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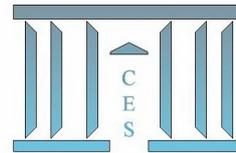


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A Generalized FKKM Theorem and Variational Inequality

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Abstract

We present a generalized FKKM Theorem and its application to the existence of solution for the variational inequalities using a generalized coercivity type condition for correspondences defined in L-space.

Key words and phrases: L-structures, L-spaces, L-KKM correspondences, L-coercing family and variational inequality.

Classification-JEL: C02, C69, C72.

The purpose of this article is to give a generalization of FKKM Theorem [KKM] and its application in variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan [F1] [F2], Dugundji and Granas [DG], Ding and Tan [DT] and Yen [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli [BC1], [BC2], in convex spaces in the sense of Lassonde [L] and in L-spaces by Chebbi, Gourdel and Hammami [CGH], [GH].

In this paper, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let A be a subset of a vector space X . We denote by $\langle A \rangle$ the family of all nonempty finite subsets of A and $\text{conv}A$ the convex hull of A . Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters $F, G, Q, S, \Gamma, \dots$, and the single valued functions will be represented by small letters. We denote by $\text{graph}F$ the graph of the correspondence F . If X and Y are two topological spaces, $\zeta(X, Y)$ denotes the set of all continuous functions from X to Y .

If n is any integer, Δ_n denotes the unit-simplex of \mathbb{R}^{n+1} and for every $J \subset \{0, 1, \dots, n\}$, Δ_J denotes the face of Δ_n corresponding to J . Let X be

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a topological space. An *L-structure* (also called *L-convexity*) on X is given by a correspondence $\Gamma : \langle X \rangle \rightarrow X$ with nonempty valued such that for every $A = \{x_0, \dots, x_n\} \in \langle X \rangle$, there exists a continuous function $f^A : \Delta_n \rightarrow \Gamma(A)$ such that for all $J \subset \{0, \dots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$. Such a pair (X, Γ) is called an *L-space*. A subset $C \subset X$ is said to be *L-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$. A subset $P \subset X$ is said to be *L-quasi-compact* if for every $A \in \langle X \rangle$, there exists a quasi-compact L-convex set D such that $A \cup P \subset D$. Clearly, if C is an L-convex subset of an L-space (X, Γ) , then the pair $(C, \Gamma|_{\langle C \rangle})$ is an L-space.

1 A Generalized FKKM Theorem

In this section, we first remind some known definitions of L-KKM correspondences and L-coercing family quoted in [CGH], then we give a generalized FKKM Theorem and we deduce a more adapted theorem to study the variational inequality.

Definition 1.1 *Let (X, Γ) be an L-space and $Z \subset X$ an arbitrary subset. A correspondence $F : Z \rightarrow X$ is called L-KKM if and only if:*

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

Definition 1.2 *Let Z be an arbitrary set of an L-space (X, Γ) , Y a topological space and $s \in \zeta(X, Y)$. A family $\{(C_a, K)\}_{a \in X}$ is said to be L-coercing for a correspondence $F : Z \rightarrow Y$ with respect to s if and only if:*

- (i) K is a quasi-compact subset of Y ,
- (ii) for each $A \in \langle Z \rangle$, there exists a quasi-compact L-convex set D^A in X containing A such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

$$(iii) \left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

Definition 1.3 *If X is a topological space, a subset B of X is called strongly compactly closed (open respectively) if for every quasi-compact subset K of X , $B \cap K$ is closed (open, respectively) in K .*

We remind the following result given in [CGH], which is an extension of a lemma in [F1] to L-spaces.

Lemma 1.1 *Let (X, Γ) be an L-space, Z a nonempty subset of X and $F : Z \rightarrow X$ an L-KKM correspondence with strongly compactly closed values. Suppose that for some $z \in Z$, the correspondence $F(z)$ is quasi-compact, then $\bigcap_{x \in Z} F(x) \neq \emptyset$.*

Proof: see [CGH].

The main result of this paper is the following generalized FKKM Theorem (see for example Theorem 4 in [F2] and Theorem 1 in [CGH]):

Theorem 1.1 *Let Z be an arbitrary set in the L-space (X, Γ) , Y an arbitrary topological space and $F, G : Z \rightarrow Y$ two correspondences such that:*

- (a) *for every $x \in Z$, $F(x)$ is strongly compactly closed,*
- (b) *for every $x \in Z$, $G(x) \subset F(x)$,*
- (c) *there is a function $s \in \zeta(X, Y)$ satisfying :*

1. *the correspondence $R : Z \rightarrow X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,*
2. *there exists an L-coercing family $\{(C_a, K)\}_{a \in X}$ for G with respect to s ,*
3. *for each quasi-compact L-convex set C in X :*

$$\bigcap_{x \in C \cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x \in C \cap Z} F(x) \cap s(C) \neq \emptyset.$$

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$ more precisely $K \cap \left(\bigcap_{x \in Z} F(x) \right) \neq \emptyset$.

