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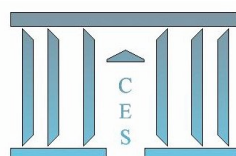
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**Borsuk's Antipodal and Fixed-Point Theorems for
Correspondences Without Convex Values**

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Borsuk's Antipodal and Fixed-Point Theorems for Correspondences Without Convex Values

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Abstract

We present an extension of Borsuk's antipodal theorem (existence of a zero) for antipodally approachable correspondences without convex values. This result is a generalization of Borsuk-Ulam Theorem and has a fixed-point equivalent formulation.

Key words and phrases: Borsuk's antipodal Theorem, balanced set, approachable selection, fixed points.

Classification-JEL: C02, C65, C69.

The aim of this note is to extend Borsuk's Theorem to the antipodally approachable correspondences without convex values. Under suitable assumption, a correspondence with convex values is antipodally approachable. This concept is stable by composition which is not the case for correspondence with convex values. Our result generalizes those that use a correspondence with convex values.

In what follows X (resp Y) is a nonempty subset of \mathbb{R}^n (resp \mathbb{R}^p), capital letters $F : X \rightarrow Y$ denote correspondences while non capital letters $f : X \rightarrow Y$ will denote single-valued functions. We denote by ∂X the boundary of the subset X and $\text{conv} X$ its convex hull. In the whole paper, we will assume that correspondences have nonempty values. Let $\text{Gr} F = \{(x, y) \mid y \in F(x), x \in X\}$ be the graph of F , $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ the unit n -sphere and $B_X(A, r)$ the open ball of X with center A and radius r . Let \mathcal{N}^m be a fundamental basis of open symmetric neighborhood of the

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origin in \mathbb{R}^m . A set $M \subset Y$ is said to be balanced if $\lambda M \subset M$ for every real number λ with $|\lambda| \leq 1$. Suppose that X is symmetric, a correspondence $F : X \rightarrow Y$ is said antipodal-preserving (resp. antipodal⁴) if for all $x \in X$, $F(x) = -F(-x)$ (resp. $F(x) \cap -F(-x) \neq \emptyset$). It is easy to see that if F is antipodal-preserving then F is antipodal. Note that if the correspondence F is antipodal and single-valued then it is antipodal-preserving.

We recall that a correspondence $F : X \rightarrow Y$, X and Y topological spaces, is upper semi-continuous (u.s.c) on X if and only if for any open subset V of Y , the set $\{x \in X : F(x) \subset V\}$ is open in X .

Definition 1 *Let X be a symmetric nonempty subset of \mathbb{R}^n , Y a nonempty subset of \mathbb{R}^p and F a correspondence from X to Y .*

- (1) *For any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, a function $s : X \rightarrow Y$ is said to be a (U, V) -approximative selection of F if for any $x \in X$, $s(x) \in (F[(x + U) \cap X] + V) \cap Y$ or equivalently $\text{Grs} \subset \text{Gr}F + (U \times V)$.*
- (2) *A correspondence $F : X \rightarrow Y$ is said to be approachable if for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists a continuous (U, V) -approximative selection for F . We denote by $\mathcal{A}(X, Y)$ the class of such correspondences and we write $\mathcal{A}(X) = \mathcal{A}(X, X)$.*

We will use the notion of approachable correspondences (see [B]).

Definition 2 (1) *A correspondence $F : X \rightarrow Y$ is said to be antipodally approachable if for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists a continuous antipodal (U, V) -approximative selection for F . We denote by $\mathcal{A}_a(X, Y)$ the class of such correspondences and we write $\mathcal{A}_a(X) = \mathcal{A}_a(X, X)$.*

- (2) *A correspondence $F : X \rightarrow Y$ is said to be antipodally approximable if its restriction to any symmetric compact subset K of X , $F|_K$, is antipodally approachable.*

Remark 1 *Let Z be a symmetric subset of \mathbb{R}^n . Let $F : Z \rightarrow \mathbb{R}^p$ be an antipodal preserving correspondence with convex value. If there exists a continuous selection of F then there exists an antipodal-preserving selection of it. Indeed, it suffices to consider $\tilde{h}(x) = \frac{h(x) - h(-x)}{2}$ where h is the continuous selection.*

⁴It is a generalization of the original single-valued antipodal function to set-valued maps.

