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ON BERGE EQUILIBRIUM

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Abstract

Based on the notion of equilibrium of a coalition P relatively to a coalition K , of Berge, Zhukovskii has introduced Berge equilibrium as an alternative solution to Nash equilibrium for non cooperative games in normal form. The essential advantage of this equilibrium is that it does not require negotiation of any player with the remaining players, which is not the case when a game has more than one Nash equilibrium. The problem of existence of Berge equilibrium is more difficult (compared to that of Nash). This paper is a contribution to the problem of existence and computation of Berge equilibrium of a non cooperative game. Indeed, using the g -maximum equality, we establish the existence of a Berge equilibrium of a non cooperative game in normal form. In addition, we give sufficient conditions for the existence of a Berge equilibrium which is also a Nash equilibrium. This allows us to get equilibria enjoying the properties of both concepts of solution. Finally, using these results, we provide two methods for the computation of Berge equilibria: the first one computes Berge equilibria; the second one computes Berge equilibria which are also Nash equilibria.

Key words: Berge equilibrium, Nash equilibrium, individually rational,
 g -maximum equality
PACS: C72, C62 and C02.

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1 Introduction

Based on the notion of equilibrium for a coalition P with respect to a coalition K introduced by Berge in 1957 [6], Zhukovskii introduced the Berge Equilibrium [24]. This equilibrium can be used as an alternative solution when Nash equilibrium [16] does not exist. In addition to this, it does not require negotiation or coordination between players in the case where there are many Nash equilibria. In this equilibrium each player chooses his strategy without consulting the others, he obtains the maximum payoff if the situation is favorable for him: by obligation or willingness, the other players choose favorable for him strategies.

Zhukovskii [23] has investigated the problem of existence of Berge equilibrium in the case of two person games and for some differential linear quadratic games. Gaidov wrote a pair of short papers ([10], [11]) in stochastic differential games. The volume entitled "Multicriteria Dynamical Problems Under Uncertainty", a Collection of Scientific Works, published in Orekhovo-Zuevo, contains three papers on different aspects of Berge equilibrium: existence theorems [9], Berge equilibrium in difference differential games [7] and Berge equilibrium in bi-matrix games [12]. Radjef [20] have also studied the problem of existence of this equilibrium. In all the mentioned works the set of players is assumed to be finite and no procedure for the computation of Berge equilibrium is proposed.

In [1], [2], [3] and [4] Abalo and Kostreva generalize the Zhukovskii's definition of Berge equilibrium. They also provide a theorems of existence of this equilibrium in the case of infinite set of players as Theorems 2, 3 in paper [2], Theorems 3.1, 3.2 in paper [1], Theorems 2, 3 in paper [3] and Theorems 3.2, 3.4 in paper [4]. It is to be noted that these theorems are based on an earlier paper of Radjef [20] providing an existence theorem of Berge equilibrium in the sense of Zhukovskii [24]. After a deep investigation we have found that the assumptions given in the Abalo and Kostreva's Theorem are not sufficient for the existence of Berge equilibrium in the sense of Zhukovskii. The same remark can be made for the Radjef's Theorem. Indeed, we provide a simple game that verifies the assumptions of Abalo and Kostreva's Theorem without Berge equilibrium in the sense of Zhukovskii, which is a particular case of Berge equilibrium in the sense of Abalo and Kostreva.

In this paper we provide general sufficient conditions for the existence of Berge equilibrium when the set of players may be infinite countable. Next, we provide a procedure for its computation. We also provide sufficient conditions for the existence of Berge equilibrium that is also Nash equilibrium (Berge Nash equilibrium) and a method for its computation. Our approach is totally different from the existing ones, we use the g-maximum equality theorem [17].

This paper is organized as follows. In Section 2 we recall the different definitions of Berge equilibrium and some of its properties. In Section 3 we provide sufficient conditions for its existence and we establish the existence of Berge-Nash equilibrium. Then from these two results, we derive two procedures for the determination of this equilibrium. We end the paper with a conclusion.

2 Different Definitions of Berge Equilibrium

Consider the following non cooperative game in normal form

$$G = (X_i, f_i)_{i \in I}. \quad (2.1)$$

where I is the set of players, which we assume to be finite or infinite countable; $X = \prod_{i \in I} X_i$ is the set of strategy profiles of the game, where X_i is the set of strategies of player i ; $X_i \subset E_i$, E_i is a vector space; $f_i : X \rightarrow \mathbb{R}$ is the payoff function of player i .

In this game the aim of each player is to maximize his objective function.

Let \mathfrak{S} denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $R \in \mathfrak{S}$, we denote by $-R$; the set $-R = \{i \in I \text{ such that } i \notin R\}$: the remaining of coalition R , if R is reduced to a singleton $\{i\}$, we denote then by $-i$ the set of $-R$. We also denote by $X_R = \prod_{i \in R} X_i$ the set of strategies of the players in coalition R . If $\{K_i\}_{i \in \{1, \dots, s\} \subset \mathbb{N}}$ is a partition of the set of players I , then any strategy profile $x = (x_1, \dots, x_n) \in X$ can be written as $x = (x_{K_1}, x_{K_2}, \dots, x_{K_s})$ where $x_{K_i} \in \prod_{j \in K_i} X_j$.

We denote by \bar{A} the closure of a set A and by ∂A its boundary. Let Y_0 be a nonempty convex subset of a convex subset Y of a vector space and $y \in Y_0$, we denote by $H_{Y_0}(y)$, $T_{Y_0}(y)$ and $Z_{Y_0}(y)$ respectively the following sets: $H_{Y_0}(y) = \bigcup_{h>0} [Y_0 - y] / h$, $T_{Y_0}(y) = \overline{H_{Y_0}(y)}$ and $Z_{Y_0}(y) = [T_{Y_0}(y) + y] \cap Y$. Note that $T_{Y_0}(y)$ is called tangent cone to Y_0 at the point y [5].

