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Cash-in-advance constraints, bubbles and monetary policy*

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Abstract

In this paper, we study the equilibrium dynamics of an overlapping generations model with capital, money and cash-in-advance constraints. At each period, the economy can experience two different regimes: either the cash-in-advance constraint is binding and money is a dominated asset, or the constraint is strictly satisfied and money has the same return as capital. When the second regime occurs, we say that the economy experiences a temporary bubble. We show the existence of temporary bubbles, and we prove that cyclical equilibria may exist. In these equilibria, the economy experiences some periods without bubbles and some periods with bubbles. We also show that monetary creation can be used in order to eliminate temporary bubbles.

JEL numbers: D9, E5 and G1.

Key words: overlapping generations model, bubbles, cash-in-advance constraint, monetary policy.

1 Introduction

The interplay between monetary policy and macroeconomic fluctuations is a standard theme in monetary analysis, including, among many authors, the famous contributions of Keynes (1964) and Friedman (1969). Recent theoretical works consider this question within general equilibrium frameworks, and particularly within the overlapping generations model. Endogenous volatility arises from the existence of multiple equilibria and indeterminacy, whose existence may depend on monetary policy.

Samuelson (1958) and Tirole (1985) have greatly contributed to the pioneering work in this field. Tirole presents a benchmark model where money is viewed as a rational bubble, which is valued only if its return is equal to the rate of return on capital. He shows that real money balances held by the agents can only be valued in an economy that is inefficient without money. In this case, an infinity of equilibria exist: one of these equilibria leads to a constant and positive value in the long run for the bubble held by each agent, and the economy converges towards the golden rule; the other trajectories converge to the stationary state of Diamond's model without bubbles. Tirole also shows that the introduction of a reserve requirement constraint precludes the existence of asymptotic trajectories without bubbles.

Hahn and Solow (1995) study a framework close to Tirole (1985). They consider the standard overlapping generation framework *à la* Diamond (1965), in which money is introduced by a cash-in-advance constraint. They study the intertemporal equilibrium under the assumption that the liquidity constraint is binding in each period: the rate of return for money must be smaller than the rate of return for capital.

More recent literature on monetary analysis explicitly models credit market frictions and financial intermediaries. Bhattacharya, Guzman, Huybens and Smith (1997) and Schreft and Smith (1998) introduce spatial separation and limited communication between agents. These assumptions provide micro-foundations for money holding: money is held even if it is a dominated asset. These authors also assume that agents are subject to stochastic relocations that act like shocks to their portfolio preferences, thereby creating an explicit role for the banking system. The main findings of these studies are obtained from analyzing a situation where the government issues both money and bonds. In Schreft and Smith (1998), monetary creation allows the government to pay back interest payments on public debt. In Bhattacharya, Guzman, Huybens and Smith (1997), money also makes it possible to finance the current government deficit. Both contributions show that these policies

can lead to the existence of many steady-state equilibria and indeterminacy¹.

Boyd and Smith (1998) and Huybens and Smith (1998) introduce another form of credit market. They assume that only the project owner can observe at no cost the return on an investment project. Such a framework allows for the existence of credit rationing and gives an explicit allocative function to financial intermediation. This leads to the existence of multiple equilibria, whose existence may depend on the rate of monetary growth.

Michel and Wigniolle (2003) bring a new argument to this literature in a simple example, which is based on Hahn and Solow's (1995) model. Hahn and Solow (1995), Bhattacharya, Guzman, Huybens and Smith (1997), Schreft and Smith (1998) and Gomis-Porqueras (2000) focus on equilibria where money is a dominated asset. On the contrary, in Tirole (1985), Boyd and Smith (1998) and Huybens and Smith (1998), money and bank deposits have the same rate of return. Therefore the literature has only focused on the case of permanent regimes, in which money is either dominated or not dominated at all dates. In contrast to this literature, Michel and Wigniolle (2003) consider that along an intertemporal equilibrium, the economy can experience both periods where money and capital have the same return ("Tirole's regime") and periods where money is a dominated asset ("Hahn and Solow's regime"). Periods in which Tirole-regime occurs are called temporary bubbles. Such temporary bubbles may exist in an economy, which would experience under-accumulation without money, when the weight of the cash-in-advance constraint is not too hard.

In this paper we propose a twofold generalization of Michel and Wigniolle (2003). Firstly we deal with general formulations for preferences and production function. Secondly, we introduce monetary policy. Then, we can prove that the existence of temporary bubbles is a general property that can arise as soon as the economy without money (the economy corresponding to Diamond's (1965) model) have a stationary state associated with under-accumulation, and when the weight of the cash-in-advance constraint is not too hard. Moreover, if the stationary state of the economy without money is associated with over-accumulation, there exists a large range of rate of monetary creation such that temporary bubbles exist. These results are reached by doing a local study within the neighborhood of an stationary equilibrium. It is proved that cyclical equilibria may exist where the economy experiences n periods of temporary bubbles and p periods without bubbles, n and p being some integers. Thus, a multiplicity of equilibria exists.

The influence of monetary creation is another new aspect that we consider

¹As stressed by Gomis-Porqueras (2000), the existence of multiple steady states in these models depends on the design of the monetary policy.

in our framework. What is the impact of monetary policy on temporary bubbles? Intuition suggests that monetary creation induces inflation, which causes a drop in the return of money detention. In that case, the return on money could no longer be equal to the return on capital. In the particular case of Cobb-Douglas functions for utility and production, we show that a high enough rate of monetary creation eliminates temporary bubbles. Thus, the economy follows the only equilibrium trajectory without bubbles and indeterminacy vanishes.

Monetary creation can be viewed as a counter-bubble weapon capable of stabilizing the economy on the bubbleless equilibrium. However, such a policy is not Pareto improving. Indeed, a one period bubble is beneficial for the generation living during the bubble period, even if it is detrimental for the following generations. Moreover, monetary creation increases the distortion between the returns on money and capital savings. But we develop two arguments in favor of such a monetary policy. For each argument, we compare a trajectory where monetary creation have eliminated bubbles with a trajectory with bubbles. Firstly, considering the long run, we show that there exist parameter values of the model such that the long run utility levels are smaller along a trajectory with the periodic appearance of bubbles than along the bubbleless trajectory. Secondly, considering the short run, we show that for any integer P , there exist parameter values such that the only appearance of a one period bubble is enough to drop the utility levels during P periods under the level reached along the bubbleless trajectory. For these two reasons, fighting bubbles can be an objective of monetary policy.

In our model, indeterminacy and multiple equilibria are the by-products of an economy with money and capital. A temporary bubble on money can be interpreted as a deflation period, where money and capital have the same return. It creates a drop in capital accumulation and production because it absorbs a share of savings. A monetary policy can eliminate such deflation periods as it precludes the appearance of bubbles. But these “Keynesian” features of monetary policy are not obtained by Keynesian mechanisms such as an increase in demand, but by bubbles elimination. Our results seems to contradict the conventional wisdom, which consider that an expansionist monetary policy can promote the appearance of bubbles. In fact, this is only an outward discrepancy as we are interested in bubbles on money, when the usual argument takes into account bubbles on asset prices. Our model shows that a too restrictive monetary policy can induce a deflationist bubble.

The paper is organized in the following way. The model is presented in the second section. The study of the intertemporal equilibrium is achieved

in section three. The fourth section studies the dynamics of the economy. Finally, a fifth section considers the counter-bubble monetary policy.

2 The model

We consider a standard overlapping generations model *à la* Allais (1947)-Diamond (1965), in which money is introduced by a cash-in-advanced constraint. Agents live two periods. They supply one unit of labor in the first period (when young), and they are retired and consume the proceeds of their savings in the second period (when old). The number of young agents at a date t , N_t , grows at the constant rate n : $N_t = (1 + n)N_{t-1}$.

2.1 Money and monetary policy

Following Hahn and Solow (1995), we assume that agents are subject to a cash-in-advance constraint: a share μ of consumption expenses in the second period must be financed by the amount of money saved during the first period,

$$M_t \geq \mu P_{t+1} d_{t+1} \quad (1)$$

μ is a parameter such that $0 < \mu < 1$, d_{t+1} is the second period amount of consumption and P_{t+1} is the price of the good in money in period $t + 1$.

The government creates money, and it gives this money as a lump sum transfer to young people. We denote the total supply of money in period t by \bar{M}_t and the rate of money creation by λ_t . Thus, we have:

$$\bar{M}_t = (1 + \lambda_t) \bar{M}_{t-1} \quad (2)$$

T_t is the lump sum transfer received by each of the N_t young agents living during period t . This transfer is financed by money creation:

$$\lambda_t \bar{M}_{t-1} = N_t T_t \quad (3)$$

2.2 The agents

Agents born in t are endowed with an intertemporal utility function:

$$U_t = U(c_t, d_{t+1}) \quad (4)$$

c_t is the first period consumption, and d_{t+1} is the second period consumption.

