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Nonlinear Inequality, Fixed Point and Nash Equilibrium

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Abstract

In this paper, we give new sufficient conditions for the existence of a solution of the g-maximum equality. As a consequence, we prove a new fixed point theorem. We ensure the existence of a fixed of a function $f : X \to E$ such that $X \subset f(X)$ is established. We also prove a new theorem of existence of Nash equilibrium.

Key words: Ky Fan inequality, g-maximum equality, fixed point, Nash equilibrium.
PACS: C61, C62 and C72.

1 Introduction

Let X and Y are a nonempty subsets of spaces E and F, respectively. Let $\Psi : X \times Y \longrightarrow \mathbb{R}$ and $g : X \longrightarrow Y$ are functions, and let $r \in \mathbb{R}$ be a constant.

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Consider the problem of finding $\overline{x}$ such that

$$\Psi(\overline{x}, y) \leq r, \quad \forall y \in Y. \tag{1.1}$$

Ky Fan [7] introduced and studied the minimax inequality problem of finding a solution $\overline{x} \in X$ of the inequality (1.1) in the case where $E = F$, $X = Y$, $g = id_X$ and $r = \sup_{x \in X} \Psi(x, x)$. The Ky Fan inequality has proven to be very useful in solving nonlinear problems in different areas. Due to various applications of the Ky Fan inequality in many areas, many researchers made efforts to generalize it. Indeed, many results have been obtained in this direction of research: we mention the results of Ding and Tan [6], Georgiev and Tanaka [11], Simons [16], Tian and Zhou [17], Yu and Yuan [18] and Yuan [19] and equilibrium problems studied by many authors as special cases, see [2], [5], [6], [8], [9], [10] and the references therein.

Note that in general all these works assume that $X = Y$ in (1.1). As far as we know there is only one result [15], where the author assumes $X \neq Y$, but considers the set X as an interval in the real line $\mathbb{R}$. In [14], the inequality (1.1) has been studied in the case where $E \neq F$ or $X \neq Y$. The same authors proved the following theorem.

Theorem 1.1 *[14] (g-Maximum Equality Theorem) Let X be a nonempty subset of a metric space E, Y be a nonempty convex, compact subset of a hausdorff locally convex vector space F and Ψ be a real-valued function defined on $X \times Y$. Suppose that there exists a nonempty compact subset X_0 of X and a continuous function g of X_0 into Y. Assume, in addition, that the following conditions are satisfied.*

(1) $g(X_0)$ is convex in Y,
(2) the function Ψ is continuous on $X_0 \times Y$,
(3) the function $y \mapsto \Psi(x, y)$ is quasi-concave on Y, for each $x \in X_0$,
(4) for each $g(x) \in \partial g(X_0)$ and for each $y \in Y$, there exists $z \in Z_{g(X_0)}(g(x))$ such that $\Psi(x, y) \leq \Psi(x, z)$ where $Z_{g(X_0)}(g(x)) = \left[\overline{\bigcup_{h>0} \frac{g(X_0)-g(x)}{h}} + g(x)\right] \cap Y$.

Then there exists $\overline{x} \in X_0$ such that

$$\sup_{y \in Y} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})). \tag{1.2}$$

The main purpose of this paper is to establish the existence of a solution of the nonlinear inequality (1.2), under assumptions different from those of Theorem 1.1. Then using the new result, we provide a new fixed point theorem

and we give an application in game theory, more precisely a new theorem of existence of Nash equilibrium.

Let us first introduce some notations and definitions.

Consider a nonempty subset X of a metric space E and Y a nonempty subset of a locally convex space F. Let 2^Y be the set of all the parts of Y.
A set-valued $C : X \to 2^Y$ is said to be closed if the corresponding graph is closed in $X \times Y$, *i.e.* the set $\{(x, y) \in X \times Y \text{ such that } y \in C(x)\}$ is closed in $X \times Y$ [3]. A function $f : Y \to \mathbb{R}$ is said to be upper semicontinuous over Y if $\forall c \in \mathbb{R}$, the set $\{x \in Y, f(x) \geq c\}$ is closed; f is said to be lower semicontinuous over Y if $-f$ is upper semicontinuous and f is said to be continuous over Y if f and $-f$ are upper semicontinuous over Y. We say that f is quasi-concave on Y if for any y_1, y_2 in Y and for any $\theta \in [0, 1]$, we have $\min\{f(y_1),\ f(y_2)\} \leq f(\theta y_1 + (1-\theta)y_2)$. And f is quasi-convexe if $-f$ is quasi-concave.

