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Universités d'Aix-Marseille II et III

Document de Travail
n°2008-18

OPTIMAL GROWTH AND COMPETITIVE EQUILIBRIUM BUSINESS CYCLES UNDER DECREASING RETURNS IN TWO-COUNTRY MODELS

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May 2008

Optimal growth and competitive equilibrium business cycles under decreasing returns in two-country models*

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Abstract: *This paper investigates the interlinkage in the business cycles of large-country economies in a free-trade equilibrium. We consider a two-country, two-good, two-factor general equilibrium model with Cobb-Douglas technologies and linear preferences. We also assume decreasing returns in both sectors. We first identify the determinants of each country's accumulation pattern in autarky equilibrium, and second we show how a country's business cycle may spread throughout the world once trade opens. We prove indeed that under free-trade, globalization and market integration may generate a contagion of the capital exporting country's business cycles and thus have destabilizing effects on the capital importing country.*

Keywords: *Two-country general equilibrium model, busines cycles, capital intensities, decreasing returns*

Journal of Economic Literature Classification Numbers: C62, E32, F11, F43, O41.

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*This paper is dedicated to the memory of Koji Shimomura. We would like to thank an anonymous referee for useful comments.

†This paper has been completed while Alain Venditti was visiting the Institute of Economic Research of Kyoto University. He thanks Professor Kazuo Nishimura and all the staff of the Institute for their kind invitation.

1 Introduction

Over the last two decades, the increasing development of international globalization and market integration have raised the question of the contagion of macroeconomic instability across countries. Indeed, international trade interlinks the business cycles of trading countries, as it relates economic activities of agents in one country to those in another. As a result, a country's business cycle may be spread throughout the world.

It is well-known since Benhabib and Nishimura [2, 3] that in multisector optimal growth models,[1] endogenous fluctuations (periodic cycles) easily arise.[2] However, these results are provided within a closed-economy framework with a single representative agent. Building on the papers of Becker [1], Bewley [4], Yano [12, 13] and Epstein [6], that have demonstrated that in a perfect foresight model with many consumers a competitive equilibrium path behaves like an optimal growth path, Nishimura and Yano [10] extend the analysis of Benhabib and Nishimura [3] to a two-country, two-sector, two-factor trade model in which consumption and capital goods are freely mobile between countries once trade opens whereas labor is internationally immobile. They analyze the interlinkage in the business cycles of large country economies in a free-trade equilibrium: starting from the identification of the determinants of each country's global accumulation pattern in an autarky equilibrium, they characterize fluctuant and monotone free-trade equilibrium paths. Their analysis being quite general, it remains however difficult to interpret in terms of the fundamentals their main conditions.

Considering a specialization of the formulation of Nishimura and Yano [10] based on Cobb-Douglas technologies and linear preferences in both countries, Nishimura, Venditti and Yano [9] give conditions on the capital intensity differences across sectors in each country to obtain a contagion of business cycles throughout the world. In order to avoid the specialization of countries, they assume decreasing returns in the consumption good sectors. Their main result consists in showing that if in each country endogenous fluctuations arise under autarky, then business cycles also arise once

[1] Going back to at least the contribution of Oniki and Uzawa [11], dynamic analysis of international trade is made within two-sector optimal growth models.

[2] See also Boldrin and Montrucchio [5] for the consideration of more complex (chaotic) behavior of optimal paths.

international trade opens. However, they consider a degenerate autarky stationary distribution in which each country exactly produces in the long run the amount of capital necessary to produce both goods. While they suggest that a non-degenerate free-trade distribution with international trade of both goods may exist and may be associated with a contagion of one country's business cycles to the world level even though the other country is characterized by stability under autarky, they do not provide clear conditions for such a result to hold.

The objective of the current paper is precisely to tackle this problem. We extend the formulation of Nishimura, Venditti and Yano [9] by considering decreasing returns in both sectors of both countries. We first show that for a stationary capital stock at the world level, two types of stationary distributions across countries may occur: an autarky distribution in which each country exactly produces in the long run the amount of capital necessary to produce both goods but trades with the other country along the transition path, and a free-trade distribution in which one country is characterized by net exports in capital and net imports in consumption while the other is characterized by net imports in capital and net exports in consumption.

Second we analyze the local stability properties of each type of stationary distribution. We start by providing factor intensities conditions for the existence of period-two cycles in a closed-economy under decreasing returns. Once international trade opens, focussing first on the autarky distribution, we prove that if both countries have optimal period-two cycles under autarky, then endogenous fluctuations also occur under free-trade. In this case, the existence of international business cycles is derived from the existence of business cycles in each country. This result generalizes the main conclusion of Nishimura, Venditti and Yano [9] to economies with decreasing returns in both sectors. Then, building on the same type of arguments, we give factor intensities conditions for the existence of period-two cycles along the free-trade distribution. However, we prove that business cycles may occur at the world level once trade opens even though the capital importing country is characterized by a saddle-point stable steady state under autarky. In this case, opening to international trade generates a contagion of the capital exporting country's business cycles and thus has a destabilizing effect on the capital importing country.

