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## Chapter 1

# FUZZY MEASURES AND INTEGRALS IN MCDA

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**Abstract** This chapter aims at a unified presentation of various methods of MCDA based on fuzzy measures (capacity) and fuzzy integrals, essentially the Choquet and Sugeno integral. A first section sets the position of the problem of multicriteria decision making, and describes the various possible scales of measurement (cardinal unipolar and bipolar, and ordinal). Then a whole section is devoted to each case in detail: after introducing necessary concepts, the methodology is described, and the problem of the practical identification of fuzzy measures is given. The important concept of interaction between criteria, central in this chapter, is explained in detail. It is shown how it leads to  $k$ -additive fuzzy measures. The case of bipolar scales leads to the general model based on bi-capacities, encompassing usual models based on capacities. A general definition of interaction for bipolar scales is introduced. The case of ordinal scales leads to the use of Sugeno integral, and its symmetrized version when one considers symmetric ordinal scales. A practical methodology for the identification of fuzzy measures in this context is given.

**Keywords:** Choquet integral, fuzzy measure, interaction, bi-capacities.

## 1. Introduction

MultiCriteria Decision Aid (MCDA) aims at modeling the preferences of a Decision Maker (DM) over alternatives described by several points of view, which are denoted by  $X_1, \dots, X_n$ . An alternative is characterized by a value w.r.t. each point of view and is thus identified with a point in the Cartesian product  $X$  of the points of view:  $X = X_1 \times \dots \times X_n$ . We denote by  $N := \{1, \dots, n\}$  the index set of points of view. The preference relation of the DM over alternatives is denoted by  $\succeq$ . For  $x, y \in X$ , “ $x \succeq y$ ” means that the DM prefers alternative  $x$  to  $y$ .

The main concern in practice is to come up with the knowledge of  $\succeq$  on  $X \times X$  from a relatively small amount of questions asked to the DM on  $\succeq$ . The information provided by the DM can be composed of examples of comparisons between alternatives, which gives  $\succeq$  on a subset of  $X \times X$ , as well as more qualitative judgments, whose modelling is more complex, and depends on the kind of representation of  $\succeq$  we choose. In general, we look for a *numerical representation* [44]  $u : X \rightarrow \mathbb{R}$  such that:

$$\forall x, y \in X, \quad x \succeq y \Leftrightarrow u(x) \geq u(y). \quad (1.1)$$

It is classical to write  $u$  in the following way [43]:

$$u(x) = F(u_1(x_1), \dots, u_n(x_n)) \quad \forall x \in X, \quad (1.2)$$

where the  $u_i$ 's :  $X_i \rightarrow \mathbb{R}$  are called the *utility functions* and  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is an *aggregation function*. A result by Krantz *et al.* gives the axioms that characterize the representation of  $\succeq$  by (1.2) [44]. As it will be detailed in Section 2.1, the *weak separability* axiom is the key axiom that justifies the construction of utility functions, that is partial preference relations over the points of view, from the overall preference relation  $\succeq$ . A criterion is defined as a preference relation  $\succeq_i$  over one point of view  $X_i$ . Thus a criterion is the association of one point of view  $X_i$  with its related utility function  $u_i$ .

In practice, we restrict ourself to a family  $\mathcal{F}$  of aggregation functions (parameterized by some coefficients). The justification of the use of a special family is based on an axiomatic approach. The axioms that characterize the family should be in accordance with the problem in consideration and the behaviour of the decision maker. The DM has then to provide the needed information to set the parameters of the model. The more restrictive the family is, the less representative it is, but the less information the DM shall give.

The most classical functions used to aggregate the criteria are the weighted sums  $F(u_1, \dots, u_n) = \sum_{i=1}^n \alpha_i u_i$ . As an aggregation operator, they are characterized by an independence axiom [73, 43]. This property implies some limitations in the way the weighted sum can model typical decision behaviours. To make this more precise, let us consider the example of two criteria having the same importance, an example which we will consider in more details in Section 3.5. We are interested in the following four alternatives:  $x$  is bad in both criteria,  $y$  is bad in the first criterion but good at the second one,  $z$  is good in the first criterion but bad in the second one, and  $t$  is good in both. Clearly  $x \prec t$  and the DM is equally satisfied by  $y$  and  $z$  since the two criteria have the same importance. However, the comparison of  $y, z$  with  $x$  and  $t$  leads to several cases. First, the DM may say that  $x \sim y \sim z \prec t$ , where  $\sim$  means indifference. This depicts a DM who is *intolerant*, since both criteria have to be satisfied in order to get a satisfactory alternative. In the opposite way, the DM may think that  $x \prec y \sim z \sim t$ , which depicts a *tolerant* DM, since only one criterion has to be satisfactory in order to get a satisfactory alternative. Finally, we may have all intermediate cases, where  $x \prec y \sim z \prec t$ . An important fact is that, due to additivity, the weighted sum is unable to distinguish among all these cases, in particular, all decision behaviours related to tolerance or intolerance are missed. These phenomena are called *interaction* between criteria. They encompass also other phenomena such as *veto*. We will show in this chapter that the notions of capacity and fuzzy integrals enable to model previous phenomena.

The construction of the utility functions and the determination of the parameters of the aggregation function are often carried out in two separate steps. The utility functions are generally set up first, that is without the knowledge of the precise aggregator  $F$  within  $\mathcal{F}$ . However, the utility functions have no intrinsic meaning to the DM and shall be determined from questions regarding only the overall preference relation  $\succeq$ . It is not assumed that the DM can isolate attributes and give information directly on  $u_i$ . This point is generally not considered in the literature. The main reason is probably that due to the use of a weighted sum as an aggregation function, the independence assumption (preferential or cardinal independence) makes it possible in some sense to separate each attribute and thus construct the utility functions directly. This becomes far more complicated when this assumption is removed. Besides, these approaches are not relevant from a theoretical standpoint. To our knowledge, the only approach that addresses this problem with the use of a weighted sum is the so-called MACBETH approach designed by Bana e Costa and Vansnick [1, 2, 3]. A generalization of this approach to more

complex aggregation operators has been proposed by the authors [33]. These approaches are considered in this chapter.

The determination of the utility function is not concerned only with measurement considerations. The main difficulty is to ensure commensurateness between criteria. Commensurateness means that one shall be able to compare any element of one point of view with any element of any other point of view. This is inter-criteria comparability:

For  $x_i \in X_i$  and  $x_j \in X_j$ , we have  $u_i(x_i) \geq u_j(x_j)$  iff  $x_i$  is considered at least as good as  $x_j$  by the DM.

Commensurateness implies the existence of a preference relation over  $\bigcup_{i=1}^n X_i$ . This assumption, considered by Modave *et al.* [57], is very strong. Taking a simple example involving two criteria (for instance consumption and maximal speed), this amounts to know whether the DM prefers a consumption of 5 liters/100km to a maximum speed of 200 km/h. This does not generally make sense to the DM, so that he or she is not generally able to make this comparison directly.

In sections 3 and 4 we push the previous method one step further by considering on top of intra-criteria information some natural inter-criteria information to determine the aggregation functions as well. We will show that the requirements induced by measurement considerations naturally imply the use of fuzzy integrals as aggregation operators. In section 5, we deal with the case of ordinal information. It will be seen that this induces difficulties, so that the previous construction no more applies.

## 2. Measurement theoretic foundations

As explained in the introduction, we focus on a model called *decomposable* given by Eq. (1.2), involving an aggregation function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ , and utility functions  $u_i : X_i \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ .

In this section we will give some considerations coming from measurement theory as well as more practical considerations coming from the MACBETH approach around this kind of model. This will help us in giving a firm theoretical basis to our construction.

### 2.1. Basic notions of measurement, scales

This section is based on [44, 64], to which the reader is referred for more details.

The fundamental aim of measurement theory is to build homomorphisms  $f$  between a relational structure  $\mathcal{A}$  coming from observation, and a relational structure  $\mathcal{B}$  based on real numbers (or more generally, some totally ordered set). Doing so, we get a numerical *representation*

of our observation. A *scale* (of measurement) is the triplet  $(\mathcal{A}, \mathcal{B}, f)$ . If no ambiguity occurs,  $f$  alone denotes the scale.

A simple example is when  $\mathcal{A} = (A, \succeq)$ , where  $\succeq$  is a binary relation expressing e.g. the preference of the DM on some set  $A$ , and  $\mathcal{B}$  is simply  $(\mathbb{R}, \geq)$ . As usual,  $\sim$  and  $\succ$  denote respectively the symmetric and asymmetric parts of  $\succeq$ , and  $A/\sim$  is the set of equivalence classes of  $\sim$  (when defined). This measurement problem is called *ordinal measurement*. The homomorphism satisfies the following condition

$$(\mathbf{Ord}[A]) \ a \succeq b \text{ iff } f(a) \geq f(b), \quad \forall a, b \in A.$$

Obviously,  $f$  is not unique since any strictly increasing transform  $\phi \circ f$  of  $f$  is also a homomorphism. Generally speaking, the set of functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\phi \circ f$  remains a homomorphism is called the *set of admissible transformations*.

Types of scale are defined by their set of admissible transformations. The most common ones are:

- *ordinal scales*, where the set of admissible transformations are all strictly increasing functions. Examples: scale of hardness, of earthquakes intensity.
- *interval scales*, where all  $\phi(t) = \alpha t + \beta$ ,  $\alpha > 0$  are admissible (positive affine transformations). Example: temperature in Celsius.
- *ratio scales*, where the admissible transformations are of the form  $\phi(t) = \alpha t$ ,  $\alpha > 0$ . Examples: temperature in Kelvin, mass.

Thus, our condition  $(\mathbf{Ord}[A])$  defines an ordinal scale. The conditions under which such a  $f$  exists are well known. A necessary condition is that  $\succeq$  is a weak order (reflexive, complete, transitive). A second condition (and then both are necessary and sufficient) is that  $A/\sim$  contains a countable order-dense subset (this is known as the Birkhoff-Milgram theorem, we do not enter further into details).

An ordinal scale is rather poor, and does not really permit to handle numbers, since usual arithmetic operations are not invariant under admissible transformations. It would be better to build an interval scale in the above sense. This is related to the *difference measurement* problem: in this case,  $\mathcal{A} = (A, \succeq^*)$ , where  $\succeq^*$  is a quaternary relation. The meaning of  $ab \succeq^* st$  is the following: the difference of intensity (e.g. of preference) between  $a$  and  $b$  is larger than the difference of intensity between  $s$  and  $t$ . Then, the homomorphism  $f$  should satisfy:

$$ab \succeq^* st \Leftrightarrow f(a) - f(b) \geq f(s) - f(t). \quad (1.3)$$

It is shown that under several conditions on  $\mathcal{A}$ , such a function  $f$  exists, and that it defines an interval scale. Thus the ratio  $\frac{f(a)-f(b)}{f(s)-f(t)}$  is meaningful (invariant under any admissible transformation).

Based on this remark, we express the interval scale condition under a form which is suitable for our purpose.

(**Inter**[ $A$ ]).  $\forall a, b, s, t \in A$  such that  $a \succ b$  and  $s \succ t$ , we have

$$\frac{f(a) - f(b)}{f(s) - f(t)} =: k(a, b, s, t), \quad k(a, b, s, t) \in \mathbb{R}_+$$

if and only if the difference of satisfaction degree that the DM feels between  $a$  and  $b$  is  $k(a, b, s, t)$  times as large as the difference of satisfaction between  $s$  and  $t$ .

The conditions of existence of  $f$  amounts to verify the following condition.

(**C-Inter**[ $A$ ]).  $\forall a, b, s, t, u, v \in A$  such that  $a \succ b$ ,  $s \succ t$  and  $u \succ v$ ,

$$k(a, b, s, t) \times k(s, t, u, v) = k(a, b, u, v).$$

We end this section by addressing the case where  $A$  is a product space, as for  $X = X_1 \times \cdots \times X_n$ . Conditions for an ordinal representation by  $u : X \rightarrow \mathbb{R}$  are given by the Birkhoff-Milgram theorem. However, we are interested in a decomposable form of  $u$  (see (1.2)). If  $F$  is one-to-one in each place, then necessarily  $\sim$  satisfies *substitutability*:

$$(x_i, z_{-i}) \sim (y_i, z_{-i}) \Leftrightarrow (x_i, z'_{-i}) \sim (y_i, z'_{-i}), \quad \forall x, y, z, z' \in X. \quad (1.4)$$

Notation  $z = (x_A, y_{-A})$  means that  $z$  is defined by  $z_i = x_i$  if  $i \in A$ , else  $z_i = y_i$  (hence,  $-A$  stands for  $N \setminus A$ ). This property implies the existence of equivalence relations  $\sim_i$  on each  $X_i$ . If  $F$  is strictly increasing, then  $\sim$  has to be replaced by  $\succeq$  in (1.4) (this is called *weak separability*), and relations  $\succeq_i$  are obtained on each  $X_i$ .

Reciprocally, substitutability (or weak separability) and the conditions of the Birkhoff-Milgram theorem lead to an *ordinal* representation: hence,  $u$  is unique up to a strictly increasing function.

This result remains of theoretical interest, since not verifiable in practice, and moreover, it does not lead to an interval scale. The MACBETH methodology will serve as a basis for such a construction, whose essence is briefly addressed below. Before that, some words on unipolar and bipolar scales are in order.

## 2.2. Bipolar and unipolar scales

Let us view scales under a different point of view. Let  $(A, \succeq)$  be a relational system, and  $f$  a scale, which is supposed to be numerical, without loss of generality. It may exist in  $A$  a particular element or level  $e$ , called *neutral level*, such that if  $a \succ e$ , then  $a$  is considered as “good”, while if  $e \succ a$ , then  $a$  is considered as “bad” for the DM. We may choose for convenience  $f$  such that  $f(e) = 0$ .

Such a neutral level exists whenever relation  $\succeq$  corresponds to two opposite notions of common language. For example, this is the case when  $\succeq$  means “more attractive than”, “better than”, etc., whose pairs of opposite notions are respectively “attractiveness/repulsiveness”, and “good/bad”. By contrast, relations as “more priority than”, “more allowed than”, “belongs more to category  $C$  than” do not clearly exhibit a neutral level.

A scale is said to be *bipolar* if  $A$  contains such a neutral level. A *unipolar scale* has no neutral level, but has a least level, i.e. an element or level  $a_0$  in  $A$  such that  $a \succeq a_0$  for all  $a \in A$ . We may for convenience choose  $f$  so that  $f(a_0) = 0$ .

A scale has a greatest element if there exists an element or level  $a_1 \in A$  such that  $a_1 \succeq a$ , for all  $a \in A$ . We say that a unipolar scale is *bounded* if it has a greatest level. A bipolar scale is bounded if it has a least and a greatest level (since there is an inherent symmetry in bipolar scales, the existence of a greatest level implies the existence of a least level).

Taking our previous examples, the relations “more attractive than”, “better than”, “more priority than” may not be bounded, while “more allowed than” and “belongs more to category  $C$  than” are clearly bounded, the greatest levels being respectively “fully authorized” and “fully belongs to  $C$ ”.

Typically,  $f$  maps on  $\mathbb{R}$  (resp.  $\mathbb{R}_+$ ) when the scale is unbounded bipolar (resp. unipolar). In the case of bounded scales,  $f$  maps respectively to a closed interval centered on 0, and an interval such as  $[0, \beta]$ .