In this paper we recall the different definitions of Berge equilibrium and we give a more general definition of Berge equilibrium in the sense that the number of players may be infinite countable, which is not the case in the definition given by Zhukovskii who considers the finite case only.

Definition 2.1 [24] *A strategy profile $\bar{x} \in X$ is a simple Berge equilibrium in the sense of Zhukovskii of the game (2.1) if*

$$f_i(\bar{x}) \geq f_i(x_{-i}, \bar{x}_i), \quad (2.2)$$

for each given $i \in I$ and $x_{-i} \in X_{-i}$.

We can see that, the definition means that when a player $i \in I$ plays his strategy \bar{x}_i from the Berge equilibrium \bar{x} , he cannot obtain a maximum payoff unless the remaining players $-i$ willingly (or obliged) play the strategy \bar{x}_{-i} from the Berge equilibrium \bar{x} . In other words, if at least one of the players of coalition $-i$ deviates from his equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to his payoff $f_i(\bar{x})$ in Berge equilibrium.

Definition 2.2 ([1],[2],[3] and [4]) *Consider the game (2.1).*

Let $R = \{R_i\}_{i \in M} \subset \mathfrak{S}$ be a partition of I and $S = \{S_i\}_{i \in M}$ be a set of subsets of I . A feasible strategy $\bar{x} \in X$ is an equilibrium point for the set R relative to the set S or simply a Berge equilibrium point for (2.1) if

$$f_{r_m}(\bar{x}) \geq f_{r_m}(x_{S_m}, \bar{x}_{-S_m}),$$

for each given $m \in M$, any $r_m \in R_m$ and $x_{S_m} \in X_{S_m}$.

A strategy profile is a simply Berge equilibrium point if no coalition S_m in S can profitably deviate from the prescribed profile relatively to the set of players R_m . Indeed, if a coalition S_m deviates from its strategy \bar{x}_{S_m} some simply Berge equilibrium point \bar{x} , then she cannot improve the earning of all the players in R_m if the rest of the players ($-S_m$) maintains its strategy \bar{x}_{-S_m} of the \bar{x} .

If we consider $R_i = \{i\}$, for any $i \in I$. Then, it is obvious that the family $R = \{R_i\}_{i \in I}$ is a partition of set of players I , and let $S_i = -i$, for all $i \in I$. In this case the definition 2.2 reduces to the definition 2.1.

Let $M = I$, consider $R_i = \{-i\}$, for any $i \in I$. It is obvious that the family $R = \{R_i\}_{i \in I}$ is not a partition of the set of players I , and let $S_i = -i$, for all $i \in I$. In this case the definition 2.2 reduces to the following definition of strong Berge equilibrium.

Definition 2.3 ([15]) *A strategy profile $\bar{x} \in X$ is said to be strong Berge equilibrium (SBE) of game (2.1), if*

$$\forall i \in I, \forall j \in -i, f_j(\bar{x}_i, y_{-i}) \leq f_j(\bar{x}), \forall y_{-i} \in X_{-i}. \quad (2.3)$$

If a player i chooses its strategy \bar{x}_i of a \bar{x} which is a SBE, then the remaining of the players ($-i$) cannot improve the earning of all his (her) players, *i.e.*, this equilibrium is stable against deviation of any player of $-i$.

Definition 2.4 *We say that a strategy profile $\bar{x} \in X$ is a Berge equilibrium of the game (2.1) if*

1. $\forall i \in I, \forall y_{-i} \in X_{-i}, f_i(\bar{x}_i, y_{-i}) \leq f_i(\bar{x})$
2. $\forall i \in I, \alpha_i = \sup_{x_i \in X_i} \inf_{y_{-i} \in X_{-i}} f_i(x_i, y_{-i}) \leq f_i(\bar{x})$.

We can see that, the first part of the definition means that when a player $i \in I$ plays his strategy \bar{x}_i from the Berge equilibrium \bar{x} , he cannot obtain a maximum payoff unless the remaining players $-i$ willingly (or obliged) play the strategy \bar{x}_{-i} from the Berge equilibrium \bar{x} . In other words, if at least one of the players of coalition $-i$ deviates from his equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to his payoff $f_i(\bar{x})$ in Berge equilibrium. The second part means that strategy profile \bar{x} is individually rational. In other words, for each player $i \in I$, Berge equilibrium \bar{x} yields a payoff that is greater or equal to his security level, denoted by α_i . We then say that Berge equilibrium is individually rational.

It is to be noted that, initially in the 1980s, Zhukovskii has introduced his definition of Berge equilibrium without the condition 2 (individual rationality) in definition 2.4. Later, he constructed examples of games that have Berge equilibria that are not rationally individual, *i.e.* do not verify the condition 2 [24]. It is also important to emphasize the fact that the condition 2 of individual rationality in definition 2.4 doesn't appear in the papers of Abalo and Kostreva [1-4], this is also a major difference between our work and their results.

