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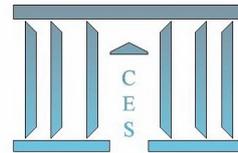


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## Applications of a Generalized Ky Fan's Matching Theorem In Minimax and Variational Inequality

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# Applications of A Generalized Ky Fan's Matching Theorem In Minimax and Variational Inequality

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## Abstract

We present some application of the generalized Ky Fan's Matching Theorem stated by Chebbi, Gourdel and Hammami in minimax and variational inequalities using a generalized coercivity type condition for correspondences defined in L-space.

**Key words and phrases:** L-structures, L-spaces, L-KKM correspondences, L-coercing family, minimax and variational inequalities.

**Classification-JEL:** C02, C69, C72.

The purpose of this paper is to give some application of the generalized Ky Fan's Matching theorem stated by Chebbi, Gourdel and Hammami [CGH] to minimax and variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan in [F2] [F3], Ding and Tan in [DT] and Yen in [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli in [BC1] and [BC2] or in convex spaces in the sense of Lassonde in [L].

In this article, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let  $A$  be a subset of a vector space  $X$ . We denote by  $\langle A \rangle$  the family of all nonempty finite subsets of  $A$  and  $\text{conv}A$  the convex hull of  $A$ . Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters  $F, G, Q, S, \Gamma, \dots$  and the single valued functions will be represented by small letters. We denote by  $\text{graph}F$  the graph of the correspondence  $F$ . If  $X$  and  $Y$  are two topological spaces,

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$\zeta(X, Y)$  denotes the set of all continuous functions from  $X$  to  $Y$ .

If  $n$  is any integer,  $\Delta_n$  denotes the unit-simplex of  $\mathbb{R}^{n+1}$  and for every  $J \subset \{0, 1, \dots, n\}$ ,  $\Delta_J$  denotes the face of  $\Delta_n$  corresponding to  $J$ . Let  $X$  be a topological space. An  $L$ -structure (also called  $L$ -convexity) on  $X$  is given by a correspondence  $\Gamma : \langle X \rangle \rightarrow X$  with nonempty valued such that for every  $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ , there exists a continuous function  $f^A : \Delta_n \rightarrow \Gamma(A)$  such that for all  $J \subset \{0, \dots, n\}$ ,  $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$ . Such a pair  $(X, \Gamma)$  is called an  $L$ -space. A subset  $C \subset X$  is said to be  $L$ -convex if for every  $A \in \langle C \rangle$ ,  $\Gamma(A) \subset C$ . A subset  $P \subset X$  is said to be  $L$ -quasi-compact if for every  $A \in \langle X \rangle$ , there is a quasi-compact  $L$ -convex set  $D$  such that  $A \cup P \subset D$ . Clearly, if  $C$  exists an  $L$ -convex subset of an  $L$ -space  $(X, \Gamma)$ , then the pair  $(C, \Gamma|_{\langle C \rangle})$  is an  $L$ -space.

## 1 A Generalized Ky Fan's Matching Theorem

In this section we remind some known definitions of  $L$ -KKM correspondences and  $L$ -coercing family quoted in [CGH] and we give a more adapted theorem than the mean result of [CGH] in order to generalize Fan's minimax inequality.

**Definition 1.1** *Let  $(X, \Gamma)$  be an  $L$ -space and  $Z \subset X$  an arbitrary subset. A correspondence  $F : Z \rightarrow X$  is called  $L$ -KKM if and only if:*

$$\forall A \in \langle Z \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

**Definition 1.2** *Let  $Z$  be an arbitrary set of an  $L$ -space  $(X, \Gamma)$ ,  $Y$  a topological space and  $s \in \zeta(X, Y)$ . A family  $\{(C_a, K)\}_{a \in X}$  is said to be  $L$ -coercing for a correspondence  $F : Z \rightarrow Y$  with respect to  $s$  if and only if:*

