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**Steady state analysis and endogenous fluctuations  
in a finance constrained model**

Thomas SEEGMULLER, EUREQua

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# Steady state analysis and endogenous fluctuations in a finance constrained model

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<sup>1</sup>Part of this paper has been done when I was at the Universidade Católica Portuguesa that I thank for its hospitality.

## Abstract

The overlapping generations model, like the one studied by Reichlin (1986) or Cazzavillan (2001), can be interpreted as an optimal growth economy where consumption is totally constrained by capital income. In this paper, we analyze steady states and dynamic properties of an extended version of such framework by considering that only a share of consumption expenditures is constrained by capital income. We notably establish that the steady state is not necessarily unique. Moreover, in contrast to the intuition, consumer welfare can increase at a steady state following a raise of the share of consumption constrained by capital income, i.e. the market imperfection. Concerning dynamics, we show that endogenous fluctuations (indeterminacy and cycles) can emerge depending on two parameters: the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution. Such fluctuations appear when these two parameters take values in accordance with empirical studies and without introducing increasing returns or imperfect competition.

*Keywords:* finance constraint, steady states, indeterminacy, endogenous cycles.

## Résumé

Le modèle à générations imbriquées, tel que celui étudié par Reichlin (1986) ou Cazzavillan (2001), peut s'interpréter comme un modèle de croissance optimale où la consommation est totalement contrainte par le revenu du capital. Dans ce papier, nous analysons les états stationnaires et les propriétés dynamiques d'une extension de ce type de modèle en considérant que seule une part de la consommation est contrainte par le revenu du capital. Nous établissons notamment que l'état stationnaire n'est pas nécessairement unique. De plus, contrairement à l'intuition, le bien-être des consommateurs à l'état stationnaire peut augmenter avec la part de la consommation contrainte par le revenu du capital, c'est-à-dire avec l'imperfection de marché. Concernant la dynamique, nous montrons que des fluctuations endogènes (indétermination et cycles) peuvent émerger, suivant la valeur de deux paramètres : l'élasticité de substitution intertemporelle pour la consommation et l'élasticité de substitution entre le capital et le travail. De telles fluctuations apparaissent lorsque ces deux paramètres prennent des valeurs en accord avec les études empiriques et en l'absence de rendements croissants et de concurrence imparfaite.

*Mots-clés:* contrainte de financement, états stationnaires, indétermination, cycles endogènes.

*JEL classification:* C62, D91, E32

# 1 Introduction

When one analyzes macroeconomic dynamics, two main models are usually used: the optimal growth model and the overlapping generations one. As it is well-known, when there is no market imperfection, one sector, constant returns to scale and standard assumptions on utility and production functions, the first one has a unique steady state characterized by saddle path stability. On the contrary, in overlapping generations models, such result is not ensured whatever the values of the economic parameters are. Indeed, considering that consumers supply labor when young, save through productive capital and consume only when old, Reichlin (1986) has established that the steady state can be locally indeterminate and endogenous cycles can emerge if the elasticity of substitution between the production factors is sufficiently weak, i.e. smaller than the capital share in total income.<sup>2</sup> Even if such conditions are quite restrictive, it proves that endogenous fluctuations occur more easily in overlapping generations economies.

Since the work of Barro (1974), a link between the overlapping generations and optimal growth models is well-known. Indeed, consider that in an overlapping generations economy agents are altruistic, more precisely their utility function depends on the preferences of their children. Then, if the bequests are positive, the overlapping generations model can behave like the optimal growth one. In particular, if the utility is separable between consumptions when young and old, the steady state is a saddle and hence there is monotonic convergence as in the optimal growth model.<sup>3</sup> However, Michel and Venditti (1997) have shown that this result is no more relevant when the utility function of overlapping generations consumers is non-separable. Indeed, in this case, they prove the existence of non-monotonic dynamic trajectories and cycles of period two.<sup>4</sup>

Another link between the overlapping generations and the optimal growth models can be exploited. In the spirit of Woodford (1988), one can deduce the overlapping generations framework studied by Reichlin (1986) or Cazzavillan (2001) from an optimal growth model with elastic labor supply where consumption is totally constrained by capital income at each period.<sup>5</sup> It implicitly means that labor income cannot be used for current consumption.

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<sup>2</sup>See also Cazzavillan (2001).

<sup>3</sup>See among others Blanchard and Fisher (1989), chapter 3.

<sup>4</sup>One can also refer to Venditti (2003) who extends these results to an economy with externalities.

<sup>5</sup>As an example, Dos Santos Ferreira and Lloyd-Braga (2003) use such a framework in order to study the existence of sunspot equilibria under imperfect competition and free entry.

In this paper, we extend and generalize this last type of model. In this way, we introduce a finance constraint in an one-sector optimal growth model with constant returns to scale. However, instead of considering that consumption is totally constrained by capital income, we rather assume that only a share of consumption expenditures is constrained by capital income. In this framework, we analyze steady states and local dynamics.

We first establish the existence of a steady state. It notably requires a not too weak share of consumption constrained by capital income. Moreover, this steady state is not necessarily unique. Indeed, considering a CES production function, we prove that two steady states can coexist when the elasticity of capital-labor substitution is smaller than one. However, this result also depends on the level of the finance constraint. In particular, when the share of consumption constrained by capital income is quite small, multiplicity of steady states requires more restrictive conditions on the substitution between capital and labor. We also give some insights concerning the welfare properties of a steady state. Evidently, a steady state is not characterized by the modified golden rule, due to the finance constraint. Moreover, an intuitive result would be that when this finance constraint becomes more important, consumer welfare would decrease. We find that this conclusion is not always satisfied.

Studying local dynamics, we show that endogenous fluctuations can occur in the neighborhood of a steady state. We discuss the results with respect to two parameters: the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution. More precisely, we establish that for a high intertemporal substitution in consumption, indeterminacy occurs when the elasticity of factor substitution is not too strong, in any cases smaller than one. On the contrary, when the elasticity of intertemporal substitution in consumption is sufficiently weak, endogenous fluctuations emerge for a higher substitution between capital and labor. In this last case, indeterminacy is compatible with an elasticity of capital-labor substitution which can be arbitrarily close to one or even equal to one (Cobb-Douglas technology).

The occurrence of indeterminacy depends on the intertemporal substitution in consumption and the substitution between capital and labor because two phenomena are important for the emergence of endogenous fluctuations due to self-fulfilling expectations: on one hand the effect of a variation of expected interest rate on labor supply, and on the other hand the effect of a variation of labor supply on savings. We also remark that the conditions for the occurrence of endogenous fluctuations are compatible with empirical studies, concerning the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution. Indeed, if these two

elasticities are often equal to one in models with infinitely lived agents, recent results established that the first one can take values smaller than one (Campbell (1999), Kocherlakota (1996)), whereas the second one can be different to one, greater or smaller values than one being admissible if they are not too far from the unit case (Duffy and Papageorgiou (2000)).