This paper is organized as follows: The next section sets up the basic model. In Section 3 we study the stability properties of the competitive equilibrium path in closed economies under decreasing returns. Section 4 provides the main results on the existence of endogenous business cycles within open economies under free-trade. Section 5 contains concluding comments. All the proofs are gathered in a final Appendix.

2 The model

We consider a simple perfect foresight trade model with two countries and two goods. Each country $i = A, B$ is characterized by an infinitely-lived representative agent with single period linear utility function given by

$$u(c^i) = c^i$$

with c^i the consumption level. We assume that the labor supply is inelastic. There are two goods: the pure consumption good, x^i, and the pure capital good, k^i. Each good is produced with a Cobb-Douglas technology. We denote by x^i and y^i the output of sectors c^i and k^i:

$$x^i = \mathcal{E}_c^i (K_c^i)^{\alpha_1^i} (L_c^i)^{\alpha_2^i}, \ y^i = \mathcal{E}_y^i (K_y^i)^{\beta_1^i} (L_y^i)^{\beta_2^i}$$

with $\mathcal{E}_c^i, \mathcal{E}_y^i > 0$ some normalization constants. We assume decreasing returns to scale in both sectors, i.e. $\beta_1^i + \beta_2^i < 1$ and $\alpha_1^i + \alpha_2^i < 1$.[3] Labor is normalized to one, $L_c^i + L_y^i = 1$, and the total stock of capital in country i is given by $K_c^i + K_y^i = k^i$. Moreover capital fully depreciates at each period. Goods x^i and k^i are assumed to be freely mobile between countries once trade opens, whereas labor is internationally immobile both before and after the opening of trade. It follows that along a free-trade equilibrium, the market clearing conditions for goods x^i and k^i are as:

$$c_t^A + c_t^B = x_t^A + x_t^B, \quad k_{t+1}^A + k_{t+1}^B = y_t^A + y_t^B \tag{1}$$

On the contrary along an autarky equilibrium, the market clearing conditions become

[3] A possible interpretation of decreasing returns would be to assume the existence of a factor in fixed supply such as land in the technology, namely

$$x^i = \mathcal{E}_c^i (K_c^i)^{\alpha_1^i} (L_c^i)^{\alpha_2^i} (\mathcal{L}_c^i)^{1-\alpha_1^i-\alpha_2^i}, \ y^i = \mathcal{E}_y^i (K_y^i)^{\beta_1^i} (L_y^i)^{\beta_2^i} (\mathcal{L}_y^i)^{1-\beta_1^i-\beta_2^i}$$

Returns to scale are therefore constant when considering this factor but decreasing with respect to capital and labor. In such a case, the income of the representative consumer is increased by the rental rate of land. Our formulation implicitly assumes a normalization $\mathcal{L}_c^i = \mathcal{L}_y^i = 1$.

$$c_t^i = x_t^i, \quad k_{t+1}^i = y_t^i \tag{2}$$

In each country, the optimal allocation of factors across sectors is obtained by solving the following program:

$$\begin{array}{rl} \max\limits_{K_c^i, L_c^i, K_y^i, L_y^i} & \mathcal{E}_c^i (K_c^i)^{\alpha_1^i} (L_c^i)^{\alpha_2^i} \\ s.t. & y^i = \mathcal{E}_y^i (K_y^i)^{\beta_1^i} (L_y^i)^{\beta_2^i} \\ & 1 = L_c^i + L_y^i \\ & k^i = K_c^i + K_y^i \end{array} \tag{3}$$

Denote by q_t^i, p_t^i, ω_t^i and r_t^i respectively the prices of the consumption good and the capital good, the wage rate of labor and the rental rate of the capital good at time t. In free-trade equilibrium, $q_t^A = q_t^B$, $p_t^A = p_t^B$ and $r_t^A = r_t^B$ must hold. On the contrary, because labor is immobile across countries, ω_t^i may differ between countries even in the free-trade case. In the following we will choose the consumption good as numeraire and thus adopt the normalization $q_t^A = q_t^B = 1$. The Lagrangian corresponding to program (3) is:

$$\begin{aligned} \mathcal{L}_t \;=\;& \mathcal{E}_c^i (K_{ct}^i)^{\alpha_1^i} (L_{ct}^i)^{\alpha_2^i} + p_t^i \Big(\mathcal{E}_y^i (K_{yt}^i)^{\beta_1^i} (L_{yt}^i)^{\beta_2^i} - y_t^i \Big) \\ +\;& \omega_t^i (1 - L_{ct}^i - L_{yt}^i) + r_t^i (k_t^i - K_{ct}^i - K_{yt}^i) \end{aligned} \tag{4}$$