- (i)  $K$  is a quasi-compact subset of  $Y$ ,
- (ii) for each  $A \in \langle Z \rangle$ , there exists a quasi-compact  $L$ -convex set  $D^A$  in  $X$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

$$(iii) \left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

**Definition 1.3** *If  $X$  is a topological space, a subset  $B$  of  $X$  is called strongly compactly closed (open respectively) if for every quasi-compact subset  $K$  of  $X$ ,  $B \cap K$  is closed (open, respectively) in  $K$ .*

We remind the generalization of Fan's matching theorem of [CGH]:

**Theorem 1.1** *Let  $Z$  be an arbitrary set in the L-space  $(X, \Gamma)$ ,  $Y$  an arbitrary topological space and  $F : Z \rightarrow Y$  a correspondence. Suppose that there is a function  $s \in \zeta(X, Y)$  such that:*

- (a) *for every  $x \in Z$ ,  $F(x)$  is strongly compactly closed,*
- (b) *the correspondence  $R : Z \rightarrow X$  defined by  $R(x) = s^{-1}(F(x))$  is L-KKM,*
- (c) *there exists an L-coercing family  $\{(C_x, K)\}_{x \in X}$  for  $F$  with respect to  $s$ .*

*Then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ , more precisely  $K \cap (\bigcap_{x \in Z} F(x)) \neq \emptyset$ .*

For any correspondence  $F : X \rightarrow Y$ , let  $F^* : Y \rightarrow X$  the "dual" correspondence of  $F$  defined, for all  $y \in Y$ , by  $F^*(y) = X \setminus F^{-1}(y)$ , where  $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ .

The following theorem can be seen as a corollary of Theorem 1.1. It will be used in order to generalize Fan's minimax inequality.

**Theorem 1.2** *Let  $(X, \Gamma)$  an L-space,  $Y$  an arbitrary topological space and  $F, G : X \rightarrow Y$  be two correspondences satisfying:*

- (a) *for every  $x \in X$ ,  $F(x)$  is strongly compactly closed,*
- (b) *for every  $x \in X$ ,  $G(x) \subset F(x)$ ,*
- (c) *there exists a function  $s \in \zeta(X, Y)$  such that:*
  1. *for every  $x \in X$ ,  $s(x) \in G(x)$ ,*
  2. *for every  $x \in X$ ,  $S^*(x)$  where  $S$  is defined by  $S(x) = s^{-1}(G(x))$  is L-convex,*
  3. *There exists an L-coercing family  $\{(C_x, K)\}_{x \in X}$  for  $F$  with respect to  $s$ .*

Then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ .

Proof : The correspondence  $F$  has strongly compactly closed values and admits an L-coercing family then in order to apply Theorem 1.1, it suffices to show that the correspondence  $R : X \rightarrow X$  defined by  $R(x) = s^{-1}(F(x))$  is L-KKM. Let  $A \subset \langle X \rangle$  and  $z \in \Gamma(A)$ , then by (c.1),  $s(z) \in G(\Gamma(A))$ . One can check that Condition (c.2) can be equivalently rewritten as  $S(\Gamma(A)) \subset S(A)$ . Moreover, by (c.1), for all  $B \subset X$ ,  $B \subset S(B)$ , in particular  $\Gamma(A) \subset S(\Gamma(A))$ . Hence we deduce,  $\Gamma(A) \subset S(A)$ . By construction,  $S \subset R$ , which implies that  $R$  is L-KKM. ■

**Remark 1.1** *If  $s$  is the identity function, the proof of the previous theorem becomes a simple application of Lemma 1 of section 4 in [H2].*

## 2 Some Generalizations of Fan's Minimax Inequality

The object of this section is to get a generalization of minimax inequality due to Fan [F3]. In the sequel of this section, for any subset  $A$  of  $\overline{\mathbb{R}}$ <sup>3</sup> and every  $z \in \mathbb{R}$ ,  $A \leq z$  denotes for all  $a \in A$ ,  $a \leq z$  and  $A \not\leq z$  means that there exists  $a \in A$  such that  $a > z$ .