It is convenient to denote by  $\mathbf{0}$  the neutral level of a bipolar scale, or the least level of a unipolar scale. We may also use  $\mathbf{1}$  to denote the greatest level when it exists, and  $-\mathbf{1}$  for the least level of a bipolar scale.

When the scale is unbounded, it may be convenient to introduce another particular level, called the *satisfactory level*, and denoted by  $\mathbf{1}$ . This level is considered as *good and completely satisfactory* if the DM could obtain it, even if more attractive elements could exist in  $A$  (due to unboundness). The existence of such a level has been the main argument of H. Simon in his theory of *satisficing bounded rationality* [69], and a fundamental assumption in the MACBETH methodology, as described in next section. For convenience, we may fix  $f(\mathbf{1}) = 1$ . If in addition the scale is bipolar, the same considerations lead to a level denoted  $-\mathbf{1}$  (unsatisfactory level).

Finally, let us remark that there is no direct relation between unipolar/bipolar scales and the types of scales given in Section 2.1 (interval, ratio, etc.). For example, the temperature scales are clearly unipolar with a least level (at least in the physical sense), but may be of the



ratio type (in Kelvin) or of the interval type (in Celsius, Fahrenheit). However, the neutral level of a bipolar scale clearly plays the role of the zero in a ratio scale, since it cannot be shifted.

### 2.3. Construction of the measurement scales and absolute references levels

The MACBETH methodology [1, 2, 3], described in Chapter 9, permits to build interval scales from a questionnaire. We limit ourselves here to necessary notions.

We consider  $A$  a finite set on which the decision maker is able to express some preference (the finiteness assumption is necessary for the method. If  $A$  is infinite, then a finite subset  $\bar{A}$  of representative objects should be chosen). The decision maker is asked for any pair  $(a, b) \in A^2$ :

- 1 Is  $a$  more attractive than  $b$ ?
- 2 If yes, is the difference of attractivity between  $a$  and  $b$  *very weak, weak, moderate, strong, very strong, or extreme*?

The first question concerns ordinal measurement: we are looking for a function  $f : A \rightarrow \mathbb{R}$  satisfying condition **(Ord[ $A$ ])**. The second question is related to difference measurement. The six ordered categories *very weak, . . . , extreme* define a quaternary relation on  $A$ , as defined in Section 2.1. MACBETH is able to test in a simple way if  $f$  as in (1.3) exists, and if yes, produces such a function, unique up to a positive affine transformation. In summary, we get an interval scale satisfying conditions **(Inter[ $A$ ])** and **(C-Inter[ $A$ ])**.

As explained in Section 2.2, we may have a unipolar or a bipolar scale, in which case a  $\mathbf{0}$  level exists. It is convenient to choose  $f$  such that  $f(\mathbf{0}) = 0$ . If several sets  $A_1, \dots, A_n$  are involved, then commensurability between the scales  $f_1, \dots, f_n$  may be required, as it will be seen later.

We say that scales  $f_i, f_j$  are *commensurate* if  $f_i(a_i) = f_j(a_j)$  means that the DM has the same intensity of attractiveness (or satisfaction, etc.) for  $a_i$  and  $a_j$ . A set of scales is commensurate if any pair is commensurate. Under the assumption that all  $f_i$ 's are interval scales, it is sufficient to find two levels on each  $A_i$ ,  $i = 1, \dots, n$  for which the DM feels an equal satisfaction for all  $i$  (they are in a sense *absolute levels*), and to impose equality of the scales for those levels.

Obviously, the levels  $\mathbf{0}_i$  of each  $A_i$  have an identical absolute meaning, provided the  $A_i$ 's are either all bipolar or all unipolar, but not mixed. We fix  $f_i(\mathbf{0}_i) = 0$ ,  $i = 1, \dots, n$ .

The second absolute levels could be the levels  $\mathbf{1}_i$  (satisfactory levels in case of unbounded scales, and greatest elements otherwise). As suggested in Section 2.2, we may fix  $f_i(\mathbf{1}_i) = 1$ ,  $i = 1, \dots, n$ .

The same considerations apply to the absolute levels  $-\mathbf{1}_i$ .

To conclude this section, let us stress the fact that the underlying assumptions on which MACBETH (and hence, the method presented here) is based is that the DM is able to deliver information concerning difference measurement, and that the DM is able to exhibit on  $A$  two elements or levels with an absolute meaning, denoted  $\mathbf{0}$  and  $\mathbf{1}$ , the precise meaning of them being dependent on the type of scale. We adopt throughout the paper the convention that

$$f(\mathbf{0}) = 0, \quad f(\mathbf{1}) = 1. \quad (1.5)$$

### 3. Unipolar scales

We address in this section the construction of our model in the case of unipolar scales. As explained in Section 2, we have on each  $X_i$  two absolute levels  $\mathbf{0}_i$  and  $\mathbf{1}_i$  given by the DM.

#### 3.1. Notion of interaction - A motivating example

To introduce more precisely the idea of interaction and show some flaws of the weighted sum, let us give an example. The director of a university decides on students who are applying for graduate studies in management where some prerequisites from school are required. Students are indeed evaluated according to mathematics (M), statistics (S) and language skills (L). All the marks with respect to the scores are given on the same scale from 0 to 20. These three criteria serve as a basis for a preselection of the candidates. The best candidates have then an interview with a jury of members of the university to assess their motivation in studying in management. The applicants have generally speaking a strong scientific background so that mathematics and statistics have a big importance to the director. However, he does not wish to favor too much students that have a scientific profile with some flaws in languages. Besides, mathematics and statistics are in some sense *redundant*, since, usually, students good at mathematics are also good at statistics. As a consequence, for students good in mathematics, the director prefers a student good at languages to one good at statistics. Consider the following student  $A$

	mathematics (M)	statistics (S)	languages (L)
student $A$	16	13	7

Student  $A$  is highly penalized by his performance in languages. Henceforth, the director would prefer a student (with the same mark in mathematics) that is a little bit better in languages even if the student would be a little bit worse in statistics. This means that the director prefers the following student to  $A$

	mathematics (M)	statistics (S)	languages (L)
student $B$	16	11	9

We have thus

$$A \prec B \tag{1.6}$$

Consider now a student that has a weakness in mathematics. In this case, since the applicants are supposed to have strong scientific skills, a student good in statistics is now preferred to one good in languages. Consider the following two students

	mathematics (M)	statistics (S)	languages (L)
student $C$	6	13	7
student $D$	6	11	9

Following above arguments,  $C$  is preferred to  $D$  even though  $C$  has poor language skills.

$$C \succ D \tag{1.7}$$

Satisfying (1.6) and (1.7) at the same time leads to the following requirement

$$F(16, 13, 7) > F(16, 11, 9) \quad \text{and} \quad F(6, 13, 7) < F(6, 11, 9).$$

No weighted sum can model such preferences since (1.6) implies that languages is more important than statistics whereas (1.7) tells exactly the contrary. There is an inversion of preferences between (1.6) and (1.7) in the sense that the relative importance of languages compared to statistics depends on the satisfaction level in mathematics. This behaviour is a typical example of *interaction* between criteria.

### 3.2. Capacities and Choquet integral

The natural generalization of giving weights on criteria is to assign weights on *coalitions* (i.e. groups, subsets) of criteria. This can be achieved by introducing particular functions on  $\mathcal{P}(N)$ , called fuzzy measures or capacities. We recall that  $N := \{1, \dots, n\}$  is the index set of criteria.

A *fuzzy measure* [70] or *capacity* [5] is a set function  $\mu : 2^N \rightarrow \mathbb{R}$  satisfying

**(FM<sub>a</sub>)**  $A \subset B \Rightarrow \mu(A) \leq \mu(B)$ ,

**(FM<sub>b</sub>)**  $\mu(\emptyset) = 0$ ,

**(FM<sub>c</sub>)**  $\mu(N) = 1$ .

Property **(FM<sub>a</sub>)** is called *monotonicity* of the capacity. In MCDA,  $\mu(A)$  is interpreted as the overall assessment of the binary alternative  $(1_A, 0_{-A})$ . A set function satisfying only **(FM<sub>b</sub>)** is called a *game* or a *non-monotonic fuzzy measure*.

The *conjugate*  $\mu^*$  of a capacity  $\mu$  is defined by  $\mu^*(S) = \mu(N) - \mu(N \setminus S)$ . The capacity is said to be *additive* if  $\mu(A \cup B) = \mu(A) + \mu(B)$ , whenever  $A \cap B = \emptyset$ , while it is said to be *symmetric* if  $\mu(A)$  depends only on  $|A|$ .

Let  $a := (a_1, \dots, a_n) \in \mathbb{R}_+^n$ . The Choquet integral [5] of  $a$  w.r.t. a capacity  $\mu$  has the following expression :

$$\mathcal{C}_\mu(a) = a_{\tau(1)} \mu(N) + \sum_{i=2}^n (a_{\tau(i)} - a_{\tau(i-1)}) \mu(\{\tau(i), \dots, \tau(n)\}), \quad (1.8)$$

where  $\tau$  is a permutation on  $N$  such that  $a_{\tau(1)} \leq a_{\tau(2)} \leq \dots \leq a_{\tau(n)}$ . Note that the Choquet integral is also well-defined w.r.t. set functions which are games.

When the capacity is additive, the Choquet integral reduces to a weighted sum.

We say that  $a, b \in \mathbb{R}_+^n$  are *comonotone* if  $a_i < a_j \Rightarrow b_i \leq b_j$  for any  $i, j \in N$ . In other words,  $a, b$  are comonotone if they belong to  $\Gamma_\tau := \{a \in \mathbb{R}_+^n \mid a_{\tau(1)} \leq a_{\tau(2)} \leq \dots \leq a_{\tau(n)}\}$  for the same permutation  $\tau$ . Thus, it is clear from (1.8) that for comonotone  $a, b$  we have  $\mathcal{C}_\mu(a + b) = \mathcal{C}_\mu(a) + \mathcal{C}_\mu(b)$ . This property, called *comonotonic additivity*, is characteristic of the Choquet integral, as shown by Schmeidler [66].

For other properties and characterizations of the Choquet integral, we refer the reader to survey papers [7, 61, 50].

Taking  $F$  as the Choquet integral, let us see whether it exists some capacity  $\mu$  such that  $\mathcal{C}_\mu$  is able to model relation (1.6) and (1.7). The modeling of (1.6) implies that  $2\mu(M, S) > \mu(M) + 1$ , while (1.7) gives  $2\mu(S) > \mu(S, L)$ . There is no contradiction between previous two inequalities, hence the Choquet integral can model the preferences of the DM.

### 3.3. General method for building utility functions

Let us describe now a general method to construct the utility functions  $u_i$  without the prior knowledge of  $F$  [33, 47]. The utility functions shall

be determined through questions regarding elements of  $X$ . Following the MACBETH approach [1, 2, 3], the subset  $X \downarrow_i$  (for  $i \in N$ ) of  $X$  will serve as a basis for the determination of  $u_i$ :

$$X \downarrow_i = \{(x_i, \mathbf{0}_{-i}) \mid x_i \in X_i\}.$$

We apply the MACBETH methodology to each set  $X \downarrow_i$ , which amounts to satisfy conditions **(Ord** $[X \downarrow_i]$ ), **(Inter** $[X \downarrow_i]$ ), **(C-Inter** $[X \downarrow_i]$ ). This gives the numerical representation  $u_{X \downarrow_i}$  of  $X \downarrow_i$ . It is uniquely determined if (1.5) is applied. Since  $\mathbf{0}_i$  is a least level of  $X_i$ , the utility function  $u_i$  is non-negative. Besides, it satisfies (1.5).

For  $(x_i, \mathbf{0}_{-i}) \in X \downarrow_i$ , one has by (1.2) and (1.5), since  $u_{X \downarrow_i}(x_i, \mathbf{0}_{-i})$  corresponds to the overall utility of the act  $(x_i, \mathbf{0}_{-i})$ :

$$u_{X \downarrow_i}(x_i, \mathbf{0}_{-i}) = F(u_i(x_i), u_{-i}(\mathbf{0}_{-i})) = F(u_i(x_i), 0_{-i}).$$

Assume that the family  $\mathcal{F}$  of aggregation functions satisfies

$$\exists \alpha_i \in \mathbb{R}_+^* \quad , \quad F(a_i, 0_{-i}) = \alpha_i a_i \quad \text{for all } a_i \in \mathbb{R}_+. \quad (1.9)$$

Since  $u_{X \downarrow_i}(\mathbf{1}_i, \mathbf{0}_{-i}) = F(1_i, 0_{-i}) = \alpha_i$ , we get for any  $x_i \in X_i$ :

$$u_i(x_i) = \frac{F(u_i(x_i), 0_{-i})}{F(1_i, 0_{-i})} = \frac{u_{X \downarrow_i}(x_i, \mathbf{0}_{-i})}{u_{X \downarrow_i}(\mathbf{1}_i, \mathbf{0}_{-i})}. \quad (1.10)$$

This shows that if all aggregation functions belonging to  $\mathcal{F}$  satisfy (1.9) then  $u_i$  can be determined by (1.10) from cardinal information related to  $X \downarrow_i$ .

Note that we do not need to assume weak separability, thanks to (1.9).

Considering the case of the Choquet integral, it is easy to see that whenever  $\mu(\{i\}) > 0$  for any  $i \in N$ , condition (1.9) is fulfilled so that the utility functions can be constructed with  $\mathcal{F}$  being equal to the Choquet integral w.r.t. capacities satisfying previous condition.

### 3.4. Justification of the use of the Choquet integral

We adopt here a slightly different approach than the one described in the introduction. We show that if we consider natural information that allow the modeling of interaction between criteria on top of information regarding  $X \downarrow_i$ , the Choquet integral comes up as a natural aggregation function. The justification of the use of the Choquet integral does not come from a pure axiomatic approach but rather from some reasonable information asked to the DM.

**3.4.1 Required information.** As said in Section 3.3, each utility function  $u_i$  is built from the set  $X \upharpoonright_i$ , which requires the satisfaction of conditions (**Ord** $[X \upharpoonright_i]$ ), (**Inter** $[X \upharpoonright_i]$ ), and (**C-Inter** $[X \upharpoonright_i]$ ), and is uniquely determined by (1.5).

Now that we have described intra-criterion information, let us give the inter-criteria information, that is data needed for the gathering of all criteria. The information regarding the aggregation of the criteria can be limited to alternatives whose scores on criteria are either  $\mathbf{0}_i$  or  $\mathbf{1}_i$ . In order to be able to model subtle interaction phenomena, all combinations of  $\mathbf{0}_i$  and  $\mathbf{1}_i$  must be considered. This leads to defining the following set:

$$X \upharpoonright_{\{0,1\}} := \{(\mathbf{1}_A, \mathbf{0}_{-A}) \text{ , } A \subset N\} \text{ ,}$$

called the set of *binary alternatives*. The application of the MACBETH methodology leads to the interval scale  $u_{X \upharpoonright_{\{0,1\}}}$ , which requires the satisfaction of conditions (**Ord** $[X \upharpoonright_{\{0,1\}}]$ ), (**Inter** $[X \upharpoonright_{\{0,1\}}]$ ), and (**C-Inter** $[X \upharpoonright_{\{0,1\}}]$ ). Applying (1.5) to this scale, it becomes uniquely determined:

$$u_{X \upharpoonright_{\{0,1\}}}(\mathbf{0}_N) = 0, \quad u_{X \upharpoonright_{\{0,1\}}}(\mathbf{1}_N) = 1. \quad (1.11)$$

The second condition in (1.11) says that an alternative which is completely satisfactory on each criteria should be completely satisfactory, and similarly for the first condition.