These results can be easily compared with existing contributions. Indeed, when consumption expenditures are totally constrained by capital income and the finance constraint is binding, the dynamics are the same than in the overlapping generations model studied by Reichlin (1986) and more recently by Cazzavillan (2001) when he assumes constant returns to scale. These two contributions only consider the case where the intertemporal substitution is high enough<sup>6</sup> and show, as we have already mentioned before, that endogenous fluctuations occur as soon as the elasticity of capital-labor substitution is smaller than the capital share in total income. Evidently, we also obtain this result in the limit case where households can only use capital income for consumption expenditures. However, it is interesting to notice that when the share of consumption expenditures financed by capital income decreases from one, indeterminacy and cycles can occur for greater values of the substitution between production factors. Moreover, when the elasticity of intertemporal substitution in consumption is not so great, endogenous fluctuations can occur for a high substitution between capital and labor in this finance constrained model without introducing an additional asset like money (Bosi, Dufourt, and Magris (2002), Bosi and Magris (2003)), heterogeneous households (Barinci (2001)), imperfect competition or increasing returns (Barinci and Chéron (2001), Cazzavillan, Lloyd-Braga, and Pintus (1998)). Finally, this paper also allows us to conclude that endogenous fluctuations can occur in the one sector growth model with infinitely lived agents without considering externalities and increasing returns (Benett and Farmer (2000), Benhabib and Farmer (1994), Farmer and Guo (1994), Harrison and Weder (2002), Hintermaier (2003), Pintus (2003a, 2003b)), imperfect competition (Gali (1994), Woodford (1991)) or counter-cyclical tax rates (Guo and Lansing (1998), Schmitt-Grohé and Uribe (1997)).

This paper is organized as follows. In the next section, we present the model. In section 3, we analyze steady states. In section 4, we study the emergence of endogenous fluctuations. Finally, we provide some concluding remarks in section 5.

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<sup>6</sup>One often introduces such a restriction in overlapping generations economies in order to have an increasing labor supply with respect to the real wage. Indeed, in contrast to optimal growth models, the elasticity of labor supply is evaluated taken into account the effects of the real wage and the labor supply on the consumption.

## 2 The model

We consider a perfectly competitive economy with discrete time,  $t = 1, 2, \dots, \infty$  and perfect foresight. The population is constant and normalized to one. So we consider a representative infinitely lived agent who supplies labor and consumes the final good. His intertemporal preferences are defined by:

$$\sum_{t=1}^{\infty} \beta^t (Bu(c_t/B) - v(l_t)) \quad (1)$$

where  $c_t$  is the consumption in period  $t$ ,  $l_t$  the labor supply,  $\beta \in (0, 1)$  the discount factor, and  $B > 0$  a scaling parameter. Moreover, we assume that  $u$  and  $v$  satisfy the following usual assumptions:

**Assumption 1** *The functions  $u(x)$  and  $v(l)$  are continuous for all  $x \geq 0$  and  $0 \leq l \leq l^*$ , where  $l^* > 1$  is the labor endowment. They have continuous derivatives of every required order for  $x > 0$  and  $0 < l < l^*$ , with  $u'(x) > 0 > u''(x)$ ,  $v'(l) > 0$  and  $v''(l) \geq 0$ .*

At each period, the representative consumer has the following budget constraint:

$$c_t + k_t = r_t k_{t-1} + w_t l_t \quad (2)$$

where  $k_{t-1}$  is the capital used in the production at period  $t$ ,  $w_t$  the real wage and  $r_t$  the real interest rate.<sup>7</sup> Moreover, the consumer faces a finance constraint.<sup>8</sup> Indeed, we assume that a share  $\mu \in (0, 1]$  of consumption has to be financed by capital income, i.e.

$$\mu c_t \leq r_t k_{t-1} \quad (3)$$

It implicitly means that the consumer cannot use labor income to finance all his current consumption. The consumer maximizes his utility function (1) under the two constraints (2) and (3). If we note  $\lambda_{0t}$  (respectively  $\lambda_{1t}$ ) the Lagrange multiplier associated to (2) (respectively to (3)), then we obtain the following first order conditions:

<sup>7</sup>For simplification, we assume that capital totally depreciates after one period of use.

<sup>8</sup>See Woodford (1986, 1988) for early works which analyze the role of a finance constraint on the occurrence of endogenous fluctuations in economies with infinitely lived agents. One can also refer to Barinci (2001), Bosi, Dufourt, and Magris (2002), Bosi and Magris (2003), Bosi, Magris, and Venditti (2003), Grandmont, Pintus, and de Vilder (1998) and Woodford (1994) for more recent results.

$$u'(c_t/B) = \lambda_{0t} + \mu\lambda_{1t} \quad (4)$$

$$v'(l_t)/w_t = \lambda_{0t} \quad (5)$$

$$\lambda_{0t} = \beta r_{t+1} (\lambda_{0t+1} + \lambda_{1t+1}) \quad (6)$$

$$\lambda_{1t} (r_t k_{t-1} - \mu c_t) = 0 \quad (7)$$

and the usual transversality condition  $\lim_{t \rightarrow +\infty} \beta^t \lambda_{0t} k_t = 0$ . In the rest of this paper, we are only interesting in the case where the finance constraint is binding. It means that  $\mu c_t = r_t k_{t-1}$  and  $\lambda_{1t} > 0$ . Then, using equations (2) – (6), we can deduce:

$$\frac{v'(l_t)}{w_t} = \beta r_{t+1} \left[ \left(1 - \frac{1}{\mu}\right) \frac{v'(l_{t+1})}{w_{t+1}} + u'\left(\frac{c_{t+1}}{B}\right) \frac{1}{\mu} \right] \quad (8)$$

$$k_t = \left(1 - \frac{1}{\mu}\right) r_t k_{t-1} + w_t l_t \quad (9)$$

At this stage, we can notice that if  $\mu = 1$ , then  $c_{t+1} = r_{t+1} k_t$ . In this case, equations (8) and (9) become exactly the same that those obtained by Reichlin (1986) or Cazzavillan (2001) who analyze the emergence of endogenous fluctuations in overlapping generations economies where consumers live two periods, supply labor when young, save through the purchase of capital and consume only when old.