**Definition 2.4** *Let  $(X, \Gamma)$  be an L-space. A correspondence  $Q : X \rightarrow \overline{\mathbb{R}}$  is said to be weakly lower semi-continuous (weakly l.s.c) on  $X$  if for each  $p \in \mathbb{R}$ , the set  $\{x \in X \mid Q(x) \leq p\}$  is closed in  $X$ <sup>4</sup> or equivalently, the set  $\{x \in X \mid Q(x) \cap ]p, +\infty] \neq \emptyset\}$  is open in  $X$ .*

**Proposition 2.1** *If  $Q$  is a lower semi-continuous correspondence then it is weakly lower semi-continuous.*

Proof : The proof is immediate: if for all  $p \in \mathbb{R}$ , we consider the closed subset  $V = \{y \in \mathbb{R} \mid y \leq p\}$ , then by l.s.c.  $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$  is a closed set.

Let  $Q$  be a l.s.c correspondence, we have to prove that for all  $p \in \mathbb{R}$ ,  $\{x \in X \mid Q(x) \leq p\}$  is a closed set. For all  $p \in \mathbb{R}$ , we consider the closed subset  $V = \{y \in \mathbb{R} \mid y \leq p\}$  consequently  $\{x \in X \mid Q(x) \leq p\} = \{x \in X \mid Q(x) \subset V\} = \{x \in X \mid Q(x) \cap V^c = \emptyset\}$ . By the l.s.c. of  $Q$ , the set  $\{x \in X \mid Q(x) \cap V^c \neq \emptyset\}$  is open then  $\{x \in X \mid Q(x) \cap V^c = \emptyset\}$  is closed and the proposition is proved. ■

<sup>3</sup>The extended real line, endowed with its usual topology, see for example Rudin [R]

<sup>4</sup>Recall that a correspondence  $Q$  is lower semi-continuous, if for each open set  $V \subset Y$ , the set  $\{x \in X : Q(x) \cap V \neq \emptyset\}$  is open in  $X$ .

**Remark 2.2** Note that the converse implication of Proposition 2.1 is false, since in order to prove that this converse implication is false, we can consider the following counter example: Let the correspondence  $Q : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $Q(x) = \{1, 2\}$  if  $x \neq 0$  and  $Q(x) = \{-1, 2\}$  if  $x = 0$ . It is easy to see that  $Q$  is weakly l.s.c but not l.s.c.

We remind a minimax inequality due to Fan [F3].

**Theorem 2.3** (Fan) Let  $E$  be a topological vector space, let  $K$  be a nonempty compact convex set in  $E$ , and let  $f$  be a real-valued function on  $K \times K$ . Suppose that

- (a) for every  $y \in K$ ,  $f(y, y) \leq 0$ ,
- (b) for each fixed  $y \in K$ , the function  $x \rightarrow f(x, y)$  is quasi-concave on  $K$ ,
- (c) for each fixed  $x \in K$ , the function  $y \rightarrow f(x, y)$  is lower semi-continuous on  $K$ .

Then there exists a vector  $y_0$  in  $K$  such that  $f(x, y_0) \leq 0$  for all  $x \in K$ .

This theorem can be extended in the following way:

**Theorem 2.4** Let  $(X, \Gamma)$  be an  $L$ -space and  $z \in \mathbb{R}$ . Let  $F$  and  $G$  be two correspondences from  $X \times X$  to  $\overline{\mathbb{R}}$  satisfying the following condition:

- (a) for every  $x \in X$ ,  $G(x, x) \leq z$ ,
- (b) for each fixed  $y \in X$ ,  $\{x \in X \mid G(x, y) \not\leq z\}$  is  $L$ -convex,
- (c) for each fixed  $x \in X$ ,  $y \rightarrow F(x, y)$  is weakly l.s.c on the quasi-compact subsets of  $X$ ,
- (d) for every  $(x, y) \in X \times X$ ,  $F(x, y) \subset G(x, y)$ ,
- (e) there exists a family  $\{(C_x, K)\}_{x \in X}$  of pairs of sets satisfying:
  - (1)  $K$  is a quasi-compact subset of  $X$ ,
  - (2) for each  $A \in \langle X \rangle$ , there exists a quasi-compact  $L$ -convex set  $D^A$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (3)  $\{y \in X, F(x, y) \leq z \text{ for all } x \in C_y\} \subset K$ .