- Proposition 1** (1) *If a correspondence $F : X \rightarrow Y$ is antipodally approachable then for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V} : X \rightarrow Y$ defined by $(F((x + U) \cap X) + V) \cap Y$ is antipodal.*
- (2) *If a correspondence $F : X \rightarrow Y$ is u.s.c. with compact values and for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V}$ is antipodal then F is antipodal.*
- (3) *If a correspondence $F : X \rightarrow Y$ is antipodally approachable and u.s.c. with compact values then F is antipodal.*
- (4) *Let Z be a symmetric compact subset of \mathbb{R}^n . If a correspondence $F : Z \rightarrow \mathbb{R}^p$ is antipodal and u.s.c. with convex value then F is antipodally approachable.*

Proof:

- (1) The correspondence F is antipodally approachable then for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists an antipodal-preserving function $s : X \rightarrow Y$ such that $s(x) \in (F((x + U) \cap X) + V) \cap Y$ for all $x \in X$. By the fact that $s(x) = -s(-x)$ for all $x \in X$ and the symmetry of U and V , it follows that $s(x) = -s(-x) \in \{(F((x + U) \cap X) + V) \cap Y\} \cap \{(-F((-x + U) \cap X) + V) \cap Y\}$, then $F^{U,V}$ is antipodal.
- (2) Let $x_0 \in X$, for all $n \geq 1$, there exists $y_n \in \{(F((x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\} \cap \{(-F((-x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\}$. Therefore, there exists $t_{\frac{1}{n}} \in x_0 + B_X(O, \frac{1}{n})$ (resp. $\tilde{t}_{\frac{1}{n}} \in -x_0 + B_X(O, \frac{1}{n})$) and $h_n \in B_Y(O, \frac{1}{n})$ (resp. $\tilde{h}_n \in B_Y(O, \frac{1}{n})$) such that $y_n \in F(t_{\frac{1}{n}}) - h_n$ (resp. $y_n \in -F(\tilde{t}_{\frac{1}{n}}) - \tilde{h}_n$) then $(t_{\frac{1}{n}}, y_n + h_n) \in GrF$ (resp. $(\tilde{t}_{\frac{1}{n}}, -y_n + \tilde{h}_n) \in GrF$). By a compactness argument, we can extract a subsequence $(t_{\varphi(n)}, y_{\varphi(n)} + h_{\varphi(n)})$ which converges to $(x_0, \bar{y}) \in GrF$ when $n \rightarrow +\infty$. Remark that $(\tilde{t}_{\varphi(n)}, -y_{\varphi(n)} + \tilde{h}_{\varphi(n)})$ is a subsequence of $(\tilde{t}_n, -y_n + \tilde{h}_n)$ which converges to $(-x_0, -\bar{y})$ when $n \rightarrow +\infty$. Consequently, $(-x_0, -\bar{y}) \in GrF$ and then $\bar{y} \in F(x_0) \cap -F(-x_0)$.
- (3) If a correspondence $F : X \rightarrow Y$ is antipodally approachable then by (2), for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V}$ is antipodal. Since, the correspondence F is u.s.c. with compact values then by (2), the correspondence F is antipodal.
- (4) For any V in \mathcal{N}^n , we define as in [CH] $F^V : Z + V \rightarrow \mathbb{R}^p$ by $F^V(x) = \text{conv}(\bigcup_{z \in (x+V) \cap Z} F(z))$. Let $W_1 \in \mathcal{N}^n$ and $W_2 \in \mathcal{N}^p$, with no loss of