**3.4.2 Measurement conditions.**  $u_{X \upharpoonright_{\{0,1\}}}$  represents the importance that the DM gives to the coalition  $A$  in the DM process for any  $A \subset N$ . It depicts the way criteria are aggregated. It leads to the definition of a capacity  $\mu$  defined by  $\mu(A) := u_{X \upharpoonright_{\{0,1\}}}(\mathbf{1}_A, \mathbf{0}_{-A})$ . Consequently, it is natural to write  $u$  as follows:

$$u(x) = F_\mu(u_1(x_1), \dots, u_n(x_n)) \text{ ,} \quad (1.12)$$

where  $F_\mu$  is the aggregation operator.  $F_\mu$  depends on  $\mu$  in a way that is not known for the moment.

The  $u_i$ 's correspond to interval scales, whose admissible transformations are the positive affine transformations (see Section 2.1). Hence, one could change all  $u_i$ 's in  $\alpha u_i + \beta$ , for any  $\alpha > 0$  and  $\beta \in \mathbb{R}$ , without any change in the model. On the other hand,  $\mu(A)$  corresponds in fact to the difference of the satisfaction degrees between the alternatives  $(\mathbf{1}_A, \mathbf{0}_{-A})$  and  $\mathbf{0}_N$ . Applying this to  $A = \emptyset$ , the value  $\mu(\emptyset)$  shall always be equal to zero, whatever the interval scale attached to  $X \upharpoonright_{\{0,1\}}$  may be. Henceforth,  $\mu$  corresponds to a ratio scale, and can be replaced by  $\gamma\mu$ ,

with  $\gamma \in \mathbb{R}_+$ , since these are the admissible transformations for ratio scales. Hence one shall have [47]:

**(Meas-Inter)** The preference relation  $\succeq$  and the ratio  $\frac{u(x)-u(y)}{u(z)-u(t)}$  for  $x, y, z, t \in X \rfloor_i$  (for all  $i \in N$ ) and for  $x, y, z, t \in X \rfloor_{\{0,1\}}$  shall not be changed if all the  $u_i$ 's are changed into  $\alpha u_i + \beta$  with  $\alpha > 0$  and  $\beta \in \mathbb{R}$ , and  $\mu$  is changed into  $\gamma \mu$  with  $\gamma \in \mathbb{R}_+$ .

From **(Ord** $[X \rfloor_i]$ ), **(Inter** $[X \rfloor_i]$ ), **(C-Inter** $[X \rfloor_i]$ ), (1.5), **(Ord** $[X \rfloor_{\{0,1\}]}$ ), **(Inter** $[X \rfloor_{\{0,1\}]}$ ), **(C-Inter** $[X \rfloor_{\{0,1\}]}$ ), (1.11) and **(Meas-Inter)**, it can be shown that [47, Lemma 2]

$$\frac{F_\mu((\alpha + \beta)_A, \beta_{-A}) - F_\mu((\alpha + \beta)_B, \beta_{-B})}{F_\mu((\alpha + \beta)_C, \beta_{-C}) - F_\mu((\alpha + \beta)_D, \beta_{-D})} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)}.$$

Taking this with  $B = D = \emptyset$  and  $C = N$ , we get

$$\frac{F_\mu((\alpha + \beta)_A, \beta_{-A}) - F_\mu(\beta_N)}{F_\mu((\alpha + \beta)_N) - F_\mu(\beta_N)} = \mu(A).$$

Since  $F_\mu$  acts on commensurate scales and returns a value on the same scale, it is natural to assume that  $F_\mu$  satisfies idempotency [15]

$$F_\mu(\beta, \dots, \beta) = \beta, \quad \forall \beta \in \mathbb{R}.$$

Plugging this into previous relation one gets

$$F_\mu((\alpha + \beta)_A, \beta_{-A}) = \alpha \mu(A) + \beta.$$

This equality with  $\alpha = 1$  and  $\beta = 0$  gives

**Properly Weighted (PW):** If  $\mu$  satisfies conditions **(FM<sub>b</sub>)** and **(FM<sub>c</sub>)**, then  $F_\mu(1_A, 0_{-A}) = \mu(A)$ ,  $\forall A \subset N$ .

Previous relation together with **(PW)** gives

**Stability for the admissible Positive Linear transformations (weak SPL):** If  $\mu$  satisfies conditions **(FM<sub>b</sub>)** and **(FM<sub>c</sub>)**, then for all  $A \subset N$ ,  $\alpha > 0$ , and  $\beta \in \mathbb{R}$ ,

$$F_\mu((\alpha + \beta)_A, \beta_{-A}) = \alpha F_\mu(1_A, 0_{-A}) + \beta$$

Since  $F_\mu$  aggregates satisfaction scales, it is natural to assume that  $x \mapsto F_\mu(x)$  is increasing. Hence  $F_\mu$  shall satisfy the following axiom.

**Increasingness (In):** If  $\mu$  satisfies conditions **(FM<sub>a</sub>)** and **(FM<sub>b</sub>)**, then  $\forall x, x' \in \mathbb{R}^n$ ,

$$x_i \leq x'_i \quad \forall i \in N \quad \Rightarrow \quad F_\mu(x) \leq F_\mu(x')$$

Measurement considerations yield linearity of the mapping  $\mu \mapsto F_\mu(x)$  [47]. Hence  $F_\mu$  shall satisfy to the following axiom.

**Linearity w.r.t. the Measure (LM):** If  $\mu$  satisfies condition **(FM<sub>b</sub>)**, then for all  $x \in \mathbb{R}^n$  and  $\gamma, \delta \in \mathbb{R}$ ,

$$F_{\gamma\mu + \delta\mu'}(x) = \gamma F_\mu(x) + \delta F_{\mu'}(x).$$

The following result can be shown.

**Theorem 1 (Theorem 1 in [47])**  $F_\mu$  satisfies **(LM)**, **(In)**, **(PW)** and **(weak SPL)** if and only if  $F_\mu \equiv \mathcal{C}_\mu$  in  $\mathbb{R}^n$ .

We have seen that the measurement conditions we have on  $u_i$  and  $u_{X \setminus \{0,1\}}$  lead naturally to axioms **(LM)**, **(In)**, **(PW)** and **(weak SPL)**. There is only one aggregation function that satisfies these axioms, namely the Choquet integral w.r.t.  $\mu$ . So the cardinal information we work with leads naturally to the use of the Choquet integral.

Let us remark that Theorem 1 is a weak version of an axiomatic characterization obtained by Marichal [50].

### 3.5. Shapley value and interaction index

By construction, the capacity  $\mu$  expresses the score of binary alternatives. Since there are  $2^n$  such alternatives, it may be difficult to analyse or explain the behaviour of the decision maker through the values taken by  $\mu$ .

A first question of interest is: “What is the importance of a given criterion for the decision?”. We may say that a criterion  $i$  is important if whenever added to some coalition  $A$  of criteria, the score of  $(\mathbf{1}_{A \cup i}, \mathbf{0}_{-(A \cup i)})$  is significantly larger than the score of  $(\mathbf{1}_A, \mathbf{0}_{-A})$ . Hence, an importance index should compute an average value  $\Delta_i$  of the quantity  $\mu(A \cup i) - \mu(A)$  for all  $A \subset N \setminus i$ . A second requirement is that the sum of importance indices for all criteria should be a constant, say 1. Lastly, the importance index should not depend on the numbering of the criteria. Strangely enough, these three requirements plus a linearity assumption, which imposes that the average  $\Delta_i$  is a weighted arithmetic mean, suffices to determine uniquely the importance index, known as the *Shapley importance index* [67]

$$\phi^\mu(i) := \sum_{K \subset N \setminus i} \frac{(n-k-1)!k!}{n!} [\mu(K \cup i) - \mu(K)] \quad (1.13)$$

with  $k := |K|$ . We omit the superscript if no ambiguity occurs. The *Shapley value* is the vector  $(\phi(1), \dots, \phi(n))$ . As said above, we have



$\sum_{i=1}^n \phi(i) = \mu(N) = 1$ . Another fundamental property is that  $\phi(i) = \mu(\{i\})$  if  $\mu$  is additive.

We have shown by an example in Section 3.1 that interaction may occur among criteria, and that the Choquet integral was able to deal with situations where interaction occurs. We define this notion more precisely. Let us consider for simplicity 2 criteria and the following alternatives (see figure 1.1):

- $x = (\mathbf{0}_1, \mathbf{0}_2)$
- $y = (\mathbf{1}_1, \mathbf{0}_2)$
- $z = (\mathbf{0}_1, \mathbf{1}_2)$
- $t = (\mathbf{1}_1, \mathbf{1}_2)$

Clearly,  $t$  is more attractive than  $x$ , but preferences over other pairs may depend on the decision maker. Due to monotonicity ( $\mathbf{FM}_a$ ), we can range from the two extremal following situations (recall that  $\mu(\{1, 2\}) = 1$  and  $\mu(\emptyset) = 0$ ):

**extremal situation 1 (lower bound):** we put  $\mu(\{1\}) = \mu(\{2\}) = 0$ , which is equivalent to the preferences  $x \sim y \sim z$  (figure 1.1, left) (strictly speaking,  $\mu(\{i\})$  cannot attain the value 0: see Section 3.3). This means that for the DM, both criteria have to be satisfactory in order to get a satisfactory alternative, the satisfaction of only one criterion being useless. We say that the criteria are *complementary*.

**extremal situation 2 (upper bound):** we put  $\mu(\{1\}) = \mu(\{2\}) = 1$ , which is equivalent to the preferences  $y \sim z \sim t$  (figure 1.1, middle). This means that for the DM, the satisfaction of one of the two criteria is sufficient to have a satisfactory alternative, satisfying both being useless. We say that the criteria are *substitutive*.

Clearly, in these two situations, the criteria are not independent, in the sense that the satisfaction of one of them acts on the usefulness of the other in order to get a satisfactory object (necessary in the first case, useless in the second). We say that there is some *interaction* between the criteria.

A situation without interaction is such that the satisfaction of each criterion brings its own contribution to the overall satisfaction, hence:

$$\mu(\{1, 2\}) = \mu(\{1\}) + \mu(\{2\}) \quad (1.14)$$

(additivity) (see Fig. 1.1, right). In the first situation,  $\mu(\{1, 2\}) > \mu(\{1\}) + \mu(\{2\})$ , while the reverse inequality holds in the second situation. This suggests that the interaction  $I_{12}$  between criteria 1 and 2

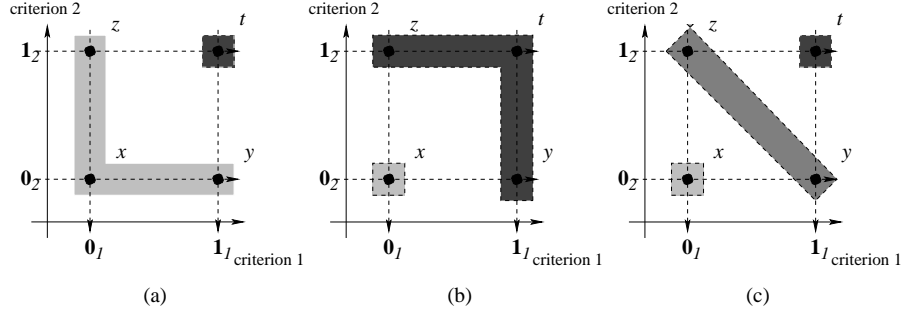


Figure 1.1. Different cases of interaction

should be defined as :

$$I_{12}^{\mu} := \mu(\{1, 2\}) - \mu(\{1\}) - \mu(\{2\}) + \mu(\emptyset). \quad (1.15)$$

This is simply the difference between binary alternatives on the diagonal (where there is strict dominance) and on the anti-diagonal (where no dominance relation exists). The interaction is positive when criteria are complementary, while it is negative when they are substitutive. This is consistent with intuition considering that when criteria are complementary, they have no value by themselves, but put together they become important for the DM.

In the case of more than 2 criteria, the definition of interaction follows the same idea as with the Shapley index, in the sense that all coalitions of  $N$  have to be taken into account. The following definition has been first proposed by Murofushi and Soneda [60], for a pair of criteria  $i, j$ :

$$I_{ij}^{\mu} := \sum_{K \subset N \setminus \{i, j\}} \frac{(n - k - 2)!k!}{(n - 1)!} [\mu(K \cup \{i, j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)], \quad (1.16)$$

The definition of this index has been extended to any coalition  $\emptyset \neq A \subset N$  of criteria by Grabisch [19]:

$$I^{\mu}(A) := \sum_{K \subset N \setminus A} \frac{(n - k - |A|)!k!}{(n - |A| + 1)!} \sum_{L \subset A} (-1)^{|A| - |L|} \mu(K \cup L), \quad \forall A \subset N, A \neq \emptyset. \quad (1.17)$$

We have  $I_{ij} = I(\{i, j\})$ . When  $A = \{i\}$ ,  $I(\{i\})$  coincides with the Shapley index  $\phi(i)$ . It is easy to see that when the fuzzy measure is additive, we have  $I(A) = 0$  for all  $A$  such that  $|A| > 1$ . Also  $I_{ij} > 0$

(resp.  $< 0, = 0$ ) for complementary (resp. substitutive, non-interactive) criteria.

The definition can be extended to  $A = \emptyset$ , just putting  $\sum_{L \subset A} (-1)^{|A|-|L|} \mu(K \cup L) = \mu(K)$ . Hence  $I$  defines a set function  $I : \mathcal{P}(N) \rightarrow \mathbb{R}$ . Properties of this set function has been studied and related to the Möbius transform [8, 34]. In particular, it is possible to recover  $\mu$  if  $I$  is given for each  $A \subset N$ , which means that the interaction index can be viewed as a particular *transform* of a fuzzy measure, which is invertible, as the Möbius transform. Also,  $I$  has been characterized axiomatically by Grabisch and Roubens [37], in a way similar to the Shapley index.

Another important property is that the interaction index can be obtained recursively from the Shapley importance index, by considering sub-problems with less criteria [37]. For  $I_{ij}^\mu$ , the relation writes:

$$I_{ij}^\mu = \phi^{\mu^{[ij]}}([ij]) - \phi^{\mu_{N \setminus i}}(j) - \phi^{\mu_{N \setminus j}}(i), \quad (1.18)$$

where  $[ij]$  stands for an artificial criterion ( $i$  and  $j$  taken together),  $\mu^{[ij]} : \mathcal{P}((N \setminus \{i, j\}) \cup \{[ij]\}) \rightarrow [0, 1]$ , with  $\mu^{[ij]}(A) := \mu(A \cup \{i, j\})$  if  $A \ni [ij]$ , and  $\mu(A)$  else, and  $\mu_{N \setminus i}$  is the restriction of  $\mu$  to  $N \setminus i$ .

