extant manuscripts of commentaries are comparatively less important. Bhāskara's commentary is known through six manuscripts, five of which are in the same library in Kerala. All are incomplete[24]. Similarly, Sūryadeva's commentary has been transmitted to us through eight south Indian recension s and copies[25], while SAB is known through only one copy[26]. The BSS is known through thirty-four manuscripts[27] and PBSS is known in seven manuscripts, two of which are fairly recent copies of two others, none of which are extant[28]. Only two of these seven manuscripts contain the commentary on chapter XII which is explicitly devoted to *gaṇita*[29]. Finally, the edited PG is known through one manuscript – the same one containing APG[30]. These details are summarized in Table 3.

Table 3: Number of manuscripts for commentaries and treatises on mathematics from the fifth to twelfth century

Text	Treatise	Commentary	Number of remaining manuscripts
Ab	x		47
BAB		x	6
SAB		x	1
BSS	x		24
PBSS		x	7
PG	x		1
APG		x	1

Therefore, in these collections, commentaries were much less numerous than the treatises and even then the commentaries were not always complete. Is this scarcity due to the hazards of preservation, or does it reflect a set of assumptions about the commentaries in common currency when the texts were copied or collected during the eighteenth and nineteenth centuries? Did the authors of commentaries themselves consider their commentaries fragmentary texts? Do these assumptions explain the way commentaries were subsequently treated by historians? To answer these questions precisely, we must inquire more deeply into the complexities of each text and how it has been transmitted to us. However, before looking at how these commentaries were transmitted, let us specify what kind of texts are subject to this analysis.

1.3 A preliminary characterization of commentaries on mathematics

As is clear from this article, much work stills needs to be done in organizing and categorizing the corpus of Sanskrit manuscripts. Consequently, there is something tricky in any attempt to isolate and

[24](K. S. Shukla, 1976), CESS IV 297b.

[25](K. V. Sarma, 1976, p. xvii to xxv).

[26](Sen et al., 1966, p. 202), CESS I-II 51a.

[27]CESS IV 254 b-255b, CESS V 239 b- 240 a.

[28]CESS IV 221 a, CESS V 224 a .

[29](Ikeyama, 2003, p. S7).

[30](K. S. Shukla, 1959). As noted previously, (Sen et al., 1966, p. 204) notes a second manuscript of the PG *ṭīkā*, incomplete at fifty-four folios, in the Descriptive Catalogue of the Oriental Mss in the Mackenzie Collection, compiled by H. H. Wilson in Madras in 1882.

identify mathematical commentaries among the other Sanskrit texts on mathematics and related subjects. Nonetheless, some useful but tentative conclusions may be derived.

The titles of texts and expressions used in them indicate that the Sanskrit scholarly tradition distinguished between treatises (*śāstra* , *tantra*, *siddhānta*) and commentaries (*vyakhyā*, *bhāṣya, ṭīkā*)[31]. A similar distinction separated the fields of *jyotiṣa* and *gaṇita*[32]. Many other kinds of texts occupied these fields, such as handbooks for making calendars (*karaṇa*), almanacs (*pañcāṅga*), compendiums (*nibandha*), etc.[33]. With this wealth of titles in mind, we can ponder whether certain kinds of text had been preferred over others in times past. Did such preferences vary over time, in different places, or according to certain sub-fields? Do these trends tell us something about the conceptions which different actors had of what an astronomical or mathematical text ought to be? What do these trends indicate about how such disciplines were practiced? Did the historians of science who identified these texts focus on one kind of text over another?

Concerning texts composed between the fifth and twelfth centuries, the Ab is referred to as a *tantra* by Bhāskara[34] but is described as a *śāstra* by Sūryadeva Yajvan. Finally, APG refers to the text for which it provides commentary as a *śāstra*[35]. Similarly, the Sanskrit authors rely on a paucity of names to title to their commentaries. Thus, Bhākcara calls his commentary a *vyākhyā*[36]. As the title given to his commentary reflects, Bhākcara invokes a similarity with the tradition of *bhāṣya*. Sūryadeva Yajvan occasionally refers to his own text as a *vyākhyā*, but more generally, he calls it a *prakāśa*, "light"[37].

[31]For an attempt to characterize "scholastic Sanskrit" of commentaries (in the case of grammatical, philosophical and logical texts), see (Tubb, 2007).

[32]In the case of commentaries, many different technical names are recorded either in the titles or by D. Pingree in the CESS. Keeping in mind that these numbers and percentages should be used cautiously, the initial results may be reported. First, *ṭīkā*s compose 549 titles, (67.2% of all commentaries), but other technical names and titles appear. *Vyākhyā*s represent 85 titles (or 10.4%); *vivṛtti*s number 50 (or, 6.1%); and *bhāṣya*s number 34 (or, 4.1%). Smaller numbers of *avacūrṇi*s, *vārtikka*s, *tippaṇaṇi*s, and *vivāraṇa*s also occur. This diversity raises the general question of how the texts were titled. Unlike the compositions examined by C. Minkowski, the names of patrons seldom appear here. Moreover, were the titles composed by authors or by later scholars? Is it possible that the titles were modified by those who copied texts? These questions can be extended to all texts in the census. I count only 205 texts (or, 5.5%) with *siddhānta* in the title. Likewise, only 18 are described as *śāstra*s and *tantra*s. In 141 cases (or, 3.8 %), the titles (*saṅgraha, jñana*) express the fact that the compositions bestow knowledge; in 118 cases (or, 3.2%), the titles (*phala, sāra*) convey the idea of providing an essence of something. Sometimes the titles indicate both of these concerns: (*sārasaṅgraha* is quite a common title compound). The metaphor of light (*dīpika*, *prakāśa*) occurs 289 times (or, 7.8%) and somewhat less frequently in the titles of 28 commentaries (or, 3.4%).

[33](Pingree, 1981). I have counted in the CESS 130 texts (or, 3.5%) bearing the name *karaṇa* or some associated title and 66 almanacs (or, 1.7%) bearing the name(*pacāṅgas*) or texts explaining how to make almanacs. The CESS further notes a number of non-standard texts, such as the *Aparājitapṛcchā* of Bhuvanadeva (fl. twelfth century), a text on architecture written as a dialogue (CESS V 264 a.).

[34]To be more specific, the three last chapters of the *Āryabhaṭīya* are discussed in this way in the conclusions of the commentaries to each of these chapters. (K. V. Sarma, 1976, p. xxv), (K. S. Shukla, 1976, p. 171; p. 239; p. 288).

[35](K. S. Shukla, 1959, p.1).

[36]See the *maṅgalācaraṇam* of BAB.2 : *vyākhyānaṃ gurupādalabdham adhunā kiñcin mayā likhyate*, (K. S. Shukla, 1976, p. 43).

[37]Thus, at the end of the introduction which begins his commentary, Yajvan writes *evam upodghātaṃ pradarśya śastraṃ vyākhyāyate.* (K. V. Sarma, 1976, p. 7) However, at the end of that chapter's commentary, he refers to his own text as a *prakāśa.* (K. V. Sarma, 1976, p. 32, p. 79 (note 11); p. 117, p. 185). Again, at the end of SYAB.2, Yajvan relies on the verbal root *vyākh-*.

Finally, the APG simply calls itself a *ṭīkā*[38]. We do not know then if these titles and names are simply synonyms or if they were intended to impart something about the kind of commentary which the authors and readers had in mind when they named them. We do not know then if these names are just synonyms or if they tell us something of the kind of commentary, authors and readers had in mind, when they read or gave them such names.

A commentary, by definition, is a secondary text, a deuteronomic text, a text that needs another text to make any sense[39]. However, is this definition sufficient to characterize a text? Sanskrit have a multiplicity of forms. This diversity is reflected somewhat in the Sanskrit field of mathematics. Some commentaries respect the original order of the text, while others do not[40]. In particular, BAB, SAB, SYAB and APG[41] respect the structure of the original text, whereas PBSS does not[42]. Alternately, SYAB develops a long introduction (*upodghata*) before the first gloss of the text, but this is not the case in the other commentaries considered here. Even though all commentaries have in common the use of a base text, the way they provide commentary varies widely. Thus, the relation between the commentary and the treatise which it elucidates must be specified.

All commentaries considered here quote the text which they purport to explain in its entirety[43]. Despite the obvious embedding of the original text, finding stylistic criteria which separate the commentaries from the treatises proves to be a difficult task. Almost as soon as a principle of separation is proposed, a counter-example can be identified. Manuscripts, did not always graphically distinguish treatises from their commentaries. For instance, consider the illustration in Figure 3. Here, the treatise appear to be an undifferentiated part of the commentary.