Assumption 1: U is strictly quasi-concave, twice continuously differentiable, and satisfies the Inada conditions. It is also assumed that c and d are normal goods.

From this assumption, we can deduce the existence of a continuously differentiable function $\sigma(w, R)$ defined on \mathbb{R}_{++}^2 by:

$$\sigma(w, R) = \arg \max_{\sigma} U(w - \sigma, R\sigma) \quad (5)$$

and that $0 < \sigma'_1 < 1$.

Agents can invest their savings in capital and in money. In real terms, the two budget constraints of a generation t agent are:

$$c_t + s_t + \frac{M_t}{P_t} = w_t + \theta_t \quad (6)$$

$$d_{t+1} = R_{t+1}s_t + \frac{M_t}{P_{t+1}} \quad (7)$$

where w_t is the real wage in period t and s_t is the amount of savings invested in capital. R_{t+1} is the real return factor expected for period $t + 1$. M_t is the money amount held in period t and P_t is the price of the good in money. $\theta_t = T_t/P_t$ is the real value of the money transfer.

2.3 Agents' behavior

Each young agent born in period t maximizes his utility given by (4) under the budget constraints (6) and (7), and the liquidity constraint (1). As in Michel and Wigniolle (2003), we must distinguish at each period t the two possible cases:

The Hahn and Solow's case (HS-regime): the liquidity constraint is binding: $M_t/P_{t+1} = \mu d_{t+1}$ and then the expected return on money is no greater than the return on financial savings $P_t/P_{t+1} \leq R_{t+1}$. In that case, using (6), (7) and the liquidity constraint to eliminate M_t and s_t , we obtain the intertemporal constraint:

$$c_t + \frac{d_{t+1}}{\rho_{t+1}} = w_t + \theta_t \quad (8)$$

$$\text{with } \frac{1}{\rho_{t+1}} = \frac{1 - \mu}{R_{t+1}} + \mu \frac{P_{t+1}}{P_t} \quad (9)$$

ρ_{t+1} is the real expected return of total savings when the liquidity constraint is binding. $1/\rho_{t+1}$ is the mean of the inverse return of money weighted by μ and the inverse return of capital weighted by $1 - \mu$.

The resolution of the consumer program leads to the expression of total savings:

$$\sigma_t = \sigma(w_t + \theta_t, \rho_{t+1}) = s_t + m_t \quad (10)$$

where m_t is the real money holding: $m_t \equiv M_t/P_t$. Using (7) and the liquidity constraint, we obtain:

$$(1 - \mu)m_t = \mu \frac{P_{t+1}}{P_t} R_{t+1} s_t \quad (11)$$

Equations (10) and (11) give s_t and m_t . Finally, by using (11), the condition (9) can be replaced by:

$$\rho_{t+1} = \frac{R_{t+1} s_t}{(1 - \mu)(m_t + s_t)} \quad (12)$$

The three conditions (10), (11) and (12) characterize the behavior of a generation t agent who is expecting a binding liquidity constraint.

The Tirole's Case (T-regime): the liquidity constraint is not binding: $M_t/P_{t+1} > \mu d_{t+1}$ and the expected return on money must be equal to the return on financial savings: $P_t/P_{t+1} = R_{t+1}$. In this case, the consumer's total savings is the same as in the Diamond's model:

$$\sigma_t = \sigma(w_t + \theta_t, R_{t+1}) = s_t + m_t \quad (13)$$

The consumer's savings can be shared by any proportion of money or capital. The only constraint is the cash-in-advance constraint, which is equivalent to:

$$(1 - \mu)m_t > \mu s_t \quad (14)$$

2.4 Firms

We assume that at each period t , there exists one competitive firm that uses neoclassical technology with constant returns to scale $F(K_t, L_t)$. F is increasing in its two arguments, concave, twice continuously differentiable. L_t is the quantity of labor used in production, paid by the real wage w_t . The profit maximization of the firm gives:

$$\begin{aligned} w_t &= F_L(K_t, L_t) \\ R_t &= F_K(K_t, L_t) \end{aligned}$$

3 Intertemporal equilibrium

3.1 Equilibrium characterization

Defining the variable k_t as K_t/N_t , the equilibrium wage and the factor of return for productive investments are:

$$w_t = F_L(k_t, 1) \equiv w(k_t) \quad (15)$$

$$R_t = F_K(k_t, 1) \equiv R(k_t) \quad (16)$$

\bar{M}_t being the total stock of money, we assume that it is held in equal shares by the agents. Equilibrium in the money market gives:

$$\bar{M}_t = N_t M_t = N_t P_t m_t \quad (17)$$

which deduces the real value of the lump sum monetary transfer:

$$\theta_t = \frac{\lambda_t \bar{M}_{t-1}}{N_t P_t} = \frac{\lambda_t}{1 + \lambda_t} m_t \quad (18)$$

Finally, we express that capital in period $t + 1$ results from savings of generation t agents:

$$K_{t+1} = N_t s_t \Leftrightarrow (1 + n)k_{t+1} = s_t \quad (19)$$

Using (17), the money gross return is given by:

$$P_t/P_{t+1} = \frac{\bar{M}_t/(N_t m_t)}{\bar{M}_{t+1}/(N_{t+1} m_{t+1})} = \frac{(1 + n)m_{t+1}}{(1 + \lambda_{t+1})m_t} \quad (20)$$

This return cannot be larger than the return of physical capital, or, with $k_{t+1} > 0$:

$$m_{t+1} \leq \frac{R_{t+1}(1 + \lambda_{t+1})}{1 + n} m_t \quad (21)$$

The two preceding cases will be studied separately.

The HS-regime: the liquidity constraint is binding between t and $t + 1$, and $P_t/P_{t+1} \leq R_{t+1}$. Using the expression of the cash-in-advance constraint (11), and equations (19) and (20), we obtain:

$$(1 - \mu)m_{t+1} = \mu(1 + \lambda_{t+1})R_{t+1}k_{t+1} \quad (22)$$

The dynamics of capital is given by replacing savings s_t in (19) by its expression (13), and in using ρ_{t+1} given by (12) and θ_t given by (18):

$$(1+n)k_{t+1} = \sigma \left(w_t + \frac{\lambda_t}{1+\lambda_t} m_t, \rho_{t+1} \right) - m_t \quad (23)$$

$$\text{with } \rho_{t+1} = \frac{(1+n)R_{t+1}k_{t+1}}{(1-\mu)(m_t + (1+n)k_{t+1})}$$

Finally, the condition, which ensures that money is a dominated asset, corresponds to equation (21). With (22), we can write:

$$\mu(1+n)k_{t+1} \leq (1-\mu)m_t \quad (24)$$

The T-regime: money is not dominated between t and $t+1$: $P_t/P_{t+1} = R_{t+1}$. In this case, (21) is verified with an equality, and we have:

$$m_{t+1} = \frac{R_{t+1}(1+\lambda_{t+1})}{1+n} m_t \quad (25)$$

The capital dynamics is always given by (23), but with $\rho_{t+1} = R_{t+1}$:

$$(1+n)k_{t+1} = \sigma \left(w_t + \frac{\lambda_t}{1+\lambda_t} m_t, R_{t+1} \right) - m_t \quad (26)$$

Finally, we must write that the cash-in-advance constraint of generation t agents is not binding, or (14). Using (19), we find:

$$\mu(1+n)k_{t+1} < (1-\mu)m_t \quad (27)$$

Here, we recognize the same relation as (24), but written with a strict inequality.

Finally, we have to make precise the equilibrium conditions for the first old agents in period $t=0$. Their budget constraint is: $d_0 = M_{-1}/P_0 + R_0 s_{-1}$ with $s_{-1} = K_0/N_{-1}$. They also must satisfy the cash-in-advance requirement: $M_{-1} \geq \mu P_0 d_0$. Thus, they must hold a real amount of money m_0 such that:

$$(1-\mu) \frac{m_0}{1+\lambda_0} \geq \mu R_0 k_0 (1+n) \quad (28)$$

It is now possible to give the following definition:

Definition 1 : Given an initial value k_0 and a sequence $(\lambda_t)_{t \geq 0}$ of rates of money creation, a sequence $(k_t, m_t)_{t \geq 0}$ with $k_t > 0$ and $m_t > 0$ that satisfy equations (15), (16) and (28), and for all $t \geq 0$,

- either the equilibrium conditions of the HS-regime: (22), (23) and (24)
- or the equilibrium conditions of the T-regime: (25), (26) and (27)

defines an intertemporal equilibrium with perfect foresight.