Let f be a real-valued function defined from a metric space E. We say that *support* of f (denoted by $supp(f)$) the smallest closed set S such that $f(x) = 0$, $\forall x \notin S$, *i.e.* $supp(f) = \overline{\{x \in E, \text{ such that } f(x) \neq 0\}}$.

Let us consider an open finite covering $\{A_i\}_{i=1,...,n}$ of a set E. We say that a continuous partition of unity associated to this finite covering, a family of continuous functions $\{f_i\}_{i=1,...,n}$ defined from E into $[0, 1]$ such that:

$$\begin{cases} 1)\ \forall x \in E,\ \sum_{i=1}^{n} f_i(x) = 1, \\ 2)\ supp(f_i) \subset A_i,\ i = 1, ..., n. \end{cases}$$

We have the following Lemma.

Lemma 1.1 *(Theorem 4.1.31. page 187, [3]) For all open finite covering of a metric space E, there exists a continuous partition of unity associated to this finite covering.*

Zeidler [20] showed that this Lemma rest true if E is a locally convex Hausdorff space.

Let us consider a set-valued C defined from X into X. A point $x \in X$ is called fixed point of F if $x \in C(x)$. If C is a single-valued function, then a fixed point x of C will be such that $x = C(x)$.

We will use the following lemma.

Lemma 1.2 *(Kakutani-Fan-Glicksberg fixed point Theorem) Let K be a subset nonempty compact convex of a locally convex Hausdorff space, and let the closed set-valued $C : X \to 2^X$ have nonempty convex values. Then the set of fixed points of C is nonempty and compact.*

2 The g-maximum Equality

Let us consider the following example.

Example 2.1 *Let $X = [0,1]$ and $Y =]-\infty, 0]$, $g(x) = -x$, $\forall x \in X$ and $\Psi(x,y) = -x^2 - y^2$.*

It is clear that Theorem 1.1 cannot be applicable because Y is not compact. Nevertheless, there exists $x = 0$ such that $\sup_{y\in Y}\Psi(x,y) = \Psi(x,-x)$. We conclude then, under other conditions, equation (1.2) has at least one solution.

In the following theorem we provide sufficient conditions for which the g-maximum equality has at least one solution.

Theorem 2.1 *Let X be a nonempty convex compact set of a locally convex Hausdorff space, and let Y be a nonempty set of a metric space. Let us consider the following functions: $g : X \to Y$ continuous over X and $\Psi : X \times Y \to \mathbb{R}$ such that*

(1) function $x \mapsto \Psi(x,y)$ is continuous over X, $\forall y \in Y$ and function $z \mapsto \Psi(x, g(z))$ is lower semicontinuous over X, $\forall x \in X$
(2) function $x \mapsto \Psi(x,y)$ is quasi-concave over X, $\forall y \in Y$
(3) $\forall (x,y) \in X \times Y$, $\exists z \in X$ such that $\Psi(x,y) \leq \Psi(z, g(x))$.

Then there exists $\overline{x} \in X$ such that

$$\sup_{y\in Y}\Psi(\overline{x},y) = \Psi(\overline{x}, g(\overline{x})). \tag{2.1}$$

Proof. Suppose that (2.1) is not true, then

$$\forall x \in X, \exists y \in Y \text{ such that } \Psi(x,y) > \Psi(x, g(x)) \tag{2.2}$$

X can then be covered by the sets

$$\theta_y = \{\ x \in X \text{ such that } \Psi(x,y) > \Psi(x,g(x))\}\ ,\ y \in Y.$$

Let us prove that $\forall y \in Y$, θ_y is open. Indeed, let $x \in \overline{X/\theta_y}$, there exists a sequence $\{x_p\}_{p\geq 1}$ in X/θ_y converging to x, hence $\forall p \geq 1$, $\Psi(x_p, y) \leq \Psi(x_p, g(x_p))$. Taking into account condition (1) of Theorem 2.1, when $p \to +\infty$, we obtain $\Psi(x, y) \leq \Psi(x, g(x))$, *i.e.* $x \in X/\theta_y$, therefore X/θ_y is closed in X.