In other cases, such as in Figure 4, the part of the original text which received commentary was colored differently or graphically separated from the commentary, but this was not a systematical rule[44].

Thus, even though the titles suggest a difference, the material culture does not always preserve such a standard. Does this indicate that copied texts were not intended to be read? A historical study of traditions of reading and copying in the Indian subcontinent could shed much needed light on the practice

[38] (K. S. Shukla, 1959, p.1).

[39] On the question of "secondary texts" in the history of mathematics, see (Netz, 1998), (Chemla, 1999), (Bernard, 2003) and also Chemla in this volume.

[40] See (Bronkhorst, 1990).

[41] One should recall however that PG, given in Shukla's edition, is known only from this single recension (K. S. Shukla, 1959).

[42] According to CESS IV 221b, the text of PBSS corresponds to I.1–3, XXI.1–XXII.3, I.4–II.68, XV.1–9, III.1–XIV.55, XV.1à–XX.19, and XXII.4–XXIV.13, respectively, with chapters denoted by roman numerals and sections within chapter indicated by arabic numerals. It remains to be clarified what logic directed the composition of PBSS in this order.

[43] However, they also all quote also other texts, albeit not completely. Thus, SYAB often quotes PG, and APG cites BSS. SYAB frequently paraphrases BAB. Therefore, commentaries may be described as composite texts, assembled from parts of previously composed texts which are sometimes rewritten or paraphrased, and intertwined with passages original to the commentary. The commentaries also frequently share versified examples and these can be considered a form of quotation as well.

[44] Perhaps this standard arose in modern (eighteenth-nineteenth century) scribal traditions, and may very well have its ultimate origin in practices introduced by European texts, or as suggested in (Plofker, 2009, A.3, 305) by Islamic ones.

Figure 3: Palm leaf manuscript of BAB in a copy of the Kerala University Oriental Manuscripts Library

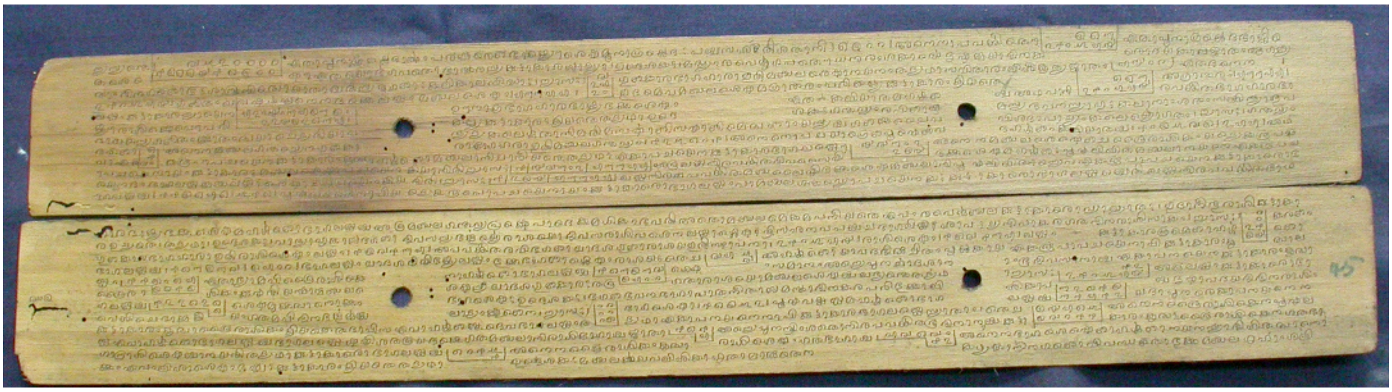
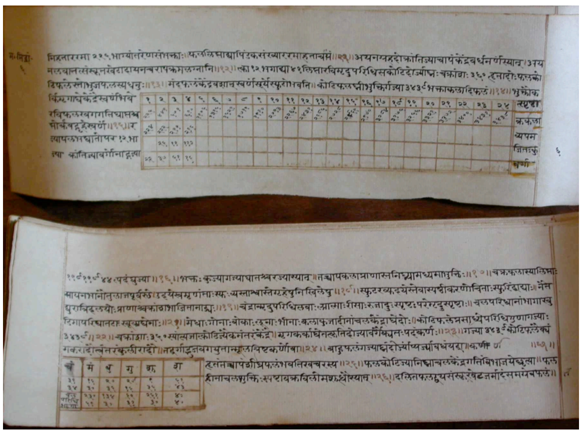


Figure 4: Paper manuscript with colored highlights of MS in a copy of the Mumbai University Library



of composing commentaries[45].

The distinction between treatises and commentaries has often been described as a written reflection of the oral tradition, wherein some portion of the text was known by heart (the treatise) and some portion contained the explanations that were given subsequently (the commentary). Whereas the treatise was essentially an oral composition, the commentary was largely a written text[46]. This distinction derives from the words of the commentators themselves. Thus, Bhāskara insists on a distinction between Āryabhaṭa who produced the treatise orally (*āh-*), and Bhāskara's own commentary which he committed to writing (*likh-*). Conversely, though, Hayashi has shown that the word *likh-* (to write) may occur as a synonym for *vac-* (to speak) in the BM[47]. In this case, there appears to be no difference between a written text and an oral one.

Not even the distinction between verse and prose passes unchallenged. Indeed, the treatises here are all versified. Thus, chapter 2 of the AB, chapter XII of the BSS, and the PG are composed as *ārya* verses. The commentaries all include prose sections, which can present great variation : for instance, they can introduce dialogs or grammatical analysis, but they all include versified parts. Thus, BAB even contains versified tables. Likewise, all commentaries contain versified examples[48].

The presence of worked examples, together with the fact that they contain non-discursive parts such as numerical tables and drawings, may be a defining characteristic of mathematical commentaries in Sanskrit. Only further systematic study of the features of Sanskrit commentaries–especially of Sanskrit commentaries in the field of *jyotiṣa*–will provide additional elements. Despite these possible defining features, the characteristics of treatises and commentaries remains a complex and difficult question, both in Sanskrit literature generally as well as in the limited case of mathematical commentaries[49].

Internal evidence may yet reveal something about how texts were understood in relation to one another. Questioning why and how texts were copied may also yield some as yet unconsidered kinds of evidence and provide clues to the historical evolution of the social conceptualization of such texts. The history of text copying, collection and reading additionally gives us contextual information on who had texts collected and copied, as we will now see.

1.4 An example: manuscripts of SYAB

K. V. Sarma has described the codex in which the SYAB was found in Kerala[50]. In an Indological context, "codex"' refers to the bundle of texts which may sometimes comprise several texts. These bundles are an indication that the texts were either copied together or had been collected at the time they were integrated in the library. The first codex under consideration contains a set of manuscripts

[45] Note also that a strong stylistic criteria which separates commentaries from treatises would help philologists who edit manuscripts to determine what portion belongs to the original text and what portion belongs the commentary.

[46] See, for instance, (K. S. Shukla, 1976, Introduction).

[47] (Hayashi, 1995, p. 85).

[48] In his edition of PG and APG, Shukla seems hesitant: are the examples part of the treatise, or part of the commentary? All editions of L consider examples part of the treatise.

[49] In several parts of her book, (Plofker, 2009) offers several rough characteristics of mathematical commentaries:their uses of proofs and language games essentially.

[50] (K. V. Sarma, 1976, Introduction, xvii-xx).

which represent part of a collection made by scholarly astronomers.

Some historical details are known about Codex C. 224-A, the codex containing Manuscript A. This codex was made for the royal family of Eḍapaḷḷī of Kerala, in 1753. Aside from Manuscript A, the codex also contained six other texts. The codex also preserved two early works by Bhāskara I (seventh century): the *Laghubhāskarīya* and *Mahābhāskarīya*. Also in the codex are two tenth-century works, the *Sūryasiddhānta* and the *Siddhāntaśekhara* of Śrīpati (fl. 1050). Obviously, the codex included the twelfth-century *Bhaṭaprakāśikā* of Sūryadeva Yajvan (SYAB). Finally, two fourteenth century compositions, the *Goladīpikā* of Parameśvara and the *Tantrasaṇgraha* of Nīlakaṇṭha Somayājin (fl. 1450) round out the codex. Although they represent nearly eight hundred years of intellectual tradition, all these texts share a common subject matter: astronomy. In a similar way, manuscript D, also originated from a codex, (C. 22475-A). This codex belonged to the same royal family of Kerala and contains both SYAB and the solitary first chapter of Ab.