It is also interesting to remark that since a binding finance constraint means  $c_t = r_t k_{t-1}/\mu$  and  $u'(c_t/B) > v'(l_t)/w_t$ , the following inequality has to be satisfied:

$$v'(l_t) < w_t u'\left(\frac{r_t k_{t-1}}{\mu B}\right) \quad (10)$$

Concerning the production sector, we assume that the final good is produced by a continuum of firms of unit size with the constant returns to scale technology  $y_t = A f(a_t) l_t$ , where  $A > 0$  is a scaling parameter,  $a_t = k_{t-1}/l_t$  denotes the capital-labor ratio and  $f$  the intensive production function. We further assume:

**Assumption 2** *The function  $f(a)$  is continuous for  $a \geq 0$ , positively valued and differentiable as many times as needed for  $a > 0$ , with  $f'(a) > 0 > f''(a)$ .*

Since all markets are perfectly competitive, the profit maximization gives us the expressions of the real wage and real interest rate:

$$w_t = A (f(a_t) - a_t f'(a_t)) \equiv w(a_t) \quad (11)$$

$$r_t = A f'(a_t) \equiv r(a_t) \quad (12)$$

We can now determine the dynamics and define the intertemporal equilibrium. Using the last two expressions and the equality  $l_t = k_{t-1}/a_t$ , we have:

$$\frac{v'(k_{t-1}/a_t)}{w(a_t)} = \beta r(a_{t+1}) \left[ \left(1 - \frac{1}{\mu}\right) \frac{v'(k_t/a_{t+1})}{w(a_{t+1})} + u' \left( \frac{r(a_{t+1}) k_t}{\mu B} \right) \frac{1}{\mu} \right] \quad (13)$$

$$k_t = \left(1 - \frac{1}{\mu}\right) r(a_t) k_{t-1} + w(a_t) k_{t-1}/a_t \quad (14)$$

with

$$v'(k_{t-1}/a_t) < w(a_t) u' \left( \frac{r(a_t) k_{t-1}}{\mu B} \right) \quad (15)$$

Furthermore, the capital stock has to be positive at each period. If we note  $s(a) \equiv f'(a) a/f(a) \in (0, 1)$  the capital share in total income, it means that:

$$\mu \geq s(a_t) \quad (16)$$

Then, we can define an intertemporal equilibrium as follows:

**Definition 1** *An intertemporal equilibrium with perfect foresight is a sequence  $(a_t, k_{t-1}) \in \mathbb{R}_{++}^2, t = 1, 2, \dots, \infty$ , such that equations (13), (14), (15) and (16) are satisfied, where  $w(a_t)$  and  $r(a_t)$  are defined by (11) and (12).*

Before to analyze steady states, it is useful to define a link between the elasticities of the real wage and the real interest rate with respect to the capital-labor ratio  $a$  and technological parameters. The elasticity of capital-labor substitution  $\sigma(a)$  is defined by the equality  $1/\sigma(a) = w'(a) a/w(a) - r'(a) a/r(a)$ . Using  $w'(a) = -ar'(a)$ , we obtain  $w'(a) a/w(a) = s(a)/\sigma(a)$  and  $r'(a) a/r(a) = -(1 - s(a))/\sigma(a)$ .

### 3 Steady state analysis

In this section, we first study the existence of a steady state. Then, using a CES production function, we determine the conditions for uniqueness or multiplicity of steady states. Finally, we establish some welfare properties of the steady state.

#### 3.1 Existence of a steady state

A steady state  $(a, k)$  is defined by the following equations<sup>9</sup>:

$$\left(1 - \frac{1}{\mu}\right) r(a) + w(a)/a = 1 \iff A \frac{f(a)}{a} (\mu - s(a)) = \mu \quad (17)$$

$$v'(k/a) [\mu + (1 - \mu) \beta r(a)] = \beta r(a) w(a) u' \left( \frac{r(a) k}{\mu B} \right) \quad (18)$$

$$v'(k/a) < w(a) u' \left( \frac{r(a) k}{\mu B} \right) \quad (19)$$

$$\mu > s(a) \quad (20)$$

Following Aloi, Dixon, and Lloyd-Braga (2000) and Cazzavillan, Lloyd-Braga, and Pintus (1998), we establish the existence of a normalized steady state  $(a, k) = (1, 1)$  by choosing appropriate values of the two scaling parameters  $A > 0$  and  $B > 0$ . Using equation (17), one can easily see that it exists a unique solution  $A^* > 0$  defined by:

$$A^* = \frac{\mu}{\mu - s(1)} \frac{1}{f(1)} \quad (21)$$

In this case, if  $\lim_{x \rightarrow +\infty} u'(x) < [\mu + (1 - \mu) \beta r(1)] v'(1) / (\beta r(1) w(1)) < \lim_{x \rightarrow 0} u'(x)$ , there is a unique  $B^* > 0$  such that the following equation is satisfied:

$$v'(1) [\mu + (1 - \mu) \beta r(1)] = \beta r(1) w(1) u' \left( \frac{r(1)}{\mu B^*} \right) \quad (22)$$

Such a steady state has to satisfy the two conditions (19) and (20). Using (18), the first one can be rewritten  $\beta r(1) < 1$ , which means that the modified

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<sup>9</sup>We can notice that at a steady state, there is a strict inequality in (20) because when we will study local dynamics in the neighborhood of a steady state, inequality (16) has to be satisfied in this neighborhood.

golden rule is not satisfied in the economy.<sup>10</sup> One can further notice that this last inequality is also equivalent to  $\mu > \tilde{\mu}$ , with:

$$\tilde{\mu} = \frac{s(1)}{1 - \beta s(1)} \quad (23)$$

Evidently, it requires  $\tilde{\mu} < 1$ , i.e.  $s(1) < 1/(1 + \beta)$ . Finally, at the steady state  $(a, k) = (1, 1)$ , the condition  $\mu > s(1)$  has to be ensured, which is always satisfied when  $\mu > \tilde{\mu}$ .

**Proposition 1** *Let  $\tilde{\mu}$  be defined by (23) and  $s(1) < 1/(1 + \beta)$ . Assuming  $\lim_{x \rightarrow +\infty} u'(x) < [\mu + (1 - \mu)\beta r(1)]v'(1) / (\beta r(1)w(1)) < \lim_{x \rightarrow 0} u'(x)$  and  $\mu > \tilde{\mu}$ ,  $(a, k) = (1, 1)$  is a steady state of the dynamic system (13)-(14) if  $A$  and  $B$  are the unique solutions of (21) and (22).*

However, such a steady state is not necessarily unique. In order to clearly analyze the number of stationary solutions, we consider in the following subsection the case of a CES technology.