3.2 Two types of equilibrium dynamics

3.2.1 Dynamics in the HS-regime

Assuming that the liquidity constraint is binding along all the dynamics, from (22) and (23) these dynamics satisfy:

$$g(k_{t+1}, k_t, \mu, \lambda_t) = 0 \quad (29)$$

$$\text{with } g(k_{t+1}, k_t, \mu, \lambda_t) \equiv (1+n)k_{t+1} - \sigma \left(w_t + \frac{\lambda_t \mu R(k_t) k_t}{1-\mu}, \eta(k_{t+1}, k_t, \mu, \lambda_t) \right) \\ + \frac{\mu(1+\lambda_t)R(k_t)k_t}{1-\mu}$$

$$\text{and } \eta(k_{t+1}, k_t, \mu, \lambda_t) \equiv \frac{(1+n)R(k_{t+1})k_{t+1}}{\mu(1+\lambda_t)R(k_t)k_t + (1-\mu)(1+n)k_{t+1}}$$

This is a one-dimensional dynamics of k_t .

Assuming λ is constant, a stationary state in the HS-regime (k^*, m^*) satisfies the following equations:

$$(1+n)k^* + \frac{\mu(1+\lambda)}{1-\mu}R(k^*)k^* = \sigma \left(w(k^*), \frac{1}{\mu \frac{1+\lambda}{1+n} + \frac{1-\mu}{R(k^*)}} \right) \\ m^* = \frac{\mu(1+\lambda)}{1-\mu}R(k^*)k^*$$

From (24), (k^*, m^*) are such that money is a dominated asset if: $R(k^*) \geq (1+n)/(1+\lambda)$.

3.2.2 Dynamics in the T-regime

In the T-regime, equations (25) and (26) define a dynamical system of dimension 2, which does not depend on the cash-in-advance constraint. Providing that this constraint (condition (27)) is satisfied, the dynamics are the same as in the Diamond's model with a bubble. This is the Tirole (1985) model with monetary creation (in the Tirole model, the stock of money is constant). The cash-in-advance constraint compels the money stock to retain a positive value. Capital is a pre-determined variable when the real value of money is a forward-looking variable.

Assuming a constant rate of money creation: $\lambda_t = \lambda \forall t$, a stationary state (k^*, m^*) must satisfy (25), or

$$R(k^*) = \frac{1+n}{1+\lambda} \quad (30)$$

equation (26),

$$(1+n)k^* = \sigma \left(w(k^*) + \frac{\lambda}{1+\lambda} m^*, \frac{1+n}{1+\lambda} \right) - m^* \quad (31)$$

and the constraint (27):

$$m^* > \frac{\mu(1+n)}{1-\mu} k^*$$

From (30), the stationary state value $k_T^*(\lambda)$ is an increasing function of λ , which is equal to the golden rule when $\lambda = 0$. Defining the function:

$$\phi_\lambda(m) = \sigma \left(w(k^*) + \frac{\lambda}{1+\lambda} m, \frac{1+n}{1+\lambda} \right) - m - (1+n)k^*$$

$\phi_\lambda(m)$ is a decreasing function of m as:

$$\phi'_\lambda(m) = \frac{\lambda}{1+\lambda} \sigma'_1 - 1 < 0$$

which becomes negative when m is high enough. Thus, a stationary state in the T-regime exists if:

$$\phi_\lambda \left(\frac{\mu(1+n)}{1-\mu} k_T^*(\lambda) \right) > 0$$

and, when it exists, it is unique.

4 Study of the dynamics

4.1 Local study of the dynamics in the HS-regime

We make a local study of convergent trajectories in the HS-regime for a small value of μ . We consider a constant rate of monetary creation λ and a stationary equilibrium of the Diamond's economy k^D , assumed to be stable, and which satisfies $R(k^D) > (1+n)/(1+\lambda)$. For $\lambda = 0$, this last condition is equivalent to under-accumulation. If $\lambda > 0$, the condition is weaker than under-accumulation. More precisely, for any value of k^D , there always exists a value of λ such this assumption 2 is satisfied.

Assumption 2: k^D is a solution of $(1+n)k^D = \sigma(w(k^D), R(k^D))$, such that:

$$R(k^D) > \frac{1+n}{1+\lambda} \text{ and } \sigma'_w w'(k^D) + \sigma'_R R'(k^D) < 1+n$$

The last condition is equivalent to:

$$\frac{dk_{t+1}}{dk_t} = \frac{\sigma'_w w'(k^D)}{1 + n - \sigma'_R R'(k^D)}$$

strictly between 0 and 1.

This assumption means that a steady state of the Diamond's economy exists, which satisfies $R(k^D) > (1+n)/(1+\lambda)$. Furthermore, the dynamics in a neighborhood of this steady state are monotonic and convergent.

Proposition 1 : *There exists a neighborhood $I = (\underline{k}, \bar{k})$ of k^D and $\varepsilon > 0$, such that, for all $\mu \in (0, \varepsilon)$ and all $k_0 \in I$, there exists a unique intertemporal equilibrium $(k_t, m_t)_{t \geq 0}$ of the economy with a liquidity constraint, with initial conditions k_0 and $m_0 = \frac{\mu(1+\lambda)}{1-\mu} R_0 k_0$. This equilibrium is located in I (i.e. $k_t \in I \forall t$), and such that at each date the liquidity constraint is binding (HS-regime). This trajectory converges towards a stationary equilibrium $k^*(\mu)$ of the HS-regime. Moreover, the sequence $k_t(\mu)$ converges uniformly towards the Diamond trajectory $k_t(0)$ starting from k_0 when μ tends to 0.*

Proof: see Appendix 1.

This proposition shows that within a neighborhood of a stationary state that satisfies assumption 2, it is possible to define an intertemporal equilibrium with money and binding liquidity constraints at each period, when μ is small enough. The trajectory of the monetary economy converges uniformly towards the non-monetary economy as μ tends to 0. We will see, however, that this intertemporal equilibrium in the HS-regime is not unique, and that other equilibria with temporary bubbles exist.

4.2 Trajectories with bubbles

4.2.1 A one-period bubble

We will now study the following question: is it possible that a trajectory that converges in the HS-regime in the long run includes bubbles at certain dates ? Let us consider an intertemporal equilibrium, which is entirely in the HS-regime, as defined in proposition 1. Is it possible to modify this trajectory in one point in order to obtain a bubble (and to experience one period in the T-regime), and then to go back to the HS-regime?

A trajectory with one period in the T-regime can be characterized by the following conditions:

1. In $t = 0$, with k_0 and $m_0 = \frac{\mu(1+\lambda)}{1-\mu}R_0k_0$, there is no expected bubble; the HS-regime occurs. Thus, we have:

$$g(k_1, k_0, \mu, \lambda) = 0$$

$$m_1 = \frac{\mu(1+\lambda)}{1-\mu}R_1k_1$$

2. In $t = 1$, there is an expected bubble, and the economy experiences one period in the T-regime.

$$(1+n)k_2 = \sigma \left(w_1 + \frac{\lambda}{1+\lambda}m_1, R_2 \right) - m_1$$

$$m_2 = \frac{R_2(1+\lambda)}{1+n}m_1$$

The liquidity constraint can be written as:

$$(1-\mu)m_1 > \mu(1+n)k_2$$

or equivalently:

$$(1+\lambda)R_1k_1 > (1+n)k_2$$

3. In $t = 2$, there is no expected bubble: we go back to the HS-regime.

$$(1+n)k_3 = \sigma \left(w_2 + \frac{\lambda}{1+\lambda}m_2, \rho_3 \right) - m_2$$

$$\text{with } \rho_3 = \frac{(1+n)R_3k_3}{(1-\mu)(m_2 + (1+n)k_3)}$$

$$m_3 = \frac{\mu(1+\lambda)}{1-\mu}R_3k_3$$

4. After that, the dynamics of the HS-regime apply: for $t \geq 3$,

$$g(k_{t+1}, k_t, \mu, \lambda) = 0$$

$$m_{t+1} = \frac{\mu(1+\lambda)}{1-\mu}R_{t+1}k_{t+1}$$

Proposition 2 : *Under Assumption 2, we can modify a trajectory of the HS-regime sufficiently close to k^D and for μ small enough in introducing a bubble during one period.*

Proof: In a stable Diamond's equilibrium, which satisfies assumption 2 and in a neighborhood of this equilibrium, all these conditions are satisfied with $\mu = 0$ as the inequality $(1 + \lambda)R_1k_1 > (1 + n)k_2$. The implicit function theorem makes it possible to determine k_1, m_1, k_2, m_2, k_3 as functions of μ , when μ is small enough, and k_0 is sufficiently close to k^D , these functions satisfying conditions 1, 2 and 3. By continuity, it is possible to use neighborhoods, such that $k_3 \in I$. Proposition 1 can be applied from k_3 .