Again, Manuscript E of the same edition belonged to a codex (C.2121 C and D) which originally came from the "library of a family of astronomical scholars, the Maṅgalappaḷḷī Illam, at Āranmuḷa, in southern Kerala"[51].Aside from the SYAB, this codex contained nine compositions. The earliest text contained by the codex are the last three chapters of Ab. Next, chronologically came the *Mahābhāskarīya* of Bhāskara I and its anonymous commentary, the *Mahābhāskarīyavyakhyā*. Likewise, the codex also contains the *Laghumānasakaraṇa* of Muñjāla (fl. 932) and its anonymous commentary. The codex includes T, and while only the commentary to chapter II and parts of chapter III of the *Sūryasiddhānta* appear, the codex preserves the whole of the *Sūryasiddhānta*. Finally, the codex contains an anonymous prose *Rāmāyaṇa*. Setting aside the prose version of the famous Indian Epic, this codex also concentrates on astronomical and mathematical lore. Here, only part of a commentary on the *Sūryasiddhānta* is copied but the treatise itself is extensively copied. Ab is known to have been transmitted in two separate parts: the first chapter was copied separately (as in the codex for Manuscript A) from the 3 other parts (as in the codex of Manuscript E).

Evidently, commentaries were not exclusively copied together with other commentaries. Rather, a codex could contain a treatise by itself, and a couple chapters of commentary by a given author. Unlike the Chinese tradition, texts of important treatises could be copied and collected without any commentaries. Different commentaries to the same text were usually copied separately, unlike the tradition of the Chinese *Nine Chapters*.

Another codex, C. 2320-A, sheds light on texts copied for another kind of social context: the *nampūtiri*, the brahmin cast of Kerala whose priests follow Vedic rituals, (*śrauta*). Of course, this codex contains SYAB, but it also contains also a text describing the horse sacrifice and detailed accounts of expenses for a ceremony carried out in 1535. The codex itself seems to be a copy of a manuscript dating from 1536. This textual justification finds some historical precedent. Sūryadeva, himself a performer of Vedic rituals, understood Āryabhaṭa's text in relation to the *śrauta*[52]. Save for the *siddhāntic* text, this small codex corresponds to the kind of *śrauta* collection which C. Minkowski describes for the

[51](K. V. Sarma, 1976, Introduction, xix)

[52]Oddly, SYAB states this clearly in its general introduction to Ab (K. V. Sarma, 1976, xxv-xxvi, 2-4). Indeed, Āryabhaṭa can scarcely be considered a ritualist, given the fame he garnered for taking puranic cosmology lightly.

Toro family. Might the surprising inclusion of the *siddhāntic* text indicate that the use of *siddhāntic* astronomy was a common phenomenon among ritualist families?

Manuscript B of Sarma's edition was preserved by a scholarly family from *vaṭṭapaḷḷi maṭham* (a religious complex) near *kanyākumārī* (the southernmost tip of the Indian sub-continent), in Tamil Nadu. Along with the *Bhaṭaprakāśikā* (SYAB), the codex contains an *Āṣṭādhyāyī-sūtrānukramaṇi*, an alphabetical index of Pāṇini's *sūtras*. In this way, the codex sets grammatical lore alongside mathematical learning.[53]. In this case, the motive for the inclusion of SYAB with literary elements seems to have been the result of an endeavor to collect general Sanskrit scholarship.

The provenance of these manuscripts, then, underscores the already well-known legacy of *Āryabhaṭa* in Kerala. The SYAB seems to have been used in three different, although probably not separate, social contexts. The SYAB appears among scholarly astronomers, among priests who perform Vedic rituals and among the general scholarly atmosphere of south Indians monasteries.

Nonetheless, little information about how these texts were integrated into the Kerala University Oriental Manuscript Library where they are now can be found. Originally created by the Government of Travencore, an autonomous state within the British Raj, since 1908 the library has taken as its mission the the preservation of local heritage[54]. The history of this library calls for further investigation, but for now, let it suffice that the structure of the collection in which different codices containing SYAB are found recalls those C. Minkowski has described.

The manuscripts of other mathematical texts do not always provide as rich information as the manuscripts of SYAB. For example, the BAB manuscripts preserved at the KUOML are not preserved within a codex. Similarly, the extant manuscripts of the PBSS tell us more about their recent accessions than about the tradition which transmitted them. To wit, the two manuscripts which contain chapter XII of the PBSS: one served as the text for the Colebrooke translation and the other served as the source for the edition which S. Dvivedin made in Benares[55].

As might be expected, treatises on mathematics have attracted more attention, study and attempts at translation than have commentaries on mathematical treatises. This fact notwithstanding, mathematics as a discipline has attracted far more scholarship in the last two centuries than other elements of *jyotiṣa* lore. Criteria to differentiate commentaries from treatises and tools to understand how the two types of texts relate to each other have been difficult to find. This relationship is complicated by the fact that each individual text seems to have a unique story of preservation and transmission. In the early modern period in South India, royal families with an interest in astronomy, ritualist families and religious groups had texts on mathematics copied. Those who copied the commentaries to those mathematical texts could copy them only partially. They seem not to have considered commentaries as independent extant texts. However, the question has not been addressed whether the authors of the commentaries considered their

[53](K. V. Sarma, 1976, xviii).

[54]The "Index of Manuscripts" of the library notes that "In 1940 it possessed 3000 manuscripts, 142 publications in Sanskrit, 63 in Malayalam. Travancore University (which became the University of Kerala) organized after its establishment (1938) a manuscript preservation and collection department. Both were amalgamated in 1940. In 1958 there was (sic) 28 000 Codices in Sanskrit; 5 000 in Malayalam."

[55]See (Pingree, 1970-1995, op.cit), (Ikeyama, 2003, op.cit) and (M. S. Dvivedin, 1902).

compositions independent texts? Were treatises even considered in this light? Furthermore, it is not certain how much the attitude of early modern copiers informed (or reciprocally was informed by) the practices and interpretations of the Europeans who read these texts at the end of the eighteenth century and the beginning of the nineteenth century.

2 The rediscovery of Ab, BSS, PG and their commentaries in the historiography of Indian mathematics

Now that an understanding of the textual traditions of Sanskrit mathematical texts, their commentaries, and the problems related to these categories has been established, the textual aspects which informed the writing of the mathematical history of India may be reconsidered. Many of the questions depend on the assumptions and conceptions of past historians: How have historians of mathematics looked at the Ab, the BSS and the PG and their commentaries written between the seventh and twelfth centuries? There is no simple trend or common understanding which unifies how different historians dealt with commentaries. However, a key difference in perspective may lie in whether the historians were mathematicians or philologists. The more strongly the historian identified as a mathematician, the less sensitivity he or she generally exhibited toward textual problems; conversely, the more strongly he or she followed in the tradition of philologists, the greater weight he or she tended to ascribe to the commentary. Because of individual variation, however, this does not mean that commentaries were always treated as secondary texts by mathematicians but deemed important sources by philologists. As the twentieth century came to a close, the general historiographical trend has been to pay more and more attention to the mathematical contents of commentaries. We cannot afford here to look closely at the shifting attitudes of all historians, but we will try to draw out some of the characteristic attitudes which Colebrooke, Datta & Sing, and K. S. Shukla exhibited toward commentaries.

As D. Raina's article in this volume explains[56], by the early seventeenth century, a number of Europeans knew of the existence of Sanskrit astronomical treatises through the testimonies of travelers, academic envoys, and Jesuit missionaries. Some time passed before curious Europeans were able to obtain these texts and study them. By the early nineteenth century, the first translations of Sanskrit texts on mathematics into European languages (especially English) had been made. Thus, in 1812, in London, Stratchey produced the first translation of the BG into English. Next, in 1816, Taylor translated the BG in Mumbai. Finally, Colebrooke translated L and BG together with the mathematical chapters of Brahmagupta's BSS and published the results in London in 1817[57].

2.1 Colebrooke and commentaries

Colebrooke's publication proved to be a landmark. A former director of the Asiatic Society of Bengal and a recognized specialist of Hindu Law, Vedic ritual and Indian languages, Colebrooke had solid credentials. His publication marked the first interest of a well-established, Indologist in mathematical

[56]See also (Raina, 2003), (Raina, 1999).

