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Pascal Gourdel, Hakim Hammami. Applications of generalized Ky Fan's matching theorem in minimax and variational inequality. 2007. halshs-00204627

HAL Id: halshs-00204627

<https://shs.hal.science/halshs-00204627v1>

Submitted on 15 Jan 2008

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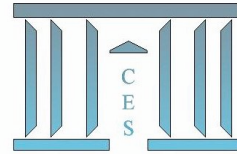


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Applications of a Generalized Ky Fan's Matching Theorem In Minimax and Variational Inequality

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2007.78



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Applications of A Generalized Ky Fan's Matching Theorem In Minimax and Variational Inequality

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Abstract

We present some application of the generalized Ky Fan's Matching Theorem stated by Chebbi, Gourdel and Hammami in minimax and variational inequalities using a generalized coercivity type condition for correspondences defined in L-space.

Key words and phrases: L-structures, L-spaces, L-KKM correspondences, L-coercing family, minimax and variational inequalities.

Classification-JEL: C02, C69, C72.

The purpose of this paper is to give some application of the generalized Ky Fan's Matching theorem stated by Chebbi, Gourdel and Hammami [CGH] to minimax and variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan in [F2] [F3], Ding and Tan in [DT] and Yen in [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli in [BC1] and [BC2] or in convex spaces in the sense of Lassonde in [L].

In this article, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let A be a subset of a vector space X . We denote by $\langle A \rangle$ the family of all nonempty finite subsets of A and $\text{conv}A$ the convex hull of A . Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters $F, G, Q, S, \Gamma, \dots$ and the single valued functions will be represented by small letters. We denote by $\text{graph}F$ the graph of the correspondence F . If X and Y are two topological spaces,

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$\zeta(X, Y)$ denotes the set of all continuous functions from X to Y .

If n is any integer, Δ_n denotes the unit-simplex of \mathbb{R}^{n+1} and for every $J \subset \{0, 1, \dots, n\}$, Δ_J denotes the face of Δ_n corresponding to J . Let X be a topological space. An *L-structure* (also called *L-convexity*) on X is given by a correspondence $\Gamma : \langle X \rangle \rightarrow X$ with nonempty valued such that for every $A = \{x_0, \dots, x_n\} \in \langle X \rangle$, there exists a continuous function $f^A : \Delta_n \rightarrow \Gamma(A)$ such that for all $J \subset \{0, \dots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$. Such a pair (X, Γ) is called an *L-space*. A subset $C \subset X$ is said to be *L-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$. A subset $P \subset X$ is said to be *L-quasi-compact* if for every $A \in \langle X \rangle$, there is a quasi-compact L-convex set D such that $A \cup P \subset D$. Clearly, if C exists an L-convex subset of an L-space (X, Γ) , then the pair $(C, \Gamma|_{\langle C \rangle})$ is an L-space.

1 A Generalized Ky Fan's Matching Theorem

In this section we remind some known definitions of L-KKM correspondences and L-coercing family quoted in [CGH] and we give a more adapted theorem than the mean result of [CGH] in order to generalize Fan's minimax inequality.

Definition 1.1 *Let (X, Γ) be an L-space and $Z \subset X$ an arbitrary subset. A correspondence $F : Z \rightarrow X$ is called L-KKM if and only if:*

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

Definition 1.2 *Let Z be an arbitrary set of an L-space (X, Γ) , Y a topological space and $s \in \zeta(X, Y)$. A family $\{(C_a, K)\}_{a \in X}$ is said to be L-coercing for a correspondence $F : Z \rightarrow Y$ with respect to s if and only if:*

- (i) K is a quasi-compact subset of Y ,
- (ii) for each $A \in \langle Z \rangle$, there exists a quasi-compact L-convex set D^A in X containing A such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

$$(iii) \left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

Definition 1.3 *If X is a topological space, a subset B of X is called strongly compactly closed (open respectively) if for every quasi-compact subset K of X , $B \cap K$ is closed (open, respectively) in K .*

We remind the generalization of Fan's matching theorem of [CGH]:

Theorem 1.1 *Let Z be an arbitrary set in the L-space (X, Γ) , Y an arbitrary topological space and $F : Z \rightarrow Y$ a correspondence. Suppose that there is a function $s \in \zeta(X, Y)$ such that:*

- (a) *for every $x \in Z$, $F(x)$ is strongly compactly closed,*
- (b) *the correspondence $R : Z \rightarrow X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,*
- (c) *there exists an L-coercing family $\{(C_x, K)\}_{x \in X}$ for F with respect to s .*

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$, more precisely $K \cap (\bigcap_{x \in Z} F(x)) \neq \emptyset$.

For any correspondence $F : X \rightarrow Y$, let $F^* : Y \rightarrow X$ the "dual" correspondence of F defined, for all $y \in Y$, by $F^*(y) = X \setminus F^{-1}(y)$, where $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

The following theorem can be seen as a corollary of Theorem 1.1. It will be used in order to generalize Fan's minimax inequality.

Theorem 1.2 *Let (X, Γ) an L-space, Y an arbitrary topological space and $F, G : X \rightarrow Y$ be two correspondences satisfying:*

- (a) *for every $x \in X$, $F(x)$ is strongly compactly closed,*
- (b) *for every $x \in X$, $G(x) \subset F(x)$,*
- (c) *there exists a function $s \in \zeta(X, Y)$ such that:*
 1. *for every $x \in X$, $s(x) \in G(x)$,*
 2. *for every $x \in X$, $S^*(x)$ where S is defined by $S(x) = s^{-1}(G(x))$ is L-convex,*
 3. *There exists an L-coercing family $\{(C_x, K)\}_{x \in X}$ for F with respect to s .*

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$.

Proof : The correspondence F has strongly compactly closed values and admits an L-coercing family then in order to apply Theorem 1.1, it suffices to show that the correspondence $R : X \rightarrow X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM. Let $A \subset \langle X \rangle$ and $z \in \Gamma(A)$, then by (c.1), $s(z) \in G(\Gamma(A))$. One can check that Condition (c.2) can be equivalently rewritten as $S(\Gamma(A)) \subset S(A)$. Moreover, by (c.1), for all $B \subset X$, $B \subset S(B)$, in particular $\Gamma(A) \subset S(\Gamma(A))$. Hence we deduce, $\Gamma(A) \subset S(A)$. By construction, $S \subset R$, which implies that R is L-KKM. ■

Remark 1.1 *If s is the identity function, the proof of the previous theorem becomes a simple application of Lemma 1 of section 4 in [H2].*

2 Some Generalizations of Fan's Minimax Inequality

The object of this section is to get a generalization of minimax inequality due to Fan [F3]. In the sequel of this section, for any subset A of $\overline{\mathbb{R}}$ ³ and every $z \in \mathbb{R}$, $A \leq z$ denotes for all $a \in A$, $a \leq z$ and $A \not\leq z$ means that there exists $a \in A$ such that $a > z$.

Definition 2.4 *Let (X, Γ) be an L-space. A correspondence $Q : X \rightarrow \overline{\mathbb{R}}$ is said to be weakly lower semi-continuous (weakly l.s.c) on X if for each $p \in \mathbb{R}$, the set $\{x \in X \mid Q(x) \leq p\}$ is closed in X ⁴ or equivalently, the set $\{x \in X \mid Q(x) \cap]p, +\infty] \neq \emptyset\}$ is open in X .*

Proposition 2.1 *If Q is a lower semi-continuous correspondence then it is weakly lower semi-continuous.*

Proof : The proof is immediate: if for all $p \in \mathbb{R}$, we consider the closed subset $V = \{y \in \mathbb{R} \mid y \leq p\}$, then by l.s.c. $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$ is a closed set.