### 3.6. $k$ -additive measures

Although we have shown that our construction is able to model in a clear way interaction, this has to be paid by an exponential complexity, since the number of binary alternatives is  $2^n$ . There exists a way to cope with complexity by defining sub-families of fuzzy measures, which require less than  $2^n$  coefficients to be defined. The first such family which has been defined is the one of *decomposable measures* [11, 75], which includes the well-known class of  $\lambda$ -measures proposed by Sugeno [70]. These fuzzy measures are defined by a kind of density function, and thus need only  $n - 1$  coefficients. However, they have a very limited ability to represent interaction since e.g.  $I_{ij}$  has the same sign for all  $i, j$ .

A second family is given by the concept of  $k$ -additive measure, which is detailed in this section.

**Definition 1** [19] *Let  $k \in \{1, \dots, n - 1\}$ . A fuzzy measure  $\mu$  is said to be  $k$ -additive if  $I(A) = 0$  whenever  $|A| > k$ , and there exists some  $A \subset N$  with  $|A| = k$  such that  $I(A) \neq 0$ .*

From the properties of interaction cited in Section 3.5, a 1-additive measure is simply an additive measure, hence the name. Also, since  $\mu$  is completely determined by the values of  $I$  on  $\mathcal{P}(N)$ , a  $k$ -additive mea-

sure is determined by  $1 + n + \binom{n}{2} + \dots + \binom{n}{k}$  parameters, among which 2 are not free.

The 2-additive measure, which needs only  $\frac{n(n+1)}{2} - 1$  parameters, permits to model interaction between pair of criteria, which is in general sufficient in practice (it is in fact fairly difficult to have a clear understanding of interaction among more than 2 criteria).

The Choquet integral can be expressed using  $I$  instead of  $\mu$  in a very instructive way when the measure is 2-additive [18]:

$$\begin{aligned} \mathcal{C}_\mu(a_1, \dots, a_n) &= \sum_{I_{ij} > 0} (a_i \wedge a_j) I_{ij} + \sum_{I_{ij} < 0} (a_i \vee a_j) |I_{ij}| \\ &\quad + \sum_{i=1}^n a_i \left( \phi_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}| \right), \quad \forall a \in [0, 1]^n, \quad (1.19) \end{aligned}$$

for all  $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ , with the property that  $\phi_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}| \geq 0$  for all  $i$ . It can be seen that the Choquet integral for 2-additive measures is the sum of a conjunctive, a disjunctive and an additive part, corresponding respectively to positive interaction indices, negative interaction indices, and the Shapley value. Equation (1.19) shows clearly the disjunctive and conjunctive effects of negative and positive interaction between criteria, which has been explained in Section 3.5. It is important to notice that, due to the normalization  $\sum_{i=1}^n \phi_i = 1$ , (1.19) is a convex combination of disjunctions, conjunctions, and a linear part. Hence, as illustrated in [21] in a graphical way, the Choquet integral is the convex closure of all conjunctions and all disjunctions of pair of criteria, and of all dictators (single criteria).

Before ending this section, we mention a third family of fuzzy measures introduced by Miranda and Grabisch, the *p-symmetric fuzzy measures* [56]. The idea is to generalize symmetric fuzzy measures (see Section 3.2), by considering a partition  $\{A_1, \dots, A_p\}$  of  $N$  into subsets of indifference: taking elements in  $A_1, \dots, A_p$ , the value of  $\mu$  does not depend on the particular elements which are chosen in each  $A_i$ , but only on their number. Hence a symmetric measure corresponds to a 1-symmetric measure (i.e. the partition is  $N$  itself). The number of parameters needed to define a  $p$ -symmetric measure is  $\prod_{i=1}^p (|A_i| + 1) - 2$ .

### 3.7. Identification of capacities

We assume here that the utility functions  $u_i$  are known. Their construction is carried out with the help of cardinal data on the sets  $X \downarrow_i$  (See section 3.4). So, we focus in this section on the determination of the capacity.

In section 3.4, we proposed to determine the aggregation function with the help of cardinal information related to binary alternatives. The main advantage of this method is that by **(PW)** each alternative is associated to one term of the capacity. However, this way is not considered in practice because of the following two reasons. The first one is that it may not be natural for a DM to give his preferences on the prototypical alternatives  $(\mathbf{1}_A, \mathbf{0}_{-A})$ . The second one is that it forces the DM to construct a ratio scale over  $2^n$  alternatives using the MACBETH approach. This requires roughly  $4^n/2$  questions to be asked to the DM. This is too much in practice.

The first idea is to replace  $X]_{\{0,1\}}$  by a set of more intuitive alternatives. The DM provides a set of learning examples  $x^1, \dots, x^p$  in  $X$ . As for  $X]_{\{0,1\}}$ , we want a numerical representation of these learning examples. In order to obtain a unique interval scale, the two prototypical alternatives  $\mathbf{0}_N$  and  $\mathbf{1}_N$  are added to the learning examples. Let

$$X] := \{x^1, \dots, x^p\} \cup \{\mathbf{0}_N, \mathbf{1}_N\}.$$

An interval scale  $u_{X]}$  representing the preference on  $X]$  can be obtained using the MACBETH methodology, if conditions **(Ord[X])**, **(Inter[X])**, **(C-Inter[X])** are satisfied. The application of (1.5) makes the scale unique, putting 0 for  $\mathbf{0}_N$  and 1 for  $\mathbf{1}_N$ . One wishes to determine the capacity  $\mu$  solution to the following set of equations:

$$\forall i \in \{1, \dots, p\}, \quad \mathcal{C}_\mu(u_1(x_1^i), \dots, u_n(x_n^i)) = u_{X]}(x^i). \quad (1.20)$$

Unfortunately, no solution may exist or there may be more than one solution. In these cases, in order to get an approximate solution, previous problem is written as a minimization problem [36, 17] in which the unknowns are the parameters of the capacity:

Minimize

$$\sum_{i=1}^p \left| \mathcal{C}_\mu(u_1(x_1^i), \dots, u_n(x_n^i)) - u_{X]}(x^i) \right|^2$$

under the constraints **(FM<sub>a</sub>)**, **(FM<sub>b</sub>)** and **(FM<sub>c</sub>)**.

It can be shown that the above problem is a quadratic minimization problem under linear constraints [36, 17]. Thanks to **(FM<sub>b</sub>)** and **(FM<sub>c</sub>)**, there are  $2^n - 2$  unknowns. Moreover, there are  $n(2^{n-1} - 1)$  monotony constraints [17]. There is generally not a unique solution to this problem [54]. Experiments on real data have shown some drawbacks of this method.

- if there is too few data, the solution is of course not unique, and the solution proposed by quadratic optimization libraries may be counterintuitive, because many coefficients are near 0 or 1.

- as  $n$  grows up, the dimensions of vectors and matrices grows exponentially, so does the memory required and the computation time.  $n = 8$  is already a large value, and  $n = 10$  is nearly infeasible.

For these reasons, some authors have looked for more heuristic methods, as Ishii and Sugeno [41] and Mori and Murofushi [58]. Based on this last one, Grabisch has proposed an optimization algorithm [16], which although sub-optimal, gives better results than previous attempts. The basic idea is that, in the absence of any information, the most non-arbitrary (least specific) way of aggregation is the arithmetic mean, thus a Choquet integral with respect to an additive equidistributed fuzzy measure. Any input of information tends to move away the fuzzy measure from this equilibrium point. This means that, in case of few data, coefficients of the fuzzy measure which are not concerned with the data are kept as near as possible to the equilibrium point, in order to ensure monotonicity.

Experiments done in classification problems show the good performance of the algorithm, even better than the optimal method when  $n$  is large. Especially, the memory and computation time required are much smaller than for the quadratic program, and it is possible to treat problems with  $n = 16$ .

The DM may not be able to give cardinal information on alternatives. So, the second idea is to use a set of examples of comparisons between alternatives provided by the DM. In other words, the DM gives two sets of alternatives  $x^1, \dots, x^p$  and  $y^1, \dots, y^p$  in  $X$  such that  $x^1 \succ y^1, \dots, x^p \succ y^p$ . One looks then for a fuzzy measure that is consistent with previous relations and thus that satisfies

$$\forall i \in \{1, \dots, p\}, \mathcal{C}_\mu(u_1(x_1^i), \dots, u_n(x_n^i)) > \mathcal{C}_\mu(u_1(y_1^i), \dots, u_n(y_n^i)). \quad (1.21)$$

Most of the time, there is a huge number of solutions. In order to reduce the solution space, additional constraints must be added. As remarked by Marichal and Roubens in [53], when the DM states that  $x^i \succ y^i$ , he or she generally means that  $x^i$  is significantly preferred to  $y^i$ . If the overall utilities of the two alternatives  $x^i$  and  $y^i$  are almost the same, it will probably not represent the DM's intention. Henceforth, among all solutions to (1.21), one should prefer the ones with the highest margin. This led Marichal and Roubens to introduce a positive coefficient  $\epsilon$  in the right-hand side of (1.21), and to maximize  $\epsilon$ :

Maximize  $\epsilon$   
 under the constraints  $(\mathbf{FM}_a)$ ,  $(\mathbf{FM}_b)$ ,  $(\mathbf{FM}_c)$ ,  $\epsilon \geq 0$  and for all  $i \in \{1, \dots, p\}$

$$\mathcal{C}_\mu(u_1(x_1^i), \dots, u_n(x_n^i)) \geq \mathcal{C}_\mu(u_1(y_1^i), \dots, u_n(y_n^i)) + \epsilon$$

This is a linear programming problem. It is a simplified version of a linear method proposed by Marichal and Roubens [53].

Other learning methods have been tried, principally using genetic algorithms (see in particular Wang [74], Kwon and Sugeno [45], and Grabisch [23]).

#### 4. Bipolar scales

We address now the construction of the model in the case of bipolar scales. As explained in Section 2, we have on each  $X_i$  one neutral level  $\mathbf{0}_i$  and another absolute level  $\mathbf{1}_i$  given by the DM.

##### 4.1. A motivating example

Let us go a little deeper in the example described in Section 3.1. We have seen in Section 3.1 that for students good in mathematics, the director prefers someone good at languages to one good at statistics. In other words, when the mark with respect to mathematics is good, the director thinks that languages is more important than statistics. This leads to the following rule

**(R1)**: For a student good at mathematics (M), L is more important than S.

The comparison between students  $A$  and  $B$  in Section 3.1 are governed by this rule. Let us consider now another set of students. Consider the following students  $E$  and  $F$

	mathematics (M)	statistics (S)	languages (L)
student $E$	14	16	7
student $F$	14	15	8

According to rule **(R1)**, the director prefers student  $F$  to  $E$

$$E \prec F \quad (1.22)$$

As justified in Section 3.1, when the score w.r.t. mathematics is bad, a student good in statistics is now preferred to one good in languages. More precisely, we have the following statement

**(R2)**: For a student bad in mathematics M, S is more important than L.

Consider the following two students

	mathematics (M)	statistics (S)	languages (L)
student $G$	9	16	7
student $H$	9	15	8

Following rule **(R2)**,  $G$  is preferred to  $H$  even though  $G$  is very bad in languages.

$$G \succ H \quad (1.23)$$

Relations (1.22) and (1.23) look similar to (1.6) and (1.7). However, we will see that they exhibit a weakness of the Choquet integral. Let us indeed try to model (1.22) and (1.23) with the help of the Choquet integral. We have  $\mathcal{C}_\mu(E) = 7 + 7\mu(\{M, S\}) + 2\mu(\{S\})$  and  $\mathcal{C}_\mu(F) = 8 + 6\mu(\{M, S\}) + \mu(\{S\})$ . This shows that (1.22) is equivalent to

$$\mu(\{M, S\}) + \mu(\{S\}) < 1.$$

Similarly, relation (1.23) is equivalent to  $\mu(\{M, S\}) + \mu(\{S\}) > 1$ , which contradicts previous inequality. Hence, the Choquet integral cannot model (1.22) and (1.23).

It is no surprise that the Choquet integral cannot model both **(R1)** and **(R2)**. This is due to the fact that the Choquet integral satisfies comonotonic additivity (see Section 3.2). In our example, the marks of the four students  $E$ ,  $F$ ,  $G$  and  $H$  are ranked in the same way: languages is the worst score, mathematics is the second best score, and statistics is the best score. Those four students are comonotonic. The Choquet integral is able to model rules of the following type :

**(R1')**: If M is the best satisfied criteria, L is more important than S.

**(R2')**: If M is the worst satisfied criteria, S is more important than L.

On the other hand, rules **(R1)** and **(R2)** make a reference to absolute values (good/bad in mathematics). The Choquet integral does not allow to model this type of property. The Choquet integral fails to represent the expertise that makes an explicit reference to an absolute value. This happens quite often in applications.

Let us study the meaning of the reference point used in rules **(R1)** and **(R2)**. In our example, the satisfaction level is either rather good (good in mathematics) or rather bad (bad in mathematics). This makes an implicit reference to a neutral level that is neither good nor bad. This suggests to construct criteria on ratio scales. In such scales, the zero element is the neutral element. It has an absolute meaning and cannot be shifted. Values above this level are attractive (good) whereas values below the zero level are repulsive (bad).

## 4.2. The symmetric Choquet integral and Cumulative Prospect Theory

**4.2.1 Definitions.** Let  $f : N \longrightarrow \mathbb{R}$  be a real-valued function, and let us denote by  $f^+(i) := f(i) \vee 0, \forall i \in N$ , and  $f^- := (-f)^+$  the positive and negative parts of  $f$ .

The *symmetric Choquet integral* [6] (also called the *Šipos̆ integral* [72]) of  $f$  w.r.t.  $\mu$  is defined by:

$$\check{\mathcal{C}}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_\mu(f^-).$$



This differs from the usual definition of Choquet integral for real-valued functions, sometimes called *asymmetric Choquet integral* [6], which is

$$\mathcal{C}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_{\bar{\mu}}(f^-).$$

The Cumulative Prospect Theory model [71] generalizes these definitions, by considering different capacities for the positive and negative parts of the integrand.

$$\text{CPT}_{\mu_1, \mu_2}(f) := \mathcal{C}_{\mu_1}(f^+) - \mathcal{C}_{\mu_2}(f^-).$$

**4.2.2 Application to the example.** Let us go back to the example of Section 4.1. In this example, value 10 for the marks seems to be the appropriate neutral value. Hence, in order to transform the regular marks given in the interval  $[0, 20]$  to a ratio scale, it is enough to subtract 10 to each mark yielding the mark 10 to the zero level. This gives :

	mathematics (M)	statistics (S)	languages (L)
student $E'$	4	6	-3
student $F'$	4	5	-2
student $G'$	-1	6	-3
student $H'$	-1	5	-2

Modeling our example with the Šipoš integral, a straightforward calculation shows that (1.22) is equivalent to  $\mu(\{S\}) < \mu(\{L\})$  whereas relation (1.23) is equivalent to  $\mu(\{S\}) > \mu(\{L\})$ , which contradicts previous inequality. Henceforth, the Šipoš integral is not able to model both (1.22) and (1.23).

Trying now the representation of our example with the CPT model, it is easy to see that (1.22) is equivalent to  $\mu_1(\{S\}) < \mu_2(\{L\})$ , and relation (1.23) is equivalent to  $\mu_1(\{S\}) > \mu_2(\{L\})$ . Henceforth, the CPT model too fails to model both (1.22) and (1.23).