### 3.2 Uniqueness versus multiplicity in a CES economy

Assume in what follows that Proposition 1 is satisfied, i.e. the steady state  $(a, k) = (1, 1)$  exists. Since the technology is CES, the intensive production function  $f(a)$  can be written:

$$f(a) = \left( sa^{\frac{\sigma-1}{\sigma}} + 1 - s \right)^{\frac{\sigma}{\sigma-1}} \quad (24)$$

with  $s \in (0, 1)$ ,  $\sigma > 0$  and  $\sigma \neq 1$ . This specification satisfies  $f(1) = 1$  and we have:

$$s(a) = \frac{s}{s + (1 - s)a^{\frac{1-\sigma}{\sigma}}} \quad (25)$$

where  $s(1) = s$ . The number of steady states is determined by equation (17). Using (21) and (25), it means that studying uniqueness or multiplicity of steady states requires to analyze the number of solutions of the following equation<sup>11</sup>:

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<sup>10</sup>Using (17), one deduces  $r(1) = \mu s(1)/(\mu - s(1))$ . It means that there is overaccumulation ( $r(1) < 1$ ) if  $\mu > s(1)/(1 - s(1))$ , whereas there is underaccumulation if the reverse inequality is satisfied.

<sup>11</sup>It is not difficult to see that when the elasticity of substitution between capital and labor  $\sigma$  is equal to 1, there is a unique steady state. Take for example  $f(a) = a^s$ . In this case, we have  $G(a) \equiv a^{s-1} = 1$  and there is only one steady state characterized by  $a = 1$ .

$$G(a) \equiv \frac{1}{\mu - s} \left( s + (1 - s) a^{\frac{1-\sigma}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \left( \mu - \frac{s}{s + (1 - s) a^{\frac{1-\sigma}{\sigma}}} \right) = 1 \quad (26)$$

Note that  $\mu > \tilde{\mu}$  implies  $\mu > s$  and we evidently have  $G(1) = 1$ . Moreover, recall that steady states have to satisfy  $\beta r(a) < 1$  and  $\mu > s(a)$ . The first inequality can be rewritten:

$$\left( s + (1 - s) a^{\frac{1-\sigma}{\sigma}} \right)^{\frac{1}{\sigma-1}} < \frac{\mu - s}{\beta s \mu} \quad (27)$$

Since the left-hand-side of this expression decreases with respect to  $a$ , it means that  $a > \underline{a}$ , with:

$$\underline{a} \equiv \frac{1}{(1 - s)^{\frac{\sigma}{1-\sigma}}} \left[ \left( \frac{\mu - s}{\beta s \mu} \right)^{\sigma-1} - s \right]^{\frac{\sigma}{1-\sigma}} \quad (28)$$

We can notice that  $\underline{a} \leq 1$  when  $\mu > \tilde{\mu}$ . The second inequality is equivalent to:

$$s + (1 - s) a^{\frac{1-\sigma}{\sigma}} > s/\mu \quad (29)$$

If we define  $\tilde{a} \equiv \left( \frac{s}{1-s} \frac{1-\mu}{\mu} \right)^{\frac{\sigma}{1-\sigma}}$ , equation (29) means that  $a > \tilde{a}$  if  $\sigma < 1$  and  $a < \tilde{a}$  if  $\sigma > 1$ . Consequently,  $a > \max\{\underline{a}, \tilde{a}\}$  when  $\sigma < 1$  and  $a \in (\underline{a}, \tilde{a})$  when  $\sigma > 1$ .<sup>12</sup> In order to analyze the number of steady states, we now compute the derivative of  $G(a)$ . It is defined by:

$$G'(a) = \frac{\mu(1-s)a^{\frac{1-\sigma}{\sigma}-2} \left[ s(1/(\sigma\mu) - 1) - (1-s)a^{\frac{1-\sigma}{\sigma}} \right]}{(\mu-s) \left[ s + (1-s)a^{\frac{1-\sigma}{\sigma}} \right]^{\frac{\sigma-2}{\sigma-1}}} \quad (30)$$

When  $\sigma \geq 1/\mu$ ,  $G'(a) < 0$  for all  $a \in (\underline{a}, \tilde{a})$ . Consequently, the steady state  $a = 1$  is unique.

Consider now the case where  $\sigma < 1/\mu$ . Then,  $G'(a) = 0$  for  $a = a_0$ , where:

$$a_0 \equiv \left[ \left( \frac{1}{\sigma\mu} - 1 \right) \frac{s}{1-s} \right]^{\frac{\sigma}{1-\sigma}} \quad (31)$$

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<sup>12</sup>One can further notice that  $\tilde{a} < 1$  for all  $\sigma < 1$ , whereas  $\tilde{a} > 1$  for all  $\sigma > 1$ .

If  $1 < \sigma < 1/\mu$ , we have  $G'(a) < 0$  for all  $a < a_0$  and  $G'(a) > 0$  for all  $a > a_0$ . Since  $G(+\infty) = -(1-\mu)s^{\frac{\sigma}{\sigma-1}}/(\mu-s) < 0$ , we conclude that it exists only one steady state  $a = 1$  for  $a \in (0, +\infty)$ , and then for  $a \in (\underline{a}, \tilde{a})$ .

If  $0 < \sigma < 1$ ,  $G'(a) > 0$  for all  $a \in (0, a_0)$  and  $G'(a) < 0$  for all  $a > a_0$ . We also have  $G(0) = -(1-\mu)s^{\frac{\sigma}{\sigma-1}}/(\mu-s) < 0$  and  $G(+\infty) = 0$ . Since the steady state  $a = 1$  exists, a second steady state  $a^*$  can appear as soon as  $G'(1) \neq 0$ , i.e.  $\sigma \neq s/\mu$ . We obtain more precisely two cases. In the first one,  $\sigma < s/\mu$  which implies that  $G'(1) > 0$ . Since  $\max\{\underline{a}, \tilde{a}\} < 1$ , we deduce that there are two steady states,  $a = 1$  and  $a = a^* > 1$ . In the second one,  $\sigma > s/\mu$ , i.e.  $G'(1) < 0$ . Then, if  $a^* > \max\{\underline{a}, \tilde{a}\}$ , there are two stationary solutions  $a = 1$  and  $a = a^* < 1$ . On the contrary, if  $a^* < \max\{\underline{a}, \tilde{a}\}$ , there is only one steady state  $a = 1$ .

These results are summarized in the next proposition:

**Proposition 2** *Assuming that  $\mu > \tilde{\mu}$ , the technology is CES and Proposition 1 is satisfied, the following holds.*

- (i) *if  $\sigma < s/\mu$ , there are two steady states  $(a, k) = (1, 1)$  and  $(a, k) = (a^*, k^*)$ , with  $a^* > 1$ ;*
- (ii) *if  $1 > \sigma > s/\mu$ , there are two steady states  $(a, k) = (1, 1)$  and  $(a, k) = (a^*, k^*)$  (with  $a^* < 1$ ) when  $a^* > \max\{\underline{a}, \tilde{a}\}$ , whereas there is only one steady state  $(a, k) = (1, 1)$  when  $a^* < \max\{\underline{a}, \tilde{a}\}$ ;*
- (iii) *if  $\sigma > 1$ , there is one steady state  $(a, k) = (1, 1)$ .*

This proposition shows that considering a CES production function, two steady states can coexist when  $\sigma < 1$ . It also suggests that  $\sigma = s/\mu$  is a non generic case. Indeed, a small increase or decrease of the elasticity of capital-labor substitution leads to the occurrence of a second steady state. We will see later when we will analyze local dynamics that such a situation corresponds to the emergence of a transcritical bifurcation.