generality, we may assume that W_2 is nonempty convex. By lemma 1 in [CH], there exists $V \in \mathcal{N}^n$ such that $Gr F^V \subset Gr F + W_1 \times W_2$ and F^V is antipodal on $Z+V$ with open lower sections. Let us now consider $F|_Z^V$ the restriction of F^V to Z , then $F|_Z^V$ is antipodal with convex values and open lower sections. Let us now consider the correspondence $G : Z \rightarrow \mathbb{R}^p$ defined by $G(x) = F|_Z^V(x) \cap -F|_Z^V(-x)$. This correspondence has nonempty convex values and open lower sections, hence with the Theorem of Michael [M], it has a continuous selection. Since the correspondence G is antipodal preserving then, in view of Remark 1, G (hence $F|_Z^V$) has a continuous antipodal selection $f : Z \rightarrow \mathbb{R}^p$. Consequently, $Gr f \subset Gr F + W_1 \times W_2$. ■

Remark 2 (1) *Under the assumptions of Proposition 2.5 in [B], the composition of two antipodally approachable correspondences is antipodally approachable.*

(2) *Remark that the class of correspondences with convex values is not stable by composition.*

We will give two examples in order to show that the u.s.c. assumption (respectively compactness) can't be dropped in the assertion (2) of Proposition 1. Let $\mathcal{C}(O, r)$ be the circle in \mathbb{R}^2 with center $O = (0, 0)$ and radius r .

Example 1 Define $F : \mathcal{C}(O, 1) \rightarrow \mathbb{R}^2$ by $F(x) = x$ if $x \neq (1, 0)$ and $F(1, 0) = (-1, 0)$. The correspondence F which can be viewed as a function has compact values but is not u.s.c. It is clear that for any $U \in \mathcal{N}^2$, $V \in \mathcal{N}^2$, the correspondence $F^{U,V}$ is antipodal (not antipodal-preserving) but F is not antipodal in $(1, 0)$.

Example 2 Define $F : \mathcal{C}(O, 1) \rightarrow \mathbb{R}^2$ by $F(x) = x$ if $x \neq (1, 0)$ and $F(1, 0) = \mathcal{C}(O, 1) \setminus (1, 0)$. The correspondence F is u.s.c with no compact values. In $(1, 0)$, it is clear that F is not antipodal but for any $U \in \mathcal{N}^2$, $V \in \mathcal{N}^2$, the correspondence $F^{U,V}$ is antipodal (not antipodal-preserving).

We recall Borsuk's antipodal theorem:

Theorem 1 Borsuk's antipodal theorem *A single-valued antipodal continuous map $f : S^n \rightarrow \mathbb{R}^n$ has a zero value.*

The following remark will be used to extend the domain in Borsuk's antipodal theorem from S^n to the boundary of any open bounded symmetric balanced subset of \mathbb{R}^{n+1} .

Remark 3 Let K be a symmetric compact subset of \mathbb{R}^{n+1} and $T : S^n \rightarrow K$ a u.s.c. antipodal correspondence with convex values. If we consider T as a correspondence from S^n to \mathbb{R}^{n+1} then, by Proposition 1 assertion (4), the correspondence T is antipodally approachable that is for all $\delta > 0$, $\eta > 0$, there exists a continuous $B_{\mathbb{R}^{n+1}}(O, \delta) \times B_{\mathbb{R}^{n+1}}(O, \eta)$ -antipodal-preserving selection $\tilde{\varphi} : S^n \rightarrow K + B_{\mathbb{R}^{n+1}}(O, \eta)$ of T .

Theorem 2 Let U be an open bounded symmetric balanced subset of \mathbb{R}^{n+1} , then any antipodal single valued continuous function $s : \partial U \rightarrow \mathbb{R}^n$ has a zero value.