Let Q be a l.s.c correspondence, we have to prove that for all $p \in \mathbb{R}$, $\{x \in X \mid Q(x) \leq p\}$ is a closed set. For all $p \in \mathbb{R}$, we consider the closed subset $V = \{y \in \mathbb{R} \mid y \leq p\}$ consequently $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$. By the l.s.c. of Q , the set $\{x \in X \mid Q(x) \cap V^c \neq \emptyset\}$ is open then $\{x \in X \mid Q(x) \cap V^c = \emptyset\}$ is closed and the proposition is proved. ■

³The extended real line, endowed with its usual topology, see for example Rudin [R]

⁴Recall that a correspondence Q is lower semi-continuous, if for each open set $V \subset Y$, the set $\{x \in X : Q(x) \cap V \neq \emptyset\}$ is open in X .

Remark 2.2 Note that the converse implication of Proposition 2.1 is false, since in order to prove that this converse implication is false, we can consider the following counter example: Let the correspondence $Q : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Q(x) = \{1, 2\}$ if $x \neq 0$ and $Q(x) = \{-1, 2\}$ if $x = 0$. It is easy to see that Q is weakly l.s.c but not l.s.c.

We remind a minimax inequality due to Fan [F3].

Theorem 2.3 (Fan) Let E be a topological vector space, let K be a nonempty compact convex set in E , and let f be a real-valued function on $K \times K$. Suppose that

- (a) for every $y \in K$, $f(y, y) \leq 0$,
- (b) for each fixed $y \in K$, the function $x \rightarrow f(x, y)$ is quasi-concave on K ,
- (c) for each fixed $x \in K$, the function $y \rightarrow f(x, y)$ is lower semi-continuous on K .

Then there exists a vector y_0 in K such that $f(x, y_0) \leq 0$ for all $x \in K$.

This theorem can be extended in the following way:

Theorem 2.4 Let (X, Γ) be an L -space and $z \in \mathbb{R}$. Let F and G be two correspondences from $X \times X$ to $\overline{\mathbb{R}}$ satisfying the following condition:

- (a) for every $x \in X$, $G(x, x) \leq z$,
- (b) for each fixed $y \in X$, $\{x \in X \mid G(x, y) \not\leq z\}$ is L -convex,
- (c) for each fixed $x \in X$, $y \rightarrow F(x, y)$ is weakly l.s.c on the quasi-compact subsets of X ,
- (d) for every $(x, y) \in X \times X$, $F(x, y) \subset G(x, y)$,
- (e) there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:
 - (1) K is a quasi-compact subset of X ,
 - (2) for each $A \in \langle X \rangle$, there exists a quasi-compact L -convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (3) $\{y \in X, F(x, y) \leq z \text{ for all } x \in C_y\} \subset K$.

Then there exists $y_0 \in X$ such that

$$F(x, y_0) \leq z \quad \forall x \in X.$$

Proof : The technique of the proof follows from the proof of Theorem 1 of Yen [Y], which is based on Fan's lemma [F1][F3], and Theorem 1.2. For each $x \in X$, let $\tilde{F}(x) = \{y \in X : F(x, y) \leq z\}$ and $\tilde{G}(x) = \{y \in X : G(x, y) \leq z\}$. Then by (c), the correspondence \tilde{F} has strongly compactly closed values. By (b), the set $\tilde{G}^*(y) = \{x \in X : G(x, y) \not\leq z\}$ is an L-convex subset of X . By (d), for each $x \in X$, $\tilde{G}(x) \subset \tilde{F}(x)$. Remark that, by (a), for each $x \in X$, $x \in \tilde{G}(x)$ and $\{(C_x, K)\}_{x \in X}$ is an L-coercing family of \tilde{F} . Then all the requirements of Theorem 1.2 with s the identity function are satisfied, hence $\bigcap_{x \in X} \tilde{F}(x) \neq \emptyset$ and the theorem is proved. \blacksquare

Remark 2.3 *If we consider the particular case where the correspondence g is a real-valued function in the previous theorem, we can deduce that condition (b) is implied by the classical quasi-concavity of the function $x \rightarrow G(x, y)$ for each fixed $y \in X$.*

Remark 2.4 *In view of Remark 2.3, it is easy to see how we can deduce Theorem 2.3 from the previous theorem, it suffices to apply Theorem 2.4 to the correspondences $F = G = f$, $X = K$ which is a nonempty compact convex set in a topological vector space and $z = 0$.*

In the next result, for sake of simplicity, we will focus on the particular case when $F = G$ (but not any more assumed to be a function), and we will weaken conditions (a) and (b) of Theorem 2.4.