Since X is compact, it can be covered by a finite number n of subsets $\{\theta_{y_1}, ..., \theta_{y_n}\}$ of type θ_y. Consider a continuous partition of unity $\{h_i\}_{i=1,...,n}$ associated to the finite covering $\{\theta_{y_1}, ..., \theta_{y_n}\}$ (Lemma 1.1), *i.e.* $\{h_i\}_{i=1,...,n}$ verify

$$\begin{cases} 1)\ \forall x \in X,\ \sum\limits_{i=1}^{n} h_i(x) = 1, \\ 2)\ supp(h_i) \subset \theta_{y_i},\ i = 1, ..., n. \end{cases}$$

Let us now consider the simplex S of $\mathbb{R}^n$

$$S = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, ..., n\}.$$

Consider the following set-valued function

$$C : X \to X$$

defined by $x \mapsto C(x) = \left\{ z \in X \text{ such that } \max\limits_{\lambda \in S} \sum\limits_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(z, g(x)) \right\}$.

Now, we will prove step by step that the function C satisfies the conditions of Lemma 1.2 (Kakutani-Fan-Glicksberg fixed point Theorem):

i) $\forall x \in X$, $C(x) \neq \emptyset$. Indeed, let be $x \in X$, the function $\lambda \mapsto \sum\limits_{i=1}^{n} \lambda_i \Psi(x, y_i)$ is linear on $\mathbb{R}^n$, so it is continuous on the compact S and by the Weierstrass Theorem, there exists $\overline{\lambda} \in S$ such that

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) = \sum_{i=1}^{n} \overline{\lambda}_i \Psi(x, y_i) \leq \sum_{i=1}^{n} \overline{\lambda}_i \max_{i=1,...,n} \Psi(x, y_i) = \Psi(x, y_{i_0}) \tag{2.3}$$

where $y_{i_0} \in \{y_1, .., y_n\}$ hence

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(x, y_{i_0}).$$

Condition (3) of Theorem 2.1 imply, $\exists z \in X$ such that

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(x, y_{i_0}) \leq \Psi(z, g(x)).$$

Therefore, $z \in C(x)$, thus $C(x) \neq \emptyset$.

ii) $\forall x \in X$, $C(x)$ is convex in X. Indeed, let $x \in X$ and $\overline{z}$, $\overline{\overline{z}}$ be two elements of $C(x)$ and $\theta \in [0,1]$.

Let us prove that $\theta\overline{z} + (1-\theta)\overline{\overline{z}} \in C(x)$.

Since $\overline{z}$ and $\overline{\overline{z}}$ are two elements in $C(x)$, we have

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(\overline{z}, g(x)) \text{ and } \max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(\overline{\overline{z}}, g(x)),$$

hence

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \min \left\{ \Psi(\overline{z}, g(x)),\ \Psi(\overline{\overline{z}}, g(x)) \right\}, \tag{2.4}$$

the condition (2) of Theorem 2.1 and the inequality (2.4) imply

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(\theta\overline{z} + (1-\theta)\overline{\overline{z}}, g(x)),\ \forall \theta \in [0,1]\,;$$

thus $\theta\overline{z} + (1-\theta)\overline{\overline{z}} \in C(x)$.

iii) C has a closed graph in $X \times X$.

We have $Graph(C) \subset X \times X$. By assumption X is compact. Let $(x, z) \in \overline{Graph(C)}$, then there exists a sequence $\{(x_p, z_p)\}_{p \geq 1}$ in $Graph(C)$ which converges to (x, z).

Hence we have $\forall p \geq 1$, $z_p \in C(x_p)$, *i.e.* $\forall p \geq 1$, $\max\limits_{\lambda \in S} \sum\limits_{i=1}^{n} \lambda_i \Psi(x_p, y_i) \leq \Psi(z_p, g(x_p))$.

Taking into account the condition (1) and the continuity of g of Theorem 2.1 when $p \to \infty$, we obtain

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(x, y_i) \leq \Psi(z, g(x)),$$

i.e. $z \in C(x)$, hence $(x, z) \in Graph(C)$, then $Graph(C)$ is closed in $X \times X$.