[57](Brahmagupta; Bhāskarācārya; Colebrooke, 1817).

texts from the Indian subcontinent. Colebrooke prefaced his translations of mathematical texts with a general "dissertation", followed by "notes and illustrations", which intended to establish these texts against the general history of mathematics. In this way, he endeavored to establish India among the ranks of mathematical cultures, a status which until then it had lacked[58]. In his introduction, Colebrook takes a strong position on the antiquity of the Indian tradition of mathematics, especially algebra. In so doing, Colebrooke raised the stakes in an ongoing controversy about the age of Indian mathematics[59]. Through the high quality of his translations, generally made in close collaboration with pandits, Colebrooke established his publication as an enduring reference.

Colebrooke included portions of commentaries in the footnotes to his translations. Thus, Colebrook included parts of the commentaries of Gaṅgadhāra (fl. 1420), Sūryadasa (1541), Gaṇeṣa (fl.1520/1554), and Ramakṛṣṇa's (1687?[60]) in his translation of L[61]'s translation; when Colebrook translated BG, he incorporated passages from the commentaries of Kṛṣṇa (ca.1615), Rangunāṭha (?), and Ramakṛṣṇa. Finally, when Colebrooke prepared a translation of the BSS, he used (and quoted) the PBSS, which he cited as "CA", an abbreviation of *Caturdeva*, part of the name of the author of the commentary. Furthermore, although neither he nor his readers had recourse to the original text, Colebrooke refers to and discusses Āryabhaṭa's works in his introduction[62]. In so doing, Colebrooke passed along second-hand references: Āryabhaṭa was indeed criticized by Brahmagupta and a diverse group of commentators quoted and discussed Āryabhaṭa's verses. Colebrooke also referred to a work by Śrīdhara, though not however our PG[63]. Thus, Colebrooke introduced his readers to Āryabhaṭa, Brahmagupta and Śrīdhara.

Colebrooke hoped to describe the mathematical tradition of India. This aspiration limited his interest to only the mathematical part of Brahmagupta's treatise, the BSS, and, consequently, only a portion of PBSS. The possibility that Colebrooke may have adopted his limited interest from the known tradition of copying (and thus showing a specialized interest in) only the mathematical chapters of astronomical treatises of the Sanskrit tradition prompts speculation. Just as Colebrooke may have continued past traditions, he may have founded others. The collation of BSS's chapters into a printed edition may also mark the beginning of an enduring historiographical trend noted in the introduction: among the old *jyotiṣa* texts, mathematical subjects have been the subject of more study than astronomy or astrology.

In his introduction, Colebrook presents the writing of the commentators as proofs of the authenticity and antiquity of the texts he is translating and discussing. As Colebrook writes[64]:

> "The genuineness of the text is established with no less certainty [than its date] by numerous commentaries in Sanskrit, besides a Persian version of it. Those commentaries comprise a perpetual gloss, in which every passage of the original is noticed and interpreted : and every word of it is repeated and explained, a comparison of them authenticates the text where they agree; and would

[58] *Op.cit.*; p. xvi.

[59] (Kejariwal, 1988, 111-112).

[60] CESS V 453 a.

[61] *Op.cit.* Note A p. xxv and p.xxvii

[62] *Op.cit.* Introduction, sections G to I pp. xxxvii-xiv.

[63] *Op.cit.* p.v Colebrooke writes that he has a copy of "Srìdhara's compendium of arithmetic", which is probably the *Triśatika*.

[64] (Brahmagupta; Bhāskarācārya; Colebrooke, 1817, iii)

> serve, where they did not, to detect any alterations of it that might have taken place, or variations, if any had crept in, subsequent to the composition of the earliest of them. A careful collation of several commentaries, and of three copies of the original work, has been made, and it will be seen in the notes to the translation how unimportant are the discrepancies. "

For Colebrooke, commentaries are thus useful and necessary in order to edit a text: they are philological tools. However, the way the commentaries are integrated in his translations reveals that he considered them far more than mere tools. Commentaries were key to understanding the treatises. They were stimulating mathematically with their examples and proofs. However, the commentary was not treated as a text in and of itself. It was given in bits and pieces. Selected.

Consider the translation, as seen in Figure 5. Typographically, the commentary is a secondary text, written in a smaller font. The commentary is fragmentary, relegated to and divided among different footnotes. Despite its status as a secondary text, Colebrook makes the importance he ascribes to the commentary visually explicit. In Colebrooke's edition, the commentary literally spills over and invades the space which is meant for the treatise.

If we recall that the codices sometimes include only portions of the commentaries, we may wonder how much of Colebrooke's attitude toward giving commentaries in bits and pieces reflects the training he received from pandits. Perhaps the pandits themselves worked in this way and helped Colebrooke understand the text through collating references in different commentaries. Here, a history of how commentaries were conceptualized and read in the Indian subcontinent would be helpful. Likewise, a precise description of how Colebrooke (and other European scholars) worked with pandits in relation to texts would be useful as well[65].

2.2 Indian Scholarship with Commentaries: Datta & Singh and K. S. Shukla

During the nineteenth century, the academic history of mathematics in India slowly opened to Indian scholars, who expounded a scholarship as much directed toward an inner audience as toward answering European interlocutors and engaging them in discussion[66]. The arrival of Indian scholars may first be perceived from articles discussing authorship of texts and was later placed in evidence by series of editions and translations of texts on mathematics. The names Datta, Sengupta and Dvivedin first appear during a period when European scholars had confused the fifth century Āryabhaṭa with his eleventh century namesake, and mistaken the seventh century Bhāskara I with his twelfth century namesake.

By the end of the nineteenth century, the movement of edition, translation and analysis of texts on mathematics in Sanskrit, which had begun at the turn of the previous century, came to a peak. In 1874[67], Kern edited the Ab for the first time [68]. Kern printed his edition with the commentary of Parameśvara (fourteenth century). Kern's introduction cites SYAB, which is quoted by Parameśvara, but no reference

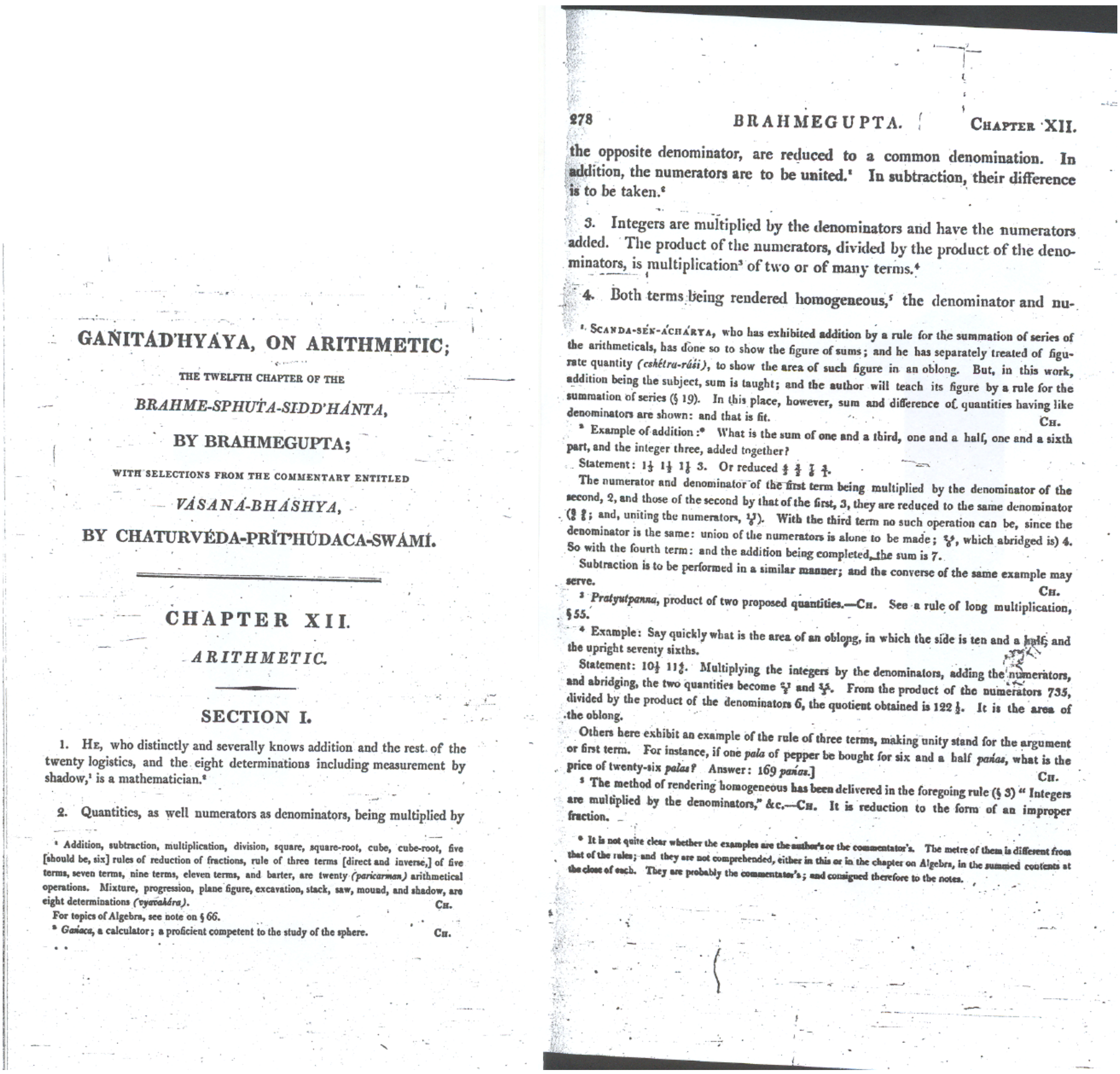
[65] The latter has been partially studied in (Kejariwal, 1988), (Aklujkar, 2001), (Bayly, 1996) among others, but little seems to have survived on precisely how European scholars and the pandits worked texts out.