Proposition 2.2 *Let (X, Γ) be an L-space, $z \in \mathbb{R}$ and $F : X \times X \rightarrow \overline{\mathbb{R}}$ a correspondence satisfying the following condition:*

- (a) *for each finite subset A of X and for each $y \in \Gamma(A)$, there exists $x_0 \in A$ such that $F(x_0, y) \leq z$,*
- (b) *for each fixed $x \in X$, $y \rightarrow F(x, y)$ is weakly l.s.c on quasi-compact subsets of X ,*
- (c) *there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:*
 - (1) *K is a quasi-compact subset of X ,*
 - (2) *for each $A \in \langle X \rangle$, there exists a quasi-compact L-convex set D^A containing A such that:*

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (3) *$\{y \in X, F(x, y) \leq z \text{ for all } x \in C_y\} \subset K$.*

Then, there exists $y_0 \in X$ such that $F(x, y_0) \leq z$ for all $x \in X$.

Proof : This proof mimics the proof of Fan Inequality: consider the correspondence $S_z : X \rightarrow X$ given by $S_z(x) = \{y \in X \mid F(x, y) \not\leq z\}$ and assume (arguing by contradiction) that for each $y \in X$ there exists $x \in X$ such that $F(x, y) \not\leq z$. Then for each $y \in X$, $S_z^{-1}(y)$ is nonempty. For each fixed $x \in X$, $y \rightarrow F(x, y)$ is weakly l.s.c. on the quasi-compact subsets of X then for each fixed $x \in X$, $S_z(x) = \{y \in X \mid F(x, y) \not\leq z\}$ is strongly compactly open in X . Consider the correspondence $\tilde{F}_z : X \rightarrow X$ given by $\tilde{F}_z(x) = X \setminus S_z(x)$ for $x \in X$. Then \tilde{F}_z is strongly compactly closed in X . It follows from (c) that $\{(C_x, K)\}_{x \in X}$ is an L-coercing family of \tilde{F}_z . Indeed let $a \in \tilde{F}_z(x)$ for all $x \in C_a \Rightarrow a \notin S_z(x)$ for all $x \in C_a \Rightarrow F(x, a) \leq z$ for all $x \in C_a \Rightarrow a \in K$. If \tilde{F}_z was L-KKM, by theorem 1.1 with s the identity function, we would have $\bigcap_{x \in X} \tilde{F}_z(x) \neq \emptyset$, in contradiction with condition : $S_z^{-1}(y)$ is nonempty for each $y \in X$. So \tilde{F}_z is not L-KKM and there exists $A \subset \langle X \rangle$ such that $\Gamma(A) \not\subset \bigcup_{x \in A} \tilde{F}_z(x) \Rightarrow \Gamma(A) \not\subset \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow \exists y_0 \in \Gamma(A)$ such that $y_0 \notin \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow y_0 \in \bigcap_{x \in A} S_z(x) \Rightarrow y_0 \in S_z(x)$ for all $x \in A$. Then there exists $A \in \langle X \rangle$ and $y_0 \in \Gamma(A)$ such that $F(x, y_0) \not\leq z$ for all $x \in A$. Which contradicts condition (a) and the proposition is proved. ■

Proposition 2.3 *Condition (a) of proposition 2.2 weaken the conditions (a) and (b) of Theorem 2.4.*