Then there exists  $y_0 \in X$  such that

$$F(x, y_0) \leq z \quad \forall x \in X.$$

Proof : The technique of the proof follows from the proof of Theorem 1 of Yen [Y], which is based on Fan's lemma [F1][F3], and Theorem 1.2. For each  $x \in X$ , let  $\tilde{F}(x) = \{y \in X : F(x, y) \leq z\}$  and  $\tilde{G}(x) = \{y \in X : G(x, y) \leq z\}$ . Then by (c), the correspondence  $\tilde{F}$  has strongly compactly closed values. By (b), the set  $\tilde{G}^*(y) = \{x \in X : G(x, y) \not\leq z\}$  is an L-convex subset of  $X$ . By (d), for each  $x \in X$ ,  $\tilde{G}(x) \subset \tilde{F}(x)$ . Remark that, by (a), for each  $x \in X$ ,  $x \in \tilde{G}(x)$  and  $\{(C_x, K)\}_{x \in X}$  is an L-coercing family of  $\tilde{F}$ . Then all the requirements of Theorem 1.2 with  $s$  the identity function are satisfied, hence  $\bigcap_{x \in X} \tilde{F}(x) \neq \emptyset$  and the theorem is proved.  $\blacksquare$

**Remark 2.3** *If we consider the particular case where the correspondence  $g$  is a real-valued function in the previous theorem, we can deduce that condition (b) is implied by the classical quasi-concavity of the function  $x \rightarrow G(x, y)$  for each fixed  $y \in X$ .*

**Remark 2.4** *In view of Remark 2.3, it is easy to see how we can deduce Theorem 2.3 from the previous theorem, it suffices to apply Theorem 2.4 to the correspondences  $F = G = f$ ,  $X = K$  which is a nonempty compact convex set in a topological vector space and  $z = 0$ .*

In the next result, for sake of simplicity, we will focus on the particular case when  $F = G$  (but not any more assumed to be a function), and we will weaken conditions (a) and (b) of Theorem 2.4.

**Proposition 2.2** *Let  $(X, \Gamma)$  be an L-space,  $z \in \mathbb{R}$  and  $F : X \times X \rightarrow \overline{\mathbb{R}}$  a correspondence satisfying the following condition:*

- (a) *for each finite subset  $A$  of  $X$  and for each  $y \in \Gamma(A)$ , there exists  $x_0 \in A$  such that  $F(x_0, y) \leq z$ ,*
- (b) *for each fixed  $x \in X$ ,  $y \rightarrow F(x, y)$  is weakly l.s.c on quasi-compact subsets of  $X$ ,*
- (c) *there exists a family  $\{(C_x, K)\}_{x \in X}$  of pairs of sets satisfying:*
  - (1)  *$K$  is a quasi-compact subset of  $X$ ,*
  - (2) *for each  $A \in \langle X \rangle$ , there exists a quasi-compact L-convex set  $D^A$  containing  $A$  such that:*

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (3)  *$\{y \in X, F(x, y) \leq z \text{ for all } x \in C_y\} \subset K$ .*

Then, there exists  $y_0 \in X$  such that  $F(x, y_0) \leq z$  for all  $x \in X$ .