Following the same argument, we also find that bubbles can be introduced during a finite number of periods. It is even possible to introduce bubbles at an infinite number of periods, which prevents the trajectory from converging towards a stationary state. To do this, all that is necessary is to introduce a bubble every time k is close enough to the stationary state of the dynamics without bubbles.

4.2.2 Cyclical bubbles

We will now consider a trajectory, along which the economy experiences an infinite number of periods with bubbles. We are looking for cyclical equilibria where the economy experiences n periods in the T-regime, and p periods in the HS-regime. Let us consider such a cycle of period $n + p$. All the dynamics can be described by the orbit:

$$\left[\begin{pmatrix} m_1 \\ k_1 \end{pmatrix} \begin{pmatrix} m_2 \\ k_2 \end{pmatrix} \cdots \begin{pmatrix} m_n \\ k_n \end{pmatrix} \begin{pmatrix} m_{n+1} \\ k_{n+1} \end{pmatrix} \begin{pmatrix} m_{n+2} \\ k_{n+2} \end{pmatrix} \cdots \begin{pmatrix} m_{n+p} \\ k_{n+p} \end{pmatrix} \right]$$

Between periods 1 and $n + 1$, the economy is in the T-regime; between periods $n + 1$ and $(n + p) + 1$, the economy is in the HS-regime,

In Appendix 2, we prove the following result:

Proposition 3 : *Under assumption 2, $\forall n$ and p , such that $n \geq 1$ and $p \geq 1$, for μ , when it is small enough, there exists a cycle of period $n + p$ in the neighborhood of k^D , such that the economy experiences n periods in the T-regime, and p periods in the HS-regime.*

The proof given in appendix 2 consists in writing the system of $2(n + p)$ equations, which defines the cyclical orbit. These equations are satisfied for $\mu = 0$ with $k_i = k^D$ and $m_i = 0$. Thus, the implicit function theorem makes it possible to show the existence of a solution $\begin{pmatrix} m_i \\ k_i \end{pmatrix}_{1 \leq i \leq n+p}$ as a function of μ , when μ is within a neighborhood of $\mu = 0$. Finally, it is possible to prove that this solution satisfies the constraints (24) and (27).

The result is obtained for μ small enough. Michel and Wigniolle (2002) study a Cobb-Douglas example without monetary creation, which makes it possible to characterize the global dynamics of the economy. They explicitly find a decreasing relationship between the admissible period of the cycle (more precisely the admissible value of n) and the limit value for μ .

5 The Counter-bubble monetary policy

We will now consider the following question: is it possible to rule out temporary bubbles with an appropriate monetary policy? The intuition suggests that monetary creation induces inflation, which causes a drop in the return of money detention. Thus, the return on money can no longer be the same as the return on capital. When a bubble appears, it absorbs a share of savings, which can no longer finance productive investments. The bubble reduces capital accumulation. Assuming the economy is in under-accumulation, fighting the possible appearance of bubbles could be an objective of the monetary policy. Another argument could be that the existence of temporary bubbles creates multiple equilibria and indeterminacy. If monetary creation precludes the appearance of bubbles, it can stabilize the economy in the only remaining equilibrium in the HS-regime.

In this part, we first prove that an appropriate monetary policy can eliminate bubbles. Then, we explain why a government which cares about agents welfare can wish to eliminate bubbles. In all the study, we now consider Cobb-Douglas functions for the utility and production functions.

5.1 The appropriate policy

The utility and production functions are now respectively given by:

$$U(c_t, d_{t+1}) = (1 - a) \ln c_t + a \ln d_{t+1} \quad (32)$$

$$F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha} \quad (33)$$

Under the assumption of a constant money stock, Michel and Wigniolle (2002) show that this formulation makes it possible to obtain an explicit form of the global dynamics followed by the economy.

Let us assume that the rate of money creation is constant: $\lambda_t = \lambda, \forall t$. We introduce the new variable:

$$z_t = \frac{w_t}{m_t}$$

Appendix 3 shows that the dynamics of the economy can be summarized as: **HS-regime between t and $t + 1$:**

$$z_{t+1} = \frac{1-\mu}{\mu} \frac{1-\alpha}{\alpha} \frac{1}{1+\lambda} \equiv \tilde{z}(\lambda) \quad (34)$$

$$z_t \leq \frac{1}{a} \left(\frac{1}{\mu} - \frac{a\lambda}{1+\lambda} \right) \equiv \bar{z}(\lambda) \quad (35)$$

T-regime between t and $t + 1$:

$$z_{t+1} = \frac{(1-\alpha)a}{\alpha(1+\lambda)} \left[z_t - \left(\frac{1}{a} - \frac{\lambda}{1+\lambda} \right) \right] \equiv \phi_\lambda(z_t) \quad (36)$$

$$z_t < \frac{1}{a} \left(\frac{1}{\mu} - \frac{a\lambda}{1+\lambda} \right) = \bar{z}(\lambda) \quad (37)$$

and for the initial condition:

$$z_0 \leq \frac{1}{1+\lambda} \frac{1-\mu}{\mu} \frac{1-\alpha}{\alpha} = \tilde{z}(\lambda) \quad (38)$$

In both regimes, the dynamics of (k_t, m_t) are given by:

$$(1+n)k_{t+1} = \frac{k_t^\alpha}{z_t} \alpha(1+\lambda) \phi_\lambda(z_t) \quad (39)$$

$$m_t = \frac{(1-\alpha)k_t^\alpha}{z_t} \quad (40)$$

From (39), z_t must satisfy another constraint to ensure that k_t remains positive in all periods: for all t ,

$$\phi_\lambda(z_t) > 0 \quad (41)$$

We assume that the economy without money ($\mu = 0$ and $\lambda = 0$) is in under-accumulation:

$$\frac{a(1-\alpha)}{\alpha} < 1$$

The dynamics of this economy is similar to Michel and Wigniolle (2003) and can be summarized by the following diagram.

$\phi_\lambda(z)$ and $\tilde{z}(\lambda)$ are shifted when λ increases. If λ is high enough, $(\phi_\lambda)^2(\tilde{z}(\lambda))$ becomes negative, and the economy cannot experience any jump into the T-regime. Thus, bubbles are ruled out.

Formally, it is sufficient to prove the two following points in order to establish this result:

- $\exists \lambda > 0$ such that $\tilde{z}(\lambda) \leq \bar{z}(\lambda)$ and $(\phi_\lambda)(\tilde{z}(\lambda)) > 0$. Indeed, from (35) and (41), when these two conditions hold, an equilibrium associated with a rate of money creation λ and experiencing all periods in the HS-regime exists.
- $(\phi_\lambda)^2(\tilde{z}(\lambda)) < 0$: the transition from the HS-regime to the T-regime during one period (or more) is impossible.

The monetary policy no longer allows for a temporary bubble to exist, and ensures the uniqueness of the equilibrium. Let us show these two points.

Proof:

1. We first prove the inequality $\tilde{z}(\lambda) \leq \bar{z}(\lambda)$. We know that $\tilde{z}(0) < \bar{z}(0)$. The inequality $\bar{z}(\lambda) - \tilde{z}(\lambda) > 0$ is equivalent to:

$$\lambda(1 - a\mu) > \frac{1 - \alpha}{\alpha}a(1 - \mu) - 1$$

As $a(1 - \alpha) - \alpha < 0$, this inequality is always satisfied for all $\lambda > 0$ and $\mu < 1$.

2. We jointly consider the two inequalities:

$$\begin{aligned} (\phi_\lambda)(\tilde{z}(\lambda)) &> 0 \\ (\phi_\lambda)^2(\tilde{z}(\lambda)) &< 0 \end{aligned}$$

As ϕ_λ is an affine function, it is easy to calculate

$$(\phi_\lambda)^t(\tilde{z}(\lambda)) = \hat{z}(\lambda) + (\tilde{z}(\lambda) - \hat{z}(\lambda)) \left(\frac{(1 - \alpha)a}{\alpha(1 + \lambda)} \right)^t$$

with

$$\hat{z}(\lambda) \equiv - \frac{\frac{1 - \alpha}{\alpha(1 + \lambda)} \left(1 - \frac{a\lambda}{1 + \lambda} \right)}{1 - \frac{(1 - \alpha)a}{\alpha(1 + \lambda)}}$$

Thus, the inequality $(\phi_\lambda)^t(\tilde{z}(\lambda)) > 0$ is equivalent to (after some calculations):

$$\frac{1}{\mu} > 1 + \frac{\left(1 - \frac{a\lambda}{1 + \lambda} \right) \left[\left(\frac{\alpha(1 + \lambda)}{a(1 - \alpha)} \right)^t - 1 \right]}{1 - \frac{(1 - \alpha)a}{\alpha(1 + \lambda)}} \equiv \frac{1}{\mu_t(\lambda)} \Leftrightarrow \mu < \mu_t(\lambda) \quad (42)$$