For any (k_t^i, y_t^i), solving the first order conditions with respect to $(K_{ct}^i, L_{ct}^i, K_{yt}^i, L_{yt}^i)$ gives inputs K_c^i, L_c^i, K_y^i and L_y^i as C^2 functions of (k_t^i, y_t^i), i.e. $\hat{K}_c^i(k_t^i, y_t^i)$, $\hat{L}_c^i(k_t^i, y_t^i)$, $\hat{K}_y^i(k_t^i, y_t^i)$ and $\hat{L}_y^i(k_t^i, y_t^i)$. We may thus define the social production function of country i as:

$$T^i(k_t^i, y_t^i) = \mathcal{E}_c^i \hat{K}_c^i(k_t^i, y_t^i)^{\alpha_1^i} \hat{L}_c^i(k_t^i, y_t^i)^{\alpha_2^i} \tag{5}$$

Using the envelope theorem we derive the equilibrium prices:

$$r_t^i = T_1^i(k_t^i, y_t^i), \quad p_t^i = -T_2^i(k_t^i, y_t^i) \tag{6}$$

where $T_1^i = \partial T^i / \partial k^i$ and $T_2^i = \partial T^i / \partial y^i$.[4]

[4] Since the technologies exhibit decreasing returns to scale, the competitive firms earn positive profits that have to be distributed back to the households who own physical capital. It can be shown as in Mino [7] that solving a planning problem in which the planner maximizes the discounted sum of utilities, under free-trade or autarky, subject to the social production function (5) and the market clearing conditions (1) or (2), is equivalent to solving a decentralized problem in which the households maximize a discounted sum of utilitites subject to some budget constraint based on given sequences of prices and the distributed profits.

3 Closed economy under decreasing returns

In a closed economy the equilibrium is derived from the following optimization program:

$$\begin{array}{rl} \max\limits_{y_t^i} & \sum\limits_{t=0}^{+\infty} \rho^t T^i(k_t^i, y_t^i) \\ s.t. & k_{t+1}^i = y_t^i \\ & k_0^i \ given \end{array}$$

with $\rho \in (0,1]$ the discount factor. The corresponding Euler equation is thus

$$T_2^i(k_t^i, k_{t+1}^i) + \rho T_1^i(k_{t+1}^i, k_{t+2}^i) = 0 \tag{7}$$

A closed-economy steady state is defined by $k_t^i = k_{t+1}^i = y_t^i = \bar{k}^i$ and is obtained by solving $T_2^i(k^i, k^i) + \rho T_1^i(k^i, k^i) = 0$.

Proposition 1. *There exists a unique closed-economy steady state $\bar{k}^i$ for country i such that:*

$$\bar{k}^i = \left(\frac{\alpha_1^i \beta_2^i}{\alpha_2^i \beta_1^i \mathcal{E}_y^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i)\rho \beta_1^i}\right)^{\frac{\beta_2^i}{1-\beta_1^i}} \left[\mathcal{E}_y^i (\rho \beta_1^i)^{\beta_1^i + \beta_2^i}\right]^{\frac{1}{1-\beta_1^i}}$$

Moreover, the stationary optimal demand for capital in the investment good sector is given by $K_y^{i*} \equiv g^i = \rho \beta_1^i \bar{y}^i = \rho \beta_1^i \bar{k}^i$

The linearization of the Euler equation around $\bar{k}^i$ gives the following characteristic polynomial:

$$\mathcal{P}_a^i(\lambda) = \rho T_{12}^i(\bar{k}^i, \bar{k}^i)\lambda^2 + \lambda \Big[T_{22}^i(\bar{k}^i, \bar{k}^i) + \rho T_{11}^i(\bar{k}^i, \bar{k}^i)\Big] + T_{12}^i(\bar{k}^i, \bar{k}^i) = 0 \tag{8}$$

As usual with Cobb-Douglas technologies, factor intensities may be determined by the exponents of the functions.

