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## A Generalized FKKM Theorem and Variational Inequality

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# A Generalized FKKM Theorem and Variational Inequality

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## Abstract

We present a generalized FKKM Theorem and its application to the existence of solution for the variational inequalities using a generalized coercivity type condition for correspondences defined in L-space.

**Key words and phrases:** L-structures, L-spaces, L-KKM correspondences, L-coercing family and variational inequality.

**Classification-JEL:** C02, C69, C72.

The purpose of this article is to give a generalization of FKKM Theorem [KKM] and its application in variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan [F1] [F2], Dugundji and Granas [DG], Ding and Tan [DT] and Yen [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli [BC1], [BC2], in convex spaces in the sense of Lassonde [L] and in L-spaces by Chebbi, Gourdel and Hammami [CGH], [GH].

In this paper, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let  $A$  be a subset of a vector space  $X$ . We denote by  $\langle A \rangle$  the family of all nonempty finite subsets of  $A$  and  $\text{conv}A$  the convex hull of  $A$ . Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters  $F, G, Q, S, \Gamma, \dots$ , and the single valued functions will be represented by small letters. We denote by  $\text{graph}F$  the graph of the correspondence  $F$ . If  $X$  and  $Y$  are two topological spaces,  $\zeta(X, Y)$  denotes the set of all continuous functions from  $X$  to  $Y$ .

If  $n$  is any integer,  $\Delta_n$  denotes the unit-simplex of  $\mathbb{R}^{n+1}$  and for every  $J \subset \{0, 1, \dots, n\}$ ,  $\Delta_J$  denotes the face of  $\Delta_n$  corresponding to  $J$ . Let  $X$  be

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a topological space. An *L-structure* (also called *L-convexity*) on  $X$  is given by a correspondence  $\Gamma : \langle X \rangle \rightarrow X$  with nonempty valued such that for every  $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ , there exists a continuous function  $f^A : \Delta_n \rightarrow \Gamma(A)$  such that for all  $J \subset \{0, \dots, n\}$ ,  $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$ . Such a pair  $(X, \Gamma)$  is called an *L-space*. A subset  $C \subset X$  is said to be *L-convex* if for every  $A \in \langle C \rangle$ ,  $\Gamma(A) \subset C$ . A subset  $P \subset X$  is said to be *L-quasi-compact* if for every  $A \in \langle X \rangle$ , there exists a quasi-compact L-convex set  $D$  such that  $A \cup P \subset D$ . Clearly, if  $C$  is an L-convex subset of an L-space  $(X, \Gamma)$ , then the pair  $(C, \Gamma|_{\langle C \rangle})$  is an L-space.

## 1 A Generalized FKKM Theorem

In this section, we first remind some known definitions of L-KKM correspondences and L-coercing family quoted in [CGH], then we give a generalized FKKM Theorem and we deduce a more adapted theorem to study the variational inequality.

**Definition 1.1** *Let  $(X, \Gamma)$  be an L-space and  $Z \subset X$  an arbitrary subset. A correspondence  $F : Z \rightarrow X$  is called L-KKM if and only if:*

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

**Definition 1.2** *Let  $Z$  be an arbitrary set of an L-space  $(X, \Gamma)$ ,  $Y$  a topological space and  $s \in \zeta(X, Y)$ . A family  $\{(C_a, K)\}_{a \in X}$  is said to be L-coercing for a correspondence  $F : Z \rightarrow Y$  with respect to  $s$  if and only if:*

- (i)  $K$  is a quasi-compact subset of  $Y$ ,
- (ii) for each  $A \in \langle Z \rangle$ , there exists a quasi-compact L-convex set  $D^A$  in  $X$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

$$(iii) \left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

**Definition 1.3** *If  $X$  is a topological space, a subset  $B$  of  $X$  is called strongly compactly closed (open respectively) if for every quasi-compact subset  $K$  of  $X$ ,  $B \cap K$  is closed (open, respectively) in  $K$ .*

We remind the following result given in [CGH], which is an extension of a lemma in [F1] to L-spaces.