As by assumption $\alpha > a(1 - \alpha)$, for $\lambda > 0$, $\alpha(1 + \lambda) > a(1 - \alpha)$. Thus, for λ given, $\mu_t(\lambda)$ is a decreasing sequence. And for a given value of t , $\mu_t(\lambda)$ is a decreasing function of λ . We have to choose λ such that $\mu < \mu_1(\lambda)$ and $\mu > \mu_2(\lambda)$, given that $\mu < \mu_2(0)$ (and $\mu < \mu_1(0)$), with:

$$\frac{1}{\mu_1(\lambda)} = 1 + \frac{\alpha[1 + (1 - a)\lambda]}{a(1 - \alpha)} \quad (43)$$

$$\frac{1}{\mu_2(\lambda)} = 1 + \frac{\alpha[1 + (1 - a)\lambda]}{a(1 - \alpha)} \left(1 + \frac{\alpha(1 + \lambda)}{a(1 - \alpha)}\right) \quad (44)$$

A simple figure allows to find the appropriate values of λ . λ must be in the following interval : $\lambda_2 < \lambda < \lambda_1$ with

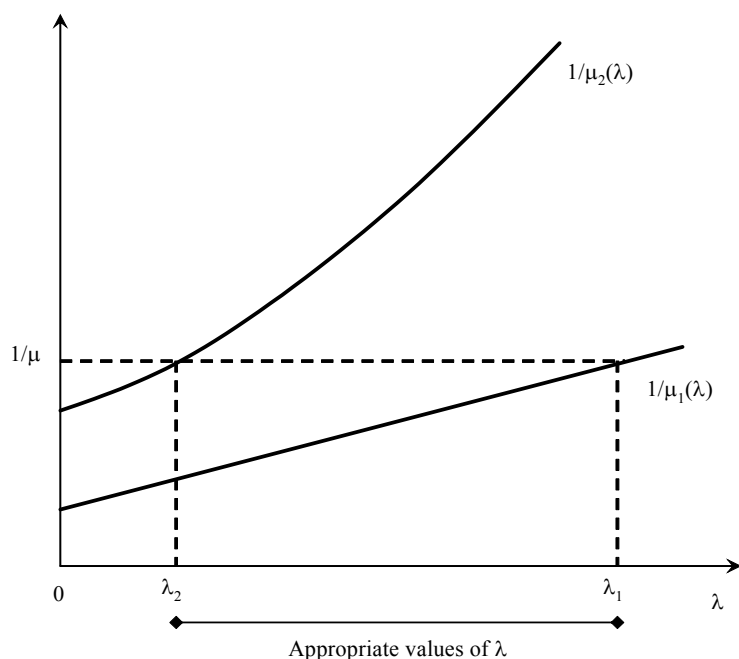
$$\lambda_1 = \frac{1}{1 - a} \left[\left(\frac{1}{\mu} - 1 \right) a \frac{1 - \alpha}{\alpha} - 1 \right]$$

$$\lambda_2 = \frac{-a_2 + \sqrt{a_2^2 - 4a_3a_1}}{2a_1}$$

$$\text{with } a_1 = (1 - a) \left[\frac{\alpha}{a(1 - \alpha)} \right]^2$$

$$a_2 = \frac{\alpha}{a(1 - \alpha)} \left[1 - a + \frac{(2 - a)\alpha}{a(1 - \alpha)} \right]$$

$$a_3 = 1 - \frac{1}{\mu} + \frac{\alpha}{a(1 - \alpha)} \left(1 + \frac{\alpha}{a(1 - \alpha)} \right)$$



Finally, we have shown that monetary policy makes it possible to obtain a unique equilibrium in the HS-regime. Temporary bubbles can no longer exist. The monetary policy reduces the return of money, and then it becomes impossible for money to have the same return as capital. This result has been obtained under the most unfavorable circumstances: those where the created money is given to young agents. This transfer plays in favor of savings, and tends to diminish the returns on capital. We have shown that this indirect effect is dominated by the direct effect of the monetary policy.

From this result, monetary policy can be viewed as a stabilization instrument. A temporary bubble on money leads to a deflation period, where money and capital have the same return. It creates a drop in capital accumulation and production because it absorbs a share of savings. We have shown that money creation can eliminate such deflation periods as it precludes the appearance of bubbles. This result has a Keynesian flavour, as it is proved that monetary creation can stabilize output fluctuations. But this effect is not provided by demand enhancing, but by bubbles elimination.

5.2 Why fighting bubbles ?

We have assumed in this section that the objective of the government was to fight bubbles. Indeed, bubbles reduce capital accumulation in an economy that is experiencing under-accumulation and create indeterminacy. Such

a government's objective, however, is partly *ad-hoc*, because it is not possible to prove that a counter-bubble policy is Pareto improving. We know that the existence of a bubble at some period is beneficial for the generation living during that period, because it increases returns on savings. It is detrimental, however, for the following generations, because it reduces capital accumulation in an economy that is experiencing under-accumulation. Thus, suppressing bubbles cannot be Pareto improving. The counter-bubble monetary policy also has a negative impact on agents' welfare, because it increases the difference between the returns on the two assets – money and capital – thus increasing the distortion related to money holding.

Nevertheless, it is possible to prove that eliminating temporary bubbles can be welfare improving in a weaker sense. Let us assume that agents who meet at the beginning of time do not know when they will be alive. We can prove that it exists a large range of basic parameters (a, α, μ) such that these agents choose to eliminate bubbles, by setting a high rate of growth of the money supply³.

More precisely we prove this property in two special cases. Indeed, when temporary bubbles may exist, there exists an infinity of equilibria that the economy can experience. Thus, we choose among these equilibria two examples of trajectories that we compare with the unique equilibrium of an economy where an appropriate policy has eliminated bubbles.

The first case is concerned with the long run of the economy. We compare the long run properties of two economies. In the first one, bubbles are ruled out by an appropriate monetary policy, and the economy experiences a constant trajectory. In the second one, without monetary policy, it experiences a limit 2-period cycle between HS and T regimes. We prove that long run utility levels are greater in the first economy for a large range of parameters (a, α, μ) . More precisely, it is possible to prove that, whatever their choice criterion is, agents at the beginning of time will prefer to eliminate bubbles for a large range of parameters, as utility levels are greater during an infinite number of periods.

The second argument is more concerned with the short run dynamics. We again compare two economies, starting from the same initial conditions. In the first one, an appropriate monetary policy has eliminated bubbles. The second one without monetary policy experiences at some date *one* period in the T-regime. We prove that for any number of periods P , there exists parameters such that utility levels in the second economy are smaller than the minimum level in the first economy during at least P periods. Thus,

³This particular notion of welfare improvement is a suggestion of an anonymous referee that we thank for this idea.

under the veil of ignorance, if agents use a Rawlsian criterion between the different periods, they will prefer to live in the first economy.

5.2.1 The long run argument

We compare the long run properties of two economies. In the first one, bubbles are ruled out by an appropriate monetary policy, and the economy experiences a constant trajectory. In the second one, without monetary policy, it experiences a limit 2-period cycle between HS and T regimes.

For the first economy, the long run value of the capital stock per young agents is given by:

$$k_{HS}^*(\lambda) = \left[\frac{\alpha}{1+n} \frac{(1+\lambda)\phi_\lambda(\tilde{z}(\lambda))}{\tilde{z}(\lambda)} \right]^{\frac{1}{1-\alpha}}$$

The second economy follows a cycle of period 2. z_t oscillates between $\tilde{z}(0)$ and $\phi_0(\tilde{z}(0))$. From the preceding part, we now that such trajectory is possible under the condition $(\phi_0)^2(\tilde{z}(0))$. Following (39), k_t oscillates in the long run between the 2 values (k_1, k_2) such that:

$$\begin{aligned} (1+n)k_2 &= \frac{\alpha\phi_0(\tilde{z}(0))}{\tilde{z}(0)} k_1^\alpha \\ (1+n)k_1 &= \frac{\alpha(\phi_0)^2(\tilde{z}(0))}{\phi_0(\tilde{z}(0))} k_2^\alpha \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} k_1 &= \left[\frac{\alpha}{1+n} [\phi_0(\tilde{z}(0))]^{\frac{-1+\alpha}{1+\alpha}} [\tilde{z}(0)]^{\frac{-\alpha}{1+\alpha}} [(\phi_0)^2(\tilde{z}(0))]^{\frac{1}{1+\alpha}} \right]^{\frac{1}{1-\alpha}} \\ k_2 &= \left[\frac{\alpha}{1+n} [\phi_0(\tilde{z}(0))]^{\frac{1-\alpha}{1+\alpha}} [\tilde{z}(0)]^{\frac{-1}{1+\alpha}} [(\phi_0)^2(\tilde{z}(0))]^{\frac{\alpha}{1+\alpha}} \right]^{\frac{1}{1-\alpha}} \end{aligned}$$