Lemma 1. *The investment (consumption) good sector of country i is capital intensive if and only if $\beta_1^i/\beta_2^i > (<) \alpha_1^i/\alpha_2^i$.*

Building on the contribution of Benhabib and Nishimura [3], we can get global monotone convergence of the optimal path when the investment good is capital intensive:

Proposition 2. *If the investment good is capital intensive, the optimal path of country i, $\{k_t^i\}_{t\geq 0}$, monotonically converges to the closed-economy steady state $\bar{k}^i$.*

In the converse capital intensity configuration, the optimal path is no longer monotone and global results are not easily derived from simple conditions on factor shares in each sectors. However we can get local results from a direct inspection of the characteristic polynomial and obtain conditions for the existence of endogenous fluctuations.

Proposition 3. *In country i, let the consumption good be capital intensive with*

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \frac{\alpha_1^i\beta_2^i(1-\alpha_1^i-\alpha_2^i)}{(1-\alpha_1^i)(1-\beta_1^i)}$$

and $\alpha_1^i > (1+\beta_1^i)/2(\beta_1^i+\beta_2^i)$. Then there exists $\bar{\rho} \in (0,1)$ such that the closed-economy steady state $\bar{k}^i$ is saddle-point stable for any $\rho \in (\bar{\rho}, 1]$. Moreover, when ρ crosses $\bar{\rho}$ from above, $\bar{k}^i$ becomes locally unstable and there exist saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\rho}$.

Notice that when compared with the results of Benhabib and Nishimura [3] derived under constant returns in both sectors, the assumption of a capital intensive consumption good is not sufficient to get optimal oscillations. The capital intensity difference needs to be strong enough to compensate the degree of decreasing returns.

4 Open economy under free trade

Under the assumption of linear utility functions, as shown in Nishimura and Yano [10], a free trade equilibrium path may be interpreted as an optimal path with respect to a linear world welfare function of the following form.

$$\begin{aligned} V(k_t, y_t) = \max_{k^A, k^B, y^A, y^B} \quad & T^A(k_t^A, y_t^A) + T^B(k_t^B, y_t^B) \\ s.t. \quad & k_t^A + k_t^B \leq k_t \\ & y_t^A + y_t^B \leq y_t \end{aligned}$$

The first order conditions give:

$$\begin{array}{rcl} T_1^A(k_t^A, y_t^A) - T_1^B(k_t^B, y_t^B) & = & 0 \\ T_2^A(k_t^A, y_t^A) - T_2^B(k_t^B, y_t^B) & = & 0 \end{array} \qquad (9)$$

The intertemporal free-trade equilibrium is finally derived from the following optimization:

$$\begin{aligned} \max_{y_t} \quad & \sum_{t=0}^{+\infty} \rho^t V(k_t, y_t) \\ s.t. \quad & k_{t+1} = y_t \\ & k_0 = k_0^A + k_0^B \ given \end{aligned}$$

The corresponding Euler equation is thus

$$V_2(k_t, k_{t+1}) + \rho V_1(k_{t+1}, k_{t+2}) = 0$$

From the first order conditions (9), the envelope theorem gives

$$\begin{aligned} V_1(k_t, k_{t+1}) &= T_1^B(k^B(k_t, k_{t+1}), y^B(k_t, k_{t+1})) = T_1^A(k^A(k_t, k_{t+1}), y^A(k_t, k_{t+1})) \\ V_2(k_t, k_{t+1}) &= T_2^B(k^B(k_t, k_{t+1}), y^B(k_t, k_{t+1})) = T_2^A(k^A(k_t, k_{t+1}), y^A(k_t, k_{t+1})) \end{aligned}$$

and the Euler equation becomes

$$\begin{aligned} & T_2^B(k^B(k_t, k_{t+1}), y^B(k_t, k_{t+1})) + \rho T_1^B(k^B(k_{t+1}, k_{t+2}), y^B(k_{t+1}, k_{t+2})) \\ = \ & T_2^A(k^A(k_t, k_{t+1}), y^A(k_t, k_{t+1})) + \rho T_1^A(k^A(k_{t+1}, k_{t+2}), y^A(k_{t+1}, k_{t+2})) \qquad (10) \\ = \ & 0 \end{aligned}$$

Let us denote k^* the steady state solution of

$$V_2(k^*, k^*) + \rho V_1(k^*, k^*) = 0 \qquad (11)$$

The steady state k^* gives a total stationary amount of capital at the world level. Contrary to the closed-economy case, an explicit computation of k^* cannot be derived from (11). Moreover, the distribution across the two countries remains to be determined.