We can also notice that if  $\mu = 1$ , our results are close to those obtained by Cazzavillan (2001) when he considers constant returns to scale. It is not so surprising since, as we have noted before, his model and our framework are closely related in this case. Notice however that in contrast to Cazzavillan (2001), multiplicity of steady states is not always ensured for  $\sigma \in (s, 1)$  in this economy.

More generally, using (28), one can note that  $\underline{a}$  decreases with respect to  $\mu$  and tends to 1 when  $\mu$  tends to  $\tilde{\mu}$ . It means that when the share of consumption expenditures constrained by capital income is close to its minimum admissible value, it only exists one steady state when  $s/\mu < \sigma <$

1. In other words, a weaker finance constraint promotes uniqueness of the stationary solution, because the existence of two steady states requires in that case  $\sigma < s/\mu$ .

### 3.3 Welfare properties

We now give some welfare properties of the steady state. As we have already noticed, a steady state is characterized by  $\beta r(a) < 1$ . Hence, this finance constrained model does not satisfy the modified golden rule ( $\beta r(a) = 1$ ) like the optimal growth model without market imperfection. Evidently, it comes from the introduction of the constraint  $\mu c \leq rk$ , which is binding at equilibria that we consider. One can further remark that the level of  $\mu$  represents the degree of market imperfection, the greater is  $\mu$  the most important is the market imperfection. Then, one could expect that a reasonable result would be that consumer welfare would decrease with respect to  $\mu$ . The following proposition shows that this intuitive conclusion is not always ensured.

**Proposition 3** *Assume that the economy is at a steady state  $(a, k)$  defined by (17) and (18) and parameters  $A$  and  $B$  are constant. Moreover, suppose that the desutility of labor is linear and note  $\epsilon_u(c/B) = -u''(c/B)(c/B)/u'(c/B)$ . Then, a slight increase of  $\mu$  increases consumer welfare for  $\mu < s(a)/(1-s(a))$  and  $\sigma(a) > s(a)/\mu$ , if  $\beta$  is not too close to 0 and  $\epsilon_u(c/B)$  is sufficiently high.*

**Proof.** Assume that the economy is at a steady state  $(a, k)$  defined by (17) and (18) and parameters  $A$  and  $B$  are given. Since we consider a linear desutility of labor,  $v(l)$  can be written  $vl$ , with  $v > 0$  a constant. Using (1) and (3), the consumer welfare  $W$  is defined by:

$$W = \frac{1}{1-\beta} \left[ Bu \left( \frac{r(a)k}{\mu B} \right) - v \frac{k}{a} \right] \quad (32)$$

From (17) and (18), one can remark that  $a$  and  $k$  can implicitly be defined as functions of  $\mu$ . It means that  $W$  can also be implicitly defined as a function of  $\mu$ . In what follows, we note  $a_\mu$  (respectively  $k_\mu$ ) the derivative of  $a$  (respectively  $k$ ) with respect to  $\mu$ . Using (17) and (18), we obtain:

$$\begin{aligned} \frac{dW}{d\mu} = & \frac{1}{\beta} \left[ \frac{r'(a)a}{r(a)} \frac{a_\mu \mu}{a} + \frac{k_\mu \mu}{k} - 1 \right] \frac{vk}{\mu w(a)} \\ & + \frac{1}{1-\beta} \frac{w(a)}{a} \left[ \frac{a_\mu \mu}{a} \left( \frac{r'(a)a}{r(a)} + 1 \right) - 1 \right] \frac{vk}{\mu w(a)} \end{aligned} \quad (33)$$

In order to determine the sign of this expression, we first compute  $a_\mu\mu/a$  and  $k_\mu\mu/k$ . Using again (17) and (18), we have:

$$\frac{a_\mu\mu}{a} = \frac{s(a)\sigma(a)}{(1-s(a))(\mu\sigma(a)-s(a))} \quad (34)$$

$$\begin{aligned} \frac{k_\mu\mu}{k} = & \left[ 1 - \frac{\mu(1-\beta r(a))}{\epsilon_u(c/B)(\mu+(1-\mu)\beta r(a))} \right] \\ & + \left[ \frac{1}{\epsilon_u(c/B)} \left( \frac{s(a)}{\sigma(a)} - \frac{1-s(a)}{\sigma(a)} \frac{\mu}{\mu+(1-\mu)\beta r(a)} \right) + \frac{1-s(a)}{\sigma(a)} \right] \frac{a_\mu\mu}{a} \end{aligned} \quad (35)$$

Finally, with this last two equations, we determine:

$$\begin{aligned} \frac{r'(a)a}{r(a)} \frac{a_\mu\mu}{a} + \frac{k_\mu\mu}{k} - 1 = & \frac{1}{\epsilon_u(c/B)} \left( \frac{s(a)}{\sigma(a)} - \frac{1-s(a)}{\sigma(a)} \frac{\mu}{\mu+(1-\mu)\beta r(a)} \right) \frac{a_\mu\mu}{a} \\ & - \frac{\mu(1-\beta r(a))}{\epsilon_u(c/B)(\mu+(1-\mu)\beta r(a))} \end{aligned} \quad (36)$$

$$\frac{a_\mu\mu}{a} \left( \frac{r'(a)a}{r(a)} + 1 \right) - 1 = \frac{s(a)(\sigma(a)-1+s(a))}{(1-s(a))(\mu\sigma(a)-s(a))} - 1 \quad (37)$$

For  $\sigma(a) > \max\{1-s(a), s(a)/\mu\}$ , the second expression is strictly positive if  $\mu < s(a)/(1-s(a))$ , whereas the first expression becomes negligible when  $\epsilon_u(c/B)$  is sufficiently high. It concludes the proof. ■

In order to give an interpretation of this last result, one can first notice that a slightly increase of  $\mu$  increases consumer welfare because  $\mu$  has a predominant effect on the capital-labor ratio  $a$  and, when capital and labor are not weak substitutes, the capital-labor ratio increases with respect to the share of consumption expenditures constrained by capital income (see equation (17)). Indeed, it induces a decrease of the real interest rate. If one considers the effect of a variation of  $\mu$  on capital as negligible, it means that a slight increase of  $\mu$  decreases consumption ( $r(a)k/\mu$ ), but also labor ( $k/a$ ). Our result comes from the fact that the effect of  $\mu$  on labor dominates.