It remains to prove that long run utility levels are greater in the first economy for some range of parameters (a, α, μ) . This property is proved in considering the limit case when $\mu \rightarrow \mu_2(0)$, with $\mu < \mu_2(0)$. In this case, the value of λ allowing to eliminate bubbles tends toward 0. And, by the definition $\mu_2(0)$, we have that $(\phi_0)^2(\tilde{z}(0))$ tends toward 0. Thus, we obtain:

$$k_{HS}^*(\lambda) \xrightarrow{\lambda \rightarrow 0} \left[\frac{\alpha}{1+n} \frac{\phi_0(\tilde{z}(0))}{\tilde{z}(0)} \right]^{\frac{1}{1-\alpha}}$$

and (k_1, k_2) tends toward $(0, 0)$. In the first economy, the long run utility level tends to some constant value, when in the second economy it tends

to $-\infty^4$. In this limit case, an agent who has to choose between these two long run states will always prefer to live in the first economy, whatever is his choice criterion. By continuity, for some choice criterion, an agent always chooses to live in the first economy for μ close to $\mu_2(0)$.

In this example, we choose for the second economy a case that can be viewed as the worst: bubbles appears all two periods. In the following one, we consider the case of an economy experiencing only one period in the T-regime.

5.2.2 The short run argument

We now consider the dynamics of two economies, starting from the same initial value of capital k_0 . In the first one, bubbles are ruled out by an appropriate monetary policy λ , and we have for all t $z_t = \tilde{z}(\lambda)$. From (39), k_t follows the following dynamics:

$$k_{t+1}^1 = \frac{\alpha}{1+n} (k_t^1)^\alpha \left[\frac{(1+\lambda)\phi_\lambda(\tilde{z}(\lambda))}{\tilde{z}(\lambda)} \right]$$

We assume that the second economy, without monetary policy, experiences all periods in the HS-regime except one: between τ and $\tau + 1$, the T-regime occurs. The dynamics of k_t are:

$$\begin{aligned} \forall t \neq \tau + 1, k_{t+1}^2 &= \frac{\alpha}{1+n} (k_t^2)^\alpha \left[\frac{\phi_0(\tilde{z}(0))}{\tilde{z}(0)} \right] \\ \text{For } t = \tau + 1, k_{\tau+2}^2 &= \frac{\alpha}{1+n} (k_t^2)^\alpha \left[\frac{(\phi_0)^2(\tilde{z}(0))}{\phi_0(\tilde{z}(0))} \right] \end{aligned}$$

When the economy experiences a period in the T-regime between τ and $\tau + 1$, the real value of money in $\tau + 1$ is higher and it absorbs higher part of savings. Thus, there is a drop of the level of capital per young in $\tau + 2$.

As in the preceding example, we study the limit case when $\mu \rightarrow \mu_2(0)$, with $\mu < \mu_2(0)$. In this case, the value of λ allowing to eliminate bubbles tends toward 0, and $(\phi_0)^2(\tilde{z}(0))$ tends toward 0. Thus, we obtain:

$$\begin{aligned} \forall t \leq \tau + 1, \lim_{\mu \rightarrow \mu_2(0)} k_t^1 &= \lim_{\mu \rightarrow \mu_2(0)} k_t^2 \\ \lim_{\mu \rightarrow \mu_2(0)} k_{\tau+2}^2 &= 0 \end{aligned}$$

⁴This last point is easy to prove, as consumption levels (c_1, c_2, d_1, d_2) are respectively bounded by $(f(k_1), f(k_2), (1+n)f(k_1), (1+n)f(k_2))$.

If we consider a given integer P , for all values of t such that $\tau + 2 \leq t \leq \tau + 2 + P$, $\lim_{\mu \rightarrow \mu_2(0)} k_t^2 = 0$. By continuity, for a given number P of periods, it is possible to find a range for the parameter μ (in a neighborhood of $\mu_2(0)$) and a corresponding value for the monetary policy λ , such that the utility in the second economy is smaller than the minimum value of utility in the first economy during P periods. Thus, under the veil of ignorance, if agents use a Rawlsian criterion between the different periods, they will prefer to live in the first economy.

6 Conclusion

In this paper, we have studied the dynamic properties of an overlapping generations model with capital and money. The medium-of-exchange role of money was taken into account by assuming that agents are subject to a cash-in-advance constraint. We have studied the intertemporal equilibrium of this economy in a general case, without excluding a priori the existence of bubbles (temporary or permanent).

In assuming that a stationary equilibrium exists in an economy without money, we have shown that a monetary equilibrium with a binding liquidity constraint for a sufficiently low liquidity constraint exists within a neighborhood of this equilibrium. This monetary equilibrium is not unique, however: we have proved that temporary bubbles on money may appear at some periods. Notably, we have shown that, within the neighborhood of an equilibrium without bubbles, cyclical equilibria exist along which the economy experiences n periods of temporary bubbles and p periods without bubbles, n and p being some integers. Thus, the existence of temporary bubbles leads to a multiplicity of equilibria.

Finally, we have assumed that the government's objective is to fight bubbles using a monetary policy. Assuming Cobb-Douglas functions for utility and production, we have shown that a high enough rate of monetary creation eliminates temporary bubbles. Thus, the economy follows the only equilibrium trajectory that does not have bubbles, and indeterminacy disappears. We develop two arguments in favor of such a policy. In the long run, we have shown that the periodic appearance of bubbles can lead to a utility level smaller than the one reached along a bubbleless trajectory. In the short run, we have shown that the only appearance of a one period bubble is enough to drop the utility levels during many periods far from the level reached along the bubbleless trajectory. For these reasons, fighting bubbles can be an objective of monetary policy.

Appendix 1: proof of proposition 1.

A trajectory of the HS-regime is characterized by the dynamical equation given by (29):

$$g(k_{t+1}, k_t, \mu, \lambda) = 0, \quad t \geq 0$$

The condition that assures that money is a dominated asset (24), with (22), is equivalent to:

$$(1+n)k_{t+1} < R(k_t)k_t(1+\lambda)$$

By assumption, we have $g(k^D, k^D, 0, \lambda) = 0$. g being continuously differentiable with respect to the first partial derivative $g'_1(k^D, k^D, 0, \lambda) \neq 0$, we can apply the implicit function theorem. There exists $\varepsilon > 0$ and I neighborhood of k^D , such that for all $k \in I$ and all μ , $|\mu| < \varepsilon$, the equation $g(x, k, \mu, \lambda) = 0$ admits a unique solution $x = h(k, \mu)$ in a neighborhood of k^{D5} . We have: $k^D = h(k^D, 0)$ and $(1+n)k^D < R(k^D)k^D(1+\lambda)$. It is possible to restrict the neighborhoods in order that the solution satisfies: $(1+n)x < R(k)k(1+\lambda)$, for all $k \in I$ and all μ , $|\mu| < \varepsilon$.

The function h is differentiable and satisfies:

$$h'_1(k^D, 0) = -g'_2/g'_1 = \frac{\sigma'_w w'(k^D)}{1+n - \sigma'_R R'(k^D)}$$

This derivative is strictly between 0 and 1. We can again restrict the neighborhoods so that the derivative $h'_1(k, \mu)$ is also between 0 and 1. Applying the implicit function theorem to the two-variable function $g(k, k, \mu, \lambda)$, we obtain a stationary equilibrium $k^*(\mu) \in I$ for all $\mu < \varepsilon$.

Let us show that for all $k_0 \in I$, the trajectory $k_t(\mu)$ such that:

$$k_{t+1}(\mu) = h(k_t(\mu), \mu), \quad k_0(\mu) = k_0$$

is defined, belongs to I and converges towards $k^*(\mu)$.

By recurrence, if $k_t(\mu) \in I$,

$$\begin{aligned} k_{t+1}(\mu) - k^*(\mu) &= h(k_t(\mu), \mu) - h(k^*(\mu), \mu) \\ &= h'_1(x_t(\mu), \mu) (k_t(\mu) - k^*(\mu)) \end{aligned}$$

with $x_t(\mu)$ between $k^*(\mu)$ and $k_t(\mu)$ and belonging to I . As $h'_1 \in (0, 1)$, the distance between $k_{t+1}(\mu)$ and $k^*(\mu)$ is strictly smaller than the one between $k_t(\mu)$ and $k^*(\mu)$. Thus, $k_{t+1}(\mu)$ is defined and belongs to I , and the sequence $k_t(\mu) - k^*(\mu)$ tends toward 0.