The concept of Berge equilibrium is totally different from the Strong Berge Equilibrium. Indeed, the Berge Strong Equilibrium has been introduced in 1957 by Berge [6]. This equilibrium is a refinement¹ of the Nash equilibrium [16] (see [15]), but in general, Berge equilibrium in the sense of Zhukovskii is not a Nash equilibrium. If only one player i adopts his strategy in a Strong Berge Equilibrium, he obliges all the other players $-i$ to choose their strategy in this equilibrium: the adoption of other strategies by any players in the coalition $-i$, would provide each of them a payoff at most equal to that they get in this equilibrium. In other words, if any player selects his strategy in a Strong Berge Equilibrium, the other players have no other choice than to follow him by choosing their strategies from the same Strong Berge Equilibrium. By contrast, if a player chooses his strategy in a Berge equilibrium in the sense of Zhukovskii, he cannot oblige the other players to follow him; he gets a maximum payoff if the other players are willing or obliged by some circumstances to choose their strategies in the same Berge equilibrium.

The Berge equilibrium in the sense of Zhukovskii is rarely mentioned (not to say used) by game theorists. One of the most important reasons for this is that Zhukovskii published his results in Russian and within former USSR with local publishers only, so his results are not known world wide. The first

¹ For more details, see the book of Van Damme [22]

paper published outside former USSR is [20]. The first papers published on Berge equilibrium in well established international journals are [1-4]. As we mentioned above, there are two main reasons that motivated Zhukovskii to introduce the Berge equilibrium as an alternative solution to Nash equilibrium [24]. The first one is the absence of a concept of solution (in pure strategies) for games where there is no Nash equilibrium; the second one is the difficulty to choose a Nash equilibrium in games where there is more than one Nash equilibrium. The Berge equilibrium can be used to study numerous non-cooperative models, more particularly coalition games. Furthermore, on the contrary to the Nash equilibrium, this concept allows to reach cooperative issues. Indeed, with this equilibrium it is no necessary to introduce behavioral assumptions to obtain cooperative issues, consequently, it becomes possible to determine cooperation in a non-cooperative framework. This property is very important for games like prisoner's dilemma.

Properties 2.1

- (1) *By definition Berge equilibrium is individually rational.*
- (2) *In general Berge equilibrium is not Pareto optimal.*

Definition 2.5 *A Berge equilibrium which is also Nash equilibrium is called Berge-Nash equilibrium.*

It would be interesting to find sufficient conditions for the existence of Berge-Nash equilibrium for such equilibrium enjoys the properties of both concepts of solution at the same time.

Let us give an example of a conflict situation where Berge equilibrium is the solution to which players will converge.

Exemple 2.1 *Consider the game illustrated by the following table.*

	<i>A</i>	<i>B</i>
<i>A</i>	<i>(-1.40, 0.94)</i>	<i>(-0.99, 0.93)</i>
<i>B</i>	<i>(-1.01, 0.98)</i>	<i>(-1, 1)</i>

there are two players I and II, and each has available two strategies. We list I's strategies as rows in the table, and II's strategies as columns. This game has no pure-strategy Nash equilibrium. On the other hand, the strategy profile (B, B) is a Berge equilibrium in the sense of Zhukovskii. The strategy A is attractive for player I because he may get his best payoff in the game, i.e -0.99, but in the case where player II chooses the strategy A, he gets his worst payoff in the game, i.e -1.40. In addition, strategy B is his maxmin strategy. Indeed, the minimum he gets by choosing A is -1.40, and by choosing B he gets

-1.01. Thus, player I will tend to choose the strategy B. He can reach the Berge equilibrium (B,B) in announcing that he has chosen the strategy B. Indeed, in this case player II will automatically choose the strategy B for which he will get his best payoff in the game, i.e 1.

3 Existence Results

In this section we establish the existence of Berge equilibrium (Definition 2.4) and Berge-Nash equilibrium (Definition 2.5). From these results we derive procedures for the computation of Berge equilibria.

3.1 Berge Equilibrium

In order to establish the existence of Berge equilibrium for the game (2.1), we will use the following generalization of the Ky Fan inequality [14] established by Nessah et al. ([17], [18] and [19]) called the g -Maximum Equality Theorem. Let us recall this theorem.

Theorem 3.1 (*g -Maximum Equality Theorem ([17], [18], [19])*) *Let X be a nonempty subset of a metric space E , Y be a nonempty, compact and convex subset of a locally convex Hausdorff space F . Let Ω be a real valued function defined on $X \times Y$. Let X_0 be a nonempty compact subset of X and g be a continuous function defined from X_0 into Y such that:*

- (1) $g(X_0)$ is a convex subset of Y ,
- (2) the function $(x, y) \mapsto \Omega(x, y)$ is continuous on $X_0 \times Y$,
- (3) for all $x \in X_0$, function $y \mapsto \Omega(x, y)$ is quasi-concave on Y ,
- (4) for all $g(x) \in \partial g(X_0)$ and for all $y \in Y$, there exists $z \in Z_{g(X_0)}(g(x))$ such that $\Omega(x, y) \leq \Omega(x, z)$.

Then, there exists $\bar{x} \in X_0$ such that

$$\sup_{y \in Y} \Omega(\bar{x}, y) = \Omega(\bar{x}, g(\bar{x})). \quad (3.1)$$

We have the following Lemmas which we will need thereafter.

Lemma 3.1 [8] *A product of convex sets is a convex set.*

Lemma 3.2 [21] *A finite or countable product of metric spaces is a metric space.*

Lemma 3.3 [21] *A product of locally convex spaces is a locally convex space.*

Lemma 3.4 [13] *A Hausdorff topological vector space, locally convex and locally bounded is a normable space.*

Let us consider the following set

$$A = \{\bar{x} \in X \text{ such that } \alpha_i = \max_{x_i \in X_i} \min_{y_{-i} \in X_{-i}} f_i(x_i, y_{-i}) \leq f_i(\bar{x}), \forall i \in I\}. \quad (3.2)$$

The set A represents the set of individually rational strategy profiles of the game (2.1). We have the following Lemma.