**Lemma 1.1** *Let  $(X, \Gamma)$  be an L-space,  $Z$  a nonempty subset of  $X$  and  $F : Z \rightarrow X$  an L-KKM correspondence with strongly compactly closed values. Suppose that for some  $z \in Z$ , the correspondence  $F(z)$  is quasi-compact, then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ .*

Proof: see [CGH].

The main result of this paper is the following generalized FKKM Theorem (see for example Theorem 4 in [F2] and Theorem 1 in [CGH]):

**Theorem 1.1** *Let  $Z$  be an arbitrary set in the L-space  $(X, \Gamma)$ ,  $Y$  an arbitrary topological space and  $F, G : Z \rightarrow Y$  two correspondences such that:*

- (a) *for every  $x \in Z$ ,  $F(x)$  is strongly compactly closed,*
- (b) *for every  $x \in Z$ ,  $G(x) \subset F(x)$ ,*
- (c) *there is a function  $s \in \zeta(X, Y)$  satisfying :*

1. *the correspondence  $R : Z \rightarrow X$  defined by  $R(x) = s^{-1}(F(x))$  is L-KKM,*
2. *there exists an L-coercing family  $\{(C_a, K)\}_{a \in X}$  for  $G$  with respect to  $s$ ,*
3. *for each quasi-compact L-convex set  $C$  in  $X$ :*

$$\bigcap_{x \in C \cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x \in C \cap Z} F(x) \cap s(C) \neq \emptyset.$$

Then  $\bigcap_{x \in Z} F(x) \neq \emptyset$  more precisely  $K \cap \left( \bigcap_{x \in Z} F(x) \right) \neq \emptyset$ .

Proof: The correspondence  $F$  has strongly compactly closed values, then in order to prove that:

$$K \cap \left( \bigcap_{x \in Z} F(x) \right) \neq \emptyset,$$

it suffices to prove that for each finite subset  $A$  of  $Z$ ,  $\left(\bigcap_{x \in A} F(x)\right) \cap K \neq \emptyset$ .

Let  $A \in \langle Z \rangle$ , by condition (ii) of Definition 1.2, there exists a quasi-compact L-convex set  $D^A$  containing  $A$  such that for all  $y \in D^A$ ,  $C_y \cap Z \subset D^A \cap Z$ . Consider now the correspondence  $R^A : D^A \cap Z \rightarrow D^A$  defined by  $R^A(x) = R(x) \cap D^A$ . By Hypothesis (c.1) and the L-convexity of  $D^A$ , it is immediate that the correspondence  $R^A$  is L-KKM. Next, by the continuity of  $s$ ,  $F(x) \cap s(D^A)$  is closed in  $s(D^A)$  then  $R^A(x) = s_0^{-1}(F(x) \cap s(D^A))$ , where  $s_0$  is the restriction of  $s$  to  $D^A$ , is closed in  $D^A$  and consequently  $R^A(x)$  is quasicompact. Since  $(D^A, \Gamma_{\langle D^A \rangle})$  is also an L-space, we deduce by Lemma 1.1 that  $\bigcap_{x \in D^A} R^A(x) \neq \emptyset$ , then  $\bigcap_{x \in D^A \cap Z} R^A(x) \neq \emptyset$ . Since for all  $x \in D^A \cap Z$ ,

$s(R^A(x)) \subset F(x) \cap s(D^A)$ , we have:  $\bigcap_{x \in D^A \cap Z} (F(x) \cap s(D^A)) \neq \emptyset$  then by (c.3),  $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \neq \emptyset$ . To finish the proof, we will show that:

$$\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x) \cap K.$$

Indeed, it is clear by (b) that  $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset \bigcap_{x \in A} F(x)$ , then it only remains to show that:  $\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A)) \subset K$ . Let  $y \in$

$\bigcap_{x \in D^A \cap Z} (G(x) \cap s(D^A))$ , then  $y \in s(D^A)$  which implies that there exists  $z \in s^{-1}(y) \cap D^A$ . By condition (ii) of Definition 1.2,  $C_z \cap Z \subset D^A \cap Z$ , it follows that  $y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} G(x)$ . Hence, by hypothesis (c.2),  $y \in K$  and the theorem is proved.  $\blacksquare$

**Remark 1.1** (1) The main result of [CGH] (Theorem 1) becomes an immediate corollary of Theorem 1.1: it suffices to take  $F = G$ .