4.1 Stationary distributions

The Euler equation along a free-trade equilibrium (10), when compared with the Euler equation along a closed-economy equilibrium (7), clearly shows that different types of distributions are compatible with a total stationary stock of capital at the world level k^*. In particular, an autarky distribution in which each country exactly produces in the long run the amount of capital necessary to produce the consumption and investment goods may occur, i.e. $k^i = y^i$. Consider indeed the closed-economy steady state given in Lemma 1 for each country, i.e. $\bar{k}^A$ and $\bar{k}^B$. Using the normalization constants $\mathcal{E}_c^i$ and $\mathcal{E}_y^i$, we can show that $\bar{k} = \bar{k}^A + \bar{k}^B$ is also a steady state of the open economy under free-trade:

Proposition 4. *Let* $\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$ *and consider* $\bar{k}^i$, $i = A, B$, *as given in Proposition 1. Then there exists* $\bar{\mathcal{E}}_c^A > 0$ *such that the autarky distribution* $\bar{k} = \bar{k}^A + \bar{k}^B$ *is a solution of equation (11), i.e.* $\bar{k} = k^*$, *if and only if* $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$.

The corresponding amount of stationary consumptions under this capital distribution can be immediately derived from (5), as

$$\bar{c}^i = T^i(\bar{k}^i, \bar{k}^i) \equiv \bar{T}^i \tag{12}$$

Notice that considering the autarky distribution does not imply that countries do not trade. They may actually trade during the transition dynamics while the long run equilibrium is characterized by autarky.

We also have to consider the existence of a free-trade distribution such that $k^* = k^{A*} + k^{B*}$ in which one country, say A, is characterized by net imports of capital, i.e. $k^{A*} > y^{A*}$, while country B is characterized by net exports of capital, i.e. $k^{B*} < y^{B*}$. Proceeding as in Proposition 4, we can use the normalization constants $\mathcal{E}_c^i$ and $\mathcal{E}_y^i$ to prove that such a free-trade distribution exists. We actually focus on a particular solution such that $k^{A*} = \theta y^{A*} > y^{A*}$ and $k^{B*} = y^{B*}/\theta < y^{B*}$ with $\theta > 1$ a given constant.

Proposition 5. *Let* $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ *and consider a constant* $\theta \in (1, 1/\rho\beta_1^B)$. *Then there exist* $\mathcal{E}_c^{A*} > 0$ *and* $\mathcal{E}_y^{A*} > 0$ *such that the free-trade distribution* $k^* = k^{A*} + k^{B*} = \theta y^{A*} + y^{B*}/\theta$ *with*

$$\begin{aligned} k^{A*} &= \theta \left(\frac{\alpha_1^A \beta_2^A}{\alpha_2^A \beta_1^A \theta + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A)\rho\beta_1^A} \right)^{\frac{\beta_2^A}{1-\beta_1^A}} \left[\mathcal{E}_y^A (\rho\beta_1^A)^{\beta_1^A + \beta_2^A} \right]^{\frac{1}{1-\beta_1^A}} \\ k^{B*} &= \left(\frac{\alpha_1^B \beta_2^B}{\alpha_2^B \beta_1^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^A)\rho\beta_1^B \theta} \right)^{\frac{\beta_2^B}{1-\beta_1^B}} \left[\frac{\mathcal{E}_y^B (\rho\beta_1^B \theta)^{\beta_1^B + \beta_2^B}}{\theta} \right]^{\frac{1}{1-\beta_1^B}} \end{aligned} \tag{13}$$

is a solution of equation (11) if and only if $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ *and* $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$.

We now have to compute the stationary consumption levels associated with this distribution of capital. At the free-trade steady state with $\theta \in (1, 1/\rho\beta_1^B)$, the country i's production of the consumption good is derived from (5) as:

$$T^{i*} = T^i(k^{i*}, k^{i*})$$

We know that country A imports capital goods while country B exports capital goods, namely

$$\mathcal{M}_y^A = (\theta - 1)y^{A*}, \quad \mathcal{X}_y^B = \left(\frac{\theta-1}{\theta}\right) y^{B*}$$

In order to have a balance of trade in equilibrium, we derive from this that country A has to export consumption goods while country B has to import consumption goods. Let $\eta > 1$ and consider the following distribution of consumption across the two countries

$$c^{A*} = \frac{T^{A*}}{\eta} < T^{A*}, \quad c^{B*} = \eta T^{B*} > T^{B*}$$

It follows that

$$\mathcal{X}_c^A = \left(\frac{\eta-1}{\eta}\right) T^{A*}, \quad \mathcal{M}_c^B = (\eta - 1)T^{B*}$$

Therefore, the balance of trade is in equilibrium in each country if

$$\mathcal{N}\mathcal{X}^A = \mathcal{X}_c^A - p\mathcal{M}_y^A = 0, \quad \mathcal{N}\mathcal{X}^B = p\mathcal{X}_y^B - \mathcal{M}_c^B = 0$$

or equivalently

$$(\theta - 1)py^{A*} = \left(\frac{\eta-1}{\eta}\right) T^{A*}, \quad \left(\frac{\theta-1}{\theta}\right) py^{B*} = (\eta - 1)T^{B*}$$

with p the relative price of the investment good. Taking the ratio of these expressions yields the following corollary:

Corollary 1. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$, $\theta \in (1, 1/\rho\beta_1^B)$ and consider the free-trade distribution of capital as given by (13). Assume also that $\alpha_1^B(\theta - \rho\beta_1^A)/\alpha_1^A(1 - \rho\beta_1^B\theta) > 1$. Then the associated free-trade distribution of consumption is $c^* = c^{A*} + c^{B*} = T^{A*}/\eta + \eta T^{B*}$ with $\eta = T^{A*}/T^{B*} = \alpha_1^B(\theta - \rho\beta_1^A)/\alpha_1^A(1 - \rho\beta_1^B\theta)$ and*

$$c^{A*} = T^{B*}, \quad c^{B*} = T^{A*} \tag{14}$$

It is worth noticing that the autarky and free-trade distributions cannot co-exist since they are respectively associated with different values for the normalization constants $\mathcal{E}_c^i, \mathcal{E}_y^i$.[5]

We may now provide a detailed stability analysis of the two possible distributions of the stationary capital stock k^* across countries.

4.2 Monotone convergence

At a steady state under free-trade we have $y = k$ and the characteristic polynomial may be written as follows:

$$\mathcal{P}_f(\lambda) = \rho V_{12}(k^*, k^*)\lambda^2 + \lambda\Big[V_{22}(k^*, k^*) + \rho V_{11}(k^*, k^*)\Big] + V_{12}(k^*, k^*) = 0 \tag{15}$$

[5]See Appendix 6.4 and 6.5 for detailed expressions.

As in the closed economy case, we can get global monotone convergence of the optimal path when the investment good is capital intensive in each country:

Proposition 6. *If the investment good is capital intensive in each country, the optimal path at the world level, $\{k_t\}_{t\geq 0}$, monotonically converges to the free-trade steady state k^*.*

This result applies both to the autarky distribution $k^* = \bar{k} = \bar{k}^A + \bar{k}^B$ and to the free-trade distribution $k^* = k^{A*} + k^{B*}$.

4.3 Endogenous fluctuations

We now focus on local stability results when the consumption good is capital intensive. As in the closed economy case, such a capital intensity configuration may be associated with endogenous fluctuations. In a first step we study the properties of the optimal path at the world level around the autarky distribution.

Proposition 7. *Let $\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$, $\theta \in (1, 1/\rho\beta_1^B)$, and consider the autarky distribution $\bar{k} = \bar{k}^A + \bar{k}^B$ as defined in Proposition 4. Assume also that in each country $i = A, B$, the consumption good is capital intensive with*

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \tfrac{\alpha_1^i\beta_2^i(1-\alpha_1^i-\alpha_2^i)}{(1-\alpha_1^i)(1-\beta_1^i)} \tag{16}$$

and $\alpha_1^i > (1+\beta_1^i)/2(\beta_1^i+\beta_2^i)$. Then there exists $\hat{\rho} \in (0,1)$ such that the autarky steady state $k^* = \bar{k}$ is saddle-point stable for any $\rho \in (\hat{\rho}, 1]$. Moreover, when ρ crosses $\hat{\rho}$ from above, $\bar{k}$ becomes locally unstable and the optimal path at the world level is characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\hat{\rho}$.

Considering Proposition 3, Proposition 7 implies that if both countries are characterized by endogenous fluctuations under autarky, then the equilibrium under free-trade is also characterized by endogenous fluctuations. Put differently, a market integration, in which international trade concerns consumption and investment goods, does not rule out periodic cycles that may exist under autarky. This result generalizes the main conclusion of Nishimura, Venditti and Yano [9] to economies with decreasing returns in both sectors.

In a last step we study the properties of the optimal path around the free-trade distribution as defined by Proposition 5

Proposition 8. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$, $\theta \in (1, 1/\rho\beta_1^B)$, and consider the free-trade distribution $k^* = k^{A*} + k^{B*}$ as defined by Proposition 5. Assume also that in each country $i = A, B$, the consumption good is capital intensive with*

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \frac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A}\frac{\alpha_1^A\beta_2^A\theta}{\theta-\beta_1^A} \tag{17}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \frac{1-\alpha_1^B-\alpha_2^B}{1-\alpha_1^B}\frac{\alpha_1^B\beta_2^B}{1-\theta\beta_1^B} \tag{18}$$