Furthermore, since the capital-labor ratio increases, the gap between the real interest rate  $r(a)$  and the modified golden rule  $1/\beta$  raises. One can also notice that the conditions established in Proposition 3 notably means that the steady state is dynamically efficient. Indeed, using equation (17), the real interest rate  $r(a)$  is equal to  $\mu s(a)/(\mu-s(a))$  at a steady state. Since

our result requires  $\mu < s(a)/(1 - s(a))$ , it means that  $r(a) > 1$ , i.e. there is underaccumulation.

Finally, the result obtained in this last proposition can be related to Bosi and Magris (2002). Indeed, considering a model with heterogeneous agents and a finance constraint in the spirit of Woodford (1986), they notably analyze the effect on welfare of the relaxation of the finance constraint. Under constant returns to scale, they obtain quite different conclusions than us, since welfare of all types of agents increases when the finance constraint is relaxing.<sup>13</sup> However, in contrast to our model, the steady states satisfy the modified golden rule in their framework.

## 4 Endogenous fluctuations

In this section, we study local dynamics, i.e. the local stability of the steady state and the occurrence of bifurcations, in order to establish the conditions for the emergence of endogenous fluctuations. In particular, we discuss our results in function of two parameters: the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution.

In this way, we consider that Proposition 1 is satisfied, i.e. the steady state  $(a, k) = (1, 1)$  exists. It notably means that  $\mu > \tilde{\mu}$ . Furthermore, we note  $s \equiv s(1)$ ,  $\sigma \equiv \sigma(1)$ ,  $\varepsilon_v \equiv v''(l)l/v'(l) \geq 0$  and  $\varepsilon_u = -u''(c/B)(c/B)/u'(c/B) > 0$  evaluated at the steady state  $(a, k) = (1, 1)$ . Note at this stage that  $1/\varepsilon_u$  represents the elasticity of intertemporal substitution in consumption. In order to simplify the analysis, we assume:

**Assumption 3**  $s < 1/2$ ,  $\varepsilon_v = 0$ .

It means that the capital share in income is smaller than one half and we consider a linear desutility of labor.<sup>14</sup> One can further notice that under Assumption 3,  $\tilde{\mu} = s/(1 - \beta s)$  is strictly smaller than 1. Now we differentiate the dynamic system (13) – (14) in the neighborhood of the steady state  $(a, k) = (1, 1)$ . We obtain:

$$\begin{bmatrix} \frac{da_{t+1}}{a} \\ \frac{dk_t}{k} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \frac{da_t}{a} \\ \frac{dk_{t-1}}{k} \end{bmatrix} \quad (38)$$

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<sup>13</sup>The same conclusion does not always apply when the returns to scale are increasing.

<sup>14</sup>Since Hansen (1985), this last assumption is often used in macroeconomic dynamic models.

with

$$K_{11} = \frac{1}{\Pi} \left[ s(\mu - s) + \varepsilon_u \frac{1-s}{\mu-s} (\sigma\mu - s)(\mu(1-\beta s) - s(1-\beta)) \right] \quad (39)$$

$$K_{12} = -\varepsilon_u \frac{\sigma}{\Pi} [\mu(1-\beta s) - s(1-\beta)] \quad (40)$$

$$K_{21} = \mu \frac{1-s}{\mu-s} \left( \frac{s}{\mu\sigma} - 1 \right) \quad (41)$$

$$K_{22} = 1 \quad (42)$$

and

$$\Pi = \mu(1-s + \beta s^2) - s(1-s + \beta s) - \varepsilon_u(1-s)(\mu(1-\beta s) - s(1-\beta)) \quad (43)$$

The characteristic polynomial associated to the Jacobian matrix can be written  $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$ , where  $T$  and  $D$  are respectively the trace and the determinant of this Jacobian matrix. Their expressions are given by the two following equations:

$$T = 1 + \frac{1}{\Pi} \left[ s(\mu - s) + \varepsilon_u \frac{1-s}{\mu-s} (\sigma\mu - s)(\mu(1-\beta s) - s(1-\beta)) \right] \quad (44)$$

$$D = \frac{1}{\Pi} s(\mu - s) \quad (45)$$

We can first notice that  $\mu > \tilde{\mu}$  does not only imply  $\mu > s$  but also  $\mu > s(1-s + \beta s) / (1-s + \beta s^2)$ . As a consequence,  $\Pi > 0 (< 0)$  when  $\varepsilon_u < \varepsilon_u^\infty (> \varepsilon_u^\infty)$ , where:

$$\varepsilon_u^\infty \equiv \frac{\mu(1-s + \beta s^2) - s(1-s + \beta s)}{(1-s)(\mu(1-\beta s) - s(1-\beta))} \quad (46)$$

Taking into account this first result, we now study the sign of  $P(1) = 1 - T + D$  and  $P(-1) = 1 + T + D$ .

**Lemma 1** *Note  $\sigma_T \equiv s/\mu$ .  $P(1) = 0$  if  $\sigma = \sigma_T$  and  $P(1) > 0$  if  $\varepsilon_u < \varepsilon_u^\infty$  and  $\sigma < \sigma_T$  or  $\varepsilon_u > \varepsilon_u^\infty$  and  $\sigma > \sigma_T$ . Otherwise,  $P(1)$  is strictly negative.*

**Proof.** Using equations (44) and (45), we first determine  $P(1)$ . We have:

$$P(1) = 1 - T + D = \frac{1}{\Pi} \varepsilon_u \frac{1-s}{\mu-s} (s - \sigma\mu) [\mu(1 - \beta s) - s(1 - \beta)] \quad (47)$$

The lemma directly comes from the observation that the sign of  $P(1)$  is determined by the sign of  $\Pi$  and the sign of  $s - \sigma\mu$ , and evidently  $1 - T + D = 0$  when  $\sigma = s/\mu$ . ■

**Lemma 2**  $P(-1) = 0$  if  $\sigma = \sigma_F$  and  $P(-1) > 0$  if  $\varepsilon_u < \varepsilon_u^\infty$  and  $\sigma > \sigma_F$  or  $\varepsilon_u > \varepsilon_u^\infty$  and  $\sigma < \sigma_F$ . Otherwise,  $P(-1)$  is strictly negative.<sup>15</sup>

**Proof.** Using (44) and (45),  $P(-1) = 1 + T + D$  can be written:

$$\begin{aligned} P(-1) = & \frac{1}{\Pi} [\sigma\mu\varepsilon_u \frac{1-s}{\mu-s} (\mu(1 - \beta s) - s(1 - \beta)) \\ & + 2(\mu(1 + \beta s^2) - s(1 + \beta s)) \\ & - \varepsilon_u(1-s)(\mu(1 - \beta s) - s(1 - \beta)) \left( \frac{s}{\mu-s} + 2 \right)] \end{aligned} \quad (48)$$