Proof: By contradiction, suppose that for all $x \in \partial U$, $s(x) \neq 0$. Let us consider the correspondence $S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ defined by $S(x) = s(x)$ if $x \in \partial U$ and $S(x) = \mathbb{R}^n$ if not. By the Dugundji extension Theorem⁵ (see [DG] p. 163), there exists a continuous function that extends s over \mathbb{R}^{n+1} , then there exists a continuous selection of S . Since the correspondence S is antipodal preserving with convex values then by Remark 1, there exists an antipodal selection \tilde{s} of S such that $\tilde{s}(x) = S(x) = s(x)$ for all $x \in \partial U$. Let $Z = \{x \in \mathbb{R}^{n+1}, \tilde{s}(x) \neq 0\}$, then Z is an open set containing ∂U . Since ∂U is compact, then there exists $\eta > 0$ such that $\partial U + B_{\mathbb{R}^{n+1}}(0, \eta) \subset Z$.

Let $\varphi : S^n \rightarrow \partial U$ defined by $\varphi(x) = \partial U \cap \mathbb{R}_+ x$. It is easy to show that the correspondence φ is antipodal-preserving and u.s.c. with nonempty (compact) values. We will prove that φ has convex values. Let a and b in $\partial U \cap \mathbb{R}_+ x$ with $a \neq b$. Without loss of generality, we may assume that there exists α in $]0, 1[$ such that $a = \alpha b$. For any c in $\text{conv}(a, b)$, there exists $\beta \in [\alpha, 1]$ such that $c = \beta b$, which leads to the existence of λ in $]0, 1]$ such that $a = \lambda c$. Let us first remark that $c \in \overline{U}$ and $c \in \mathbb{R}_+ x$, since \overline{U} is a balanced set and $b \in \overline{U}$. Moreover if $c \in U$ then $a \in U$ which is absurd. Consequently $c \in \partial U$ and then φ has convex values. By Remark 3, there exists a continuous antipodal selection $\tilde{\varphi} : S^n \rightarrow \partial U + B_{\mathbb{R}^{n+1}}(0, \eta)$ of φ . Finally, for all $x \in S^n$, $\tilde{s}(\tilde{\varphi}(x))$ is a continuous antipodal function without zero. This is a contradiction to Borsuk's antipodal Theorem. ■

The main result of this paper is an extension of Theorem 2 to a correspondence. This extension generalizes Borsuk antipodal, Borsuk-Ulam Theorem⁶ and Theorem 4 in [CH].

Theorem 3 Let U be an open bounded symmetric balanced subset of \mathbb{R}^{n+1} and let $F : \partial U \rightarrow \mathbb{R}^n$ be u.s.c antipodally approachable correspondence with nonempty closed values. Then F has a zero on ∂U .

⁵Let X be any metrizable space and $A \subset X$ a closed subset. Then any continuous function $f : A \rightarrow \mathbb{R}$ has an extension $F : X \rightarrow \mathbb{R}$.

⁶see [DG] for the statement of Borsuk-Ulam Theorem and the equivalence with Borsuk antipodal Theorem.

Proof: Let $\mathbf{0}$ denote the zero function from ∂U into $\{0\}$. Suppose that F does not have zero, then $d(Gr(\mathbf{0}), Gr(F)) > 0$. Let $\varepsilon = \frac{1}{3}d(Gr(\mathbf{0}), Gr(F))$ then

$$Gr(\mathbf{0}) \cap [Gr(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^n}(O, \varepsilon)] = \emptyset.$$

The correspondence F is antipodally approachable then for $V = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$ and $W = B_{\mathbb{R}^n}(O, \varepsilon)$, there exists $f^{V,W}$ such that $Gr(f^{V,W}) \subset [Gr(F) + V \times W]$ which imply that $Gr(f^{V,W}) \cap Gr(\mathbf{0}) = \emptyset$. Consequently, $f^{V,W}$ is a continuous antipodal function with zero free, this is a contradiction to Theorem 2. \blacksquare

We will give two examples of u.s.c. antipodal-preserving correspondence $F : \partial B_{\mathbb{R}^3} \rightarrow \mathbb{R}^2$ with nonempty closed values. We assume that $F(x, y, z)$ depend only on z . The following two examples coincide when $z \neq 0$ but in the second example, the correspondence is antipodally approachable which is not the case in the first example.