If the following conditions also hold

$$\alpha_1^A > \frac{\theta(1-\beta_1^A-\beta_2^A)+2\beta_1^A+\beta_2^A}{2(\beta_1^A+\beta_2^A)} \tag{19}$$

and

$$\alpha_1^B > \frac{1-\beta_1^B-\beta_2^B+\theta(2\beta_1^B+\beta_2^B)}{2\theta(\beta_1^B+\beta_2^B)} \tag{20}$$

there exists $\hat{\rho} \in (0, 1)$ such that the free-trade steady state k^* is saddle-point stable for any $\rho \in (\hat{\rho}, 1]$. Moreover, when ρ crosses $\hat{\rho}$ from above, k^* becomes locally unstable and the optimal path at the world level is characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\hat{\rho}$

Proposition 8 provides conditions on the technologies of both countries for the existence of endogenous fluctuations at the free-trade steady state which are similar to those given in Proposition 7. However, notice that for country A condition (17) in Proposition 8 may hold while condition (16) in Proposition 7 does not, whereas for country B condition (18) implies condition (16). As a result, we derive the following Corollary.

Corollary 2. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$, $\theta \in (1, 1/\rho\beta_1^B)$, and consider the free-trade distribution $k^* = k^{A*} + k^{B*}$ as defined by Proposition 5. Assume also that in each country $i = A, B$, the consumption good is capital intensive with*

$$\frac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A}\frac{\alpha_1^A\beta_2^A}{1-\rho\beta_1^A} > \alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \frac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A}\frac{\alpha_1^A\beta_2^A\theta}{\theta-\rho\beta_1^A}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \frac{1-\alpha_1^B-\alpha_2^B}{1-\alpha_1^B}\frac{\alpha_1^B\beta_2^B}{1-\theta\beta_1^B}$$

for any given $\rho \in (0,1]$. If the following conditions also hold

$$\alpha_1^A > \frac{\theta(1-\beta_1^A-\beta_2^A)+2\beta_1^A+\beta_2^A}{2(\beta_1^A+\beta_2^A)} \tag{21}$$

and

$$\alpha_1^B > \frac{1-\beta_1^B-\beta_2^B+\theta(2\beta_1^B+\beta_2^B)}{2\theta(\beta_1^B+\beta_2^B)} \tag{22}$$

there exists $\hat{\rho} \in (0,1)$ such that the free-trade steady state k^* is saddle-point stable for any $\rho \in (\hat{\rho},1]$. Moreover, when ρ crosses $\hat{\rho}$ from above, k^* becomes locally unstable and the optimal path at the world level is characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\hat{\rho}$, while the steady state under autarky in economy A is saddle-point stable with monotone convergence.

We have thus proved that business cycles may occur at the world level once trade opens even though the capital importing country is characterized by a saddle-point stable steady state under autarky. In this case, international globalization and market integration generate a contagion of the capital exporting country's business cycles and thus has a destabilizing effect on the capital importing country.

5 Concluding comments

In a perfect foresight model with two countries characterized by Cobb-Douglas technologies and decreasing returns, we have investigated the way endogenous business cycles of countries may spread all over the world through international trade.

We have first identified the determinants of each country's global accumulation pattern in the closed economy configuration. As in the case with constant returns studied by Benhabib and Nishimura [3], endogenous fluctuations require a capital intensive consumption good. However, the capital intensity difference needs to be strong enough to compensate the degree of decreasing returns in the consumption good sector.

Secondly, we have shown how real business cycles may occur at the world level once trade opens. We have proved that two types of stationary distributions across countries compatible with a global stationary capital stock may occur: an autarky distribution which is associated with countries that do not trade in the long run but trade along the transition path, and a free-trade distribution in which one country is characterized by net capital imports while the other is characterized by net capital exports. Dealing in a first step with the autarky distribution, we have shown that if the consumption good in each country is sufficiently capital intensive as in the closed economy configuration, then endogenous business cycles occur at the world level. In this case, the existence of endogenous fluctuations in both countries under autarky implies the existence of endogenous fluctuations at the world level.

Dealing finally with the free-trade distribution, we have proved that endogenous fluctuations may occur at the world level once trade opens even though one country is characterized by monotone convergence of the optimal path under autarky. In this case, globalization and market integration generate a contagion of macroeconomic instability across countries as the business cycles of one country spreads throughout the world.

6 Appendix

6.1 Proof of Proposition 1

We start by characterizing the first partial derivatives of the social production function.