From (i)-(iii), we conclude that the function C satisfies all conditions of Lemma 1.2. Consequently, $\exists \widetilde{x} \in X$ such that $\widetilde{x} \in C(\widetilde{x})$, *i.e.*

$$\max_{\lambda \in S} \sum_{i=1}^{n} \lambda_i \Psi(\widetilde{x}, y_i) \leq \Psi(\widetilde{x}, g(\widetilde{x}))$$

hence $\forall \lambda \in S$, $\sum\limits_{i=1}^{n} \lambda_i \Psi(\widetilde{x}, y_i) \leq \Psi(\widetilde{x}, g(\widetilde{x}))$.

Using the continuous partition of unity $\{h_i\}_{i=1,...,n}$ given above, let $\widetilde{\lambda} = (h_1(\widetilde{x}), .. ., h_n(\widetilde{x}))$, we have $\widetilde{\lambda} \in S$ because $h_i(\widetilde{x}) \geq 0$ and $\sum\limits_{i=1}^{n} h_i(\widetilde{x}) = 1$, therefore,

$$\sum_{i=1}^{n} h_i(\widetilde{x})\Psi(\widetilde{x}, y_i) \leq \Psi(\widetilde{x}, g(\widetilde{x})).$$

Let $J = \{i \in \{1, ..., n\} \text{ such that } h_i(\widetilde{x}) > 0\}$, then $J \neq \emptyset$.
Note that $\sum\limits_{i=1}^{n} h_i(\widetilde{x})\ \Psi(\widetilde{x}, y_i) = \sum\limits_{i \in J} h_i(\widetilde{x})\Psi(\widetilde{x}, y_i)$.
We have $\forall i \in J$, $h_i(\widetilde{x}) > 0$, therefore $\widetilde{x} \in supp(h_i) \subset \theta_{y_i}\ \forall i \in J$, *i.e.*

$$\forall i \in J, \quad \Psi(\widetilde{x}, y_i) > \Psi(\widetilde{x}, g(\widetilde{x})).$$

It follows that $\sum\limits_{i \in J} h_i(\widetilde{x})\Psi(\widetilde{x}, y_i) > \sum\limits_{i \in J} h_i(\widetilde{x})\Psi(\widetilde{x}, g(\widetilde{x})) = \Psi(\widetilde{x}, g(\widetilde{x}))$ and then

$$\Psi(\widetilde{x}, g(\widetilde{x})) < \sum_{i \in J} h_i(\widetilde{x})\Psi(\widetilde{x}, y_i) = \sum_{i=1}^{n} h_i(\widetilde{x}).\Psi(\widetilde{x}, y_i) \leq \Psi(\widetilde{x}, g(\widetilde{x})),$$

i.e. we obtain a contradiction,

$$\Psi(\widetilde{x}, g(\widetilde{x})) > \Psi(\widetilde{x}, g(\widetilde{x})).$$

Therefore (2.2) is not true, hence

$$\exists \overline{x} \in X \text{ such that } \Psi(\overline{x}, y) \leq \Psi(\overline{x}, g(\overline{x})),\ \forall y \in Y,$$

i.e. $\sup\limits_{y \in Y} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x}))$. ■

Consider again Example 2.1. We have $X = [0, 1]$ and $Y =]-\infty, 0]$, $g(x) = -x$, $\forall x \in X$ and $\Psi(x, y) = -x^2 - y^2$.

It is clear that the assumptions and conditions (1)-(2) of Theorem 2.1 are satisfied. Let us verify condition (3) of Theorem 2.1. Indeed, let be $(x, y) \in X \times Y$, we have $\Psi(x, y) = -x^2 - y^2$ and $\Psi(z, g(x)) = -z^2 - x^2$. Since $-y^2 \leq 0$, $\forall y \in Y$, then there exists $z = 0 \in X$ such that $\Psi(x, y) = -x^2 - y^2 \leq -z^2 - x^2 = \Psi(z, g(x))$. Consequently $\exists x = 0$ such that $\sup\limits_{y \in Y} \Psi(x, y) \leq \Psi(z, g(x))$. Indeed, $x = 0$ is such a point.

If the sets X and Y are identical and if we consider $g = id_X$, we obtain the following inequality similar to the Ky Fan inequality under other conditions.

Corollary 2.1 *Let X be a nonempty, convex and compact set in a locally convex Hausdorff space E and Ψ a real valued function defined on $X \times X$. Suppose that the following conditions are satisfied*

(1) function $x \mapsto \Psi(x,y)$ is continuous over X, $\forall y \in X$ and function $y \mapsto \Psi(x,y)$ is lower semicontinuous over X, $\forall x \in X$
(2) function $x \mapsto \Psi(x,y)$ is quasi-concave over X, $\forall y \in X$
(3) $\forall (x,y) \in X \times X$, $\exists z \in X$ such that $\Psi(x,y) \leq \Psi(z,x)$.