⁵As λ is assumed to be fixed in this part, it is not useful to make explicit the dependence of x with respect to λ .

The sequence $k_t(\mu)$ belongs to the HS-regime as it satisfies:

$$(1+n)k_{t+1}(\mu) < R(k_t(\mu))k_t(\mu)(1+\lambda)$$

from the preceding restrictions taken on I .

The uniform convergence of this sequence when $\mu \rightarrow 0$ results from its convergence and the convergence of $k^*(\mu)$ towards $k^*(0) = k^D$. Indeed, we can write:

$$|k_t(\mu) - k_t(0)| \leq |k_t(\mu) - k^*(\mu)| + |k^*(\mu) - k^D| + |k_t(0) - k^D|$$

From the preceding recurrence, the first and third terms can be bounded above by a sequence $z_t \geq 0$, which has a null limit and does not depend on μ . Thus, $\forall \varepsilon > 0$, when t is high enough ($t \geq T$), these two terms can be made smaller than ε . We also have: $\lim_{\mu \rightarrow 0} |k^*(\mu) - k^D| = 0$. Finally, for $t < T$, the continuity with respect to the variable μ makes it possible to obtain the convergence: $\sup_{t < T} |k_t(\mu) - k_t(0)| \rightarrow 0$ when $\mu \rightarrow 0$.

Appendix 2: proof of proposition 3.

We define the following notation:

$$\chi(k, m, \mu) = \frac{(1+n)R(k)k}{(1-\mu)(m + (1+n)k)}$$

The cyclical orbit of period $n+p$

$$\left[\begin{pmatrix} m_1 \\ k_1 \end{pmatrix} \quad \begin{pmatrix} m_2 \\ k_2 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} m_n \\ k_n \end{pmatrix} \quad \begin{pmatrix} m_{n+1} \\ k_{n+1} \end{pmatrix} \quad \begin{pmatrix} m_{n+2} \\ k_{n+2} \end{pmatrix} \quad \cdots \quad \begin{pmatrix} m_{n+p} \\ k_{n+p} \end{pmatrix} \right]$$

has to satisfy the following equations:

$$\left\{ \begin{array}{l}
\text{Between periods } n + p \text{ and } 1 : \text{ HS-regime} \\
m_1 - \frac{\mu(1+\lambda)}{1-\mu} R(k_1) k_1 = 0 \\
(1+n)k_1 - \sigma \left[w(k_{n+p}) + \frac{\lambda}{1+\lambda} m_{n+p}, \chi(k_1, m_{n+p}, \mu) \right] + m_{n+p} = 0 \\
\\
\text{Between periods } 1 \text{ and } 2 : \text{ T-regime} \\
m_2 - \frac{(1+\lambda)R(k_2)}{1+n} m_1 = 0 \\
(1+n)k_2 - \sigma \left[w(k_1) + \frac{\lambda}{1+\lambda} m_1, R(k_2) \right] + m_1 = 0 \\
\\
\text{Between periods } 2 \text{ and } 3 : \text{ T-regime} \\
m_3 - \frac{(1+\lambda)R(k_3)}{1+n} m_2 = 0 \\
(1+n)k_3 - \sigma \left[w(k_2) + \frac{\lambda}{1+\lambda} m_2, R(k_3) \right] + m_2 = 0 \\
\vdots \\
\text{Between periods } n \text{ and } n + 1 : \text{ T-regime} \\
m_{n+1} - \frac{(1+\lambda)R(k_{n+1})}{1+n} m_n = 0 \\
(1+n)k_{n+1} - \sigma \left[w(k_n) + \frac{\lambda}{1+\lambda} m_n, R(k_{n+1}) \right] + m_n = 0 \\
\\
\text{Between periods } n + 1 \text{ and } n + 2 : \text{ HS-regime} \\
m_{n+2} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+2}) k_{n+2} = 0 \\
(1+n)k_{n+2} - \sigma \left[w(k_{n+1}) + \frac{\lambda}{1+\lambda} m_{n+1}, \chi(k_{n+2}, m_{n+1}, \mu) \right] + m_{n+1} = 0 \\
\\
\text{Between periods } n + 2 \text{ and } n + 3 : \text{ HS-regime} \\
m_{n+3} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+3}) k_{n+3} = 0 \\
(1+n)k_{n+3} - \sigma \left[w(k_{n+2}) + \frac{\lambda}{1+\lambda} m_{n+2}, \chi(k_{n+3}, m_{n+2}, \mu) \right] + m_{n+2} = 0 \\
\vdots \\
\text{Between periods } n + p - 1 \text{ and } n + p : \text{ HS-regime} \\
m_{n+p} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+p}) k_{n+p} = 0 \\
(1+n)k_{n+p} - \sigma \left[w(k_{n+p-1}) + \frac{\lambda}{1+\lambda} m_{n+p-1}, \chi(k_{n+p}, m_{n+p-1}, \mu) \right] + m_{n+p-1} = 0
\end{array} \right. \quad (45)$$

In addition to these equations, the periodic equilibrium has to satisfy $n + p$ constraints that have the same expression in both types of regimes (cf. constraints (24) and (27)):

$$\forall t = 1, \dots, n + p, (1 - \mu)m_t > \mu(1 + n)k_{t+1} \quad (46)$$

(with the convention that $k_{n+p+1} = k_1$).

We first show the existence of a unique solution to the system (45) for each value of μ within a neighborhood of 0. We then prove that such a solution satisfies all the constraints (46) when μ is sufficiently small.

All conditions in the system (45) are satisfied for $\mu = 0$, $k_i = k^D$ and $m_i = 0$. We denote the vector of dimension $2(n+p)$ by X

$$X = (m_1, m_2, \dots, m_n, m_{n+1}, \dots, m_{n+p}, k_1, k_2, \dots, k_n, k_{n+1}, \dots, k_{n+p})$$

and we note:

$$X_0 = \left(\underbrace{0, 0, \dots, 0, 0, \dots, 0}_{n+p \text{ times}}, \underbrace{k^D, k^D, \dots, k^D, k^D, \dots, k^D}_{n+p \text{ times}} \right)$$

The $2(n+p)$ preceding equations can be expressed by defining function $Z : \mathbb{R}^{2(n+p)+1} \rightarrow \mathbb{R}^{2(n+p)}$, such that:

$$Z(X, \mu) = 0$$

with: $Z(X, \mu) \equiv$

$$\left(\begin{array}{l} m_1 - \frac{\mu(1+\lambda)}{1-\mu} R(k_1) k_1 \\ m_2 - \frac{(1+\lambda)R(k_2)}{1+n} m_1 \\ m_3 - \frac{(1+\lambda)R(k_3)}{1+n} m_2 \\ \vdots \\ m_{n+1} - \frac{(1+\lambda)R(k_{n+1})}{1+n} m_n \\ m_{n+2} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+2}) k_{n+2} \\ m_{n+3} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+3}) k_{n+3} \\ \vdots \\ m_{n+p} - \frac{\mu(1+\lambda)}{1-\mu} R(k_{n+p}) k_{n+p} \\ (1+n)k_2 - \sigma \left[w(k_1) + \frac{\lambda}{1+\lambda} m_1, R(k_2) \right] + m_1 \\ (1+n)k_3 - \sigma \left[w(k_2) + \frac{\lambda}{1+\lambda} m_2, R(k_3) \right] + m_2 \\ \vdots \\ (1+n)k_{n+1} - \sigma \left[w(k_n) + \frac{\lambda}{1+\lambda} m_n, R(k_{n+1}) \right] + m_n \\ (1+n)k_{n+2} - \sigma \left[w(k_{n+1}) + \frac{\lambda}{1+\lambda} m_{n+1}, \chi(k_{n+2}, m_{n+1}, \mu) \right] + m_{n+1} \\ (1+n)k_{n+3} - \sigma \left[w(k_{n+2}) + \frac{\lambda}{1+\lambda} m_{n+2}, \chi(k_{n+3}, m_{n+2}, \mu) \right] + m_{n+2} \\ \vdots \\ (1+n)k_{n+p} - \sigma \left[w(k_{n+p-1}) + \frac{\lambda}{1+\lambda} m_{n+p-1}, \chi(k_{n+p}, m_{n+p-1}, \mu) \right] + m_{n+p-1} \\ (1+n)k_1 - \sigma \left[w(k_{n+p}) + \frac{\lambda}{1+\lambda} m_{n+p}, \chi(k_1, m_{n+p}, \mu) \right] + m_{n+p} \end{array} \right)$$

We have chosen this particular order for the different equations, because it makes easier the calculations. We know that $Z(X_0, 0) = 0$ and that Z is continuously differentiable within a neighborhood of $(X_0, 0)$. If the differential

$d_X Z(X_0, 0)$ is bijective from $\mathbb{R}^{2(n+p)}$ on $\mathbb{R}^{2(n+p)}$, the equation $Z(X, \mu) = 0$ defines an implicit function within a neighborhood of $\mu = 0$. More precisely, $\exists \alpha > 0$ and $\beta > 0$, such that $\forall \mu, 0 \leq \mu < \alpha, \exists ! X \in B(X_0, \beta)$, such that: $Z(X, \mu) = 0$.