[66] (Keller, 2006b).

[67] (Kern, 1874).

[68] The tradition of copying manuscripts can, of course, also be understood as the editorial tradition of classical India, but printed books are referred to here.

Figure 5: The BSS and PBSS in Colebrooke's translation



GANITAD'HYAYA, ON ARITHMETIC;

THE TWELFTH CHAPTER OF THE

BRAHME-SPHUTA-SIDD'HANTA,

BY BRAHMEGUPTA;

WITH SELECTIONS FROM THE COMMENTARY ENTITLED

VÁSANÁ-BHASHYA,

BY CHATURVÉDA-PRĬT'HÚDACA-SWÁMÍ.

CHAPTER XII.

ARITHMETIC.

SECTION I.

1. HE, who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations including measurement by shadow,[1] is a mathematician.[2]

2. Quantities, as well numerators as denominators, being multiplied by

[1] Addition, subtraction, multiplication, division, square, square-root, cube, cube-root, five [should be, six] rules of reduction of fractions, rule of three terms [direct and inverse,] of five terms, seven terms, nine terms, eleven terms, and barter, are twenty *(paricarman)* arithmetical operations. Mixture, progression, plane figure, excavation, stack, saw, mound, and shadow, are eight determinations *(vyavahára)*. CH.

For topics of Algebra, see note on § 66.

[2] *Ganaca*, a calculator; a proficient competent to the study of the sphere. CH.

the opposite denominator, are reduced to a common denomination. In addition, the numerators are to be united.[1] In subtraction, their difference is to be taken.[2]

3. Integers are multiplied by the denominators and have the numerators added. The product of the numerators, divided by the product of the denominators, is multiplication[3] of two or of many terms.[4]

4. Both terms being rendered homogeneous,[5] the denominator and nu-

[1] SCANDA-SÉN-ÁCHÁRYA, who has exhibited addition by a rule for the summation of series of the arithmeticals, has done so to show the figure of sums; and he has separately treated of figurate quantity *(cshétra-rási)*, to show the area of such figure in an oblong. But, in this work, addition being the subject, sum is taught; and the author will teach its figure by a rule for the summation of series (§ 19). In this place, however, sum and difference of quantities having like denominators are shown: and that is fit. CH.

[2] Example of addition:* What is the sum of one and a third, one and a half, one and a sixth part, and the integer three, added together?

Statement: $1\frac{1}{3}$ $1\frac{1}{2}$ $1\frac{1}{6}$ 3. Or reduced $\frac{4}{3}$ $\frac{3}{2}$ $\frac{7}{6}$ $\frac{3}{1}$.

The numerator and denominator of the first term being multiplied by the denominator of the second, 2, and those of the second by that of the first, 3, they are reduced to the same denominator ($\frac{8}{6}$ $\frac{9}{6}$; and, uniting the numerators, $\frac{17}{6}$). With the third term no such operation can be, since the denominator is the same: union of the numerators is alone to be made; $\frac{24}{6}$, which abridged is) 4. So with the fourth term: and the addition being completed, the sum is 7.

Subtraction is to be performed in a similar manner; and the converse of the same example may serve. CH.

[3] *Pratyutpanna*, product of two proposed quantities.—CH. See a rule of long multiplication, § 55.

[4] Example: Say quickly what is the area of an oblong, in which the side is ten and a half; and the upright seventy sixths.

Statement: $10\frac{1}{2}$ $11\frac{2}{3}$. Multiplying the integers by the denominators, adding the numerators, and abridging, the two quantities become $\frac{21}{2}$ and $\frac{35}{3}$. From the product of the numerators 735, divided by the product of the denominators 6, the quotient obtained is $122\frac{1}{2}$. It is the area of the oblong.

Others here exhibit an example of the rule of three terms, making unity stand for the argument or first term. For instance, if one *pala* of pepper be bought for six and a half *paṅas*, what is the price of twenty-six *palas*? Answer: 169 *paṅas*.] CH.

[5] The method of rendering homogeneous has been delivered in the foregoing rule (§ 3) "Integers are multiplied by the denominators," &c.—CH. It is reduction to the form of an improper fraction.

* It is not quite clear whether the examples are the author's or the commentator's. The metre of them is different from that of the rules; and they are not comprehended, either in this or in the chapter on Algebra, in the summed contents at the close of each. They are probably the commentator's; and consigned therefore to the notes.

is made to BAB. In 1879, Leon Rodet translated chapter 2 of the Ab into French and conducted an analysis of the text[69]. In 1896, Dikshit edited the BSS together with his own commentary[70]. These two editions were followed by a number of translations in English, which in turn prompted the first studies in this language. Thus, in 1907 and 1908, G. R. Kaye published his controversial work, *Notes on Indian Mathematics*, part two of which is devoted to Āryabhaṭa[71]. Sengupta published the first English translation of the Ab[72], followed by Clark in 1930[73]. In 1937, B. Datta and S. N. Singh published the enduring classic "Hindu Mathematics"[74]. They aimed to provide a general description of all the variations by which Hindu mathematicians practiced elementary and sometimes higher mathematics, each Sanskrit author contributing elements of his own expertise. Datta & Singh also wanted to refute G. R. Kaye's claims for an Arabic or European origin of Indian mathematics generally and his attribution of the mathematics in Ab to a foreign source specifically. In order to accomplish these ends, they dedicated part of their effort to the comparison of the history of mathematics in Europe with what had been discovered about the history of mathematics on the Indian subcontinent. To a certain extent, Datta & Singh also wrote a history of mathematics in India in praise of its great mathematicians and the important discoveries recorded in their treatises. As trained mathematicians, Datta and Singh largely focused on the mathematical contents of the texts[75]. Throughout their publications, B. Datta and A. N. Singh essentially consider mathematical commentaries to be mathematical texts. In this context, they occasionally referred to Bhāskara I, whose text was known but was not published, as an astronomer who dealt with mathematics[76]. Datta & Singh sometimes quoted Bhāskara I's commentary to explain or interpret the verses in Ab[77]. Occasionally, they seem to share Colebrooke's model and relegate these citations to footnotes on the text[78]. Datta & Singh often collated these citation with other commentaries on Ab, such as that of Nīlakaṇṭha (fourteenth century), even though these commentaries were not yet widely available as edited texts. Most frequently, however, they cite BAB for its mathematical contents, with little attention paid to its relationship with Ab[79]. At times, Datta & Singh mingle the substance of BAB with the contents of Bhāskara's other astronomical texts[80].

With political independence and the creation of institutions for the study of the history of science in

[69](Rodet, 1879)

[70](Dikshit, 1896).

[71](Kaye, March 1908).

[72](Sengupta, 1927).

[73](Clark, 1930).

[74](Datta, 1935).

[75]Despite this tendency, they also embraced the dying tradition of mastering Sanskrit texts on *jyotiṣa*. Within an Indian context, there results were successful and, at the end of his life, B. Datta was addressed as *pandit*. The enduring quality of their translations reflect their mastery of the Indian intellectual tradition. Indeed, "Hindu Mathematics" effectively blends two traditions, the lore of *jyotiṣa* and the modern history of mathematics. A detailed scrutiny of Datta & Singh's works could probably yield much useful information about the tradition of *jyotiṣa*.

[76](Datta, 1935, Volume I, 125).