### 4.3. Bi-capacities and the corresponding integral

The Choquet, Šipoš and CPT models are limited by the fact that they are constructed on the notion of capacity. The idea is thus to generalize the notion of capacity. Such generalizations have first been introduced in the context of game theory. The concept of ternary voting games has recently been defined by D. Felsenthal and M. Machover as a generalization of binary voting games [14]. Binary voting games model the result of a vote when some voters are in favor of the bill and the other voters are against [68]. The main limitation of such games is that they cannot represent decision rules in which *abstention* is an alternative

option to the usual *yes* and *no* opinions. This led D. Felsenthal and M. Machover to introduce *ternary voting games* [14]. These voting games can be represented by a function  $\nu$  with two arguments, one for the *yes* voters and the other one for the *no* voters. This concept of ternary voting game has been generalized by J.M. Bilbao *et al.* in [4], yielding the definition of *bi-cooperative game*. Let

$$\mathcal{Q}(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\}.$$

A bi-cooperative game is a function  $\nu : \mathcal{Q}(N) \rightarrow \mathbb{R}$  satisfying  $\nu(\emptyset, \emptyset) = 0$ . In the context of game theory, the first argument  $A$  in  $\nu(A, B)$  is called the *defender* part, and the second argument  $B$  in  $\nu(A, B)$  is called the *defeater* part.

This generalization has recently been rediscovered independently by the authors in the context of MCDA [29, 49]. A bi-capacity is a function  $\nu : \mathcal{Q}(N) \rightarrow \mathbb{R}$  satisfying

$$\text{(BFM}_a\text{)} \quad A \subset A' \Rightarrow \nu(A, B) \leq \nu(A', B),$$

$$\text{(BFM}_b\text{)} \quad B \subset B' \Rightarrow \nu(A, B) \geq \nu(A, B'),$$

$$\text{(BFM}_c\text{)} \quad \nu(\emptyset, \emptyset) = 0,$$

$$\text{(BFM}_d\text{)} \quad \nu(N, \emptyset) = 1, \nu(\emptyset, N) = -1$$

Conditions **(BFM<sub>a</sub>)** and **(BFM<sub>b</sub>)** together define *monotonic* bi-capacities. Bi-capacities are special cases of bi-cooperative games. In MCDA,  $\nu(A, B)$  is interpreted as the overall assessment of the ternary alternative  $(1_A, -1_B, 0_{-(A \cup B)})$ . Thanks to that interpretation, the first argument  $A$  in  $\nu(A, B)$  is called the *positive* part, and the second argument  $B$  in  $\nu(A, B)$  is called the *negative* part.

The *conjugate* or *dual*  $\nu^*$  of a bi-capacity  $\nu$  can be defined by  $\nu^*(S, T) = -\nu(T, S)$  for all  $(S, T) \in \mathcal{Q}(N)$  [46, 48]. In the context of Game Theory, it means that the defenders and the defeaters are switched, and the abstentionists are untouched. This definition of dual bi-capacity coincides with that proposed in [14] for ternary voting games.

A bi-capacity  $\nu$  is of the *CPT type* if it can be written  $\nu(A, B) = \mu_1(A) - \mu_2(B)$ , for all  $(A, B) \in \mathcal{Q}(N)$ , where  $\mu_1, \mu_2$  are capacities. If  $\mu_1 = \mu_2$ , we say that the bi-capacity is *symmetric*. If  $\mu_1$  and  $\mu_2$  are additive, then  $\nu$  is said to be *additive*.

A similar concept has also been introduced by S. Greco *et al.* leading to the concept of *bipolar capacity* [39]. A bipolar capacity is a function  $\zeta : \mathcal{Q}(N) \rightarrow [0, 1] \times [0, 1]$  with  $\zeta(A, B) =: (\zeta^+(A, B), \zeta^-(A, B))$  such that

- If  $A \supset A'$  and  $B \subset B'$  then  $\zeta^+(A, B) \geq \zeta^+(A', B')$  and  $\zeta^-(A, B) \leq \zeta^-(A', B')$ .
- $\zeta^-(A, \emptyset) = 0$ ,  $\zeta^+(\emptyset, A) = 0$  for any  $A \subset N$ .
- $\zeta(N, \emptyset) = (1, 0)$  and  $\zeta(\emptyset, N) = (0, 1)$ .

$\zeta^+(A, B)$  can be interpreted as the importance of coalition  $A$  of criteria in the presence of  $B$  for the positive part.  $\zeta^-(A, B)$  can be interpreted as the importance of coalition  $B$  of criteria in the presence of  $A$  for the negative part.

The Choquet integral w.r.t. a bi-capacity  $\nu$  proposed in [29] is now given. For any  $a \in \mathbb{R}^n$ ,

$$\mathcal{BC}_\nu(a) := \mathcal{C}_{\mu_{N^+}}(|a|)$$

where  $\mu_{N^+}(C) := \nu(C \cap N^+, C \cap N^-)$ ,  $N^+ = \{i \in N \mid a_i \geq 0\}$ ,  $N^- := N \setminus N^+$ , and  $|a|$  stands for  $(|a_1|, \dots, |a_n|)$ . Note that  $\mu_{N^+}$  is a non-monotonic capacity.

The Choquet integral w.r.t. a bipolar capacity can also be defined [39]. For  $a \in \mathbb{R}^n$ , let  $\tau$  be a permutation on  $N$  such that

$$|a_{\tau(1)}| \leq \dots \leq |a_{\tau(n)}|. \quad (1.24)$$

Let

$$\begin{aligned} A_i^+ &:= \{\tau(j), j \in \{i, \dots, n\} \text{ such that } a_{\tau(j)} \geq 0\} \\ A_i^- &:= \{\tau(j), j \in \{i, \dots, n\} \text{ such that } a_{\tau(j)} < 0\} \end{aligned}$$

and

$$\begin{aligned} C^+(a; \zeta) &= \sum_{i \in N} \left( a_{\tau(i)}^+ - a_{\tau(i-1)}^+ \right) \zeta^+(A_i^+, A_i^-) \\ C^-(a; \zeta) &= \sum_{i \in N} \left( a_{\tau(i)}^- - a_{\tau(i-1)}^- \right) \zeta^-(A_i^+, A_i^-) \end{aligned}$$

where  $a_{\tau(0)} := 0$  and for  $a \in \mathbb{R}$  we set  $a^+ = \max(a, 0)$  and  $a^- = (-a)^+$ . Finally the Choquet integral w.r.t.  $\zeta$  is defined by

$$C(a; \zeta) := C^+(a; \zeta) - C^-(a; \zeta).$$

For  $a \in \mathbb{R}^n$  for which several permutations  $\tau$  satisfy (1.24), it is easy to see that the previous expression depends on the choice of the permutation. This is not the case of the usual Choquet integral or the Choquet integral w.r.t. a bi-capacity. Enforcing that the results are the same for all permutations satisfying (1.24), we obtain the following constraints on the bipolar capacity:

$$\forall (A, B) \in \mathcal{Q}(N), \quad \zeta^+(A, B) - \zeta^-(\emptyset, B) = \zeta^+(A, \emptyset) - \zeta^-(A, B).$$

It can be shown then that the bipolar capacity  $\zeta$  reduces exactly to a bi-capacity  $\nu$  defined by

$$\nu(A, B) := \zeta^+(A, B) - \zeta^-(\emptyset, B).$$

One has indeed  $\zeta^+(A, B) = \nu(A, B) - \nu(\emptyset, B)$  and  $\zeta^-(A, B) = \nu(A, \emptyset) - \nu(A, B)$ . Moreover, it can be shown that the Choquet integral w.r.t.  $\zeta$  is equal to  $\mathcal{BC}_\nu$ . As a consequence, the concept of bipolar capacity reduces to bi-capacities when the Choquet integral is used. For this reason, we will consider only bi-capacities from now on. Note however that the concept of bipolar capacities has some interests in itself for other domains than MCDA.

The concept of bi-capacities is now applied to the example of section 4.2.2.

Let us try to model (1.22) and (1.23) with the extension of the Choquet integral to bi-capacities. We have  $\mathcal{BC}_\nu(4, 6, -3) = \mathcal{C}_{\mu_{N^+}}(4, 6, 3) = 3\mu(\{M, S, L\}) + \mu(\{M, S\}) + 2\mu(\{S\}) = 3\nu(\{M, S\}, \{L\}) + \nu(\{M, S\}, \emptyset) + 2\nu(\{S\}, \emptyset)$  and  $\mathcal{BC}_\nu(4, 5, -2) = 2\nu(\{M, S\}, \{L\}) + 2\nu(\{M, S\}, \emptyset) + \nu(\{S\}, \emptyset)$ . Hence (1.22) is equivalent to

$$\nu(\{M, S\}, \emptyset) - \nu(\{M, S\}, \{L\}) > \nu(\{S\}, \emptyset)$$

Similarly, relation (1.23) is equivalent to

$$\nu(\{S\}, \{L\}) > 0.$$

There is no contradiction between these two inequalities. Henceforth,  $\mathcal{BC}_\nu$  is able to model the example. This aggregation operator models the expertise that makes an explicit reference to an absolute value.

Before ending this section, we would like to stress that bi-capacities cannot account for *all* decision behaviours involving bipolar scales. To illustrate this, let us change the scores of  $E'$  and  $F'$  as follows.

	mathematics (M)	statistics (S)	languages (L)
student $E''$	2	6	-4
student $F''$	2	5	-3

It is easy to check that maintaining  $E'' \prec F''$  is equivalent to

$$\nu(\{S\}, \{L\}) < 0,$$

a contradiction with  $G' \succ H'$ . The fact is that with  $E'', F''$ , the score on mathematics is now too weak with respect to the score on languages. Hence  $E''$  should be preferred to  $F''$  since the latter one is better in statistics.

#### 4.4. General method for building utility functions

Let us now describe a general method to construct the utility functions  $u_i$  without the prior knowledge of  $F$ . It is possible to extend the method described in Section 3.3 in a straightforward way. Due to the existence of a neutral level, utility functions can now take positive and negative values. Hence assumption (1.9) is replaced by the following one:

$$\exists \alpha_i \in \mathbb{R}_+^* \quad , \quad F(a_i, 0_{-i}) = \alpha_i a_i \quad \text{for all } a_i \in \mathbb{R}. \quad (1.25)$$

Then the utility function can be derived from (1.10). It has been shown in [33] that the Šipoš integral satisfies (1.25). However, this condition is too restrictive since the usual Choquet does not fulfill it [33]. As a consequence, we are looking for a more general method.

Since the neutral level has a central position, the idea is to process separately elements which are “above” the neutral level (attractive part), and “below” it (repulsive part). Doing so, we may avoid difficulties due to some asymmetry between attractive and repulsive parts [29, 49]. The positive part of the utility function of  $X_i$  will be based on the two absolute levels  $\mathbf{0}_i$  and  $\mathbf{1}_i$ , while the negative part is based on the absolute levels  $\mathbf{0}_i$  and  $-\mathbf{1}_i$ , as defined in Section 2.3.

Generalizing (1.5), we set

$$u_i(\mathbf{0}_i) = 0, \quad u_i(\mathbf{1}_i) = 1 \quad \text{and} \quad u_i(-\mathbf{1}_i) = -1. \quad (1.26)$$

The two values 1 and  $-1$  are opposite to express the symmetry between  $\mathbf{1}_i$  and  $-\mathbf{1}_i$ .

The construction of the positive and negative parts of the utility function  $u_i$  is performed through the MACBETH methodology from the following two sets  $X_i^+$  and  $X_i^-$ :

$$X_i^\pm = \{(x_i, \mathbf{0}_{-i}) \mid x_i \in X_i^\pm\},$$

where  $X_i^+ = \{x_i \in X_i \mid (x_i, \mathbf{0}_{-i}) \succeq \mathbf{0}_N\}$  and  $X_i^- = \{x_i \in X_i \mid (x_i, \mathbf{0}_{-i}) \preceq \mathbf{0}_N\}$ . Interval scales  $u_{X_i^+}, u_{X_i^-}$  are obtained for  $i = 1, \dots, n$ , provided conditions **(Ord** $[X_i^+]$ **)**, **(Inter** $[X_i^+]$ **)**, **(C-Inter** $[X_i^+]$ **)**, **(Ord** $[X_i^-]$ **)**, **(Inter** $[X_i^-]$ **)**, and **(C-Inter** $[X_i^-]$ **)** are satisfied for  $i = 1, \dots, n$ . Now the scales are uniquely determined if one applies (1.5) to all positive scales, and the symmetric condition

$$u_{X_i^-}(\mathbf{0}_N) = 0 \quad \text{and} \quad u_{X_i^-}(-\mathbf{1}_i, \mathbf{0}_{-i}) = -1. \quad (1.27)$$

to all negative scales. Like for interval scales, one has for  $x_i \in X_i^\pm$

$$u_{X_i^\pm}(x_i, \mathbf{0}_{-i}) = F(u_i(x_i), 0_{-i}).$$

The assumption on the family  $\mathcal{F}$  becomes

$$\exists \alpha_i^\pm \in \mathbb{R}_+^* \quad , \quad F(a_i, 0_{-i}) = \alpha_i^\pm a_i \quad \text{for all } a_i \in \mathbb{R}_\pm. \quad (1.28)$$

Hence by (1.26), one has for any  $x_i \in X_i^\pm$ :

$$u_i(x_i) = \frac{F(u_i(x_i), 0_{-i})}{F(\pm \mathbf{1}_i, 0_{-i})} = \frac{u_{X_i^\pm}(x_i, \mathbf{0}_{-i})}{u_{X_i^\pm}(\pm \mathbf{1}_i, \mathbf{0}_{-i})}. \quad (1.29)$$

Hence, under assumption (1.28), the positive and negative parts of the utility functions can be constructed in two separate steps by (1.29) from cardinal information related to  $X_i^\pm$ .

It can be shown that the Choquet integral, Šipoš integral, the CPT model and the generalized Choquet integral fulfills (1.28).

## 4.5. Justification of the use of the generalized Choquet integral

**4.5.1 Required information.** For any  $i \in N$ , the utility function  $u_i$  is built from  $u_{X_i^+}$  and  $u_{X_i^-}$  like in Section 4.4.

Inter-criteria information is a generalization of the set  $X_{\{0,1\}}$ . The three reference levels  $-\mathbf{1}_i$ ,  $\mathbf{0}_i$  and  $\mathbf{1}_i$  are now used to build the set of *ternary alternatives*:

$$X_{\{-1,0,1\}} := \{(\mathbf{1}_A, -\mathbf{1}_B, \mathbf{0}_{-(A \cup B)}) \quad , \quad (A, B) \in \mathcal{Q}(N)\}.$$

Let  $u_{X_{\{-1,0,1\}}}$  be a numerical representation of  $X_{\{-1,0,1\}}$ . In the previous set, three special points can be exhibited:  $\mathbf{1}_N$ ,  $\mathbf{0}_N$  and  $-\mathbf{1}_N$ . Thanks to commensurateness between the  $\mathbf{1}_i$  levels, between the  $\mathbf{0}_i$  levels and between the  $-\mathbf{1}_i$  levels, it is natural to set

$$u_{X_{\{-1,0,1\}}}(-\mathbf{1}_N) = -1 \quad , \quad u_{X_{\{-1,0,1\}}}(\mathbf{0}_N) = 0 \quad \text{and} \quad u_{X_{\{-1,0,1\}}}(\mathbf{1}_N) = 1. \quad (1.30)$$

Relation  $u_{X_{\{-1,0,1\}}}(\mathbf{1}_N) = 1$  means that the alternative which is satisfactory on all attributes is also satisfactory. Relation  $u_{X_{\{-1,0,1\}}}(\mathbf{0}_N) = 0$  means that the alternative which is neutral on all attributes is also neutral. Finally, relation  $u_{X_{\{-1,0,1\}}}(-\mathbf{1}_N) = -1$  means that the alternative which is unsatisfactory on all attributes is also unsatisfactory. Since there are only two degrees of freedom in a scale of difference, one of these

three points must be removed for the practical construction of the scale. We decide to remove the act  $-\mathbf{1}_i$ . Let  $X^{\uparrow}_{\{-1,0,1\}} := X^{\uparrow}_{\{-1,0,1\}} \setminus \{-\mathbf{1}_N\}$ .