Let us consider the following spiral ζ :

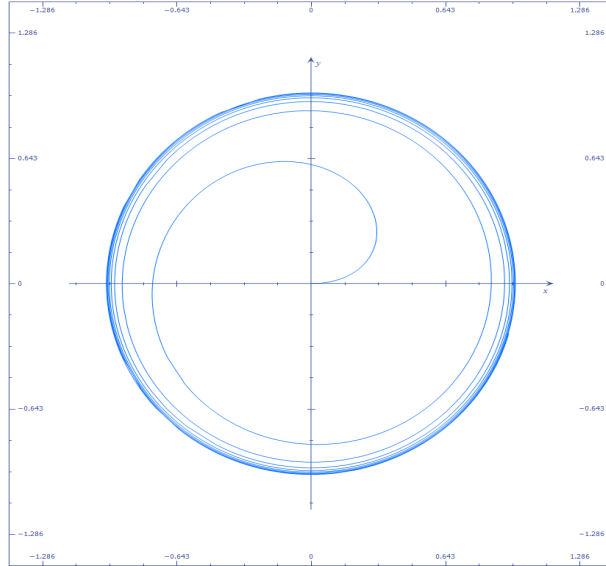
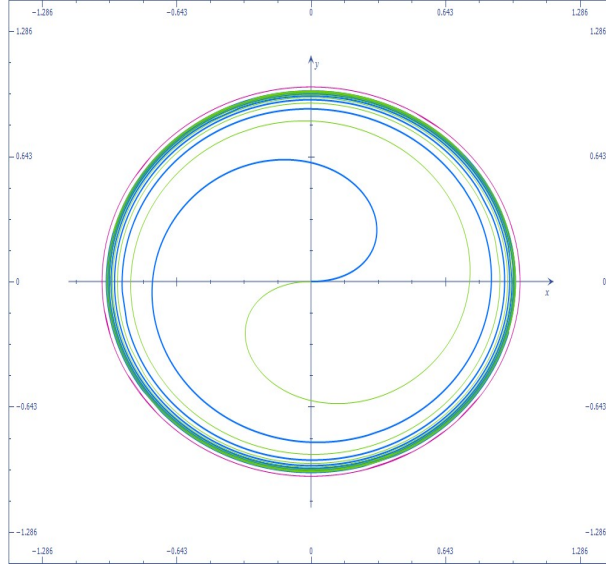


Figure 1: ζ spiral with polar equation $r = \frac{\theta}{1+\theta}$, $\theta \geq 0$.

Example 3 We define the correspondence $F : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$ by:

$$F(x, y, z) = \varphi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1 - z^2}) \cap \zeta & \text{if } z > 0 \\ \mathcal{C}(O, \sqrt{1 - z^2}) \cap -\zeta & \text{if } z < 0 \\ \mathcal{C}(O, 1) & \text{if } z = 0 \end{cases}$$



We deduce that the correspondence F is antipodal-preserving, u.s.c. with closed nonempty non convex values ($\varphi(0)$ is not convex). Note that it is easy to check that φ is a continuous antipodal-preserving function on $[-1, 0[\cup]0, 1]$ and that $\varphi(0)$ is the limit superior in the sense of Painlevé Kuratowski of the family $(C_n)_{n>0}$ where $C_n = \varphi(]0, \frac{1}{n}[)$ ⁷. Let us now prove that F is not antipodally approachable. Suppose by contradiction that the correspondence F is antipodally approachable then for all $\varepsilon > 0$, there exists a continuous antipodal function $f^\varepsilon : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$ such that $Gr f^\varepsilon \subset Gr F + B_{\mathbb{R}^2}(O, \varepsilon) \times B_{\mathbb{R}^2}(O, \varepsilon)$. Let us now fix $\varepsilon \in]0, \frac{1}{2}[$, for each fixed $z \in [-1, 1]$, we define the closed path $\gamma_z : [0, 2\pi] \rightarrow \mathbb{R}^2$ by

$$\gamma_z(t) = f^\varepsilon(\cos(t)\sqrt{1 - z^2}, \sin(t)\sqrt{1 - z^2}, z),$$