Proof: The correspondence F has strongly compactly closed values, then in order to prove that:

$$K \cap \left(\bigcap_{x \in Z} F(x) \right) \neq \emptyset,$$

it suffices to prove that for each finite subset A of Z , $\left(\bigcap_{x \in A} F(x)\right) \cap K \neq \emptyset$.

Let $A \in \langle Z \rangle$, by condition (ii) of Definition 1.2, there exists a quasi-compact L-convex set D^A containing A such that for all $y \in D^A$, $C_y \cap Z \subset D^A \cap Z$. Consider now the correspondence $R^A : D^A \cap Z \rightarrow D^A$ defined by $R^A(x) = R(x) \cap D^A$. By Hypothesis (c.1) and the L-convexity of D^A , it is immediate that the correspondence R^A is L-KKM. Next, by the continuity of s , $F(x) \cap s(D^A)$ is closed in $s(D^A)$ then $R^A(x) = s_0^{-1}(F(x) \cap s(D^A))$, where s_0 is the restriction of s to D^A , is closed in D^A and consequently $R^A(x)$ is quasicompact. Since $(D^A, \Gamma_{\langle D^A \rangle})$ is also an L-space, we deduce by Lemma 1.1 that $\bigcap_{x \in D^A} R^A(x) \neq \emptyset$, then $\bigcap_{x \in D^A \cap Z} R^A(x) \neq \emptyset$. Since for all $x \in D^A \cap Z$,

$s(R^A(x)) \subset F(x) \cap s(D^A)$, we have: $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \neq \emptyset$ then by (c.3), $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \neq \emptyset$. To finish the proof, we will show that:

$$\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x) \cap K.$$

Indeed, it is clear by (b) that $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x)$, then

it only remains to show that: $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset K$. Let $y \in$

$\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A))$, then $y \in s(D^A)$ which implies that there exists

$z \in s^{-1}(y) \cap D^A$. By condition (ii) of Definition 1.2, $C_z \cap Z \subset D^A \cap Z$, it follows that $y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} G(x)$. Hence, by hypothesis (c.2), $y \in K$

and the theorem is proved. \blacksquare

Remark 1.1 (1) The main result of [CGH] (Theorem 1) becomes an immediate corollary of Theorem 1.1: it suffices to take $F = G$.

(2) In view of our approach, it is possible to state a weakened version of Theorem 1 in [CGH] by replacing the coercivity on F by a coercivity on G together with condition (c.3) of our Theorem 1.1.

Corollary 1.1 Under the conditions of Theorem 1.1, if we assume in addition that X is a quasi-compact set and s is a surjective function, then we can reinforce the conclusion:

$$\bigcap_{x \in Z} G(x) \neq \emptyset.$$

Proof: All the requirements of Theorem 1.1 are satisfied then $\bigcap_{x \in Z} F(x) \neq \emptyset$. By the definition of L-space, it is clear that X is an L-convex set. In addition, X is a quasi-compact set and $s(X) = Y$, then by assumption (c.3), $\bigcap_{x \in Z} G(x) \neq \emptyset$. ■

Remark 1.2 *It is obvious as in [DG] that if we add the following condition:*

$$\bigcap_{x \in Z} F(x) \neq \emptyset \Leftrightarrow \bigcap_{x \in Z} G(x) \neq \emptyset$$

in Theorem 1.1 then in addition to $\bigcap_{x \in Z} F(x) \neq \emptyset$ we have $\bigcap_{x \in Z} G(x) \neq \emptyset$.

The next theorem is more specially adapted to the study of variational inequality. It can be seen as a corollary of Theorem 1.1 and it is a generalization of Theorem II [L] and Corollary 1.4 [DG].

Theorem 1.2 *Let Z be an arbitrary set in the L-space (X, Γ) , Y an arbitrary topological space and $F, G : Z \rightarrow Y$ two correspondences such that:*

- (a) *for every $x \in Z$, $F(x)$ is strongly compactly closed,*
- (b) *for every $x \in Z$, $G(x) \subset F(x)$,*
- (c) *there is a surjective function $s \in \zeta(X, Y)$ satisfying :*

1. *the correspondence $R : Z \rightarrow X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,*
2. *there exists an L-coercing family $\{(C_a, K)\}_{a \in X}$ for G with respect to s ,*
3. *for each L-convex set C in X :*

$$\bigcap_{x \in C \cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x \in C \cap Z} F(x) \cap s(C) \neq \emptyset.$$

Then $\bigcap_{x \in Z} G(x) \neq \emptyset$.