The numerical representation  $u_{X^{\uparrow}_{\{-1,0,1\}}}$  on  $X^{\uparrow}_{\{-1,0,1\}}$  is ensured by  $(\mathbf{Ord}[X^{\uparrow}_{\{-1,0,1\}}])$ ,  $(\mathbf{Inter}[X^{\uparrow}_{\{-1,0,1\}}])$ ,  $(\mathbf{C-Inter}[X^{\uparrow}_{\{-1,0,1\}}])$  and the last two conditions in (1.30).  $u_{X^{\uparrow}_{\{-1,0,1\}}}$  is uniquely determined by previous requirements. In summary

$$u_{X^{\uparrow}_{\{-1,0,1\}}}(\mathbf{1}_A, -\mathbf{1}_B, \mathbf{0}_{-(A \cup B)}) = \begin{cases} u_{X^{\uparrow}_{\{-1,0,1\}}}(\mathbf{1}_A, -\mathbf{1}_B, \mathbf{0}_{-(A \cup B)}) & \text{if } (A, B) \neq (\emptyset, N) \\ -1 & \text{otherwise} \end{cases}$$

**4.5.2 Measurement conditions.**  $u_{X^{\uparrow}_{\{-1,0,1\}}}$  can be described by a bi-capacity  $\nu$  defined by:  $\nu(A, B) := u_{X^{\uparrow}_{\{-1,0,1\}}}(\mathbf{1}_A, -\mathbf{1}_B, \mathbf{0}_{-(A \cup B)})$ . Consequently, it is natural to write  $u$  as follows:

$$u(x) = F_{\nu}(u_1(x_1), \dots, u_n(x_n)), \quad (1.31)$$

where  $F_{\nu}$  is the aggregation function.

We introduce the following axioms.

**(Bi-LM):** For any bi-capacities  $\nu, \nu'$  on  $\mathcal{Q}(N)$  satisfying  $(\mathbf{BFM}_c)$ , for all  $x \in \mathbb{R}^n$  and  $\gamma, \delta \in \mathbb{R}$ ,

$$F_{\gamma\nu + \delta\nu'}(x) = \gamma F_{\nu}(x) + \delta F_{\nu'}(x)$$

**(Bi-In):** For any bi-capacity  $\nu$  on  $\mathcal{Q}(N)$  satisfying  $(\mathbf{BFM}_a)$ ,  $(\mathbf{BFM}_b)$  and  $(\mathbf{BFM}_c)$ ,  $\forall x, x' \in \mathbb{R}^n$ ,

$$x_i \leq x'_i, \forall i \in N \Rightarrow F_{\nu}(x) \leq F_{\nu}(x')$$

**(Bi-PW):** For any bi-capacity  $\nu$  satisfying  $(\mathbf{BFM}_a)$ ,  $(\mathbf{BFM}_b)$ ,  $(\mathbf{BFM}_c)$ ,  $F_{\nu}(1_A, -1_{A'}, 0_{-A \cup A'}) = \nu(A, A')$ ,  $\forall (A, A') \in \mathcal{Q}(N)$ .

**(Bi-weak SPL<sup>+</sup>):** For any bi-capacity  $\nu$  on  $\mathcal{Q}(N)$  satisfying  $(\mathbf{BFM}_a)$ ,  $(\mathbf{BFM}_b)$ ,  $(\mathbf{BFM}_c)$ , for all  $A, C \subset N$ ,  $\alpha > 0$ , and  $\beta \geq 0$ ,

$$F_{\nu}((\alpha + \beta)_{A, \beta_{-A}}) = \alpha F_{\nu}(1_A, 0_{-A}) + \beta \nu(N, \emptyset).$$

These axioms are basically deduced from the measurement conditions on  $u_{X^{\uparrow}_i}$  and  $\nu$ . This is done exactly as in Section 3.4.2 [49, 29].

For  $A \subset N$ , consider the following application  $\Pi_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $(\Pi_A(x))_i = x_i$  if  $i \in A$  and  $-x_i$  otherwise. By **(Bi-PW)**,  $\nu(B, B')$  corresponds to the point  $(1_B, -1_{B'}, 0_{(B \cup B')})$ . Define  $\Pi_A \circ \nu(B, B')$  as the term of the bi-capacity associated to the point  $\Pi_A(1_B, -1_{B'}, 0_{(B \cup B')}) = (1_{(B \cap A) \cup (B' \setminus A)}, -1_{(B \setminus A) \cup (B' \cap A)}, 0_{(B \cup B')})$ . Hence we set

$$\Pi_A \circ \nu(B, B') := \nu((B \cap A) \cup (B' \setminus A), (B \setminus A) \cup (B' \cap A)).$$

By symmetry arguments, it is reasonable to have  $F_{\Pi_A \circ \nu}(\Pi_A(x))$  being equal to  $F_\nu(x)$ .

**(Bi-Sym)**: For any  $\nu : \mathcal{Q}(N) \rightarrow \mathbb{R}$  satisfying **(BFM<sub>c</sub>)**, we have for all  $A \subset N$

$$F_\nu(x) = F_{\Pi_A \circ \nu}(\Pi_A(x)).$$

We have the following result.

**Theorem 2 (Theorem 1 in [49])**  $\{F_\nu\}_\nu$  satisfies **(Bi-LM)**, **(Bi-In)**, **(Bi-PW)**, **(Bi-weak SPL<sup>+</sup>)** and **(Bi-Sym)** if and only if for any  $\nu : \mathcal{Q}(N) \rightarrow \mathbb{R}$  satisfying **(BFM<sub>a</sub>)**, **(BFM<sub>b</sub>)**, **(BFM<sub>c</sub>)** and **(BFM<sub>d</sub>)**, and for any  $a \in \mathbb{R}^n$ ,

$$F_\nu(a) = \mathcal{BC}_\nu(a).$$

The measurement conditions we have on  $u_i$  and  $u_X \upharpoonright_{\{-1,0,1\}}$  lead to axioms **(Bi-LM)**, **(Bi-In)**, **(Bi-PW)**, **(Bi-weak SPL<sup>+</sup>)** and **(Bi-Sym)**. The Choquet integral w.r.t a bi-capacity  $\nu$  is the only aggregation operator satisfying the previous set of axioms. So the generalized Choquet integral comes up very naturally when one works with information related to a bi-capacity.

#### 4.6. Shapley value, interaction index and $k$ -additive bi-capacities

As for capacities, due to the complexity of the model, involving  $3^n$  coefficients, it is necessary to be able to analyze a bi-capacity in terms of decision behaviour, namely importance of criteria and interaction among them.

We address first the importance index. Keeping the same rationale than for capacities, we may say that a criterion  $i$  is important if whenever it is added to a coalition of satisfied criteria, or dropped from a coalition of unsatisfied criteria, there is a significant improvement. In terms of the bi-capacity, it means that the importance index should be an average of the quantities  $\nu(A \cup i, B) - \nu(A, B)$  and  $\nu(A, B) - \nu(A, B \cup i)$  over all  $(A, B) \in \mathcal{Q}(N \setminus i)$ . Summing up these two expressions gives  $\nu(A \cup i, B) - \nu(A, B \cup i)$ , where the term where  $i$  is a criterion with neutral value has disappeared. We choose here to take as basis of the importance index this last expression, making the assumption that the importance index of  $i$  should not depend on situations where  $i$  is neutral (an alternative way taken by Felsenthal and Machover [14] is to keep separate the two expressions above in the average; see a detailed discussion of this issue in [48]).



As for capacities, under a linearity assumption, it suffices to impose a symmetry condition (the result should not depend on the numbering of criteria) and a normalization condition (the sum of importance indices over all criteria is constant) to determine uniquely the importance index, we call by analogy the *Shapley importance index for bi-capacities*, which writes

$$\phi^\nu(i) = \sum_{K \subset N \setminus \{i\}} \frac{(n-k-1)!k!}{n!} [\nu(K \cup \{i\}, N \setminus (K \cup \{i\})) - \nu(K, N \setminus K)].$$

The expression is very similar to the original Shapley index (see (1.13)). Observe that only vertices of  $\mathcal{Q}(N)$  (i.e. elements of the form  $(A, A^c)$ ) are used. We have given in [46, 48] an axiomatization of this Shapley index in the spirit of the original axiomatization of Shapley.

The normalization property writes  $\sum_{i=1}^n \phi(i) = \nu(N, \emptyset) - \nu(\emptyset, N) = 2$ . If  $\nu$  is of the CPT type with  $\nu(A, B) := \mu_1(A) - \mu_2(B)$ , then  $\phi^\nu(i) = \phi^{\mu_1}(i) + \phi^{\mu_2}(i)$ .

Let us turn to the notion of interaction. As for the case of bi-capacities, we may define an interaction index  $I(A)$ ,  $A \subset N$ , obtained recursively from the Shapley importance index for bi-capacities, as with Eq. (1.18) [29]. However, due to bipolarity, it seems more natural to distinguish criteria which are satisfied from those which are not. Denoting  $A, B$  the coalitions of satisfactory and unsatisfactory criteria, we are led to an interaction index with 2 arguments  $I_{A,B}$  (this is called *bi-interaction* in [29]). Let us explain this in the case of  $n = 2$ , following the same argument than for capacities (see Section 3.5). Due to bipolarity, we have now 9 ternary alternatives, as given on Figure 1.2. In each subsquare of  $[-1, 1]^2$ , it suffices to apply the classical interaction index for capacities, i.e. Formula (1.14). This gives, using our notation:

$$\begin{aligned} I_{\{1,2\},\emptyset} &:= \nu(\{1,2\}, \emptyset) - \nu(\{1\}, \emptyset) - \nu(\{2\}, \emptyset) + \nu(\emptyset, \emptyset) & (1.32) \\ I_{\emptyset,\{1,2\}} &:= \nu(\emptyset, \emptyset) - \nu(\emptyset, \{1\}) - \nu(\emptyset, \{2\}) + \nu(\emptyset, \{1,2\}) \\ I_{1,2} &:= \nu(\{1\}, \emptyset) - \nu(\emptyset, \emptyset) - \nu(\{1\}, \{2\}) + \nu(\emptyset, \{2\}) \\ I_{2,1} &:= \nu(\{2\}, \emptyset) - \nu(\{2\}, \{1\}) - \nu(\emptyset, \emptyset) + \nu(\emptyset, \{1\}). \end{aligned}$$

Based on this principle, the general formula is the following

$$I^\nu(A, B) = \sum_{K \subset N \setminus (A \cup B)} \frac{(n-a-b-k)!k!}{(n-a-b+1)!} \Delta_{A,B} \nu(K, N \setminus (A \cup K)),$$

with  $\Delta_{A,B} \nu(S, T) := \sum_{K \subset A, L \subset B} (-1)^{(a-k)+(b-l)} \nu(S \cup K, T \setminus (K \cup L))$ . It is easy to check that our previous Shapley index writes

$$\phi(i) = I_{i,\emptyset} + I_{\emptyset,i}$$

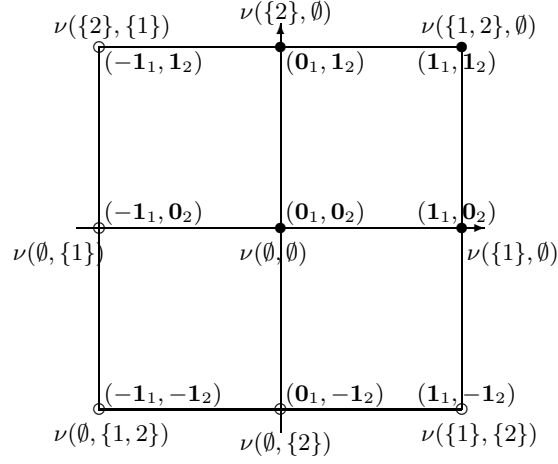


Figure 1.2. Ternary alternatives for  $n = 2$

suggesting that the Shapley index too could be divided into an index for satisfied criteria, and one for unsatisfied criteria.

If  $\nu$  is of CPT type with  $\nu(S, T) := \mu_1(S) - \mu_2(T)$ , the interaction is expressed by:

- (i)  $I_{S,T}^\nu = 0$  unless  $S = \emptyset$  or  $T = \emptyset$ .
- (ii) denoting  $I^{\mu_i}$  the interaction index of capacity  $\mu_i$ , we have:

$$I_{S,\emptyset}^\nu = I^{\mu_1}(S), \quad \forall \emptyset \neq S \subseteq N$$

$$I_{\emptyset,T}^\nu = I^{\overline{\mu_2}}(T), \quad \forall T \subseteq N.$$

Property (i) clearly expresses the fact that for a CPT model, there is no interaction between the positive part and the negative part. Property (ii) explains the relation between the interaction for bi-capacities and for capacities.

Since the complexity of bi-capacities is of order  $3^n$ , the necessity to have simplified models is yet more crucial than with capacities. The concept of  $k$ -additive bi-capacities can be defined in a way similar to the case of capacities. We refer the reader to [28] for the reasons underlying the definition hereafter.

**Definition 2** A bi-capacity is said to be  $k$ -additive for some  $k$  in  $\{1, \dots, n-1\}$  if the interaction index is such that  $I_{A,B} = 0$  whenever  $|B| < n-k$ , and there exists  $(A, B)$  with  $|B| = n-k$  such that  $I_{A,B} \neq 0$ .

As for capacities, a bi-capacity is completely determined by the values of  $I$  on  $\mathcal{Q}(N)$ , hence a  $k$ -additive bi-capacity is determined by  $1 + 2\binom{n}{n-1} + 2^2\binom{n}{n-2} + \dots + 2^k\binom{n}{n-k}$  coefficients, among which three are not free. Again, the case of 2-additive bi-capacities seems of particular interest, the number of coefficients being  $2n^2 - 3$ .

The expression of the Choquet integral for 2-additive bi-capacities is however complex (see [32]). This is not surprising since the expression contains as particular case the one of the symmetric Choquet integral [30], which is already complex compared to (1.19).

The concept of  $p$ -symmetry, as well as decomposable bi-capacities, has also been generalized to bi-capacities [28, 55].