[77](Datta, 1935, Volume I, 66-67; 196; 211. Volume II, 93-95).

[78](Datta, 1935, Volume I, 170). See also, SYAB as in *op. cit.*[Volume II, 91, footnote 4].

[79](Datta, 1935, Volume I, 80, 82, 87, 130, 204, 239; Volume II, 87, 238).

[80]Thus, the whole portion of Volume 2 devoted to the *kuṭṭaka* quotes all the different texts which appear in Bhāskara I, even though these passages sometimes present Bhāskara's own algorithms, and at other times only explain algorithms in Ab.

India, a new wave of editions appeared. K. S. Shukla took center stage in this movement. In 1959, he published an edition of PG together with APG and an English translation of PG at the University of Lucknow[81]. He then turned to Bhāskara's work, first editing and translating his treatises, which can be interpreted as elaborations of Āryabhaṭa's astronomy, the *Laghubhāskarīya* and the *Mahābhāskarīya*[82]. In 1976, K. S. Shukla and S. R. Sharma jointly published an edition and translation of the AB and editions of BAB and SYAB under the aegis of the Indian National Science Academy[83].

By the 1980s, a new generation had taken up the study of commentaries on mathematical subjects. On the one hand, publications on the Mādhava school of mathematics called the attention of historians of mathematics to the scholiasts of Āryabhaṭa; on the other hand, the Japanese students of D. Pingree, particularly T. Hayashi, published articles on the mathematical contents in different commentaries on Ab, BG and L. To this can be added the publications of F. Patte's Ph.D. and my own, which exemplify the growing interest in mathematical commentaries in Sanskrit at the end of the twentieth century[84].

In his editions, Shukla uses commentaries in three separate ways: First, following Colebrooke's model, commentaries provide a philological assessment of the original text to be edited. Secondly, the commentaries are also used to explain the text. Thus, although Shukla did not translate APG, he refers to it in the written comments that accompany his translation of PG. At times, Shukla seems to believe that the commentators give a peculiar interpretation of the treatise but at other times he seems to suppose that the commentators serve to explain the contents. In his joint edition and translation of the Ab, commentators are consulted to add mathematical depth to the algorithms. At times, Shukla combed the commentaries in order to quote their conflicting interpretations. Finally, in some instances, especially in the case of BAB, Shukla esteems the commentaries as mathematical texts in their own right[85]. The tendency to accept commentaries not only as philological aids but also as independent mathematical texts has been growing. This inclination no doubt coordinates with a general trend within the history of mathematics at large, because such approaches seem characteristic of modern approaches to the Chinese corpus as well.

In summary, commentaries have been used, read and analyzed but were seldom translated by modern historians of mathematics. This decision made sense when commentaries where thought of as philological tools, useful in the editing and understanding of the texts they treated, as was often the case. However, the lack of translations becomes surprising when commentaries are considered as independent texts on mathematics in their own right. Furthermore, if commentaries and treatises are seen as ballroom dancing

[81](K. S. Shukla, 1959).

[82](K. S. Shukla, 1963), (K. S. Shukla, 1960)

[83](K. V. Shukla & Sharma, 1976), (K. S. Shukla, 1976), (K. V. Sarma, 1976).

[84]How closely can these attitudes toward commentaries be linked to developments in the field of Indology generally? Indeed, Indology has developed a special emphasis on the study of treatises and the contents of important commentaries but has somewhat neglected any reflection on the commentary as a specific kind of text. In the last five-to-ten years, however, a renewed interest in this kind of text has surfaced, as illustrated by the recent conference titled "Forms and Uses of the Commentary in the Indian world", held at Pondicherry in February 2005. See http://www.ifpindia.org/Forms-and-Uses-of-the-Commentary-in-the-Indian-World.html., or the previously cited publication (Tubb, 2007).

[85]Before publishing his edition of BAB, Shukla published a number of analyses, which pinpointed the mathematical relevance of the text. (K. S. Shukla, 1972)

couples, sometimes the focus had been essentially on the treatise without its partner, sometimes on the commentary without its partner, and sometimes on details of how their steps follow each other. However, the global picture of how they danse together has not been drawn, or even aimed at.

3 Relating commentaries to their treatises: the example of square root extractions

In studies on the history of mathematics in India, the relationship of the commentaries to the text they explicated has generally been left unresolved. Reflections on the relation of one text to another one have usually been restricted a simplistic "right or wrong" interpretation of the treatise by the commentary. Scholarship on the mathematical contents of commentaries has focused on the mathematical ideas they contained, essentially their proofs[86]. The role of the commentary, then, is implicitly limited to providing mathematical justifications of the treatise. Through the following examples of square root extractions, the relationship between the commentary and the treatise may be shown to be more complex

In the following, rules to extract square roots, which are found in Ab and PG will be examined. BSS provides only a rule to extract cube roots, and will thus remain outside the scope of the present study.

3.1 Extracting square roots along different lines

The procedures for extracting square roots considered here rely on the decimal place-value notation. Moreover, unlike the interpolations commonly described in astronomical parts of treatises, these algorithms are not "useful" procedures to extract square roots. Rather, these procedures all presume that the root is to be extracted from a perfect square[87].

The basic idea underlying the procedure is to recognize the hidden development of a square expansion in a number written in the decimal place-value notation. In other words, the procedure relies on recognizing a square of the kind $(b_n \times 10^n + \ldots + b_i \times 10^i + \ldots + c)^2$ $(i < n)$ in a number written as $a_{2n} \times 10^{2n} + a_{2n-1} \times 10^{2n-1} + \ldots + a_1 \times 10^2 + a_0 \times 10 + c^2$. Crucial to this algorithm, then, is the ability to distinguish between powers of ten that are squares (the even powers), and those that are not (the odd powers).[88]

Let us note that the algorithm in Ab returns the square root directly, while the algorithm in PG first extracts the double of square root and then says that the result should be halved. Thus, although they are founded on the same idea, the two procedures differ in their intermediary steps. Here is not

[86] (Srinivas, 1990), (Patte, 2004), (Plofker, 2009). Strangely enough, few reflections on comments connected to the definition of the field of *gaṇita* have been published. Such reflections might help explain why chapters on *gaṇita* contain algorithms with little astronomical application, although they are included in treatises on astronomy. See (Pingree, 1981), (Keller, 2007), (Plofker, 2009).

[87] Why, then, were such procedures given? In the Sanskrit mathematical tradition, the validity of an algorithm is sometimes verified by inverting the algorithm and finding the initial input. Perhaps a procedure which inverts the squaring procedure may have seemed useful for such verifications. In commentaries to these rules, all the worked examples undo the squares illustrated in the squaring procedure.

[88] For a general explanation of the different methods, see (Datta, 1935, Volume I, 170-171), (Bag, 1979, 78-79).

the place to expose their respective algorithms, but rather to concentrate on what treatises reveal that commentaries omit, and vice-versa.

This is the rule for the extraction of square roots as given by Ab[89]:

> **Ab.2.4. One should divide, constantly, the non-square ⟨place⟩ by twice the square root|**
> **When the square has been subtracted from the square ⟨place⟩, the quotient is the root in a different place||**

Without any commentary, the algorithm is hard to understand. Part of the difficulty springs from the fact that the verse begins in the the algorithm's middle and thus emphasizes its iterative aspect. Another part of the difficulty arises from Āryabhaṭa's pun: the same name (*varga*) is given to squared digits and also to the specific positions where digits are found, "square positions". This pun highlights the mathematical idea behind the root extraction precisely: the "square places" of the decimal place value notation are positions give hints about the squaring of which digits produced the number at hand.

The two commentaries BAB and SYAB introduce a grid to smooth the difficulty caused by the pun. Instead of considering square and non-square places, even and odd places are used. In both cases, this clarification is achieved quite simply by the commentary: one word is merely substituted for another. Thus, BAB states[90]:

> In this computation (*gaṇita*), the square (*varga*) is the odd (*viṣama*) place.

Likewise, SYAB states:[91]:

> In places where numbers are set-down (*vinyāsa*), the odd places have the technical name (*samjñā*) "square".

We will return to the different grids for positions in the decimal place value notation below. For now, let us consider the act of substituting a noun given in the treatise by another noun, often called a "gloss". These substitutions are common in the commentaries. In this case, the substitution serves two functions: first, it explains the literal meaning of the verse; second, it points out the mathematical meaning of the pun. Furthermore, this substitution of words indicates the standard by which the commentators believed the *sūtra* had been composed: a mathematical pun had been used to pin down the mathematical idea

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> *bhāgaṃ hared avargān nityaṃ dviguṇena vargamūlena|*
> *vargād varge śuddhe labdhaṃ sthānāntare mūlam||*

For an explanation of the algorithm, see (K. S. Shukla, 1976, 36-37).