Then, there exists $\overline{x} \in X$ such that

$$\sup_{y \in X} \Psi(\overline{x}, y) = \Psi(\overline{x}, \overline{x}) \leq \sup_{y \in X} \Psi(y, y).$$

Remark 2.1 *If the function Ψ is semi-symmetrical, i.e. $\Psi(x,y) \leq \Psi(y,x)$, then the condition (3) of Corollary 2.1 is satisfied.*

3 Applications

In this section, we establish two applications of Theorem 2.1. In first, we prove a new theorem of existence of fixed point; in the second, we show a new theorem of existence of Nash equilibrium.

3.1 ***Fixed point problem***

Let us consider the following example.

Example 3.1 *Let f be the following function*

$$f : X = [\tfrac{6}{5}, 2] \to \mathbb{R}$$
$$x \mapsto f(x) = 1/(x-1).$$

We have $\max_{x \in [\frac{6}{5}, 2]} |f'(x)| = 25$, then f is a 25-lipschitz and also $f([\frac{6}{5}, 2]) \nsubseteq [\frac{6}{5}, 2]$ because $f(\frac{6}{5}) = 5 \notin [\frac{6}{5}, 2]$. Therefore the classical fixed point Theorems (Cauchy's, Banach-Cacciopoli-Picard's, Brouwer's, Browder's fixed point Theorem, ...) are not applicable.

The following theorem ensures the existence of a fixed point for this type of functions.

Theorem 3.1 *Let X be a nonempty convex compact of normed space $(E, \|.\|_E)$. Let f be a continuous function over X into E such that*

(1) function $x \mapsto \|f(x) - y\|_E$ is quasi-convexe over X, $\forall y \in E$,
(2) $X \subset f(X)$.

Then f has a fixed point.

Proof. Let us consider the functions Ψ and g defined as follows:

$$\Psi : X \times E \to \mathbb{R}$$
$$(x, y) \mapsto \Psi(x, y) = -\|f(x) - y\|_E,$$

$$g : X \to E$$
$$x \mapsto g(x) = x.$$

The function $\|.\|_E$ is uniformly continuous over E, then the function Ψ is continuous over $X \times E$, and $x \mapsto \Psi(x, y)$ is quasi-concave over X (condition (1)), $\forall y \in E$.

Let us prove that $\forall (x, y) \in X \times E$, there exists $z \in X$ such that $\Psi(x, y) \leq \Psi(z, x)$. Indeed, according condition (2), we have $X \subset f(X)$, then $\forall x \in X$, $\exists z \in X$ such that $x = f(z)$, which implies $\|f(z) - x\|_E = 0$ and since $\forall x \in X$, $\forall y \in E$, we have $\|f(x) - y\|_E \geq 0$. Thus,

$$\forall x \in X,\ \forall y \in E,\ \exists z \in X \text{ such that } 0 = \|f(z) - x\|_E \leq \|f(x) - y\|_E,$$

i.e.

$$\forall x \in X,\ \forall y \in E,\ \exists z \in X \text{ such that } \Psi(x, y) \leq \Psi(z, x) = 0.$$

Since X is a nonempty, convex and compact subset of a normed space E, then according to Theorem 2.1, $\exists \overline{x} \in X$ such that

$$\|f(\overline{x}) - y\|_E \geq \|f(\overline{x}) - \overline{x}\|_E, \quad \forall y \in E.$$

Thus, if we let $y = f(\overline{x})$ in the last inequality, we obtain

$$\|\overline{x} - f(\overline{x})\|_E \leq 0.$$

Therefore $f(\overline{x}) = \overline{x}$, *i.e.* $\overline{x}$ is a fixed point of function f. ■

Consider again the Example 3.2. We have the function $x \mapsto |1/(x-1) - y|$ is quasi-convexe over $[\frac{6}{5}, 2]$, $\forall y \in \mathbb{R}$ (see FIGURE 1.).

Since f is not increasing order, then $f([\frac{6}{5}, 2]) = [f(2), f(\frac{6}{5}] = [1, 5] \supset [\frac{6}{5}, 2]$. Thus according to Theorem 3.1, f has a fixed point in $[\frac{6}{5}, 2]$. Indeed, $\overline{x} = (1 + \sqrt{5})/2$ is such a point.