Lemma 3.5 *Suppose that the following conditions are verified:*

1. *for all $i \in I$, the set X_i is non empty, convex and compact in the Hausdorff locally convex space E_i ,*
2. *for all $i \in I$, the function f_i is continuous and quasi-concave on X .*

Then the set A defined in (3.2) is nonempty, convex and compact.

Proof.

- (1) A is nonempty set.

The conditions 1) and 2) of Lemma 2.5 imply that $\forall i \in I, \alpha_i = \sup_{x_i \in X_i} \inf_{y_{-i} \in X_{-i}} f_i(x_i, y_{-i})$ exists. Since the functions $f_i, i \in I$ are continuous over the compact X , then $\forall i \in I, \exists \tilde{x}_i \in X_i$ such that $\alpha_i = \sup_{x_i \in X_i} \inf_{y_{-i} \in X_{-i}} f_i(x_i, y_{-i}) = \inf_{y_{-i} \in X_{-i}} f_i(\tilde{x}_i, y_{-i})$.

Let be $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in X$, we have then:

$$\forall i \in I, f_i(\tilde{x}) = f_i(\tilde{x}_i, \tilde{x}_{-i}) \geq \inf_{y_{-i} \in X_{-i}} f_i(\tilde{x}_i, y_{-i}) = \alpha_i.$$

Thus $A \neq \emptyset$.

- (2) A is convex in X .

Let be \bar{x} and $\bar{\bar{x}}$ two elements in A and let be $\lambda \in [0, 1]$.

Let us show that $\lambda\bar{x} + (1 - \lambda)\bar{\bar{x}} \in A$.

We have $\bar{x}, \bar{\bar{x}}$ two elements in A , then $\alpha_i \leq f_i(\bar{x})$ and $\alpha_i \leq f_i(\bar{\bar{x}}), \forall i \in I$ thus

$$\alpha_i \leq \min\{f_i(\bar{x}), f_i(\bar{\bar{x}})\}, \forall i \in I.$$

Since the functions $f_i, i \in I$ are quasi-concave over X , then

$$\alpha_i \leq f_i(\lambda\bar{x} + (1 - \lambda)\bar{\bar{x}}), \forall i \in I, \forall \lambda \in [0, 1].$$

Therefore $\lambda\bar{x} + (1 - \lambda)\bar{\bar{x}} \in A$.

- (3) A is compact in X .

Since X is compact, then it sufficient to prove that A is closed.

Let $\{x^p\}_{p \geq 1}$ a sequence of elements in A converging to \bar{x} .

Let us show that $\bar{x} \in A$.

We have $\forall p \geq 1, x^p \in A$, then

$$\forall p \geq 1, \forall i \in I, \alpha_i \leq f_i(x^p).$$

Taking into account the condition 1) and the continuity of f_i of Lemma 3.5 when $p \rightarrow \infty$, we obtain: $\forall i \in I, \alpha_i \leq f_i(\bar{x})$, i.e. $\bar{x} \in A$.

■

Let us introduce the following functions

$$g : A \rightarrow \widehat{X}$$

defined by $x \mapsto g(x) = (x_{-1}, \dots, x_{-n}, \dots)$.

$$\Gamma : A \times \widehat{X} \rightarrow \mathbb{R}$$

defined by $(x, \hat{y}) \mapsto \Gamma(x, \hat{y}) = \sum_{i \in I} \{f_i(x_i, y_{-i}) - f_i(x)\}$ where $\hat{y} = (y_{-1}, \dots, y_{-n}, \dots) \in \widehat{X} = \prod_{i \in I} X_{-i}$, where $X_{-i} = \prod_{j \in -i} X_j, \forall i \in I$.

Remark 3.1 For all $x \in A$, we have

$$\sup_{\hat{y} \in \widehat{X}} \Gamma(x, \hat{y}) \geq 0.$$

We have the following Lemma.

Lemma 3.6 If for all $i \in I$, the set X_i is nonempty, convex and compact in the Hausdorff locally convex space E_i , then the following propositions are true.

1. The function g is continuous on A .
2. If A is convex and compact, then $g(A)$ is also convex and compact.

Proof. The fact that the function g is continuous is a consequence of its definition and the construction of the set \widehat{X} . The compactness of the set $g(A)$ is a consequence of Weierstrass Theorem. The convexity of $g(A)$ is a consequence of the linearity of g , which can be easily verified. ■

The following Lemma gives the relation between Berge equilibria of the game (2.1) and the functions Γ and g .

Lemma 3.7 The following two propositions are equivalent.

1. \bar{x} is a Berge equilibrium of the game (2.1).

2. $\bar{x} \in A$ and $\sup_{\hat{y} \in \widehat{X}} \Gamma(\bar{x}, \hat{y}) = 0$.

Proof. Let $\bar{x} \in X$ be a Berge equilibrium of the game (2.1). The second condition of definition 2.4 implies that $\bar{x} \in A$. The first condition of definition 2.4 implies $f_i(\bar{x}_i, t_{-i}) \leq f_i(\bar{x}), \forall t_{-i} \in X_{-i}, \forall i \in I$, hence $\Gamma(\bar{x}, \hat{y}) = \sum_{i \in I} \{f_i(\bar{x}_i, \hat{y}_{-i}) - f_i(\bar{x})\} \leq 0, \forall \hat{y} \in \widehat{X}$, i.e. $\max_{\hat{y} \in \widehat{X}} \Gamma(\bar{x}, \hat{y}) \leq 0$.