Proof: It is obvious to see that Assumption (c.3) of Theorem 1.2 imply Assumption (c.3) of Theorem 1.1, then $\bigcap_{x \in Z} F(x) \neq \emptyset$. By the definition of L-space, it is clear that X is an L-convex set and hence, for the particular case where $C = X$, Assumption (c.3) implies that $\bigcap_{x \in Z} G(x) \neq \emptyset$ and the theorem is proved. ■

2 Application to variational inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 1.2.

Let E and P denote two real topological vector space, X a nonempty convex set in E and $\langle \cdot, \cdot \rangle$ a bilinear form on $P \times E$ whose for each fixed $v \in P$, the restriction of $\langle v, \cdot \rangle$ on any quasi-compact subset Q of X is continuous² (the natural example is between a normed topological vector space E and its dual space equipped with the strong topology).

Definition 2.4 *A non empty valued correspondence $T : X \rightarrow P$ is said to be monotone if for each (x, u) and (y, v) in the graph of T , $\langle u - v, x - y \rangle \geq 0$.*

Remark 2.3 *One checks easily that if a correspondence T is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then the following condition used by Lassonde [L] for monotone correspondences is satisfied³:*

For any $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \rightarrow \mathbb{R}$ defined, for all $t \in [0, 1]$, by: $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$ is lower semi-continuous at point $t = 0$, (resp. the function $\tilde{h}_{xy} : [0, 1] \rightarrow \mathbb{R}$ defined by for all $t \in [0, 1]$, $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$ is upper semi-continuous at point $t = 0$).

The following theorem is a general version of one of the basic facts in the theory of variational inequalities (see for example [HS], [DG] and [L]).

Theorem 2.3 *Let $T : X \rightarrow P$ be a non empty monotone correspondence, $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a convex function lower semi-continuous on any quasi-compact subset of X^A . Let us suppose that there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:*

- (a) K is a quasi-compact subset of X ,
- (b) for each $A \in \langle X \rangle$, there exists a quasi-compact convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

²Which is equivalent, if we suppose that for all $x \in Z$, $\varphi_v(x) = \langle v, x \rangle$, to : for every closed subset F of \mathbb{R} , $\varphi^{-1}(F)$ is a strongly compactly closed subset.

³It suffices to consider p equal to the (continuous) linear form $\langle \cdot, y - x \rangle$ in the following definition given by Cornet: a correspondence $F : X \rightarrow P$ is said upper hemi-continuous in a point $x_0 \in X$ in the sense of Cornet if for any continuous linear function p , the function $h : x \rightarrow \sup_{y \in \varphi(x)} p(y)$ (resp. $\tilde{h} : x \rightarrow \inf_{y \in \varphi(x)} p(y)$) is upper semi-continuous (resp. lower semi-continuous) at the point x_0 .

⁴Or equivalently: for every $\alpha \in \mathbb{R}$, $\varphi^{-1}(] - \infty, \alpha])$ is a strongly compactly closed set.

$$(c) \left\{ y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle \text{ for all } x \in C_y \right\} \subset K,$$

(d) for each $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \rightarrow \mathbb{R}$ given for $t \in [0, 1]$ by $h_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$ is upper semi-continuous at $t = 0$.

Then there is a point $y_0 \in X$ such that,

$$\varphi(y_0) \leq \varphi(x) + \sup_{v \in T(y_0)} \langle v, x - y_0 \rangle \quad \forall x \in X.$$

Proof: The proof is similar to the proof of [DG] and [L]. For each $x \in X$, let

$$G(x) = \{y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle\},$$

we have to show that Theorem 1.2 can be applied in order to get $\bigcap_{x \in X} G(x) \neq \emptyset$. Let us now consider the correspondence

$$F(x) = \{y \in X \mid \varphi(x) \geq \varphi(y) + \sup_{u \in T(x)} \langle u, y - x \rangle\}.$$

We will verify that G and F satisfies requirements of Theorem 1.2 (with $Z = X$ and $s =$ the identity function).

(i.1) From the l.s.c assumption of φ and the ‘‘regularity’’ assumption of the bilinear form $\langle u, \cdot \rangle$, it follows that $F(x)$ is strongly compactly closed in X for each $x \in X$.