Proof : Indeed let us show that Conditions (a) and (b) of Theorem 2.4 imply Condition (a) of Proposition 2.2. Let (X, Γ) be an L-space, $z \in \mathbb{R}$ and F a correspondences from $X \times X$ to \mathbb{R} . Let us consider the correspondence $S_z : X \rightarrow X$ given by $S_z(y) = \{x \in X \mid F(x, y) \not\leq z\}$ and suppose that Condition (b) of Theorem 2.4 hold then for each $y \in X$, $S_z(y)$ is L-convex. Let A be a finite subset of X and $\tilde{y} \in \Gamma(A)$, then $S_z(\tilde{y})$ is an L-convex set. By Assumption (a) of Theorem 2.4, for all $x \in X$, $F(x, x) \leq z$ then $\tilde{y} \notin S_z(\tilde{y})$ and thereby $\Gamma(A) \not\subset S_z(\tilde{y})$. By the L-convexity, $A \not\subset S_z(\tilde{y})$ then there exists $x_0 \in A$ such that $F(x_0, \tilde{y}) \leq z$.

Remark 2.5 *In order to prove that Condition (a) of proposition 2.2 do not imply Conditions (a) and (b) of Theorem 2.4, we can consider the following counter example. Let $X = [0, \pi]$, $z = 0$ and for all $A \in \langle X \rangle$, $\Gamma(A) = co(A)$. The function $f : [0, \pi]^2 \rightarrow \mathbb{R}$ given by $f(x, y) = -y \sin(x)$ satisfies condition (a) of proposition 2.2 but f is not quasi-concave in x , (note that f is quasi-convex).*

3 Variational Inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 2.4.

Let E and P denote two real topological vector space, X a nonempty convex set in E and $\langle \cdot, \cdot \rangle$ a bilinear form on $P \times E$ whose for each fixed $v \in P$, the restriction of $\langle v, \cdot \rangle$ on any quasi-compact subset Q of X is continuous⁵.

Definition 3.5 *A non empty valued correspondence $T : X \rightarrow P$ is said to be monotone if for each $(x_1, u_1), (x_2, u_2) \in \text{graph}T$ we have $\langle u_1 - u_2, x_1 - x_2 \rangle \geq 0$.*

Theorem 3.5 *Let $T : X \rightarrow P$ be a monotone correspondence, $\varphi : X \rightarrow \mathbb{R}$ a quasi-convex function lower semi-continuous on any quasi-compact subset of X ⁶. Let us suppose that there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:*

- (a) K is a quasi-compact subset of X ,
- (b) for each $A \in \langle X \rangle$, there exists a quasi-compact convex set D^A containing A such that:
$$x \in D^A \Rightarrow C_x \subset D^A,$$
- (c) $\left\{ y \in X, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \} \leq 0 \text{ for all } x \in C_y \right\} \subset K$.

Then there is a point $y_0 \in X$ such that

$$\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X.$$

Proof : The proof is similar to the proof of Yen [Y]. For each $(x, y) \in X \times X$, let's consider the correspondences F and G defined by

$$G(x, y) =] - \infty, \inf_{v \in T(y)} \{ \langle v, y - x \rangle + \varphi(y) - \varphi(x) \},$$

$$F(x, y) =] - \infty, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \}.$$

The monotonicity of T ensures that for each $(x, y) \in X \times X$, $F(x, y) \subset G(x, y)$. By the quasi-convexity of φ , it follows that for all $p \in \mathbb{R}$, $\{x \in X \mid \varphi(x) < p\}$ is a convex subset of X then for each $y \in X$, $\{x \in X \mid G(x, y) \not\leq$

⁵Which is equivalent, if we denote for all $x \in Z$, $\varphi_v(x) = \langle v, x \rangle$, to : for every closed subset F of \mathbb{R} , $\varphi^{-1}(F)$ is a strongly compactly closed subset.

⁶Or equivalently: for every $\alpha \in \mathbb{R}$, $\varphi^{-1}(] - \infty, \alpha])$ is a strongly compactly closed set.