Consequently, it remains to be proved that $d_X Z(X_0, 0)$ is bijective. We set:

$$\begin{aligned} A &= -\sigma'_w [w(k^D), R(k^D)] w'(k^D) \\ B &= 1 + n - \sigma'_R [w(k^D), R(k^D)] R'(k^D) \\ C &= 1 + \frac{\sigma'_R [w(k^D), R(k^D)] R(k^D)}{(1+n)k^D} - \frac{\lambda}{1+\lambda} \sigma'_w [w(k^D), R(k^D)] \\ D &= 1 - \frac{\lambda}{1+\lambda} \sigma'_w [w(k^D), R(k^D)] \end{aligned}$$

In the calculus of $d_X Z(X_0, 0)$, it is important to note that:

$$\begin{aligned} \frac{\partial \chi}{\partial k}(k^D, 0, 0) &= R'(k^D) \\ \frac{\partial \chi}{\partial m}(k^D, 0, 0) &= -\frac{R(k^D)}{(1+n)k^D} \end{aligned}$$

We define $I_{j,j}$ as the identity matrix of dimension (j, j) . We also define $N_{j,j}^h$ and $N_{j,j}^l$ as the matrixes of dimension (j, j) such that:

$$N_{j,j}^h = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix}$$

$$N_{j,j}^l = (N_{j,j}^h)' = \begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & & \\ 0 & 1 & \ddots & \\ & & \ddots & 0 & 0 \\ 0 & & & 1 & 0 \end{bmatrix}$$

Thus, $d_X Z(X_0, 0)$ can be written:

$$d_X Z(X_0, 0) = \begin{bmatrix} E_{n,n} & 0_{n,p} & 0_{n,n} & 0_{n,p} \\ F_{p,n} & I_{p,p} & 0_{p,n} & 0_{p,p} \\ DI_{n,n} & 0_{n,p} & G_{n,n} & H_{n,p} \\ 0_{p,n} & CI_{p,p} & H_{p,n} & J_{p,p} \end{bmatrix}$$

where:

$$\begin{aligned}
E_{n,n} &= I_{n,n} - \frac{(1+\lambda)R(k^D)}{1+n} N_{n,n}^l \\
F_{p,n} &= \begin{bmatrix} 0 & \dots\dots\dots & 0 & \frac{-(1+\lambda)R(k^D)}{1+n} \\ 0 & \dots\dots\dots & 0 & 0 \\ \vdots & \dots\dots\dots & \vdots & \vdots \\ 0 & \dots\dots\dots & 0 & 0 \end{bmatrix} \\
G_{n,n} &= AI_{n,n} + BN_{n,n}^h \\
H_{i,j} &= \begin{bmatrix} 0 & 0 & \dots\dots\dots & 0 \\ 0 & & \dots\dots\dots & 0 \\ \vdots & & \dots\dots\dots & \vdots \\ B & 0 & \dots\dots\dots & 0 \end{bmatrix} \\
J_{p,p} &= AI_{p,p} + BN_{p,p}^h
\end{aligned}$$

Developing the determinant of this matrix by the lines $1, \dots, n+p$, we find:

$$\det d_X Z(X_0, 0) = \det \begin{bmatrix} G_{n,n} & H_{n,p} \\ H_{p,n} & J_{p,p} \end{bmatrix}$$

Then, developing this last determinant by the first column, we have:

$$\det d_X Z(X_0, 0) = A^{n+p} + (-1)^{1+n+p} B^{n+p} = A^{n+p} - (-B)^{n+p}$$

We know that $A < 0$. From assumption 2, we know that $B > 0$ and that $A + B > 0$ or $A > -B$. Thus, $\det d_X Z(X_0, 0) \neq 0$ and has the sign of $(-1)^{n+p+1}$.

The second part of the demonstration needs to prove that (46) holds when μ is small enough.

For t , such that $n+2 \leq t \leq n+p$ or $t=1$, m_t is such that:

$$m_t = \frac{\mu(1+\lambda)}{1-\mu} R(k_t) k_t$$

Replacing m_t in (46) gives:

$$\forall t \text{ s. t. } n+2 \leq t \leq n+p \text{ or } t=1, \frac{(1+\lambda)R(k_t)}{1+n} k_t > k_{t+1} \quad (47)$$

For t , such that $2 \leq t \leq n+1$, it is easy with (45) to obtain:

$$m_t = \frac{(1+\lambda)R(k_t)}{1+n} \frac{(1+\lambda)R(k_{t-1})}{1+n} \dots \frac{(1+\lambda)R(k_2)}{1+n} \frac{\mu(1+\lambda)}{1-\mu} R(k_1) k_1$$

Replacing m_t in (46) gives, $\forall t$ such that $2 \leq t \leq n + 1$,

$$\frac{(1 + \lambda)R(k_t)}{1 + n} \frac{(1 + \lambda)R(k_{t-1})}{1 + n} \dots \frac{(1 + \lambda)R(k_2)}{1 + n} \frac{(1 + \lambda)R(k_1)}{1 + n} k_1 > k_{t+1} \quad (48)$$

Thus, we have proved that the system (45) with all the constraints (46) is equivalent to the system (45) with (47) for $n + 2 \leq t \leq n + p$ or $t = 1$ and (48) for $2 \leq t \leq n + 1$.

Finally, from assumption 2, (47) and (48) are satisfied in X_0 , because $R(k^D) > (1 + n)/(1 + \lambda)$. Thus, (47) and (48) are satisfied when μ is small enough.

Appendix 3: the intertemporal equilibrium in a Cobb-Douglas economy

With a Cobb-Douglas utility function, the utility maximization under the budgetary constraints gives the same expression of total savings in both regimes:

$$\sigma_t = a(w_t + \theta_t) = s_t + m_t \quad (49)$$

Indeed, savings no longer depend on their return.

As total savings σ_t is the same in the two regimes, it is possible to express (23) and (26) by the same equation:

$$(1 + n)k_{t+1} = aw_t - \left(1 - \frac{a\lambda_t}{1 + \lambda_t}\right) m_t \quad (50)$$

Finally, we write the specific conditions in each regime. We introduce the new variable

$$z_t = \frac{w_t}{m_t}$$

HS-regime:

Real balances of the agents, given by (22), now become:

$$(1 - \mu)m_{t+1} = (1 + \lambda_{t+1})\mu\alpha k_{t+1}^\alpha = (1 + \lambda_{t+1})\mu \frac{\alpha}{1 - \alpha} w_{t+1} \quad (51)$$

thus, (51) can be written:

$$z_{t+1} = \frac{1 - \mu}{\mu} \frac{1 - \alpha}{\alpha} \frac{1}{1 + \lambda_{t+1}} \equiv \tilde{z}(\lambda_{t+1}) \quad (52)$$

The condition ensuring that the return on money does not exceed the one on capital (24) was:

$$\mu(1 + n)k_{t+1} \leq (1 - \mu)m_t$$

In using (50) for eliminating the variable k_{t+1} , this condition becomes:

$$z_t \leq \frac{1}{a} \left(\frac{1}{\mu} - \frac{a\lambda_t}{1 + \lambda_t} \right) \quad (53)$$

The T-regime:

In using (50) and the variable z_t , equation (25) can be written:

$$z_{t+1} = \frac{1 - \alpha}{\alpha(1 + \lambda_{t+1})} \left[az_t - \left(1 - \frac{a\lambda_t}{1 + \lambda_t} \right) \right] \quad (54)$$

The condition ensuring that the liquidity constraint is satisfied remains:

$$z_t < \frac{1}{a} \left(\frac{1}{\mu} - \frac{a\lambda_t}{1 + \lambda_t} \right) \quad (55)$$

Finally, the liquidity constraint for the first old agents (28) can be written:

$$(1 - \mu) \frac{m_0}{1 + \lambda_0} \geq \mu R_0 k_0 = \mu \frac{\alpha}{1 - \alpha} w_0$$

or:

$$z_0 \leq \frac{1}{1 + \lambda_0} \frac{1 - \mu}{\mu} \frac{1 - \alpha}{\alpha} = \tilde{z}(\lambda_0) \quad (56)$$

Finally, from (50), in both regimes the corresponding dynamics of k_t can be written:

$$(1 + n)k_{t+1} = \frac{w_t}{z_t} \left[az_t - \left(1 - \frac{a\lambda_t}{1 + \lambda_t} \right) \right]$$

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