Theorem 3.1 can be generalized as follows.

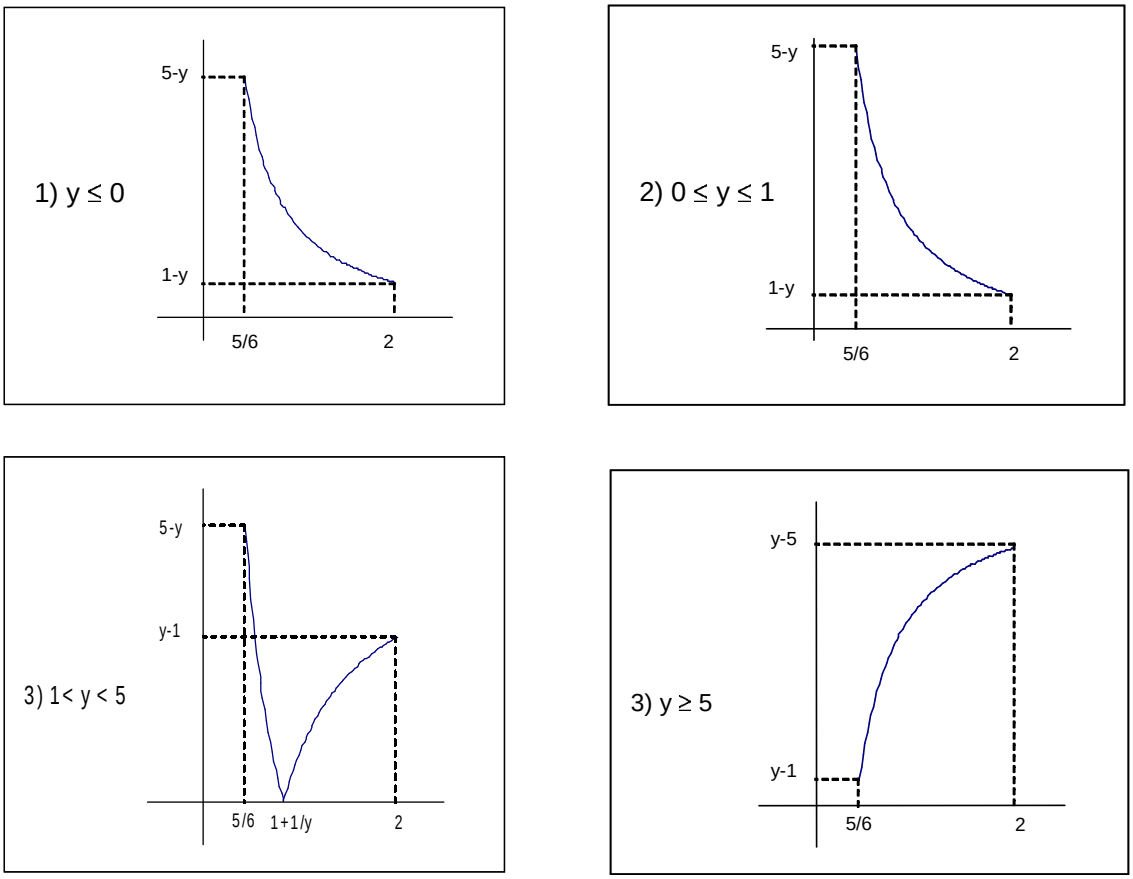

Fig. 1. The graph of function $x \mapsto |1/(x-1) - y|$, $\forall y \in \mathbb{R}$

Definition 3.1 *Let X be a nonempty convex subsets of a vector space E and let Y be a nonempty subsets of a metric space (F, d). Let us consider a set-valued $C : X \to 2^Y$. C is said to be d-quasi-convexe over X if $x \mapsto d(C(x), y) = \inf_{u \in C(x)} d(u, y)$ is quasi-convexe over X, $\forall y \in Y$.*

We have the following theorem.

Theorem 3.2 *Let X be a nonempty convex compact set of a locally convex Hausdorff space, Y be a nonempty set of a normed space and g be a continuous function defined from X into Y. Let us consider a nonempty closed set-valued $C : X \to 2^Y$ such that*

(1) $d(C(x), y)$ is continuous over $X \times Y$, where $d(C(x), y) = \inf_{u \in C(x)} \|u - y\|$,
(2) C is a d-quasi-convexe over X,
(3) $\forall x \in X$, $\exists z \in X$ such that $g(x) \in C(z)$.