We first remark that  $\mu > \tilde{\mu}$  implies  $\mu > s(1 + \beta s) / (1 + \beta s^2)$ . Furthermore, the numerator of  $P(-1)$  increases with respect to  $\sigma$  and is equal to 0 for  $\sigma = \sigma_F$ , where:

$$\begin{aligned} \sigma_F \equiv & \\ & \frac{\varepsilon_u(1-s)(\mu(1 - \beta s) - s(1 - \beta)) \left( \frac{s}{\mu-s} + 2 \right) - 2(\mu(1 + \beta s^2) - s(1 + \beta s))}{\mu\varepsilon_u \frac{1-s}{\mu-s} (\mu(1 - \beta s) - s(1 - \beta))} \end{aligned} \quad (49)$$

We conclude the proof by noting that the sign of  $P(-1)$  is determined by the sign of  $\Pi$  and  $\sigma - \sigma_F$ .<sup>16</sup> ■

In what follows, it will be useful to know the conditions such that  $P(1)$  and  $P(-1)$  are both strictly positive. In this way, we remark that  $\sigma_F$  is strictly greater (smaller) than  $\sigma_T$  if  $\varepsilon_u > \varepsilon_u^0$  ( $\varepsilon_u < \varepsilon_u^0$ ), where  $\varepsilon_u^0$  is defined by the following expression:

<sup>15</sup> $\sigma_F$  is given in the proof.

<sup>16</sup>We can notice that  $\sigma_F$  becomes negative if  $\varepsilon_u$  is weak enough, i.e.  $\varepsilon_u < \frac{2(\mu-s)}{2\mu-s} \frac{\mu(1+\beta s^2)-s(1+\beta s)}{(1-s)[\mu(1-\beta s)-s(1-\beta)]}$ .

$$\varepsilon_u^0 \equiv \frac{\mu(1 + \beta s^2) - s(1 + \beta s)}{(1 - s)[\mu(1 - \beta s) - s(1 - \beta)]} > \varepsilon_u^\infty \quad (50)$$

Finally, we analyze the value of the determinant  $D$  in function of two parameters,  $\varepsilon_u$  and  $\mu$ . The results are summarized in the next lemma.

**Lemma 3** *Taken as given  $\mu^H$  and  $\varepsilon_u^H$ ,<sup>17</sup> we have:*

- (i) *if  $\mu \in (\tilde{\mu}, \mu^H]$ ,  $D > 1$  for  $0 < \varepsilon_u < \varepsilon_u^\infty$  and  $D < 0$  for  $\varepsilon_u > \varepsilon_u^\infty$ ;*
- (ii) *if  $\mu > \mu^H$ ,  $0 < D < 1$  for  $0 < \varepsilon_u < \varepsilon_u^H$ ,  $D = 1$  for  $\varepsilon_u = \varepsilon_u^H$ ,  $D > 1$  for  $\varepsilon_u^H < \varepsilon_u < \varepsilon_u^\infty$  and  $D < 0$  for  $\varepsilon_u > \varepsilon_u^\infty$ .*

**Proof.** Using equations (43) and (45), we first notice that the determinant  $D$  increases with respect to  $\varepsilon_u$ . In particular,  $D > 0$  when  $\varepsilon_u < \varepsilon_u^\infty$  and  $D < 0$  when  $\varepsilon_u > \varepsilon_u^\infty$ . Moreover when  $\varepsilon_u = 0$ , the determinant is given by  $D = s(\mu - s) / [\mu(1 - s + \beta s^2) - s(1 - s + \beta s)]$ . This expression can never be strictly smaller than 1 if  $\mu \leq \mu^H \equiv s(1 - 2s + \beta s) / (1 - 2s + \beta s^2) \in (\tilde{\mu}, 1)$ , whereas it is strictly smaller than 1 if  $\mu > \mu^H$ . In this last case,  $D = 1$  if  $\varepsilon_u = \varepsilon_u^H$ , where  $\varepsilon_u^H$  is defined by:

$$\varepsilon_u^H \equiv \frac{\mu(1 - 2s + \beta s^2) - s(1 - 2s + \beta s)}{(1 - s)[\mu(1 - \beta s) - s(1 - \beta)]} \quad (51)$$

It concludes the proof of the lemma. ■

Since the dynamics are determined by a two dimensional system with one predetermined variable, capital, the steady state is locally indeterminate when  $P(1) = 1 - T + D > 0$ ,  $P(-1) = 1 + T + D > 0$  and  $D < 1$ . Moreover, when a parameter varies, the local stability of the steady state can change and local bifurcations can occur. Indeed, when one crosses  $P(1) = 1 - T + D = 0$ , a transcritical bifurcation generically occurs, i.e. there is an exchange of stability between two steady states. When one crosses  $P(-1) = 1 + T + D = 0$ , a flip bifurcation generically occurs, i.e. a cycle of period two appears around the steady state. Finally, when  $P(1) = 1 - T + D > 0$ ,  $P(-1) = 1 + T + D > 0$  and one crosses  $D = 1$ , a Hopf bifurcation generically occurs, i.e. an invariant closed curve appears around the steady state.

Using these remarks and Lemma 1-3, we now establish the main results concerning the emergence of endogenous fluctuations:

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<sup>17</sup>The values of  $\mu^H$  and  $\varepsilon_u^H$  are given in the proof.

**Proposition 4** Consider that  $\mu > \tilde{\mu}$ , Assumptions 1-3 and Proposition 1 are verified. Then, the steady state  $(a, k) = (1, 1)$  is locally indeterminate if one of the following conditions is satisfied:

- (i)  $\mu > \mu^H$ ,  $\varepsilon_u < \varepsilon_u^H$  and  $\sigma_F < \sigma < \sigma_T$ ;
- (ii)  $\varepsilon_u > \varepsilon_u^0$  and  $\sigma_T < \sigma < \sigma_F$ .

Moreover, a Hopf bifurcation generically occurs for  $\mu > \mu^H$ ,  $\sigma_F < \sigma < \sigma_T$  and  $\varepsilon_u = \varepsilon_u^H$ , a flip bifurcation generically occurs for  $\sigma = \sigma_F$ , and a transcritical bifurcation generically occurs for  $\sigma = \sigma_T$ .

The results obtained in this proposition enlighten that the existence of indeterminacy and endogenous cycles depends on two important parameters: the elasticity of substitution between production factors  $\sigma$  and the elasticity of intertemporal substitution in consumption  $1/\varepsilon_u$ .

In configuration (i), the occurrence of endogenous fluctuations requires a high enough elasticity of intertemporal substitution in consumption and an elasticity of factor substitution smaller than  $s/\mu$ .