Taking into account Remark 3.1, we obtain $\max_{\hat{y} \in \widehat{X}} \Gamma(\bar{x}, \hat{y}) = 0$.

Conversely, let $\bar{x} \in A$ such that $\max_{\hat{y} \in \widehat{X}} \Gamma(\bar{x}, \hat{y}) = 0$, this equality implies $\forall \hat{y} \in \widehat{X}$, $\Gamma(\bar{x}, \hat{y}) = \sum_{i \in I} \{f_i(\bar{x}_i, \hat{y}_{-i}) - f_i(\bar{x})\} \leq 0$. For an arbitrarily fixed $i \in I$, we have $\forall \hat{y} \in \widehat{X}$, $\Gamma(\bar{x}, \hat{y}) = \{f_i(\bar{x}_i, \hat{y}_{-i}) - f_i(\bar{x})\} + \sum_{\substack{j \neq i \\ j \in I}} \{f_j(\bar{x}_j, \hat{y}_{-j}) - f_j(\bar{x})\} \leq 0$.

For $\hat{y} \in \widehat{X}$ such that \hat{y}_{-i} is arbitrarily chosen in X_{-i} and $\hat{y}_{-j} = \bar{x}_{-j}, \forall j \neq i$, we have $\sum_{\substack{j \neq i \\ j \in I}} \{f_j(\bar{x}_j, \hat{y}_{-j}) - f_j(\bar{x})\} = 0$. Then from the last inequality we deduce that $\forall \hat{y}_{-i} \in X_{-i}, f_i(\bar{x}_i, \hat{y}_{-i}) \leq f_i(\bar{x})$. Since i is arbitrarily chosen in I , we have $\forall i \in I, \forall y_{-i} \in X_{-i}, f_i(\bar{x}_i, y_{-i}) \leq f_i(\bar{x})$, hence, taking into account the fact that $\bar{x} \in A$, we deduce that \bar{x} is a Berge equilibrium of the game (2.1). ■

Remark 3.2 Lemma 3.7 transforms the problem of finding Berge equilibria of the game (2.1) into a problem of finding strategy profiles $\bar{x} \in A$ verifying $\sup_{\hat{y} \in \widehat{X}} \Gamma(\bar{x}, \hat{y}) = 0$.

We will now establish the existence of Berge equilibria by g -Maximum Equality Theorem (Theorem 3.1).

Theorem 3.2 Assume that

- (1) the sets $X_i, i \in I$ are non empty compact and convex subsets of locally convex Hausdorff spaces,
- (2) $\forall i \in I$, the function f_i is continuous and quasi-concave on X . The function Γ is continuous on $A \times \widehat{X}$,
- (3) $\forall g(x) \in \partial g(A), \forall \hat{y} \in \widehat{X}, \exists \hat{z} \in Z_{g(A)}(g(x))$ such that $\Gamma(x, \hat{y}) \leq \Gamma(x, \hat{z})$.

Then the game (2.1) has at least one Berge (Definition 2.4).

Proof. Taking into account conditions (1) and (2) of Theorem 3.2 and Lemma 3.5, we deduce that the set A is nonempty, convex and compact. The condition (2) of Theorem 3.2 implies that the function $\hat{y} \mapsto \Gamma(x, \hat{y})$ is quasi-concave

on \widehat{X} . Taking into account the conditions (1) and (3) of Theorem 3.2, from Lemmas 3.1-3.4 and the non emptiness, convexity and compactness of A , we conclude that all the conditions of the g -maximum equality Theorem (Theorem 3.1) are satisfied. Consequently,

$$\exists \bar{x} \in A \text{ such that } \sup_{\widehat{y} \in \widehat{X}} \Gamma(\bar{x}, \widehat{y}) = 0. \quad (3.3)$$

Then according to Lemma 3.7, \bar{x} is a Berge equilibrium of the game (2.1). ■

Taking into account Remark 3.1 and Lemma 3.7, we deduce the following proposition for games with a finite number of players.

Proposition 3.1 *Suppose that, in the game (2.1), the set of players is finite, i.e. $I = \{1, 2, \dots, n\}$, the function Γ is continuous on $A \times \widehat{X}$ and the sets X_j , $j = \overline{1, n}$ are compact.*

Let

$$\mu = \min_{x \in A} \left[\max_{\widehat{y} \in \widehat{X}} \Gamma(x, \widehat{y}) \right]. \quad (3.4)$$

Then the following propositions are equivalent:

- (1) the game (2.1) has at least one Berge equilibrium.
- (2) $\mu = 0$.

Remark 3.3 *The Proposition 3.1 remains true if the set of player is infinite countable.*

Remark 3.4 *If all conditions of Theorem 3.2 are verified, then the condition $\mu = 0$ is satisfied.*

Suppose that all conditions of the Proposition 3.1 are verified.

Step 1 Determine the security levels $\alpha_i, \forall i \in I$.

Step 2 Calculate the value μ in (3.4).

Step 3

- If $\mu > 0$, then game (2.1) has no Berge equilibrium.
- If $\mu = 0$, then the strategy profile $\bar{x} \in A$ verifying $\max_{\widehat{y} \in \widehat{X}} \Gamma(\bar{x}, \widehat{y}) = 0$ are Berge equilibria of the game (2.1).

Fig. 1. Procedure for the Computation of a Berge Equilibrium.

From this Proposition we deduce the method presented in Figure 1 for the computation of Berge equilibria of the game (2.1). Note that the step 1 of this method can be difficult: the calculation of some α_i values for $i \in I$ may be difficult to achieve, depending on the form of the payoff functions.