(2) In view of our approach, it is possible to state a weakened version of Theorem 1 in [CGH] by replacing the coercivity on  $F$  by a coercivity on  $G$  together with condition (c.3) of our Theorem 1.1.

**Corollary 1.1** Under the conditions of Theorem 1.1, if we assume in addition that  $X$  is a quasi-compact set and  $s$  is a surjective function, then we can reinforce the conclusion:

$$\bigcap_{x \in Z} G(x) \neq \emptyset.$$

Proof: All the requirements of Theorem 1.1 are satisfied then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ . By the definition of L-space, it is clear that  $X$  is an L-convex set. In addition,  $X$  is a quasi-compact set and  $s(X) = Y$ , then by assumption (c.3),  $\bigcap_{x \in Z} G(x) \neq \emptyset$ . ■

**Remark 1.2** *It is obvious as in [DG] that if we add the following condition:*

$$\bigcap_{x \in Z} F(x) \neq \emptyset \Leftrightarrow \bigcap_{x \in Z} G(x) \neq \emptyset$$

*in Theorem 1.1 then in addition to  $\bigcap_{x \in Z} F(x) \neq \emptyset$  we have  $\bigcap_{x \in Z} G(x) \neq \emptyset$ .*

The next theorem is more specially adapted to the study of variational inequality. It can be seen as a corollary of Theorem 1.1 and it is a generalization of Theorem II [L] and Corollary 1.4 [DG].

**Theorem 1.2** *Let  $Z$  be an arbitrary set in the L-space  $(X, \Gamma)$ ,  $Y$  an arbitrary topological space and  $F, G : Z \rightarrow Y$  two correspondences such that:*

- (a) *for every  $x \in Z$ ,  $F(x)$  is strongly compactly closed,*
- (b) *for every  $x \in Z$ ,  $G(x) \subset F(x)$ ,*
- (c) *there is a surjective function  $s \in \zeta(X, Y)$  satisfying :*

1. *the correspondence  $R : Z \rightarrow X$  defined by  $R(x) = s^{-1}(F(x))$  is L-KKM,*
2. *there exists an L-coercing family  $\{(C_a, K)\}_{a \in X}$  for  $G$  with respect to  $s$ ,*
3. *for each L-convex set  $C$  in  $X$ :*

$$\bigcap_{x \in C \cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x \in C \cap Z} F(x) \cap s(C) \neq \emptyset.$$

*Then  $\bigcap_{x \in Z} G(x) \neq \emptyset$ .*

Proof: It is obvious to see that Assumption (c.3) of Theorem 1.2 imply Assumption (c.3) of Theorem 1.1, then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ . By the definition of L-space, it is clear that  $X$  is an L-convex set and hence, for the particular case where  $C = X$ , Assumption (c.3) implies that  $\bigcap_{x \in Z} G(x) \neq \emptyset$  and the theorem is proved. ■

## 2 Application to variational inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 1.2.

Let  $E$  and  $P$  denote two real topological vector space,  $X$  a nonempty convex set in  $E$  and  $\langle \cdot, \cdot \rangle$  a bilinear form on  $P \times E$  whose for each fixed  $v \in P$ , the restriction of  $\langle v, \cdot \rangle$  on any quasi-compact subset  $Q$  of  $X$  is continuous<sup>2</sup> (the natural example is between a normed topological vector space  $E$  and its dual space equipped with the strong topology).

**Definition 2.4** *A non empty valued correspondence  $T : X \rightarrow P$  is said to be monotone if for each  $(x, u)$  and  $(y, v)$  in the graph of  $T$ ,  $\langle u - v, x - y \rangle \geq 0$ .*

**Remark 2.3** *One checks easily that if a correspondence  $T$  is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then the following condition used by Lassonde [L] for monotone correspondences is satisfied<sup>3</sup>:*

*For any  $(x, y) \in X \times X$ , the function  $h_{xy} : [0, 1] \rightarrow \mathbb{R}$  defined, for all  $t \in [0, 1]$ , by:  $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$  is lower semi-continuous at point  $t = 0$ , (resp. the function  $\tilde{h}_{xy} : [0, 1] \rightarrow \mathbb{R}$  defined by for all  $t \in [0, 1]$ ,  $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$  is upper semi-continuous at point  $t = 0$ ).*

The following theorem is a general version of one of the basic facts in the theory of variational inequalities (see for example [HS], [DG] and [L]).