⁷The limit superior of a sequence of set $(C_n)_{n>0}$ in the sense of Painlevé Kuratowski [AF] is defined by:

$$\limsup_{n \rightarrow \infty} C_n := \{x \in \mathbb{R}^2 \mid \liminf_{n \rightarrow \infty} d(x, C_n) = 0\} \text{ where } d(x, X) = \inf_{y \in X} d(x, y).$$

we denote by $\tilde{\gamma}_z = \{\gamma_z(t) \mid t \in [0, 2\pi]\}$. For all $0 \leq z < 1 - 2\varepsilon$, the origin $O \notin \tilde{\gamma}_z$ then let $I(\gamma_z, O) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{dz}{z}$ be the index of γ_z with respect to O . For a fixed $z \in [0, 1 - 2\varepsilon]$, $\tilde{\gamma}_z$ is a subset of $F(z) + B_{\mathbb{R}^2}(O, 2\varepsilon)$ which is an open simply connected set⁸ then γ_z is homotopic to one point a_z . Recall that in Theorem 2 p. 60 [CA], if a path $\bar{\gamma}_z$ is homotopic to a path γ_z then, $I(\gamma_z, O) = I(\bar{\gamma}_z, O)$, in particular $I(\gamma_z, O) = I(a_z, O) = 0$. By the continuity of f^ε and the fact that the index $I(\gamma_z, O)$ is a constant when γ_z is continuously deformed, we deduce that $I(\gamma_0, O) = 0$. Since $\gamma_0(t + \pi) = -\gamma_0(t)$, a simple computation of index proves that $I(\gamma_0, O)$ is odd then it can't be equal to 0.

Example 4 Let the correspondence $F : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$ defined by:

$$F(x, y, z) = \psi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1 - z^2}) \cap \zeta & \text{if } z > 0 \\ \mathcal{C}(O, \sqrt{1 - z^2}) \cap -\zeta & \text{if } z < 0 \\ \mathcal{C}(O, 1) \cup ([-1, 1] \times \{0\}) & \text{if } z = 0 \end{cases}$$

Note that $\psi(0)$ is not convex. It is clear that the correspondence F is u.s.c. with nonempty closed values. Let us prove that the correspondence F is antipodally approachable: For every $\varepsilon > 0$, $\delta > 0$, let us consider $\eta = \min(\varepsilon, \delta)$ and

$$Q = \{(x, y) \in \overline{B}_{\mathbb{R}^2}(O, 1) \text{ such that } d((x, y), \psi(0)) < 1 - |\psi(2\eta)|\},$$

then $\psi(\eta) \in Q$. It is easy to see that Q is a path-connected subset of \mathbb{R}^2 and $O \in Q$. Then there exists a path $\gamma : [0, 1] \rightarrow Q$ such that $\gamma(0) = O$ and $\gamma(1) = \psi(\eta)$. We define the function $f^\eta : \partial B_{\mathbb{R}^3}(O, 1) \rightarrow \mathbb{R}^2$ by:

$$f^\eta(x, y, z) = h^\eta(z) = \begin{cases} \psi(z) & \text{if } |z| \geq \eta \\ \gamma(\frac{z}{\eta}) & \text{if } 0 \leq z \leq \eta \\ -\gamma(\frac{-z}{\eta}) & \text{if } -\eta \leq z \leq 0 \end{cases}$$

Then f^η is a continuous, antipodal-preserving function with $\text{Gr } f^\eta \subset \text{Gr } F + B_{\mathbb{R}^2}(O, \varepsilon) \times B_{\mathbb{R}^2}(O, \delta)$. Consequently the correspondence F is antipodally approachable.