Proof : This proof mimics the proof of Fan Inequality: consider the correspondence  $S_z : X \rightarrow X$  given by  $S_z(x) = \{y \in X \mid F(x, y) \not\leq z\}$  and assume (arguing by contradiction) that for each  $y \in X$  there exists  $x \in X$  such that  $F(x, y) \not\leq z$ . Then for each  $y \in X$ ,  $S_z^{-1}(y)$  is nonempty. For each fixed  $x \in X$ ,  $y \rightarrow F(x, y)$  is weakly l.s.c. on the quasi-compact subsets of  $X$  then for each fixed  $x \in X$ ,  $S_z(x) = \{y \in X \mid F(x, y) \not\leq z\}$  is strongly compactly open in  $X$ . Consider the correspondence  $\tilde{F}_z : X \rightarrow X$  given by  $\tilde{F}_z(x) = X \setminus S_z(x)$  for  $x \in X$ . Then  $\tilde{F}_z$  is strongly compactly closed in  $X$ . It follows from (c) that  $\{(C_x, K)\}_{x \in X}$  is an L-coercing family of  $\tilde{F}_z$ . Indeed let  $a \in \tilde{F}_z(x)$  for all  $x \in C_a \Rightarrow a \notin S_z(x)$  for all  $x \in C_a \Rightarrow F(x, a) \leq z$  for all  $x \in C_a \Rightarrow a \in K$ . If  $\tilde{F}_z$  was L-KKM, by theorem 1.1 with  $s$  the identity function, we would have  $\bigcap_{x \in X} \tilde{F}_z(x) \neq \emptyset$ , in contradiction with condition :  $S_z^{-1}(y)$  is nonempty for each  $y \in X$ . So  $\tilde{F}_z$  is not L-KKM and there exists  $A \subset \langle X \rangle$  such that  $\Gamma(A) \not\subset \bigcup_{x \in A} \tilde{F}_z(x) \Rightarrow \Gamma(A) \not\subset \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow \exists y_0 \in \Gamma(A)$  such that  $y_0 \notin \bigcup_{x \in A} X \setminus S_z(x) \Rightarrow y_0 \in \bigcap_{x \in A} S_z(x) \Rightarrow y_0 \in S_z(x)$  for all  $x \in A$ . Then there exists  $A \in \langle X \rangle$  and  $y_0 \in \Gamma(A)$  such that  $F(x, y_0) \not\leq z$  for all  $x \in A$ . Which contradicts condition (a) and the proposition is proved. ■

**Proposition 2.3** *Condition (a) of proposition 2.2 weakens the conditions (a) and (b) of Theorem 2.4.*

Proof : Indeed let us show that Conditions (a) and (b) of Theorem 2.4 imply Condition (a) of Proposition 2.2. Let  $(X, \Gamma)$  be an L-space,  $z \in \mathbb{R}$  and  $F$  a correspondences from  $X \times X$  to  $\mathbb{R}$ . Let us consider the correspondence  $S_z : X \rightarrow X$  given by  $S_z(y) = \{x \in X \mid F(x, y) \not\leq z\}$  and suppose that Condition (b) of Theorem 2.4 hold then for each  $y \in X$ ,  $S_z(y)$  is L-convex. Let  $A$  be a finite subset of  $X$  and  $\tilde{y} \in \Gamma(A)$ , then  $S_z(\tilde{y})$  is an L-convex set. By Assumption (a) of Theorem 2.4, for all  $x \in X$ ,  $F(x, x) \leq z$  then  $\tilde{y} \notin S_z(\tilde{y})$  and thereby  $\Gamma(A) \not\subset S_z(\tilde{y})$ . By the L-convexity,  $A \not\subset S_z(\tilde{y})$  then there exists  $x_0 \in A$  such that  $F(x_0, \tilde{y}) \leq z$ .

**Remark 2.5** *In order to prove that Condition (a) of proposition 2.2 do not imply Conditions (a) and (b) of Theorem 2.4, we can consider the following counter example. Let  $X = [0, \pi]$ ,  $z = 0$  and for all  $A \in \langle X \rangle$ ,  $\Gamma(A) = \text{co}(A)$ . The function  $f : [0, \pi]^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = -y \sin(x)$  satisfies condition (a) of proposition 2.2 but  $f$  is not quasi-concave in  $x$ , (note that  $f$  is quasi-convex).*

### 3 Variational Inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 2.4.

Let  $E$  and  $P$  denote two real topological vector space,  $X$  a nonempty convex set in  $E$  and  $\langle \cdot, \cdot \rangle$  a bilinear form on  $P \times E$  whose for each fixed  $v \in P$ , the restriction of  $\langle v, \cdot \rangle$  on any quasi-compact subset  $Q$  of  $X$  is continuous<sup>5</sup>.