$p\} = \{x \in X \mid p < \inf_{v \in T(y)} \{\langle v, y - x \rangle + \varphi(y) - \varphi(x)\}\}$ is a convex set. Since for each fixed $x \in X$, the function $y \rightarrow \sup_{u \in T(x)} \{\langle u, y - x \rangle + \varphi(y) - \varphi(x)\}$ is lower semi-continuous on quasi-compact subsets of X , then F is a weakly l.s.c correspondence on the quasi-compact subsets of X . Consequently, the correspondences F and G are satisfying all the assumptions of Theorem 2.4 with X a convex subset of the topological vector space E and $z = 0$. Hence, there exists $y_0 \in X$ such that $F(x, y_0) \leq 0, \forall x \in X$ then

$$\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X.$$

■

Remark 3.6 *In view of the monotony of T , it is easy to show that:*

$$\exists y_0 \text{ such that } \inf_{u \in T(y_0)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X \quad (1)$$

↓

$$\exists y_0 \text{ such that } \sup_{v \in T(x)} \langle v, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X \quad (2)$$

In the following proposition, we give the sufficient condition in order to get the converse implication:

Proposition 3.4 *If a monotone correspondence $T : X \rightarrow P$ satisfies the following condition:*

- (a) *for each $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \rightarrow \mathbb{R}$ given for $t \in [0, 1]$ by $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$ is lower semi-continuous at $t = 0$ (resp. the function $\tilde{h}_{xy} : [0, 1] \rightarrow \mathbb{R}$ given for $t \in [0, 1]$ by $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$ is upper semi-continuous at $t = 0$),*

and the function $\varphi : X \rightarrow \mathbb{R}$ is convex then (2) \Rightarrow (1) in Remark 3.6.

Proof : Suppose that there exists $y_0 \in X$ such that $\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X$. For each x' in X , let $x_r = y_0 - r(y_0 - x')$, for all $0 < r < 1$. By the convexity of X , $x_r \in X$ then $\sup_{u \in T(x_r)} \langle u, y_0 - x_r \rangle \leq \varphi(x_r) - \varphi(y_0)$. The convexity of φ implies that $\varphi(x_r) - \varphi(y_0) \leq r(\varphi(x') - \varphi(y_0))$ for all $0 < r < 1$. Hence, $\inf_{u \in T(x_r)} \langle u, y_0 - x' \rangle \leq \varphi(x') - \varphi(y_0)$. If r tends to 0 then by condition (a), we get $\inf_{u \in T(y_0)} \langle u, y_0 - x' \rangle \leq \varphi(x') - \varphi(y_0)$. ■

Remark 3.7 One check easily that if a correspondence T is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then condition (a) of proposition 3.4, used by Lassonde [L] in Theorem 2.11., is satisfied⁷:

For any $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \rightarrow \mathbb{R}$ defined by or all $t \in [0, 1]$, $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$ is lower semi-continuous at the point $t = 0$.

In view of Proposition 3.4, the following corollary is deduced from Theorem 3.5.

Corollary 3.1 Let $T : X \rightarrow P$ be a monotone correspondence, $\varphi : X \rightarrow \mathbb{R}$ a convex function lower semi-continuous on the quasi-compact subsets of X . Let us suppose that there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:

- (a) K is a quasi-compact subset of X ,
- (b) for each $A \in \langle X \rangle$, there exists a quasi-compact convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (c) $\left\{ y \in X, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \} \leq 0 \text{ for all } x \in C_y \right\} \subset K$,

- (d) for each $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \rightarrow \mathbb{R}$ given for $t \in [0, 1]$ by $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$ is l.s.c. at $t = 0$.

Then there exists point $y_0 \in X$ such that

$$\inf_{u \in T(y_0)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X.$$

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⁷It suffices to consider p equal to the (continuous) linear form $\langle \cdot, y - x \rangle$ in the following definition given by Cornet: a correspondence $F : X \rightarrow P$ is said upper hemi-continuous in a point $x_0 \in X$ in the sense of Cornet if for any continuous linear function p , the function $h : x \rightarrow \sup_{y \in \varphi(x)} p(y)$ (resp. $\tilde{h} : x \rightarrow \inf_{y \in \varphi(x)} p(y)$) is upper semi-continuous (resp. lower semi-continuous) at the point x_0 .

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