Let us now illustrate this procedure by examples.

Exemple 3.1 Let us consider the following game: $I = \{1, 2, 3\}$, $X_1 = [0, +1]$, $X_2 = [1, 2]$, $X_3 = [-1, +1]$, $x = (x_1, x_2, x_3)$.

$$f_1(x) = -x_2^2 - x_3^2,$$

$$f_2(x) = -x_3^2 + x_2,$$

$$f_3(x) = -x_3^3 x_1 - 3x_1^2 - x_3^2 x_2^2.$$

The conditions (1)-(2) of Theorem 3.2 are verified. Let us verify the condition (3).

- a) $\forall x \in X$, with $x_1 \in X_1$, $x_2 \in X_2$ and $-1 \leq x_3 < 0$, $\exists \bar{y} = (\frac{-x_3^3}{6}, 1, 0) \in X$, such that $f_i(x_i, t_{-i}) \leq f_i(x_i, \bar{y}_{-i})$, $\forall t_{-i} \in X_{-i}$, $\forall i \in I$.
- b) $\forall x \in X$, with $x_1 \in X_1$, $x_2 \in X_2$ and $0 \leq x_3 \leq 1$, $\exists \bar{y} \in X$, $\bar{y} = (0, 1, 0)$ such that $f_i(x_i, t_{-i}) \leq f_i(x_i, \bar{y}_{-i})$, $\forall t_{-i} \in X_{-i}$, $\forall i \in I$.

Hence, a) and b) imply

$$\forall x \in X, \exists y \in X, f_i(x_i, t_{-i}) \leq f_i(x_i, y_{-i}), \forall t_{-i} \in X_{-i}, \forall i \in I. \quad (3.5)$$

Now let us prove that both $\bar{y} = (0, 1, 0)$ and $\bar{y} = (\frac{-x_3^3}{6}, 1, 0)$ with $-1 \leq x_3 < 0$, are in the set A .

Indeed, we have $f_1(0, 1, 0) = -1$, $f_1(\frac{-x_3^3}{6}, 1, 0) = -1$ and $\alpha_1 = -5$,

$f_2(0, 1, 0) = 1$, $f_2(\frac{-x_3^3}{6}, 1, 0) = 1$ and $\alpha_2 = 1$,

$f_3(0, 1, 0) = -3$, $f_3(\frac{-x_3^3}{6}, 1, 0) = -3\frac{x_3^6}{36}$ and $\alpha_3 = -3$.

Hence both $\bar{y} = (0, 1, 0)$ and $\bar{y} = (\frac{-x_3^3}{6}, 1, 0)$ are in the set A . Taking into account (3.7) we deduce that the condition (3) of Theorem 3.2 is verified. Thus, according to Theorem 3.2, this game has at least one Berge equilibrium. From the preceding result we have

$$\max_{y_{-1}} f_1(0, y_{-1}) = f_1(0, 1, 0),$$

$$\max_{y_{-2}} f_2(1, y_{-2}) = f_2(0, 1, 0),$$

$$\max_{y_{-3}} f_3(0, y_{-3}) = f_3(0, 1, 0).$$

Hence, $\sum_{i=1}^3 \max_{y_{-i}} f_i(\bar{x}_i, y_{-i}) = \sum_{i=1}^3 f_i(\bar{x})$ with $\bar{x} = (0, 1, 0)$ which is equivalent to

$$\max_{\hat{y} \in \hat{X}} \sum_{i=1}^3 f_i(\bar{x}_i, \hat{y}_{-i}) = \sum_{i=1}^3 f_i(\bar{x}), \text{ i.e. } \max_{\hat{y} \in \hat{X}} \Gamma(\bar{x}, \hat{y}) = 0.$$

Since we have proved above that $\bar{x} = (0, 1, 0) \in A$, then according to step 3, $\bar{x} = (0, 1, 0)$ is a Berge equilibrium of this game.

Exemple 3.2 Suppose that $X_i = [0, 1]$, $i = \overline{1, 3}$.

$$f_1(x) = x_2^2 + x_1 + x_3,$$

$$\begin{aligned} f_2(x) &= x_1 - x_3^3, \\ f_3(x) &= x_1 - 3x_2^2. \end{aligned}$$

We have, $\max_{y_{-1}} f_1(x_1, y_{-1}) = x_1 + 2, \forall x_1 \in [0, 1]$.

$$\max_{y_{-2}} f_2(x_2, y_{-2}) = 1, \forall x_2 \in [0, 1].$$

$$\max_{y_{-3}} f_3(x_3, y_{-3}) = 1, \forall x_3 \in [0, 1].$$

$$\begin{aligned} \mu &= \min_{x \in X} \max_{\hat{y} \in \hat{X}} \hat{\phi}(x, \hat{y}) = \min_{x \in X} \left[\sum_{i=1}^3 \left(\max_{y_{-i}} f_i(x_i, \hat{y}_{-i}) - f_i(x) \right) \right] \\ &= \min_{x \in X} (4 + 2x_2^2 - x_3 - 2x_1 + x_3^3) = 2 - \frac{2}{3\sqrt{3}} \simeq 1,615 > 0. \end{aligned}$$

Since $\mu > 0$ then this game has no Berge equilibrium.

3.2 Berge-Nash Equilibrium

In this section, we establish the existence of Berge equilibrium that is also Nash equilibrium of the game (2.1), we will use the Theorem 3.1. From this approach, we deduce a procedure for the computation of Berge-Nash equilibria.