We will now focus on the correspondence from \mathbb{R}^{n+1} to \mathbb{R}^{n+1} . The following theorem is a generalization of Theorem 6 in [CH]. Indeed, the correspondence are assumed to be approachable which is more general than a correspondence with convex values (see [CE], Remark 1, Proposition 1 assumption (4) and Remark 2).

⁸An open simply connected set D is a connected set in which every closed path is homotopic to one point in D . (see [CA] p. 61)

Theorem 4 *Let U be an open bounded symmetric balanced subset of \mathbb{R}^{n+1} . Let $F : \bar{U} \rightarrow \mathbb{R}^{n+1}$ be an u.s.c. correspondence with nonempty closed values and approachable on \bar{U} by a function which is antipodal on ∂U . Then F has a zero value and a fixed point on \bar{U} .*

Proof: Let $\mathbf{0}$ denote the zero map from ∂U into $\{0\}$. Suppose that F does not have zero, then $d(Gr(\mathbf{0}), Gr(F)) > 0$. Let $\varepsilon = \frac{1}{3}d(Gr(\mathbf{0}), Gr(F))$, then $Gr(\mathbf{0}) \cap \{Gr(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^{n+1}}(O, \varepsilon)\} = \emptyset$. The correspondence F is antipodally approachable then for $V = W = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$, there exists a continuous (V, W) -approximative selection $f^{V,W}$ such that $Gr(f^{V,W}) \subset Gr(F) + V \times W$ and $f^{V,W}(x) = -f^{V,W}(-x)$ for all $x \in \partial U$. Consequently $Gr(f^{V,W}) \cap Gr(\mathbf{0}) = \emptyset$ then $f^{V,W}$ is a continuous antipodal function on ∂U with zero free. This is a contradiction to Theorem 6 [CH].

It is clear that the correspondence G defined by $G(x) = F(x) - x$ for all $x \in \bar{U}$ have a zero value then the correspondence F have a fixed point. ■

In a topological vector space, it is classical to extend the usual notion of bounded subset of a normed space using the following definition (see for example [K]):

Definition 3 *A subset Q of a topological vector space E is said to be bounded if for each neighbourhood U of 0 there is a $\rho > 0$ with $Q \subset \rho U$.*

Theorem 4 is easily extends to any finite dimensional vector space and this is an intermediate step towards topological vector spaces:

Proposition 2 *Let U be an open bounded symmetric balanced subset in a finite dimensional vector space E . Let $F : \bar{U} \rightarrow E$ be an u.s.c. correspondence with nonempty closed values and approachable on \bar{U} by a selection which is antipodal on ∂U . Then F has a zero value and a fixed point on \bar{U} .*

Proof: Let $\mathcal{B} = \{x_1, \dots, x_{n+1}\}$ be a basis of E , which allows to consider Φ the usual linear homeomorphism between \mathbb{R}^{n+1} and E . If we define $V = \Phi^{-1}(U)$, it is easy to show that it is bounded in the usual sense. Moreover, letting $G = \Phi^{-1} \circ F \circ \Phi$, it is routine to check that (V, G) satisfies the assumptions of Theorem 4, which leads to the conclusion. ■

We now extend our result to infinite dimensional space. Note that in view of Proposition 1 (4), this allows to generalize Theorem 7 of [CH].

Theorem 5 *Let M be a closed bounded symmetric balanced set in a Hausdorff locally convex topological vector space E . Let $F : M \rightarrow E$ be u.s.c. correspondence with nonempty closed values such that the closure of $F(M)$ is compact. Assume that F is approximable on M by a selection which is antipodal on ∂M , then F has at least one fixed point.*