**Theorem 2.3** *Let  $T : X \rightarrow P$  be a non empty monotone correspondence,  $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  a convex function lower semi-continuous on any quasi-compact subset of  $X^A$ . Let us suppose that there exists a family  $\{(C_x, K)\}_{x \in X}$  of pairs of sets satisfying:*

- (a)  $K$  is a quasi-compact subset of  $X$ ,
- (b) for each  $A \in \langle X \rangle$ , there exists a quasi-compact convex set  $D^A$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

<sup>2</sup>Which is equivalent, if we suppose that for all  $x \in Z$ ,  $\varphi_v(x) = \langle v, x \rangle$ , to : for every closed subset  $F$  of  $\mathbb{R}$ ,  $\varphi^{-1}(F)$  is a strongly compactly closed subset.

<sup>3</sup>It suffices to consider  $p$  equal to the (continuous) linear form  $\langle \cdot, y - x \rangle$  in the following definition given by Cornet: a correspondence  $F : X \rightarrow P$  is said upper hemi-continuous in a point  $x_0 \in X$  in the sense of Cornet if for any continuous linear function  $p$ , the function  $h : x \rightarrow \sup_{y \in \varphi(x)} p(y)$  (resp.  $\tilde{h} : x \rightarrow \inf_{y \in \varphi(x)} p(y)$ ) is upper semi-continuous (resp. lower semi-continuous) at the point  $x_0$ .

<sup>4</sup>Or equivalently: for every  $\alpha \in \mathbb{R}$ ,  $\varphi^{-1}(] - \infty, \alpha])$  is a strongly compactly closed set.



$$(c) \left\{ y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle \text{ for all } x \in C_y \right\} \subset K,$$

(d) for each  $(x, y) \in X \times X$ , the function  $h_{xy} : [0, 1] \rightarrow \mathbb{R}$  given for  $t \in [0, 1]$  by  $h_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$  is upper semi-continuous at  $t = 0$ .

Then there is a point  $y_0 \in X$  such that,

$$\varphi(y_0) \leq \varphi(x) + \sup_{v \in T(y_0)} \langle v, x - y_0 \rangle \quad \forall x \in X.$$

Proof: The proof is similar to the proof of [DG] and [L]. For each  $x \in X$ , let

$$G(x) = \{y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle\},$$

we have to show that Theorem 1.2 can be applied in order to get  $\bigcap_{x \in X} G(x) \neq \emptyset$ . Let us now consider the correspondence

$$F(x) = \{y \in X \mid \varphi(x) \geq \varphi(y) + \sup_{u \in T(x)} \langle u, y - x \rangle\}.$$

We will verify that  $G$  and  $F$  satisfies requirements of Theorem 1.2 (with  $Z = X$  and  $s =$  the identity function).

(i.1) From the l.s.c assumption of  $\varphi$  and the ‘‘regularity’’ assumption of the bilinear form  $\langle u, \cdot \rangle$ , it follows that  $F(x)$  is strongly compactly closed in  $X$  for each  $x \in X$ .

(i.2) Let us prove that for every  $x \in X$ ,  $G(x) \subset F(x)$ : Let  $y \in G(x)$ , then  $\varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x - y \rangle$ . By the monotonicity of  $T$ , we have: for all  $u \in T(x)$  and  $v \in T(y)$ ,  $\langle u, x - y \rangle \geq \langle v, x - y \rangle$  then

$$\inf_{u \in T(x)} \langle u, x - y \rangle \geq \sup_{v \in T(y)} \langle v, x - y \rangle$$

consequently

$$- \sup_{u \in T(x)} \langle u, y - x \rangle \geq \sup_{v \in T(y)} \langle v, x - y \rangle$$

which implies  $\sup_{u \in T(x)} \langle u, y - x \rangle + \varphi(y) \leq \varphi(x)$ , i.e.  $y \in F(x)$ .