(i.2) Let us prove that for every $x \in X$, $G(x) \subset F(x)$: Let $y \in G(x)$, then $\varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle$. By the monotonicity of T , we have: for all $u \in T(x)$ and $v \in T(y)$, $\langle u, x - y \rangle \geq \langle v, x - y \rangle$ then

$$\inf_{u \in T(x)} \langle u, x - y \rangle \geq \sup_{v \in T(y)} \langle v, x - y \rangle$$

consequently

$$- \sup_{u \in T(x)} \langle u, y - x \rangle \geq \sup_{v \in T(y)} \langle v, x - y \rangle$$

which implies $\sup_{u \in T(x)} \langle u, y - x \rangle + \varphi(y) \leq \varphi(x)$, i.e. $y \in F(x)$.

(ii.1) We will prove that F is KKM. Let $y \in \text{conv}\{x_1, \dots, x_n\}$, then there exists $\alpha_i \in [0, 1]$ for $i = 1, \dots, n$ such that $y = \sum_{i=1}^n \alpha_i x_i$ and $\sum_{i=1}^n \alpha_i = 1$

1. By the monotonicity of T , for all $u \in T(x_i)$ and $v \in T(y)$, $\langle u, x_i - y \rangle \geq \langle v, x_i - y \rangle$, then

$$\inf_{u \in T(x_i)} \langle u, x_i - y \rangle \geq \langle v, x_i - y \rangle$$

consequently

$$\sup_{u \in T(x_i)} \langle u, y - x_i \rangle \leq -\langle v, x_i - y \rangle$$

which implies $\sum_{i=1}^n \alpha_i \sup_{u \in T(x_i)} \langle u, y - x_i \rangle \leq 0$. It follows from the convexity of φ that $\varphi(y) \leq \sum_{i=1}^n \alpha_i \varphi(x_i)$. The two previous inequalities allows

us to deduce that: $\sum_{i=1}^n \alpha_i \left(\sup_{u \in T(x_i)} \langle u, y - x_i \rangle + \varphi(y) - \varphi(x_i) \right) \leq 0$.

Therefore, there exists $i \in \{1, \dots, n\}$ such that $\sup_{u \in T(x_i)} \langle u, y - x_i \rangle +$

$\varphi(y) \leq \varphi(x_i)$, then $y \in \bigcup_{i=1}^n F(x_i)$ and F is KKM.

(ii.2) The assumptions (a), (b) and (c), mean exactly that $\{(C_x, K)\}_{x \in X}$ is a coercing family of the correspondence G .

(b.3) Let C be any non-empty convex subset of X . Due to the inclusion between F and G , it is enough to show $\bigcap_{x \in C} (F(x) \cap C) \subset \bigcap_{x \in C} (G(x) \cap C)$.

Let $y \in \bigcap_{x \in C} (F(x) \cap C)$. Let us fix z in C and prove that $y \in G(z)$.

Obviously, we may assume $\varphi(z) < +\infty$. Since $y \in F(z)$, this implies that $\varphi(y)$ is also finite. For each $0 < t < 1$, let $z_t = (1-t)y + tz$. Since C is convex, then $z_t \in C$ and recalling that $y \in \bigcap_{x \in C} F(x)$, we can de-

duce that $y \in F(z_t)$ or equivalently, $\sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle + \varphi(y) \leq \varphi(z_t)$,

$\forall t \in]0, 1[$.

Using the convexity of φ , it implies:

$$\sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle \leq t(\varphi(z) - \varphi(y)) \quad \forall t \in]0, 1[$$

By the convexity of the function $y \rightarrow \sup_{u \in T(z_t)} \langle u, y - z_t \rangle$, it follows that

$$0 = \sup_{u_t \in T(z_t)} \langle u_t, z_t - z_t \rangle \leq (1-t) \sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle + t \sup_{u_t \in T(z_t)} \langle u_t, z - z_t \rangle.$$

Consequently

$$0 \leq t(1-t)(\varphi(z) - \varphi(y)) + t \sup_{u_t \in T(z_t)} \langle u_t, z - z_t \rangle,$$

or

$$0 \leq t(1-t) \left(\varphi(z) - \varphi(y) + \sup_{u_t \in T(z_t)} \langle u_t, z - y \rangle \right).$$

Let us first simplify by $t(1-t)$ and let t tend to 0, then from Assumption (d), it follows that,

$$0 \leq \varphi(z) - \varphi(y) + \sup_{v \in T(y)} \langle v, z - y \rangle$$

and the theorem is proved. ■

Remark that together with the monotonicity of the correspondence T , Assumption (c) of Corollary 3.1 in [GH] implies assumption (c) of the previous theorem. Then, Corollary 3.1 of [GH] is an immediate corollary of the previous theorem.

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