On the contrary in configuration (ii), local indeterminacy occurs if the elasticity of intertemporal substitution is not too high and for a range of elasticities of capital-labor substitution which is close to 1 or which can even contain the unit case (Cobb-Douglas technology). Indeed, the lower bound of this range  $\sigma_T$  is smaller than one. Moreover, one can observe that the upper bound  $\sigma_F$  increases with respect to  $\varepsilon_u$  and tends to  $2 - s/\mu$  when  $\varepsilon_u$  goes to  $+\infty$ . In particular,  $\sigma_F$  is greater than 1 if:

$$\varepsilon_u > \frac{2(\mu(1 + \beta s^2) - s(1 + \beta s))}{(1 - s)(\mu(1 - \beta s) - s(1 - \beta))} \quad (52)$$

As we have already noticed, in the limit case where  $\mu = 1$ , this model describes the same dynamics than the overlapping generations model studied by Cazzavillan (2001), when he considers constant returns to scale, or Reichlin (1986). These contributions only analyze the case where  $\varepsilon_u < 1$ .<sup>18</sup> It excludes configuration (ii) since  $\varepsilon_u^0$  is equal to  $1/(1 - s)$  when  $\mu = 1$ , but configuration (i) can apply. Since  $\mu^H$  is strictly smaller than one and  $\sigma_F$  is strictly negative, indeterminacy occurs if  $\sigma < s$  and  $\varepsilon_u < (1 - 2s)/(1 - s)$ .

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<sup>18</sup>In overlapping generations models, the elasticity of labor supply is evaluated, at the steady state, taken into account that consumption is equal to  $c = Rwl$ . Then, the labor supply increases (decreases) with respect to the real wage if  $\varepsilon_u < 1$  ( $> 1$ ). It is why one often only considers the case where  $\varepsilon_u < 1$ , which also means that future consumption and leisure are gross substitutes. In economies with infinitely lived agents, the elasticity of labor supply is rather determined considering the consumption as given.

The results established in Proposition 4 show that for small values of  $\varepsilon_u$ , indeterminacy is compatible with elasticities of capital-labor substitution closer to 1 when the share of consumption expenditures constrained by capital income  $\mu$  decreases from 1. Furthermore, the configuration (ii) of Proposition 4 proves that when the elasticity of intertemporal substitution in consumption is weak enough, indeterminacy can occur in a finance constrained economy for a high substitution between capital and labor without introducing an additional asset like money (Bosi, Dufourt, and Magris (2002), Bosi and Magris (2003)), heterogeneous households (Barinci (2001)), imperfect competition or increasing returns (Barinci and Chéron (2001), Cazzavillan, Lloyd-Braga, and Pintus (1998)).

One can also notice that indeterminacy occurs for elasticities of capital-labor substitution within a range in accordance with empirical studies. Indeed, Duffy and Papageorgiou (2000) provide estimates such that this elasticity can take values greater or smaller than one, as soon as they are not too far from the unit case. Moreover, our conclusions are not contradicted by empirical analysis on the elasticity of intertemporal substitution. Indeed, if this elasticity is often assumed to be equal to one in models with infinitely lived agents, recent empirical results rather find smaller values, which provides a support for case (ii) of Proposition 4.<sup>19</sup> Hence, this paper shows that endogenous fluctuations can occur in the one sector growth model under quite not too restrictive conditions, and without considering externalities and increasing returns (Benett and Farmer (2000), Benhabib and Farmer (1994), Farmer and Guo (1994), Harrison and Weder (2002), Hintermaier (2003), Pintus (2003a, 2003b)), imperfect competition (Gali (1994), Woodford (1991)) or counter-cyclical tax rates (Guo and Lansing (1998), Schmitt-Grohé and Uribe (1997)), but rather a finance constraint.

We will now give a more intuitive interpretation of the occurrence of local indeterminacy. Recall that the dynamics are defined by the two following equations:

$$\frac{v}{w_t} = \beta r_{t+1} \left[ \left(1 - \frac{1}{\mu}\right) \frac{v}{w_{t+1}} + \frac{1}{\mu} u' \left( \frac{r_{t+1} k_t}{\mu B} \right) \right] \quad (53)$$

$$k_t = \left(1 - \frac{1}{\mu}\right) r_t k_{t-1} + w_t l_t \quad (54)$$

Consider first that  $\varepsilon_u$  is small enough. If  $\mu = 1$ , an increase of the future expected real interest rate will decrease the real wage at the current period. Since the labor demand is negatively sloped, it will raise labor. Then,

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<sup>19</sup>See Campbell (1999) and Kocherlakota (1996).

the labor income will decrease only if  $1 - s/\sigma < 0$ . In such a case, it will lead to a decrease of savings, which will imply an increase of the future real interest rate. Hence, under such conditions, expectations are self-fulfilling. When  $\mu < 1$ , the mechanism for indeterminacy is quite similar. However, indeterminacy can occur for greater elasticities of capital-labor substitution because the increase of labor will raise capital income which has an additional negative effect on savings since  $1 - 1/\mu < 0$ .

Now, consider that  $\varepsilon_u$  is sufficiently high. As in the previous explanation, we begin by assuming  $\mu = 1$ . In this case, an increase of the future expected real interest rate will increase the current real wage and then decrease the labor, since the labor demand has a negative slope. As a consequence, the labor income will reduce if  $1 - s/\sigma > 0$ . It will imply a decrease of savings and hence an increase of the future real interest rate. Then, expectations are self-fulfilling. When  $\mu < 1$ , as before there is an additional effect due to the capital income. Indeed, a smaller labor supply decreases the current real interest rate and has a positive impact on savings because  $1 - 1/\mu < 0$ . It explains that when  $\mu < 1$ , local indeterminacy requires a higher lower bound for the range of elasticities of capital-labor substitution.

## 5 Concluding remarks

In this paper, we consider a one-sector model with infinitely lived agents and constant returns to scale. Moreover, we assume that consumption is partially constrained by capital income. This market imperfection is the only departure from the optimal growth model.

In this framework, we notably show that when a steady state exists, it is not necessarily unique. Furthermore, analyzing consumer welfare at a steady state, we establish a quite non intuitive result. Indeed, we prove that consumer welfare does not always decrease with respect to the degree of market imperfection.

Moreover, local indeterminacy and endogenous cycles can occur in the neighborhood of a steady state. It depends on two parameters: the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution. Indeed, two phenomena are important for the emergence of such fluctuations: on one hand, the effect of a variation of the future expected real interest rate on labor and on the other hand, the influence of a variation of labor on savings. We show that endogenous fluctuations can appear when these two elasticities take values in accordance with empirical studies and our results does not depend on the introduction of an additional market imperfection, like externalities or imperfect competition.

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