Proof: We will construct this fixed point as a limit of “approximated fixed point”. Let \mathcal{B} denote a closed bounded symmetric convex neighborhood base at 0 in E . Since the closure of $F(M)$ is compact, then for each V in \mathcal{B} , there exists a finite subset S_V of $F(M)$ such that $(y + V) \cap S_V \neq \emptyset$ for each $y \in F(M)$. Let H_{S_V} the vector space spanned by S_V . In the following of this proof we will refer to the topology of H_{S_V} . Define $F_V : M \cap H_{S_V} \rightarrow H_{S_V}$ by $F_V(x) = (F(x) + V) \cap H_{S_V}$, then the correspondence F_V is u.s.c. with nonempty compact values. Note that $M \cap H_{S_V}$ (resp $\partial_{H_{S_V}}(M \cap H_{S_V})$, the boundary of $M \cap H_{S_V}$ with respect to the topology of H_{S_V}) is a compact subset of M (resp ∂M). Since F is approximable on M by a selection which is antipodal on ∂M then F_V is approachable on $M \cap H_{S_V}$ by a selection which is antipodal on $\partial_{H_{S_V}}(M \cap H_{S_V})$. let us remark that $0 \in M \cap H_{S_V}$, consequently either $0 \in \text{int}_{H_{S_V}}(M \cap H_{S_V})$ or $0 \in \partial_{H_{S_V}}(M \cap H_{S_V})$. In the first case, we can apply Proposition 2 with $U = \text{int}_{H_{S_V}}(M \cap H_{S_V})$ and there exists $x_V \in F_V(x_V)$. In the second case, F_V is antipodally approachable on $\partial_{H_{S_V}}(M \cap H_{S_V})$, then there exists an antipodal approximative selection s_V of F_V such that $s_V(x) \in F_V(x+V)+V$, in particular $s_V(0) = 0 \in F_V(0+V)+V$.

In both case, for each $V \in \mathcal{B}$, there exists $x_V \in M$ such that $x_V \in F_V(x_V + V) + V$. Since V is symmetric, there exists $(v, w) \in V^2$ such that $y_V \in F(z_V)$ where $y_V = x_V + v$ and $z_V = x_V + w$. A standard argument based on the compactness of $\overline{F(M)}$, the upper semicontinuity of F and the closedness of its values ends the proof (see for example [BI]). ■

References

- [AC] J.-P. Aubin and A. Cellina, *Differential inclusions*, (1984), Springer-Verlag . Springer.
- [AF] J.-P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhäuser, (1990).
- [B] H. Ben-El-Mechaiekh, *Continuous approximations of multifunctions, fixed points and coincidences*. Proceedings of the Second International Conference on Approximation and Optimization in the Carribean, M. Florenzano et al. Eds., Approximation and Optimization **Vol. 8**, Peter Lang, Frankfurt, (1995), 69-97.
- [BI] H. Ben-El-Mechaiekh and A. Idzik, *A Leray-Schauder Type Theorem For Approximable Maps*, Proceeding Amer. Math. Soc. **122** (1994), 105-109.
- [CA] H. Cartan, *Théorie élémentaire des fonctions analytiques d’une ou plusieurs variables complexes*, (1961), Hermann.

- [CE] A. Cellina, *A theorem on the approximation of compact multi-valued mappings*, Atti Accad. Naz. Lincei, **8** (1969), 149-153.
- [CH] S.Y. Chang, *Borsuk's Antipodal and Fixed-Point Theorems For Set-Valued Maps*, Proc. Amer. Math. Soc. **121** (1994), 937-941.
- [DG] J. Dugundji and A. Granas, *A fixed point theory*, Monograph. Mat., **vol. 61**, PWN, Warsaw, (1982).
- [K] G. Köthe, *Topological Vector Space I*, (1969), Springer-Verlag .
- [M] E. Michael, *Continuous selections I*. Annals of Mathematics. **63** (1956), 361-382.
- [M1] E. Michael, *Continuous selections III*, Ann. of Math. **65** (1957), 375-390.
- [M2] E. Michael, *A survey of continuous selections. in E. Dold and B. Eckman, eds, Set-valued mappings, selections and topological properties of 2^X* , Lecture Notes in Mathematics **n 171**, Springer-Verlag, Berlin, Heidelberg and New York, (1970), 54-58.