## 4.7. Identification of bi-capacities

For  $v \in \mathbb{R}^n$  fixed, the mapping  $\nu \mapsto \mathcal{BC}_\nu(v)$  is linear. Henceforth, the methods described in section 3.7 for the determination of a capacity can be extended with no change to the case of bi-capacities. In particular, this enables the determination of  $\nu$  with a quadratic method from a set of alternatives with the associated scores, and with a linear method from a set of comparisons between alternatives. The constraints on the bi-capacity are composed of conditions **(BFM<sub>a</sub>)**, **(BFM<sub>b</sub>)**, **(BFM<sub>c</sub>)** and **(BFM<sub>d</sub>)**.

However, we are faced here to another difficulty. A bi-capacity contains  $3^n$  unknowns which makes its determination quite delicate. As an example, with 5 criteria, a capacity has  $2^5 = 32$  coefficients whereas a bi-capacity holds  $3^5 = 243$  coefficients. Ten well-chosen learning examples are generally enough to determine a capacity with 5 criteria. It would require maybe 80 learning examples to determine a bi-capacity with 5 criteria. This is obviously beyond what a human being could stand.

The way out to this problem is to reduce the complexity of the model. The first idea is to restrict to sub-classes of bi-capacities, such as the  $k$ -additive bi-capacities described above. For instance, there are  $2n^2 - 3 = 47$  unknowns for a 2-additive bi-capacity with 5 criteria. Other approaches are also possible.

## 5. Ordinal scales

### 5.1. Introduction

So far, we have supposed that the quantities we deal with (score, utilities, ...) are defined on some numerical scale, either an interval or a ratio scale, let us say a *cardinal scale*. In practical applications, most of the time it is not possible to have directly cardinal information, but

merely ordinal information. The MACBETH methodology we presented in Section 2.3 is a well-founded means to produce cardinal information from ordinal information. In some situations, this method may not apply, the decision maker being not able to give the required amount of information or being not consistent. In such a case, there is nothing left but to use the ordinal information as such, coping with the poor structure behind ordinal scales. We try in this section to define a framework and build tools as close as possible to those existing in the cardinal case, although many difficulties arise. All problems are not solved in this domain, we will present a state of the art, indicating main difficulties.

In the sequel, ordinal scales are denoted by  $L$  or similar, and are supposed to be finite totally ordered sets, with top and bottom denoted  $\mathbb{1}$  and  $\mathbb{0}$ .

Since ordinal scales forbid the use of usual arithmetic operations (see Section 2.1), minimum ( $\wedge$ ) and maximum ( $\vee$ ) become the main operations. Hence, decision models are more or less limited to combinations of these operations. We call *Boolean polynomials* expressions  $P(a_1, \dots, a_n)$  involving  $n$  variables and coefficients valued in  $L$ , linked by  $\wedge$  or  $\vee$  in an arbitrary combination of parentheses, e.g.  $((\alpha \wedge a_1) \vee (a_2 \wedge (\beta \vee a_3))) \wedge a_4$ . An important result by Marichal [52] says that the *Sugeno integral* w.r.t. a capacity coincides with the class of Boolean polynomials such that  $P(\mathbb{0}, \mathbb{0}, \dots, \mathbb{0}) = \mathbb{0}$ ,  $P(\mathbb{1}, \mathbb{1}, \dots, \mathbb{1}) = \mathbb{1}$ , and  $P$  is non-decreasing w.r.t. each variable. Since these conditions are natural in decision making, this shows that the Sugeno integral plays a central role when scales are ordinal, and the whole section is devoted to it.

Before entering into details, we wish to underline the fact that however, this is not the only way to deal with ordinal information. Roubens has proposed a methodology based on the Choquet integral (which has far better properties than the Sugeno integral, as we will show), where scores of an alternative on criteria are related to the number of times this alternative is better or worse than the others on the same criteria (see Chapter 11 by Roubens in this book, and [65]).

Let us begin by pinpointing fundamental difficulties linked to the ordinal context.

- **finiteness of scales:** sticking to a decomposable model of the type (1.2), the function  $F$  is now defined from  $L^n$  to  $L$ . Clearly it is impossible that  $F$  be strictly increasing due to the finiteness of  $L$ . A solution may be to map  $F$  on  $L'$ , with  $|L'| \geq |L|^n$ . Anyway, most measurement theoretic results are based on a solvability condition and Archimedean axioms, which cannot hold on a finite set.

- **ordinal nature:** the Sugeno integral, even defined as a function from  $\mathbb{R}^n$  to  $\mathbb{R}$ , can never be strictly increasing, and large domains of indifference exist. Hence, the decomposable model cannot satisfy weak separability (see Section 2.1). Specifically, Marichal [52] has shown that the Sugeno integral satisfies weak separability if and only if there is a dictator criterion. However, any Sugeno integral induces a preference relation  $\succeq$  which satisfies *directional weak separability*, defined by:

$$(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, z'_{-i}) \succeq (y_i, z'_{-i}), \quad \forall x, y, z, z' \in X.$$

This weaker condition ensures that no preference reversal occurs.

- **construction of utility functions  $u_i$ :** since on ordinal scales arithmetical operations are not permitted, the method described in Sections 3.3 and 4.4 cannot be applied directly. The ordinal counterpart of the multiplication being the minimum operator ( $\wedge$ ), Equation (1.9) becomes:

$$F(a_i, \mathbb{O}_{-i}) = \alpha_i \wedge a_i.$$

The term  $\alpha_i$  acts as a saturation level, hiding all utilities  $a_i$  larger than  $\alpha_i$ . Hence relation (1.9) cannot be satisfied and the previous method cannot be applied to build the utility functions.

To our knowledge, there is no method that enables the construction of utility functions in an ordinal framework. However, Greco *et al.* [40] have shown from a theoretical standpoint that this is possible (see Section 5.2). As a consequence, to avoid this problem most of works done in this area suppose that the attributes are defined on a common scale  $L$ , although this is not in general a realistic assumption.

## 5.2. Making decision with the Sugeno integral

We consider a capacity  $\mu$  on  $N$  taking its value in  $L$ , with  $\mu(\emptyset) = \mathbb{O}$  and  $\mu(N) = \mathbb{1}$ . Let  $a := (a_1, \dots, a_n)$  be a vector of scores in  $L^n$ . The *Sugeno integral* of  $a$  w.r.t.  $\mu$  is defined by [70]:

$$\mathcal{S}_\mu(a) := \bigvee_{i=1}^n [a_{\tau(i)} \wedge \mu(A_{\tau(i)})], \quad (1.33)$$

where  $\tau$  is a permutation on  $N$  so that  $a_{\tau(1)} \leq a_{\tau(2)} \leq \dots \leq a_{\tau(n)}$ , and  $A_{\tau(i)} := \{\tau(i), \dots, \tau(n)\}$ . One can notice the similarity with the Choquet integral. Taking  $L = [0, 1]$ , Choquet and Sugeno integrals coincide when

either the capacity or the integrand is 0-1 valued, specifically:

$$\begin{aligned}\mathcal{S}_\mu(1_A, 0_{-A}) &= \mu(A) = \mathcal{C}_\mu(1_A, 0_{-A}), \quad \forall A \subset N \\ \mathcal{S}_\mu(a) &= \mathcal{C}_\mu(a) \quad \forall a \in [0, 1]^n \text{ iff } \mu(A) \in \{0, 1\} \quad \forall A \subset N.\end{aligned}$$

We refer the reader to survey papers [10, 61] and to [52, 51] for properties of the Sugeno integral, especially in a decision making perspective. We mention that in the context of decision under uncertainty, an axiomatic construction similar to the one of Savage has been done by Dubois *et al.* [12, 13].

We cite here an interesting result by Greco *et al.* [40], giving a very simple characterization of the Sugeno integral in MCDA. Assuming finiteness of  $X$  (or  $X/\sim$  contains a countable order-dense subset), they have shown that the preference relation  $\succeq$  on  $X$  is representable by a Sugeno integral (i.e. there exist utility functions  $u_i : X_i \rightarrow [0, 1]$  and a capacity  $\mu$  such that  $x \succeq y$  iff  $\mathcal{S}_\mu(u_1(x_1), \dots, u_n(x_n)) \geq \mathcal{S}_\mu(u_1(y_1), \dots, u_n(y_n))$ ) iff  $\succeq$  is a weak order and satisfies

$$[(x_i, a_{-i}) \succeq w \text{ and } (y_i, b_{-i}) \succeq t] \Rightarrow [(z_i, a_{-i}) \succeq w \text{ or } (x_i, b_{-i}) \succeq t]$$

for  $i = 1, \dots, n$  and  $x, y, z, a, b \in X$ .

As said in the introduction, making decision with the Sugeno integral has some drawbacks, which are clearly put into light with the following results [51, 59]. Let  $\succeq$  be a weak order (complete, reflexive, transitive) on  $[0, 1]^n$ , and for  $a, b \in [0, 1]^n$ , denote  $a \geq b$  if  $a_i \geq b_i$  for all  $i \in N$ , and  $a > b$  if  $a \geq b$  and  $a_i > b_i$  for some  $i \in N$ , and  $a \gg b$  if  $a_i > b_i$  for all  $i \in N$ . We say that  $\succeq$  satisfies *monotonicity* if  $a \geq b$  implies  $a \succeq b$ , the *strong Pareto condition* if  $a > b$  implies  $a \succ b$ , and the *weak Pareto condition* if  $a \gg b$  implies  $a \succ b$ . Then the following holds.

**Proposition 1** *Let  $\mu$  be a capacity on  $N$ , and  $\succeq_\mu$  the weak order induced by the Sugeno integral  $\mathcal{S}_\mu$ .*

- (i)  $\succeq_\mu$  always satisfies *monotonicity*.
- (ii)  $\succeq_\mu$  satisfies the *weak Pareto condition* iff  $\mu$  is 0-1 valued.
- (iii)  $\succeq_\mu$  never satisfies the *strong Pareto condition*.

Note that the Choquet integral always satisfies the weak Pareto condition, and the strong one iff  $\mu$  is strictly monotone.

Since arithmetic operations cannot be used with ordinal scales, our definitions of importance and interaction indices cannot work, and alternatives must be sought. Grabisch [20] has proposed definitions which more or less keep mathematical properties of the original Shapley value and interaction index. However, these indices, especially the interaction index, do not seem to convey the meaning they are supposed to have.

### 5.3. Symmetric ordinal scales and the symmetric Sugeno integral

This section introduces *bipolar ordinal scales*, i.e. ordinal scales with a central neutral level, and a symmetry around it, and is based on [25, 24, 22]. The aim is to have a structure similar to cardinal bipolar scales, so as to build a counterpart of the CPT model, using a Sugeno integral for the “positive” part (above the neutral level), and another one for the “negative” part (below the neutral level):

$$\text{OCPT}_{\mu_1, \mu_2}(a) := \mathcal{S}_{\mu_1}(a^+) \ominus \mathcal{S}_{\mu_2}(a^-)$$

(“O” stands for “ordinal”) where  $a^+ := a \vee 0$ ,  $a^- := (-a)^+$ , and  $\ominus$  is a suitable difference operator. We will show that this task is not easy.

Let us call  $L^+$  some ordinal scale, and define  $L := L^+ \cup L^-$ , where  $L^-$  is a reversed copy of  $L^+$ , i.e. for any  $a, b \in L^+$ , we have  $a \leq b$  iff  $-b \leq -a$ , where  $-a, -b$  are the copies of  $a, b$  in  $L^-$ . We want to endow  $L$  with operations  $\oplus, \otimes$  satisfying (among possible other conditions):

(C1)  $\oplus, \otimes$  coincide with  $\vee, \wedge$  respectively on  $L^+$

(C2)  $-a$  is the symmetric of  $a$ , i.e.  $a \oplus (-a) = \ominus$ .

Hence we may extend to  $L$  what exists on  $L^+$  (e.g. the Sugeno integral), and a difference operation could be defined. The problem is that conditions (C1) and (C2) imply that  $\oplus$  would be non-associative in general. Take  $\ominus < a < b$  and consider the expression  $(-b) \oplus b \oplus a$ . Depending on the place of parentheses, the result differs since  $((-b) \oplus b) \oplus a = \ominus \oplus a = a$ , but  $(-b) \oplus (b \oplus a) = (-b) \oplus b = \ominus$ .

It can be shown that the best solution (i.e. associative on the largest domain) for  $\oplus$  is given by:

$$a \oplus b := \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } = -b \\ \ominus & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases} \quad (1.34)$$

Except for the case  $b = -a$ ,  $a \oplus b$  equals the absolutely larger one of the two elements  $a$  and  $b$ .

The extension of  $\wedge$ , viewed as the counterpart of multiplication, is simply done on the principle that the rule of sign should hold:  $-(a \otimes b) = (-a) \otimes b$ ,  $\forall a, b \in L$ . It leads to an associative operator, defined by:

$$a \otimes b := \begin{cases} -(|a| \wedge |b|) & \text{if } \text{sign } a \neq \text{sign } b \\ |a| \wedge |b| & \text{else.} \end{cases} \quad (1.35)$$

Based on these definitions, the OCPT model writes:

$$\text{OCPT}_{\mu_1, \mu_2}(a) := \mathcal{S}_{\mu_1}(a^+) \otimes (-\mathcal{S}_{\mu_2}(a^-)).$$

When  $\mu_1 = \mu_2 =: \mu$ , we get the *symmetric Sugeno integral*, denoted  $\tilde{\mathcal{S}}_\mu$ .

Going a step further, it is possible to define the Sugeno integral w.r.t. bi-capacities, following the same way as with the Choquet integral. One can show that, defining  $\mathcal{BS}_\nu(a) := \mathcal{S}_{\mu_{N^+}}(|a|)$ , with same notations as in Section 4.3 and replacing in the definition of Sugeno integral  $\vee, \wedge$  by  $\otimes, \oplus$ , the expression is [31] (see also Greco *et al.* [39] for a similar definition):

$$\mathcal{S}_\nu(a) = \left\langle \bigotimes_{i=1}^n \left[ |a_{\tau(i)}| \otimes \nu(A_{\tau(i)} \cap N^+, A_{\tau(i)} \cap N^-) \right] \right\rangle, \quad (1.36)$$

where  $\tau$  is a permutation on  $N$  so that  $|a_{\tau(1)}| \leq \dots \leq |a_{\tau(n)}|$ ,  $N^+ := \{i \in N \mid a_i \geq \mathbf{0}\}$ ,  $N^- := N \setminus N^+$ , and the expression  $\left\langle \bigotimes_{i=1}^n b_i \right\rangle$  is a shorthand for  $\left( \bigotimes_{i=1}^n b_i^+ \right) \otimes \left( - \bigotimes_{i=1}^n b_i^- \right)$ . It can be shown that if  $\nu$  is of the CPT type, one recovers the OCPT model.

Lastly, we mention Denneberg and Grabisch, who have proposed a general formulation of the Sugeno integral on arbitrary bipolar spaces [9].

## 5.4. Building a model from preferences

The previous sections have shown many difficulties underlying the construction. We try in this section to build a model from preferences, in a spirit close to the one of Sections 3 and 4, and based on the symmetric model [26]. We assume the existence on each attribute  $X_i$  of a neutral element  $\mathbf{0}_i$  and a greatest element  $\mathbf{1}_i$  in the sense that  $(\mathbf{1}_i, \mathbf{0}_{-i}) \succeq (x_i, \mathbf{0}_{-i})$ , for all  $x_i \in X_i$ . We suppose in addition that  $(\mathbf{1}_i, x_{-i}) \succeq (\mathbf{0}_i, x_{-i})$  for each  $x_{-i}$ . We consider as in Section 4.4 the sets  $X_i^+ := \{x_i \in X_i \mid (x_i, \mathbf{0}_{-i}) \succeq \mathbf{0}_N\}$  and  $X_i^- := \{x_i \in X_i \mid (x_i, \mathbf{0}_{-i}) \preceq \mathbf{0}_N\}$ .