(ii.1) We will prove that  $F$  is KKM. Let  $y \in \text{conv}\{x_1, \dots, x_n\}$ , then there exists  $\alpha_i \in [0, 1]$  for  $i = 1, \dots, n$  such that  $y = \sum_{i=1}^n \alpha_i x_i$  and  $\sum_{i=1}^n \alpha_i = 1$

1. By the monotonicity of  $T$ , for all  $u \in T(x_i)$  and  $v \in T(y)$ ,  $\langle u, x_i - y \rangle \geq \langle v, x_i - y \rangle$ , then

$$\inf_{u \in T(x_i)} \langle u, x_i - y \rangle \geq \langle v, x_i - y \rangle$$

consequently

$$\sup_{u \in T(x_i)} \langle u, y - x_i \rangle \leq -\langle v, x_i - y \rangle$$

which implies  $\sum_{i=1}^n \alpha_i \sup_{u \in T(x_i)} \langle u, y - x_i \rangle \leq 0$ . It follows from the convexity of  $\varphi$  that  $\varphi(y) \leq \sum_{i=1}^n \alpha_i \varphi(x_i)$ . The two previous inequalities allows

us to deduce that:  $\sum_{i=1}^n \alpha_i \left( \sup_{u \in T(x_i)} \langle u, y - x_i \rangle + \varphi(y) - \varphi(x_i) \right) \leq 0$ .

Therefore, there exists  $i \in \{1, \dots, n\}$  such that  $\sup_{u \in T(x_i)} \langle u, y - x_i \rangle +$

$\varphi(y) \leq \varphi(x_i)$ , then  $y \in \bigcup_{i=1}^n F(x_i)$  and  $F$  is KKM.

(ii.2) The assumptions (a), (b) and (c), mean exactly that  $\{(C_x, K)\}_{x \in X}$  is a coercing family of the correspondence  $G$ .

(b.3) Let  $C$  be any non-empty convex subset of  $X$ . Due to the inclusion between  $F$  and  $G$ , it is enough to show  $\bigcap_{x \in C} (F(x) \cap C) \subset \bigcap_{x \in C} (G(x) \cap C)$ .

Let  $y \in \bigcap_{x \in C} (F(x) \cap C)$ . Let us fix  $z$  in  $C$  and prove that  $y \in G(z)$ .

Obviously, we may assume  $\varphi(z) < +\infty$ . Since  $y \in F(z)$ , this implies that  $\varphi(y)$  is also finite. For each  $0 < t < 1$ , let  $z_t = (1-t)y + tz$ . Since  $C$  is convex, then  $z_t \in C$  and recalling that  $y \in \bigcap_{x \in C} F(x)$ , we can de-

duce that  $y \in F(z_t)$  or equivalently,  $\sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle + \varphi(y) \leq \varphi(z_t)$ ,

$\forall t \in ]0, 1[$ .

Using the convexity of  $\varphi$ , it implies:

$$\sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle \leq t(\varphi(z) - \varphi(y)) \quad \forall t \in ]0, 1[$$

By the convexity of the function  $y \rightarrow \sup_{u \in T(z_t)} \langle u, y - z_t \rangle$ , it follows that

$$0 = \sup_{u_t \in T(z_t)} \langle u_t, z_t - z_t \rangle \leq (1-t) \sup_{u_t \in T(z_t)} \langle u_t, y - z_t \rangle + t \sup_{u_t \in T(z_t)} \langle u_t, z - z_t \rangle.$$

Consequently

$$0 \leq t(1-t) (\varphi(z) - \varphi(y)) + t \sup_{u_t \in T(z_t)} \langle u_t, z - z_t \rangle,$$

or

$$0 \leq t(1-t) \left( \varphi(z) - \varphi(y) + \sup_{u_t \in T(z_t)} \langle u_t, z - y \rangle \right).$$

Let us first simplify by  $t(1-t)$  and let  $t$  tend to 0, then from Assumption (d), it follows that,

$$0 \leq \varphi(z) - \varphi(y) + \sup_{v \in T(y)} \langle v, z - y \rangle$$

and the theorem is proved.  $\blacksquare$

Remark that together with the monotonicity of the correspondence  $T$ , Assumption (c) of Corollary 3.1 in [GH] implies assumption (c) of the previous theorem. Then, Corollary 3.1 of [GH] is an immediate corollary of the previous theorem.

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