Then $\exists \overline{x} \in X$ such that $g(\overline{x}) \in C(\overline{x})$.

Proof. It is a straightforward consequence, if we consider the following functions $\Psi(x,y) = -d(C(x),y) = -\inf_{u\in C(x)}\|u-y\|$, $\forall (x,y)\in X\times Y$ in Theorem 2.1. ■

We deduce the following corollary in case where the set-valued C is an ordinary function (singleton).

Corollary 3.1 *Let X be a nonempty convex compact set of a locally convex Hausdorff space, Y be a nonempty set of a normed space $(F,\|.\|_F)$ and g be a continuous function defined from X into Y. Let us consider a continuous function $f: X\to Y$ such that*

(1) $x\mapsto \|f(x)-y\|_F$ is quasi-convexe over X, $\forall y\in Y$
(2) $g(X)\subset f(X)$.

Or

(1') $x\mapsto \|g(x)-y\|_F$ is quasi-convexe over X, $\forall y\in Y$
(2') $f(X)\subset g(X)$.

Then $\exists \overline{x}\in X$ such that $g(\overline{x}) = f(\overline{x})$.

3.2 ***Nash equilibrium***

As an application of Theorem 2.1 to game theory, in this section, we establish a new theorem of existence of Nash equilibrium.

Consider the following noncooperative game

$$< I,\ X,\ f(x) > \tag{3.1}$$

where I is the set of players, which may be an infinite set; X_i is the set of strategies of the player $i\in I$, X_i is a subset of a locally convex vector space E_i, $f_i: X\to \mathrm{R}$ is the payoff function of the player $i\in I$, $X=\prod_{i\in I}X_i\subset E=\prod_{i\in I}E_i$ is the set of strategy profiles; $x=(x_i)_{i\in I}\in X$ is a strategy profile of the game where $x_j\in X_j$ is the strategy of the player $j\in I$. If a strategy profile $x\in X$ is played, each player $i\in I$ receives his payoff $f_i(x)$.
The aim of each player in this game is to maximize his payoff function.

We will use the following notations. Let $i\in I$ be any player, then $-i = I-\{i\} = \{j\in I \text{ such that } j\neq i\}$, $X_{-i}=\prod_{j\in -i}X_j$ and $x_{-i}=(x_j)_{j\in -i}$.

Definition 3.2 *[12] An issue $\overline{x}\in X$ is said to be a Nash equilibrium of game*

(3.1) *if*

$$\forall i \in I,\ \forall y_i \in X_i,\ f_i(\overline{x}_{-i}, y_i) \leq f_i(\overline{x}).$$

First let us recall the some results of existence of this equilibrium.

Theorem 3.3 *(Nash Theorem [12]) Let I be an indexed finite or infinite countable set. Let $\{E_i\}_{i\in I}$ be a family of locally convex Hausdorff spaces and let X_i be a nonempty convex and compact subset of E_i such that*

(1) f_i is continuous over X,
(2) $y_i \mapsto f_i(x_{-i}, y_i)$ is quasi-concave over X_i, for each $x_{-i} \in X_{-i}$.

Then game (3.1) *has a Nash equilibrium.*

Theorem 3.4 *(Abalo-Kostreva Theorem [1]) Let I be an indexed finite or infinite countable set. Let $\{E_i\}_{i\in I}$ be a family of metrizable locally convex topological vector spaces and let X_i be a nonempty convex and weakly compact subset of E_i such that*

(1) f_i is continuous over X,
(2) $arg\max_{y_i\in X_{-i}} f_i(x_{-i}, y_i)$ is a singleton for each $x_{-i} \in X_{I-S_i}$.

Then game (3.1) *has a Nash equilibrium.*

Let us consider the following example.

Example 3.2 *Assume that in game* (3.1) $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, $x = (x_1, x_2)$ *and*

$$f_1(x) = x_1^2 - x_2^3$$
$$f_2(x) = -x_1^3 + x_2^2.$$

For this example, we have $\forall i \in I$, the function $y_i \mapsto f_i(x_{-i}, y_i)$ is not quasi-concave and $arg\max_{y_i\in X_{-i}} f_i(y_i, x_{-i}) = \{-1, +1\}$, for each $i = 1, 2$. Therefore, the Nash Theorem (Theorem 3.4) and Abalo-Kostreva Theorem (Theorem 3.5) are not applicable.