Our aim is to represent the preference of the DM on  $X$  by a (symmetric) Sugeno integral with respect to some capacity  $\mu$ , that is:

$$a \succeq b \text{ iff } \tilde{\mathcal{S}}_\mu(u_1(a_1), \dots, u_n(a_n)) \geq \tilde{\mathcal{S}}_\mu(u_1(b_1), \dots, u_n(b_n)) \quad (1.37)$$

where  $u_i : X_i \rightarrow L, i = 1, \dots, n$  are commensurable utility functions, defined on some scale  $L$ , which we will build. As explained in Section 5.1, one cannot build separately the utility functions and the capacity. In our approach, we need to determine the capacity first.

The determination of the capacity is done through the set of binary alternatives  $(\mathbf{1}_A, \mathbf{0}_{-A})$ , denoted as before  $X \upharpoonright_{\{0,1\}}$ . We suppose that  $\succeq$



restricted to this set is reflexive, transitive, and complete, and in addition that it satisfies monotonicity in the following sense: if  $A \subset B$ , then  $(\mathbf{1}_A, \mathbf{0}_{-A}) \preceq (\mathbf{1}_B, \mathbf{0}_{-B})$ .

Let us denote by  $m$  the number of equivalence classes of  $\sim$  on  $X \upharpoonright_{\{0,1\}}$ . From this, we build the ordinal scale  $L^+ = \{e_0, \dots, e_{m-1}\}$ , with  $e_0 < e_1 < \dots < e_{m-1}$ , assigning to each equivalence class a degree of the scale, which reflects the rank of the equivalence class. Then, due to monotonicity:

- $e_0$ , denoted  $\mathbb{O}$ , corresponds to  $(\mathbf{0}_1, \dots, \mathbf{0}_n) = \mathbf{0}_N$ .
- $e_{m-1}$ , denoted  $\mathbb{1}$ , corresponds to  $(\mathbf{1}_1, \dots, \mathbf{1}_n) = \mathbf{1}_N$ .

We define  $\mu(A) := u(\mathbf{1}_A, \mathbf{0}_{A^c})$ , where  $u : X \upharpoonright_{\{0,1\}} \longrightarrow L^+$  assigns to each binary alternative the value on  $L^+$  of its equivalence class. By monotonicity,  $\mu$  is a capacity on  $L^+$ .

We turn to the identification of the utility functions. The approach is related to the one proposed by Marichal [52].  $u_i$  should be a representation of the preference of the DM among alternatives in  $X \downharpoonright_i := \{(x_i, \mathbf{0}_{-i}), x_i \in X_i\}$ , i.e.

$$u_i(x_i) \geq u_i(y_i) \text{ iff } (x_i, \mathbf{0}_{-i}) \succeq (y_i, \mathbf{0}_{-i}),$$

supposing that  $\succeq$  is a weak order when restricted to each  $X \downharpoonright_i$ . In order to ensure commensurability, we impose

$$u_i(\mathbf{1}_i) = \mathbb{1}, \quad u_i(\mathbf{0}_i) = \mathbb{O}, \quad \forall i = 1, \dots, n.$$

We suppose to be in the bipolar case (otherwise we just need  $L^+$  and an ordinary Sugeno integral), hence we build the symmetrized scale  $L = L^+ \cup L^- = \{e_{-m+1}, \dots, e_{-1}, e_0, e_1, \dots, e_{m-1}\}$ , which we equip with  $\mathbb{O}, \mathbb{1}$ . We denote naturally  $e_{-m+1}$  by  $-\mathbb{1}$ . From now on, all  $u_i$ 's are from  $X_i$  to  $L$ .

We first try to determine  $u_i(x_i)$  for all  $x_i \in X_i^+$ . Suppose the DM assigns  $(x_i, \mathbf{0}_{-i})$  to  $e_k$  (more exactly, the DM thinks that  $(x_i, \mathbf{0}_{-i})$  is indifferent with any alternative from the equivalence class assigned to  $e_k$ ). Then, from (1.37) and the definition of the Sugeno integral, we necessarily have

$$e_k = u_i(x_i) \wedge \mu(i) \leq \mu(i).$$

We have two possible cases.

- if  $e_k = \mu(i)$ , then  $u_i(x_i) \geq e_k = \mu(i)$
- if  $e_k < \mu(i)$ , then  $u_i(x_i) = e_k$ .

Suppose the DM assigns  $(x_i, \mathbf{1}_{-i})$  to  $e_l$ . Then, from the representation condition by the Sugeno integral, we should have

$$e_l = u_i(x_i) \vee \mu(N \setminus i) \geq \mu(N \setminus i),$$

with again two possible cases.

- if  $e_l = \mu(N \setminus i)$ , then  $u_i(x_i) \leq e_l = \mu(N \setminus i)$
- if  $e_l > \mu(N \setminus i)$ , then  $u_i(x_i) = e_l$ .

By a repeated application of the assumption  $(\mathbf{1}_i, x_{-i}) \succeq (\mathbf{0}_i, x_{-i})$ , we deduce that  $e_l \geq e_k$ . Equality means that  $(x_i, \mathbf{0}_{-i}) \sim (x_i, \mathbf{1}_{-i})$ , which can be interpreted as a dictatorship of attribute  $X_i$ . Combining the above and supposing  $e_k < e_l$ , three cases can happen:

- **Case 1:**  $e_k = \mu(i)$ ,  $e_l > \mu(N \setminus i)$ , which entails

$$\left\{ \begin{array}{l} (x_i, \mathbf{0}_{-i}) \sim (\mathbf{1}_i, \mathbf{0}_{-i}) \\ (x_i, \mathbf{1}_{-i}) \succ (\mathbf{0}_i, \mathbf{1}_{-i}) \end{array} \right.$$

This could be interpreted as  $x_i$  “close to”  $\mathbf{1}_i$ , and in this case  $u_i(x_i) = e_l$ .

- **Case 2:**  $e_l = \mu(N \setminus i)$ ,  $e_k < \mu(i)$ , which entails

$$\left\{ \begin{array}{l} (x_i, \mathbf{1}_{-i}) \sim (\mathbf{0}_i, \mathbf{1}_{-i}) \\ (x_i, \mathbf{0}_{-i}) \prec (\mathbf{1}_i, \mathbf{0}_{-i}) \end{array} \right.$$

This could be interpreted as  $x_i$  “close to”  $\mathbf{0}_i$ , and in this case  $u_i(x_i) = e_k$ .

- **Case 3:**  $e_k = \mu(i)$ ,  $e_l = \mu(N \setminus i)$ , which entails

$$\left\{ \begin{array}{l} (x_i, \mathbf{0}_{-i}) \sim (\mathbf{1}_i, \mathbf{0}_{-i}) \\ (x_i, \mathbf{1}_{-i}) \sim (\mathbf{0}_i, \mathbf{1}_{-i}) \end{array} \right.$$

This causes the indetermination of  $u_i(x_i)$  since  $e_k \leq u_i(x_i) \leq e_l$ .

The last case corresponds to  $e_k < \mu(i)$  and  $e_l > \mu(N \setminus i)$ , which implies  $e_k = e_l$ , a case we have eliminated since it corresponds to a dictatorship of  $X_i$ .

The same procedure can be applied to “negative” values  $x_i \in X_i^-$ . Let us assume that the DM assigns  $(x_i, \mathbf{0}_{-i})$  to  $e_{-k}$ . Then, by the symmetric Sugeno integral, one should satisfy

$$e_{-k} = u_i(x_i) \otimes \mu(i).$$

Then, if  $e_{-k} = -\mu(i)$ , we have  $u_i(x_i) \leq e_{-k}$ , and if  $e_{-k} > -\mu(i)$ , we get  $u_i(x_i) = e_{-k}$ .

Now we suppose that the DM assigns to  $(x_i, -\mathbf{1}_{-i})$  the value  $e_{-l}$ . We find that

$$e_{-l} = u_i(x_i) \otimes (-\mu(N \setminus i))$$

Then, if  $e_{-l} = -\mu(N \setminus i)$ , we have  $u_i(x_i) \geq e_{-l}$ , and if  $e_{-l} < -\mu(N \setminus i)$ , then  $u_i(x_i) = e_{-l}$ .

As before, we have three cases.

- **Case 1:**  $e_{-k} = -\mu(i)$ ,  $e_{-l} < -\mu(N \setminus i)$ . Then  $u_i(x_i) = e_{-l}$ .
- **Case 2:**  $e_{-l} = -\mu(N \setminus i)$ ,  $e_{-k} > -\mu(i)$ . Then  $u_i(x_i) = e_{-k}$ .
- **Case 3:**  $e_{-k} = -\mu(i)$ ,  $e_{-l} = -\mu(N \setminus i)$ . Then  $u_i(x_i) \in [e_{-l}, e_{-k}]$ .

The above methodology can be easily extended to have a representation by an OCPT model, the case of the bipolar Sugeno integral being more tricky. Remark that the procedure may leave some indetermination for the utility functions, hence several solutions are possible. Also, the set of equivalence classes can be enriched if necessary when utility functions are built, e.g. if the DM thinks that some alternative  $(x_i, \mathbf{0}_{-i})$  is strictly between two consecutive equivalence classes.

## 5.5. Identification of capacities

In situations where utility functions are known, the problem of the identification of capacities when the model is a Sugeno integral (or OCPT, bipolar Sugeno integral) in an ordinal context, or even when  $L = [0, 1]$  or  $[-1, 1]$ , appears to be rather different from the case of the Choquet integral. The main reason is that we are not able to write the identification problem as a minimization problem *stricto sensu* (see Section 3.7), since the notion of difference between values, hence of error, is not defined in a way which is suitable on an ordinal scale, to say nothing about “squared errors” and “average values”.

Even if we take  $L$  as a real interval, which permits to define a squared error criterion as for the Choquet integral, the minimization problem obtained is not easy to solve, since it involves non-linear, non-differentiable operations  $\vee, \wedge, \otimes, \oplus$ . In such cases, only meta-heuristic methods can be used, as genetic algorithms, simulated annealing, etc. There exist some works in this direction, although most of the time used for the Choquet integral, which is questionable [74, 23].

What can be done without error criterion to minimize? The second option, also used for the Choquet integral (see Section 3.7), is to find capacities which enable the representation of the preference of the DM over

a set of alternatives of interest by the Sugeno integral (or OCPT, ...). A detailed study of this problem has been done by Rico *et al.* [63] for the Sugeno integral. We mention also the work of Greco *et al.* based on decision rules, which can be found in Chapter 12 of this book (see also [38]). We give a short description of the work by Rico *et al.*

Since utility functions are assumed to be known and commensurable, defined on some scale  $L$  (supposed to be unipolar here), the preference relation  $\succeq$  of the DM is expressed directly on  $L^n$ . We call  $\mathcal{A} \subset L^n$  the set of alternatives of interest. We distinguish two levels of representation.

- the *strong representation*, where the capacity  $\mu$  must satisfy  $\mathcal{S}_\mu(a) \geq \mathcal{S}_\mu(b)$  if and only if  $a \succeq b$ .
- the *weak representation*, where we merely forbid a reversal:  $a \succ b$  implies  $\mathcal{S}_\mu(a) \geq \mathcal{S}_\mu(b)$ .

We can guess by properties of the Sugeno integral (see. e.g. weak separability vs. directional weak separability) that the weak representation is more appropriate.

Let us suppose that the alternatives in  $O$  can be put into  $p$  equivalence classes  $[a^1], \dots, [a^p]$  by  $\sim$ , assuming  $a^1 \prec \dots \prec a^p$ . The strong representation problem amounts to find  $p$  values  $\alpha_1 < \alpha_2 < \dots < \alpha_p$  in  $L$  such that there exists a capacity  $\mu$  satisfying  $\mathcal{S}_\mu(a) = \alpha_i$ , for all  $a \in [a^i]$ ,  $i = 1, \dots, p$ . For the weak representation problem, it suffices to find  $p - 1$  numbers  $0 =: \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_p := 1$  in  $L$  such that there exists a capacity  $\mu$  satisfying  $\alpha_{i-1} \leq \mathcal{S}_\mu(a) \leq \alpha_i$ , for all  $a \in [a^i]$ ,  $i = 1, \dots, p$ .

The set of capacities such that  $\mathcal{S}_\mu(a) = \alpha$  is non-empty iff  $a_{(n)} < \alpha$  or  $a_{(1)} > \alpha$ , and is the interval  $[\check{\mu}^{a,\alpha}, \hat{\mu}^{a,\alpha}]$ , where for all  $A \neq \emptyset, N$

$$\hat{\mu}^{a,\alpha}(A) := \begin{cases} \alpha & \text{if } A \subset A_{(i_{a,\alpha}^>)} \\ \mathbb{1} & \text{otherwise} \end{cases}$$

$$\check{\mu}^{a,\alpha}(A) := \begin{cases} \alpha & \text{if } A_{(i_{a,\alpha}^{\geq})} \subset A \\ \mathbb{0} & \text{otherwise} \end{cases}$$

with  $i_{a,\alpha}^> \in N$  such that  $a_{(i_{a,\alpha}^>-1)} < \alpha \leq a_{(i_{a,\alpha}^>)}$ , and  $i_{a,\alpha}^{\geq} \in N$  such that  $a_{(i_{a,\alpha}^{\geq}-1)} \leq \alpha < a_{(i_{a,\alpha}^{\geq})}$ . The set of solutions for the strong representation is then the intersection of all these intervals for all  $\alpha_i$ .

The set of capacities solution of the weak representation problem is empty iff  $\exists i$  such that  $a_{(1)} > \alpha_i$  for some  $a \in [a^i]$  or  $\exists i$  such that  $b_{(n)} < \alpha_i$

for some  $b \in [a^{i+1}]$ , and otherwise is the interval  $[\check{\mu}, \hat{\mu}]$ , with

$$\check{\mu}(A) = \bigvee_{i=1}^{p-1} \bigvee_{a \in [a^{i+1}]} \check{\mu}^{a, \alpha_i}(A), \quad \hat{\mu}(A) = \bigwedge_{i=1}^{p-1} \bigwedge_{a \in [a^i]} \hat{\mu}^{a, \alpha_i}(A).$$

## 6. Concluding remarks

This chapter has tried to give a unified presentation of MCDA methods based on fuzzy integrals. It has shown that the concepts of capacity and bi-capacity naturally arise as overall utility of binary and ternary alternatives, and that the Choquet integral appears to be the unique solution for aggregating criteria, under a set of natural axioms.

This methodology has been applied in various fields of MCDA from a long time, particularly in subjective evaluation, and seems to receive more and more attention. Following the pionnering works of Sugeno [70], many researchers in the eighties in Japan have applied in practical problems the Sugeno integral, for example to opinion poll [62], and later the Choquet integral (see a summary of main works in [36]). More recent applications can be found in [35, 23], see also [27, 42].

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