**Definition 3.5** *A non empty valued correspondence  $T : X \rightarrow P$  is said to be monotone if for each  $(x_1, u_1), (x_2, u_2) \in \text{graph}T$  we have  $\langle u_1 - u_2, x_1 - x_2 \rangle \geq 0$ .*

**Theorem 3.5** *Let  $T : X \rightarrow P$  be a monotone correspondence,  $\varphi : X \rightarrow \mathbb{R}$  a quasi-convex function lower semi-continuous on any quasi-compact subset of  $X$ <sup>6</sup>. Let us suppose that there exists a family  $\{(C_x, K)\}_{x \in X}$  of pairs of sets satisfying:*

- (a)  $K$  is a quasi-compact subset of  $X$ ,
- (b) for each  $A \in \langle X \rangle$ , there exists a quasi-compact convex set  $D^A$  containing  $A$  such that:
$$x \in D^A \Rightarrow C_x \subset D^A,$$
- (c)  $\left\{ y \in X, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \} \leq 0 \text{ for all } x \in C_y \right\} \subset K$ .

Then there is a point  $y_0 \in X$  such that

$$\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X.$$

Proof : The proof is similar to the proof of Yen [Y]. For each  $(x, y) \in X \times X$ , let's consider the correspondences  $F$  and  $G$  defined by

$$G(x, y) = ] - \infty, \inf_{v \in T(y)} \{ \langle v, y - x \rangle + \varphi(y) - \varphi(x) \},$$

$$F(x, y) = ] - \infty, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \}.$$

The monotonicity of  $T$  ensures that for each  $(x, y) \in X \times X$ ,  $F(x, y) \subset G(x, y)$ . By the quasi-convexity of  $\varphi$ , it follows that for all  $p \in \mathbb{R}$ ,  $\{x \in X \mid \varphi(x) < p\}$  is a convex subset of  $X$  then for each  $y \in X$ ,  $\{x \in X \mid G(x, y) \not\leq$

<sup>5</sup>Which is equivalent, if we denote for all  $x \in Z$ ,  $\varphi_v(x) = \langle v, x \rangle$ , to : for every closed subset  $F$  of  $\mathbb{R}$ ,  $\varphi^{-1}(F)$  is a strongly compactly closed subset.

<sup>6</sup>Or equivalently: for every  $\alpha \in \mathbb{R}$ ,  $\varphi^{-1}(] - \infty, \alpha])$  is a strongly compactly closed set.

$p\} = \{x \in X \mid p < \inf_{v \in T(y)} \{\langle v, y - x \rangle + \varphi(y) - \varphi(x)\}\}$  is a convex set. Since for each fixed  $x \in X$ , the function  $y \rightarrow \sup_{u \in T(x)} \{\langle u, y - x \rangle + \varphi(y) - \varphi(x)\}$  is lower semi-continuous on quasi-compact subsets of  $X$ , then  $F$  is a weakly l.s.c correspondence on the quasi-compact subsets of  $X$ . Consequently, the correspondences  $F$  and  $G$  are satisfying all the assumptions of Theorem 2.4 with  $X$  a convex subset of the topological vector space  $E$  and  $z = 0$ . Hence, there exists  $y_0 \in X$  such that  $F(x, y_0) \leq 0, \forall x \in X$  then

$$\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X.$$

■

**Remark 3.6** *In view of the monotony of  $T$ , it is easy to show that:*

$$\exists y_0 \text{ such that } \inf_{u \in T(y_0)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X \quad (1)$$

↓

$$\exists y_0 \text{ such that } \sup_{v \in T(x)} \langle v, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \quad \forall x \in X \quad (2)$$

In the following proposition, we give the sufficient condition in order to get the converse implication:

**Proposition 3.4** *If a monotone correspondence  $T : X \rightarrow P$  satisfies the following condition:*

- (a) *for each  $(x, y) \in X \times X$ , the function  $h_{xy} : [0, 1] \rightarrow \mathbb{R}$  given for  $t \in [0, 1]$  by  $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$  is lower semi-continuous at  $t = 0$  (resp. the function  $\tilde{h}_{xy} : [0, 1] \rightarrow \mathbb{R}$  given for  $t \in [0, 1]$  by  $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x - y \rangle$  is upper semi-continuous at  $t = 0$ ),*

*and the function  $\varphi : X \rightarrow \mathbb{R}$  is convex then (2)  $\Rightarrow$  (1) in Remark 3.6.*

**Proof :** Suppose that there exists  $y_0 \in X$  such that  $\sup_{u \in T(x)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X$ . For each  $x'$  in  $X$ , let  $x_r = y_0 - r(y_0 - x')$ , for all  $0 < r < 1$ . By the convexity of  $X$ ,  $x_r \in X$  then  $\sup_{u \in T(x_r)} \langle u, y_0 - x_r \rangle \leq \varphi(x_r) - \varphi(y_0)$ . The convexity of  $\varphi$  implies that  $\varphi(x_r) - \varphi(y_0) \leq r(\varphi(x') - \varphi(y_0))$  for all  $0 < r < 1$ . Hence,  $\inf_{u \in T(x_r)} \langle u, y_0 - x' \rangle \leq \varphi(x') - \varphi(y_0)$ . If  $r$  tends to 0 then by condition (a), we get  $\inf_{u \in T(y_0)} \langle u, y_0 - x' \rangle \leq \varphi(x') - \varphi(y_0)$ . ■

**Remark 3.7** One check easily that if a correspondence  $T$  is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then condition (a) of proposition 3.4, used by Lassonde [L] in Theorem 2.11., is satisfied<sup>7</sup>:

For any  $(x, y) \in X \times X$ , the function  $h_{xy} : [0, 1] \rightarrow \mathbb{R}$  defined by or all  $t \in [0, 1]$ ,  $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$  is lower semi-continuous at the point  $t = 0$ .

In view of Proposition 3.4, the following corollary is deduced from Theorem 3.5.

**Corollary 3.1** Let  $T : X \rightarrow P$  be a monotone correspondence,  $\varphi : X \rightarrow \mathbb{R}$  a convex function lower semi-continuous on the quasi-compact subsets of  $X$ . Let us suppose that there exists a family  $\{(C_x, K)\}_{x \in X}$  of pairs of sets satisfying:

- (a)  $K$  is a quasi-compact subset of  $X$ ,
- (b) for each  $A \in \langle X \rangle$ , there exists a quasi-compact convex set  $D^A$  containing  $A$  such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

- (c)  $\left\{ y \in X, \sup_{u \in T(x)} \{ \langle u, y - x \rangle + \varphi(y) - \varphi(x) \} \leq 0 \text{ for all } x \in C_y \right\} \subset K$ ,

- (d) for each  $(x, y) \in X \times X$ , the function  $h_{xy} : [0, 1] \rightarrow \mathbb{R}$  given for  $t \in [0, 1]$  by  $h_{xy}(t) = \inf_{u \in T((1-t)y+tx)} \langle u, y - x \rangle$  is l.s.c. at  $t = 0$ .

Then there exists point  $y_0 \in X$  such that

$$\inf_{u \in T(y_0)} \langle u, y_0 - x \rangle \leq \varphi(x) - \varphi(y_0), \forall x \in X.$$

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<sup>7</sup>It suffices to consider  $p$  equal to the (continuous) linear form  $\langle \cdot, y - x \rangle$  in the following definition given by Cornet: a correspondence  $F : X \rightarrow P$  is said upper hemi-continuous in a point  $x_0 \in X$  in the sense of Cornet if for any continuous linear function  $p$ , the function  $h : x \rightarrow \sup_{y \in \varphi(x)} p(y)$  (resp.  $\tilde{h} : x \rightarrow \inf_{y \in \varphi(x)} p(y)$ ) is upper semi-continuous (resp. lower semi-continuous) at the point  $x_0$ .

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