We have the following theorem.

Theorem 3.5 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i\in I}$ be a family of locally convex Hausdorff spaces and let X_i be a nonempty convex and compact subset of E_i such that*

(1) f_i is continuous over X,

(2) $x_{-i} \mapsto f_i(x_{-i}, y_i)$ *is quasi-concave over* X_{-i}*, for each* $y_i \in X_i$*.*
(3) $\forall (x, y) \in X \times X$*,* $\exists z \in X$ *such that* $\sum_{i=1}^{n} f_i(x_{-i}, y_i) \leq \sum_{i=1}^{n} f_i(z_{-i}, x_i)$*.*

Then game (3.1) *has a Nash equilibrium.*

Proof. It is a straightforward consequence of Theorem 2.1, if we consider the following functions $g = id_X$ and $\Psi(x, y) = \sum_{i=1}^{n} f_i(x_{-i}, y_i)$, $\forall (x, y) \in X \times X$. ■

Let us consider again the Example 3.3, we have for each $x = (x_1, x_2)$ and $y = (y_1, y_2))$, $\Psi(x, y) = -x_1^3 - x_2^3 + y_1^2 + y_2^2$. Since $\max_{u \in [-1,1]} \{-u^3 - u^2\} = 0$ and $\max_{u \in [-1,1]} \{u^2\} = 1$, hence, $\max_{x_1, x_2 \in [-1,1]} \{-x_1^3 - x_1^2 - x_2^3 - x_2^2\} + \max_{y_1, y_2 \in [-1,1]} \{y_1^2 + y_2^2\} = 2$. Therefore,

$$-x_1^3 - x_2^3 + y_1^2 + y_2^2 \leq 2 + x_1^2 + x_2^2 = \Psi((-1, -1), x),\ \forall x_i, y_i \in [-1, 1],\ i = 1, 2.$$

Then we conclude that $\forall (x, y) \in X$, $\exists z = (-1, -1)$ such that $\Psi(x, y) \leq \Psi(z, x)$. Since $x_{-i} \mapsto f_i(x_{-i}, y_i)$ is quasi-concave over X_{-i} and f_i is continuous over the convex compact $[-1, 1]$. According to Theorem 3.6, this game has a Nash equilibrium. In fact, we can easily verify that (-1,-1), (1,-1).., are such equilibria.

In Theorem 3.6, if the set of players is an infinite countable set, we deduce the following corollary.

Corollary 3.2 *Let* I *be an infinite countable index set. Let* $\{E_i\}_{i \in I}$ *be a family of locally convex Hausdorff spaces and for each* $i \in I$ *let* X_i *be a nonempty, convex and compact subset of* E_i *such that*

(1) f_i *is uniformly continuous over* X*,*
(2) $\{f_i\}_{i \in I}$ *is uniformly bounded over* X*, i.e.* $\exists M \in \mathbb{R}$ *such that* $f_i(x) \leq M$*,* $\forall i \in I$*, and* $\forall x \in X$*,*
(3) $x_{-i} \mapsto f_i(x_{-i}, y_i)$ *is quasi-concave over* X_{-i}*, for each* $y_i \in X_i$*,*
(4) $\forall (x, y) \in X \times X$*,* $\exists z \in X$ *such that* $\sum_{i \in I} \frac{1}{2^i} f_i(x_{-i}, y_i) \leq \sum_{i \in I} \frac{1}{2^i} f_i(z_{-i}, x_i)$*.*

Then game (3.1) *has a Nash equilibrium.*

4 Conclusion

In this paper, through Theorem 2.1, we have established that the g-maximum equality has a least one solution under new conditions. This new Theorem

(Theorem 2.1) is complimentary to Theorem 1.1. As an application of Theorem 2.1, we have proved a new interesting fixed point theorem. We have provided new sufficient conditions for the existence of Nash equilibrium. We have exhibited examples where our results are applicable, but the well known theorems are not applicable. This shows that our results enlarge the class of functions for which a fixed point exists and also the class of games for which a Nash equilibrium exists. Finally, we hope that our results will be useful for solving theoretical and practical problems from various domains.

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