90 (K. S. Shukla, 1976, 52)

> *atra gaṇite viṣamam sthānam vargaḥ*

91 (K. V. Sarma, 1976, p. 36, line 15).

> *saṃkhyā vinyāsasthāneṣu viṣamasthānāni vargasaṃjñāni*

behind the algorithm. Ab gives the core mathematical idea of the algorithm; BAB and SYAB highlight it.

Ab's confusing pun was not taken up by PG[92]:

> **PG.24. Having removed the square from the odd term,**
> **one should divide the remainder by twice the root that has trickled down to a place**
> **⟨And⟩ insert the quotient on a line||**
> **PG.25. Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. ⟨Finally⟩ one should halve what has been doubled.||**

In PG, the formulation of the algorithm avoids all puns. In comparison to the formulation in Ab, the algorithm here is neither dense nor confused. The formulation does, however, give precise elements which lead to the concrete execution of the algorithm on a working surface. A line (*paṅkti*) is evoked. The metaphor of the movement of trickling down as a drop of water (*cyuta*) is used to describe the digit-by-digit appearance of the partial double root.

These precise descriptions are further developed in APG, which multiplies graphical depictions of partial roots, evokes operations as carried above (*uparita*) and below (*adhas*) and even states that the extracted partial double root slides like a snake (*sarpaṇa*, *sarpita*) to the next position[93]. In a solved example, the practical details of how the procedure it is to be carried out on a working surface are described. The intricacies of the positional system are precisely indicated by APG. However, no reference is made to the idea of partial squares found in Ab's treatise.

We see then the commentaries help us unveil how they perceived the different nature of Ab and PG as treatises. The emphasis is, on one side, on the idea behind the procedure (in Ab), on the other, on expressing all the different steps of the algorithm, including the fact that the double square is obtained on a separate line (for PG). And indeed, two different kind of treatises are involved here. Ab is a theoretical astronomical treatise while PG is explicitly devoted to practical mathematics (*vyavahāra*). Clearly commentaries differ according to the type of texts they explicate. Although all commentaries include illustrated examples, APG is the only one to detail precisely how the intermediary operations are carried out, even countenancing the possibility that a doubled number might become bigger than 10 during the steps of the algorithm[94]. On the other hand, neither BAB nor SYAB insist on these

[92] See (K. S. Shukla, 1959, 18 for the Sanskrit, 9-10 of the part in English for an explanation of the procedure as described in APG)

> *viṣamāt padas tyaktyvā vargaṃ sthānacyutena mūlena|*
> *dviguṇena bhajec cheṣaṃ labdhaṃ viniveśayet paṅktau ||*
> *tadvargaṃ saṃśodhya dviguṇaṃ kurvīt purvaval labdham|*
> *utsārya tato vibhajec śeṣaṃ dviguṇīkṛtaṃ dalayet ||*

[93] (K. S. Shukla, 1959, p.18-19 of the Sanskrit, p. 9-10 of the English version. Taken together, they reveal Shukla's interpretation of how numbers are initially disposed and how they change during the execution of an algorithm, according to APG).

[94] (K. S. Shukla, 1959, 18, line 15-16).

intermediary steps, but, on the contrary, highlight only the essential idea behind the algorithm. PG concentrates on whole numbers, whereas SYAB and BAB emphasize the fact that the square root of fractions is the fraction of the square roots. The fact that commentaries correspond to the kind of text they explicate and give only what they deem appropriate in such circumstances, is especially clear in the case of SYAB: SYAB draws on a knowledge of PG. In fact, SYAB actually quotes PG in this very verse commentary, not for the details of how the procedure is executed on whole numbers or the positioning of digits during the procedure, but for a rule concerning the square-root of fractions.

Thus, commentaries are not systematic, practical explanations of the general cases formulated in the treatises. Instead, commentaries follow closely what they deem is the aim of the treatise, explaining the text, linking it to other considerations, but not going into details which would not be appropriate for the kind of treatise at hand. A commentary of a a theoretical treatise will not detail the execution of a procedure. Conversely the commentary of a practical text will not reflect on abstract ideas, even if authors of commentaries know better. Furthermore, eventhought the non textual practice of tabular computations seems to be the realm of commentaries, PG shows that treatises can also testify of these practices, although probably not in detail. Consequently, the style of commentaries depends on styles of treatises, or, more precisely, on how the authors of the commentaries read the intentions behind the treatises. An analysis of how commentaries, by word substitutions, has read the treatises can furthermore give us insights onto practices and thoughts that have until now not been analyzed: the use of the decimal place value as a formal notation.

3.2 Positional notation and extracting square roots

Although historians of mathematics in the Indian subcontinent have long insisted on showing that the decimal place value notation came from India, they have reflected comparatively little on how this concept varied among different authors, particularly how they conceived of (and used) the idea of position. Indeed, for BAB, SYAB and APG, decimal place value is a conventional notation by which the positions containing the digits used to make a number are an ordered set on a horizontal line[95]. This is how a "place" becomes a "position", although no new Sanskrit word is introduced to express this conceptual change. Is this positional notation thought of as a positional *system*? A close look at the way commentators treated the extraction of square roots enables us to understand better their conception of position.

Let us recall the rule Ab gives to extract square roots[96]:

> **Ab.2.4. One should divide, constantly, the non-square ⟨place⟩ by twice the square root|**

[95] (Keller, 2006b).

[96]

> *bhāgaṃ hared avargān nityaṃ dviguṇena vargamūlena|*
> *vargād varge śuddhe labdhaṃ sthānāntare mūlam||*

For an explanation of the algorithm, see (K. S. Shukla, 1976, 36-37).

When the square has been subtracted from the square ⟨place⟩, the quotient is the root in a different place||

Previously, we noted that one of the difficult aspects of this verse originates in a pun which conflates the square digits (*varga*) and the digits noted in "square positions". Ab considers the decimal place-value notation an ordered line of places for incrementally increasing powers of ten. To this reading, he adds a new grid to qualify the positions and distinguishes square powers of ten from the powers of ten that are not squares. Ab's interpretation of positions simultaneously addresses the mathematical dimension of decimal place value notation and the mathematical idea on which the algorithm rests. The commentary BAB, as well as SYAB, both help us understand Ab's verse by giving new names to these positions. Both commentaries start with the decimal place value notation, but consider decimal place value notation more broadly. They count the positions in which the digits are noted, starting on the right, from the lowest power of ten, and continue toward the left. All the even numbers of this enumeration indicate "even places", and odd numbers denote "odd places".

Given that 10^0 starts this enumeration, a table may illustrate the correspondence of what Ab calls "square places" with the "uneven places" of BAB and SYAB and the equivalence of what Ab calls "non-square places" with the "even places" of BAB and SYAB:

10^5	10^4	10^3	10^2	10^1	10^0
non-square place (*avarga*)	square place (*varga*)	non-square place	square place	non-square place	square place
6	5	4	3	2	1
even place (*sama*)	odd place (*viṣama*)	even place	odd place	even place	odd place

Commentators then add their own grid to the ordered list of places that define the decimal place value notation. This grid considers the notation outside of its mathematical content, as a tabular form with a numbered list of items on a line. The commentators assess this tabular form mathematically through odd and even numbers[97]. This mathematical assessment is not directly related to the algorithm, but the commentators link their grid to the one used by Ab through a simple substitution of one word for another. These substitutions of names are summarized in Table 3.2.

Places are used and qualified in different classifications: some underline their values, others their positions in an ordered line, others again pinpoint their mathematical qualities (as squares). These multiple qualifications point to the fact that all considered authors, do indeed use the decimal place value as a *system of positions* which can be qualified in as many different ways an algorithms requires.

[97]BAB introduces this assessment through a linguistic analysis of the term *avarga*, noting: (K. S. Shukla, 1976, 52)

tasya eva nañā viṣamatve pratisiddhe avargaḥ iti samam sthānam yataḥ hi viṣamam samam ca sthānam

Since a non-square ⟨takes place⟩ when oddness is denied, by means of ⟨the affix⟩ *nañ* ⟨the expression refers to⟩ an even (*sama*) place, because, indeed, a place is either odd or even.