Let us consider the following functions:

$$\tilde{g} : X \rightarrow \hat{X} \times X$$

defined by $x \mapsto \tilde{g}(x) = ((x_{-1}, \dots, x_{-n}, \dots), x)$ and

$$\tilde{\Gamma} : X \times (\hat{X} \times X) \rightarrow \mathbb{R}$$

defined by $(x, (\hat{y}, z)) \mapsto \tilde{\Gamma}(x, (\hat{y}, z)) = \sum_{i \in I} [f_i(x_i, y_{-i}) + f_i(x_{-i}, z_i)]$.

Remark 3.5 By definition, for all $x \in X$, we have

$$\sup_{(\hat{y}, z) \in \hat{X} \times X} \tilde{\Gamma}(x, (\hat{y}, z)) \geq \tilde{\Gamma}(x, \tilde{g}(x)).$$

Lemma 3.8 The function \tilde{g} is continuous on X . If $\forall i \in I, X_i$ is convex and compact, then $\tilde{g}(X)$ is convex and compact.

Proof. The continuity of \tilde{g} is obvious from its definition and the construction of the set \hat{X} . The compactness of the set $\tilde{g}(X)$ is a consequence of the compactness of the set \hat{X} (Tychonoff's Theorem). To establish the convexity of $\tilde{g}(X)$, it suffices to verify that the function \tilde{g} is linear. ■

The following Lemma gives the relation between Berge-Nash equilibria of the game (2.1) and the functions $\tilde{\Gamma}$ and \tilde{g} .

Lemma 3.9 *The following two propositions are equivalent:*

1. $\sup_{(\hat{y}, z) \in \hat{X} \times X} \tilde{\Gamma}(\bar{x}, (\hat{y}, z)) = \tilde{\Gamma}(\bar{x}, \tilde{g}(\bar{x}))$.
2. \bar{x} is a Berge-Nash equilibrium of the game (2.1).

Proof.

1. Suppose that $\sup_{(\hat{y}, z) \in \hat{X} \times X} \tilde{\Gamma}(\bar{x}, (\hat{y}, z)) = \tilde{\Gamma}(\bar{x}, \tilde{g}(\bar{x}))$, *i.e.*

$$\sum_{i \in I} [f_i(\bar{x}_i, y_{-i}) + f_i(\bar{x}_{-i}, z_i)] \leq \sum_{i \in I} [f_i(\bar{x}) + f_i(\bar{x})], \quad \forall (\hat{y}, z) \in \hat{X} \times X \quad (3.6)$$

If we take $y_{-i} = \bar{x}_{-i}$, $\forall i \in I$ in (3.6), we conclude that $\sum_{i \in I} f_i(\bar{x}_{-i}, z_i) \leq \sum_{i \in I} f_i(\bar{x})$, $\forall z \in X$, which implies that \bar{x} is Nash equilibrium of the game (2.1).

If we take $z = \bar{x}$ in (3.6), we conclude that \bar{x} verifies the property 1) of definition 2.1 and since \bar{x} is a Nash equilibrium, it is also individually rational. We conclude that \bar{x} is a Berge equilibrium of the game (2.1).

2. Suppose that \bar{x} is a Berge-Nash equilibrium of the game (2.1).
The fact that \bar{x} is a Nash equilibrium of the game (2.1) implies

$$\max_{z \in X} \sum_{i \in I} f_i(\bar{x}_{-i}, z_i) = \sum_{i \in I} f_i(\bar{x}). \quad (3.7)$$

The fact that \bar{x} is a Berge equilibrium of the game (2.1) implies

$$\max_{\hat{y} \in \hat{X}} \sum_{i \in I} f_i(\bar{x}_i, y_{-i}) = \sum_{i \in I} f_i(\bar{x}). \quad (3.8)$$

The two equalities (3.7) and (3.8) imply

$$\max_{(\hat{y}, z) \in \hat{X} \times X} \tilde{\Gamma}(\bar{x}, (\hat{y}, z)) = \tilde{\Gamma}(\bar{x}, \tilde{g}(\bar{x}))$$

■

It is to be noted that in lemma 3.9 we have deliberately omitted the condition $\bar{x} \in A$ of individual rationality for it is well known that a Nash equilibrium is always individually rational. We have the following Theorem.

Theorem 3.3 *Suppose that*

- (1) *the sets X_i , $i \in I$ are nonempty, compact and convex subsets of Hausdorff locally convex vector spaces,*

- (2) the function $\tilde{\Gamma}$ is continuous on $X \times (\widehat{X} \times X)$ and the functions $y_{-i} \mapsto f_i(x_i, y_{-i})$ and $z_i \mapsto f_i(z_i, x_{-i})$ are quasi-concave on X_{-i} and on X_i , respectively, $\forall x \in X$ and $\forall i \in I$,
- (3) $\forall \tilde{g}(x) \in \partial \tilde{g}(X)$, $\forall (\hat{y}, z) \in \widehat{X} \times X$, $\exists (\hat{p}, q) \in Z_{\tilde{g}(x)}(\tilde{g}(x)) = [T_{\tilde{g}(x)}(\tilde{g}(x)) + \tilde{g}(x)] \cap (\widehat{X} \times X)$ such that $\tilde{\Gamma}(x, (\hat{y}, z)) \leq \tilde{\Gamma}(x, (\hat{p}, q))$.

Then the game (2.1) has at least one Berge-Nash equilibrium.