Table 4: Names of places in the algorithm to extract square roots

Texts	Even powers of ten	Uneven powers of ten
Ab	*varga*	*avarga*
BAB	*viṣama*	*sama*
PG	*viṣama*	nihil
APG	*viṣama*	*sama*
SYAB	*viṣama*	*sama*

Places are used and qualified according to different classifications: some classification emphasize the value of the places, others classifications highlight the positions of the places in an ordered line, and still others accentuate their mathematical qualities (as squares). These multiple qualifications indicate that all the authors considered here do indeed use the decimal place value as a *system of positions*, and this system can be qualified in as many different ways the algorithm requires.

Previously, we have seen that glosses, the act of substitution common to the commentaries, not only explain the literal meaning of the verse but also highlights the mathematical meaning of a pun. The substitution also creates a new grid, a new system of positions, and appends this grid to the grid from the treatise which the commentary has just elucidated.

PG uses the same vocabulary as BAB[98]:

> **PG.24. Having removed the square from the odd term,**
> **one should divide the remainder by twice the root that has trickled down to a place**
> **⟨And⟩ insert the quotient on a line||**
> **PG.25. Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. ⟨Finally⟩ one should halve what has been doubled.||**

In regard to questions of place, APG supplies the following comment to the verse [99]:

> One should subtract a possible square[100], from the *viṣama* place of the square quantity, ⟨in other words⟩ from what is called odd (*oja*), that is from the first, third, fifth, seventh etc.,

[98]See (K. S. Shukla, 1959, 18 for the Sanskrit, 9-10 of the part in English for an explanation of the procedure as described in APG):

> *viṣamāt padas tyaktyvā vargaṃ sthānacyutena mūlena|*
> *dviguṇena bhajec cheṣaṃ labdhaṃ viniveśayet paṅktau ||*
> *tadvargaṃ saṃśodhya dviguṇaṃ kurvīt purvaval labdham|*
> *utsārya tato vibhajec śeṣaṃ dviguṇīkṛtaṃ dalayet ||*

[99](K. S. Shukla, 1959, 18, line 10-12)

> *vargarāśer viṣamāt padād ojākhyād ekatṛtīyapañcamasaptamāder ekaśatāyutaprayutādisthānebhyo 'nyatamasthānād antyāt padāt sambhavinaṃ vargaṃ tyajet*

[100]The adjective *saṃbhavin* conveys both the meaning "appropriate" and "conjectured", the subtext is thus understood

⟨place⟩, from the places for one, one hundred, ten thousand, one million, etc. from the last among other places.

In the case of finding the square root of 188624, APG adds[101]:

> In due order starting from the first place which consists of four, producing the technical names (*saṃjñā karana*): "odd (*viṣama*), even (*sama*), odd (*viṣama*), even (*sama*)".
>
> Setting down:
>
> | | | | | | |
> |---|---|---|---|---|---|
> | sa | vi | sa | vi | sa | vi |
> | 1 | 8 | 6 | 6 | 2 | 4 |
>
> In this case, the odd terms which are the places one, a hundred, ten thousands, consist of four, six and eight. Their last odd term is the ten thousand place which consists of eight.

APG expounds precisely the mathematical background relative to the decimal place value notation just alluded to in the treatise. Indeed, the anonymous commentator uses different possible expressions to name a position: its value within a power of ten, the place it has in the row of numbers noted on the line, and the digit which is noted in this position. The values of power of tens that a position stands for are denoted with a *tatpuruṣa* compound ending in *sthāna*. Inside the compound, an enumerative *dvandva* gives the particular powers of ten in increasing order (*ekaśatāyutasthānāni*, "the places for one, a hundred, ten thousands"). The place within the row of numbers may be described in several ways. APG numbers the positions, starting with one for the lowest power of ten and increasing successively. These places are enumerated by a *dvandva* which gives the ordinal forms of the particular positions (*ekatṛtīyapañcamasaptamāder*, "for the first[102], the third, the fifth, the seventh, etc."). In conformity with PG, APG then reproduces the *viṣama* / *sama* (odd /even) terminology found in BAB and SYAB. In contrast to BAB and SYAB, however, PG and APG insist that positions be used within an ordered set, or *series* of numbers, for which there is a first and a last term. PG describes the terms of a series as *pada*, a word which APG glosses with the expression *anyatamasthānāt antyāt*, "the last among all other places". Finally, the particular digits are understood as tools to be used within these positions. This understanding may explain why the names of numbers end with the suffix *-ka*, such as *catuḥṣaḍaṣṭakāni* , ("consists of four, six and eight").

A close connection between commentary and treatise may explain the various ways a place becomes a position on an elaborate grid, resting not only upon the decimal place value notation and the algorithm

an "appropriately conjectured" square.

[101](K. S. Shukla, 1959, 18, line 19-22)

> *ānulomyena ekasthānāc catuṣkāt prabhṛti viṣamaṃ samaṃ viṣamaṃ samam iti saṃjñākaraṇam /*
>
> | | | | | | |
> |---|---|---|---|---|---|
> | sa | sa | sa | vi | sa | vi |
> | 1 | 8 | 6 | 6 | 2 | 4 |
>
> *atra catuḥṣaḍaṣṭakāni ekaśatāyutasthānāni viṣamapadāni tebhyo 'yutasthānastham aṣṭakam antyaṃ viṣamapadaṃ*

[102]*eka* however is used here and not *prathāma*.

to executed, but also depending on the formal system created by the notation itself. The line of place value can be ordered in many different ways.

Different lines on which the extraction of square roots may be executed are recalled, and different grids are correspondingly described by APG: the horizontal expansion of the decimal place value notation is extended into a table, with some operations conducted in columns and others on rows, as is the case for other elementary arithmetical operations. A chronological perspective might elucidate the historical evolution: over time, the decimal place value notation seems to have slowly taken on a formal aspect, namely, over time the decimal place value notation, slowly developed into a tabular form which could be used not referring systematically to what positions meant in terms of values of powers of ten.

Conclusion

Investigations into how commentators read mathematical treatises can lead to conceptual insights into the practice of the decimal place value notation. A simple word substitution, such as that used in BAB, SYAB and APG, shows how the decimal place value notation may be considered a horizontal line of a table in which formal operations can be carried out.

Two different kind of *śāstra*s have been studied here: one emphasized mathematical ideas, the other focused on mathematical practices. The first belongs to a chapter of an astronomical treatise; the second belongs to a practical mathematical text. Each has different ways of describing algorithms. The study of these different descriptive practices enables a deeper understanding of the use of these texts and what they intended to convey.

The purpose of the treatises, as the commentators understood them, directed the aims and composition of the commentaries. The commentators' interpretation of the treatises determines whether their commentaries concentrate on abstract mathematical ideas or emphasize how an algorithm is executed out on a working surface. Treatises then can hint to "pratical knowledge", that commentaries can choose to spell out or not. However in all cases, the commentarial work consists in establishing relations, integrating what is hinted in the verse into a network of other systems.

If the ballroom dancing metaphor is taken up again, the treatise seems to lead the danse, but this is what the commentator wants us to believe. Like a virtuose yet discrete partner, the commentator relies on his techniques and knowledge to showcase the treatise. Such an attitude, which has its counter examples, helps us understand why a stylistic criteria to distinguish commentaries from treatises is so difficult to find.

A late tradition may have considered commentaries as mere fragmentary explanations and not full texts. The examples studied here show that a late seventeenth century tradition in South India considered commentaries as fragmentary explanations and not full texts. Looking at texts themselves, one can wonder wether they were made to be read, verse by verse, verse-commentary by verse-commentary in due order or if they were conceived to be read in any order, separately, while looking for a specific explanation. In the case of the three commentaries on square root extraction, the decimal place value notation and the rules for elementary operations are used. With this limited pre-requisite knowledge, the commentary may be read as an autonomous text. Such an autonomous verse commentary reading may not hold for all algorithms and would further discard the more integrated vision of what the treatise was about that a full commentary could provide. It remains to be determined wether authors who wrote commentaries thought of them as texts to be subjected to partial readings. Should an oral culture of texts be imagined, where commentaries and treatises are known entirely but can be quoted and mobilized in fragmentary ways?

Along the way, some information on who caused a text to be copied has been gathered. In the case of SYAB, scholarly astronomers, ritualist families and religious institutions of South India had the texts copied. This information may prove useful in confirming or denying our hypothetical constructions. Coming back to how commentaries were read, a hypothetical evolution can be suggested: An early modern pandit tradition may have informed the way Colebrooke and other European scholars treated Sanskrit commentaries. Such attitudes could have mingled with European traditions which esteemed mathematical texts not for their textual characteristics but rather for their mathematical content. This attitude has resulted in the kind of historiography prevalent today when studying mathematical commentaries.

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