Proof. The conditions of Theorem 3.3 imply that the function $\tilde{\Gamma}$ verifies all conditions of Theorem 3.1 (g-maximum equality Theorem), consequently, $\exists \bar{x} \in X$ such that $\sup_{(\hat{y}, z) \in \widehat{X} \times X} \tilde{\Gamma}(\bar{x}, (\hat{y}, z)) = \tilde{\Gamma}(\bar{x}, \tilde{g}(\bar{x}))$.

according to Lemma 3.9, the strategy profile \bar{x} is a Berge-Nash equilibrium of the game (2.1). ■

From Remark 3.5 and Lemma 3.9, we deduce the following proposition.

Proposition 3.2 *Suppose that the function $\tilde{\Gamma}$ is continuous on $X \times (\widehat{X} \times X)$ and the sets X_j are compact. Let*

$$\beta = \min_{x \in X} \left[\max_{(\hat{y}, z) \in \widehat{X} \times X} \tilde{\Gamma}(x, (\hat{y}, z)) - \tilde{\Gamma}(x, \tilde{g}(x)) \right]. \quad (3.9)$$

Then the following two assertions are equivalent:

1. $\beta = 0$.
2. The game (2.1) has at least one Berge-Nash equilibrium.

Since the function $\tilde{\Gamma}$ is a series of functions, the calculation of the value β is generally difficult, but in the case where the set of players is finite, this Proposition can be used to verify if a Berge-Nash equilibrium exists or not. From this Proposition we deduce the method presented in Figure 2 for the computation of a Berge-Nash equilibrium of the game (2.1).

Suppose that the conditions of Proposition 3.2 are verified and the set of players is finite.

Step 1 Calculate the value β in (3.9).

Step 2

- If $\beta > 0$, then the game (2.1) has no Berge-Nash equilibrium.
- If $\beta = 0$, then the strategy profiles $\bar{x} \in X$ verifying $\sup_{(\hat{y}, z) \in \widehat{X} \times X} \tilde{\Gamma}(\bar{x}, (\hat{y}, z)) = \tilde{\Gamma}(\bar{x}, \tilde{g}(\bar{x}))$ are Berge-Nash equilibria of the game (2.1).

Fig. 2. Procedure for the determination of a Berge-Nash equilibrium

Remark 3.6 *If all conditions of Theorem 3.3 are verified then the condition $\beta = 0$ is satisfied. It is interesting to notice that this Method doesn't necessitate the calculation of the values $\alpha_i, \forall i \in I$ for the determination of Berge equilibria of the game (2.1), in addition to this, the Berge equilibria found are also Nash equilibria of this game.*

3.3 Simple Berge Equilibrium Point

In this section, we establish the existence of a simple Berge equilibrium point of the game (2.1), we will use the Theorem 3.1.

Let us consider the following functions:

$$h : X \rightarrow \widetilde{X}$$

defined by $x \mapsto h(x) = \overbrace{((x_{S_m}, \dots, x_{S_m}))}^{r_m\text{-times}}, m \in M$ and

$$F : X \times \widetilde{X} \rightarrow \mathbb{R}$$

defined by $(x, \widetilde{y}) \mapsto F(x, \widetilde{y}) = \sum_{m \in M} \sum_{j \in R_m} \{f_j(x_{-S_m}, y_{S_m}) - f_j(x)\}$, where $\widetilde{X} = \prod_{m \in M} \prod_{j \in R_m} X_{S_m}^j$ and $X_{S_m}^j = X_{S_m}, \forall j \in R_m$.

Lemma 3.10 *The function h is continuous on X . If $\forall i \in I, X_i$ is convex and compact, then $h(X)$ is convex and compact.*

The following Lemma gives the relation between simple Berge equilibria point of the game (2.1) and the functions F and h .

Lemma 3.11 *The following two propositions are equivalent:*

1. $\sup_{\substack{\bar{x} \\ \widetilde{y} \in \widetilde{X}}} F(\bar{x}, \widetilde{y}) = 0$.
2. \bar{x} is a simple Berge equilibrium point of the game (2.1).

Proof. The proof of this lemma is similar to that of Lemma 3.7. ■

We have the following Theorem.

Theorem 3.4 *Suppose that*

- (1) *the sets $X_i, i \in I$ are nonempty, compact and convex subsets of Hausdorff locally convex vector spaces,*
- (2) *the function F is continuous on $X \times \widetilde{X}$ and the functions $y_{S_m} \mapsto \sum_{i \in R_m} f_i(x_{-S_m}, y_{S_m})$ are quasi-concave on $X_{S_m}, \forall x_{-S_m} \in X_{-S_m}$ and $\forall m \in M,$*

(3) $\forall h(x) \in \partial h(X), \forall \tilde{y} \in \tilde{X}, \exists \tilde{p} \in Z_{h(X)}(h(x)) = [T_{h(X)}(h(x)) + h(x)] \cap \tilde{X}$
such that $F(x, \tilde{y}) \leq F(x, \tilde{p})$.

Then the game (2.1) has at least one simple Berge equilibrium point.

Proof. The proof of this lemma is similar to that of Theorem 3.2. ■

4 Conclusion

In this paper we dealt with the problem of existence and computation of Berge equilibrium. For the general case with an infinite countable number of players we have used the g -Maximum Equality Theorem to derive general sufficient conditions for its existence (Theorem 3.2 and Theorem 3.3). From this result we deduced a method for the determination of Berge equilibria. Using a special function we provided sufficient conditions for the existence of Berge equilibrium, then we derived an effective procedure for its computation. In Theorem 3.3, we established the existence of Berge equilibria which are also a Nash equilibria. Many things remain to do in this field of research, among them, we can mention the study of Berge equilibrium in differential games, namely, it would be interesting to derive similar results to those of Theorems 3.2,3.3,3.4 and Lemmas 3.7,3.9 for such games.

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