Lemma 6.1. *The first partial derivatives of $T^i(k^i, y^i)$ are given by:*

$$\begin{aligned} T_1^i(k^i, y^i) &= \mathcal{E}_c^i \alpha_1^i (\alpha_2^i \beta_1^i)^{\alpha_2^i} (k^i - g^i)^{\alpha_1^i + \alpha_2^i - 1} (\Delta^i)^{-\alpha_2^i} \\ T_2^i(k^i, y^i) &= -\tfrac{T_1^i(k^i, y^i)}{\mathcal{E}_y^i \beta_1^i} (\alpha_1^i \beta_2^i / \Delta^i)^{-\beta_2^i} (g^i)^{1-\beta_1^i - \beta_2^i} \end{aligned}$$

where

$$\begin{aligned} \Delta^i &= \alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i \\ g^i &= g^i(k^i, y^i) = \left\{ K_y^i \in [0, k^i] \ / \ y^i = \tfrac{\mathcal{E}_y^i (\alpha_1^i \beta_2^i)^{\beta_2^i} (K_y^i)^{\beta_1^i + \beta_2^i}}{[\alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) K_y^i]^{\beta_2^i}} \right\} \end{aligned}$$

Proof: From the Lagrangian (4) we derive the first order conditions:

$$\mathcal{E}_c^i \alpha_1^i (K_c^i)^{\alpha_1^i - 1} (L_c^i)^{\alpha_2^i} - r^i = 0 \tag{23}$$

$$\mathcal{E}_c^i \alpha_2^i (K_c^i)^{\alpha_1^i} (L_c^i)^{\alpha_2^i - 1} - \omega^i = 0 \tag{24}$$

$$\mathcal{E}_y^i p^i \beta_1^i (K_y^i)^{\beta_1^i - 1} (L_y^i)^{\beta_2^i} - r^i = 0 \tag{25}$$

$$\mathcal{E}_y^i p^i \beta_2^i (K_y^i)^{\beta_1^i} (L_y^i)^{\beta_2^i - 1} - \omega^i = 0 \tag{26}$$

Using $K_c^i = k^i - K_y^i$, $L_y^i = 1 - L_c^i$, and merging (23)-(26) we obtain:

$$\begin{aligned} L_c^i &= \frac{\alpha_2^i \beta_1^i (k^i - K_y^i)}{(\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) K_y^i + \alpha_2^i \beta_1^i k^i} &(27)\\ L_y^i &= \frac{\alpha_1^i \beta_2^i K_y^i}{(\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) K_y^i + \alpha_2^i \beta_1^i k^i} &(28)\\ K_c^i &= k^i - K_y^i &(29)\\ K_y^i &= g^i(k^i, y^i) \equiv g^i &(30) \end{aligned}$$

where

$$g^i(k^i, y^i) = \left\{ K_y^i \in [0, (k^i)^{\beta_1^i}] \ / \ y^i = \frac{\mathcal{E}_y^i (\alpha_1^i \beta_2^i)^{\beta_2^i} (K_y^i)^{\beta_1^i + \beta_2^i}}{[\alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) K_y^i]^{\beta_2^i}} \right\} \tag{31}$$

To simplify notation let:

$$\Delta^i = \alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i \tag{32}$$

From (23), (27) and (29) we obtain:

$$T_1^i(k^i, y^i) = r^i = \mathcal{E}_c^i \alpha_1^i (k^i - g^i)^{\alpha_1^i + \alpha_2^i - 1} (\alpha_2^i \beta_1^i / \Delta^i)^{\alpha_2^i} \tag{33}$$

and from (25), (28), (30) and (33):

$$-T_2^i(k^i, y^i) = p^i = \frac{\mathcal{E}_c^i \alpha_1^i}{\mathcal{E}_y^i \beta_1^i} \frac{(\alpha_2^i \beta_1^i / \Delta^i)^{\alpha_2^i}}{(\alpha_1^i \beta_2^i / \Delta^i)^{\beta_2^i}} (k^i - g^i)^{\alpha_1^i + \alpha_2^i - 1} (g^i)^{1 - \beta_1^i - \beta_2^i} \tag{34}$$

From (33) and (34) we finally derive

$$T_2^i(k^i, y^i) = -\frac{T_1^i(k^i, y^i)}{\mathcal{E}_y^i \beta_1^i} (\alpha_1^i \beta_2^i / \Delta^i)^{-\beta_2^i} (g^i)^{1 - \beta_1^i - \beta_2^i}$$

□

We may now prove Proposition 1. Using (31) we derive

$$T_2^i(k^i, y^i) = -T_1^i(k^i, y^i) \frac{g^i}{y^i \beta_1^i} \tag{35}$$

It follows that at the closed-economy steady state $g^i = \rho \beta_1^i y^i = \rho \beta_1^i \bar{k}^i$. The expression of the steady state is finally obtained by solving $T_2^i(k^i, k^i) + \rho T_1^i(k^i, k^i) = 0$. □

6.2 Proof of Proposition 2

We start by characterizing the second partial derivatives of $T^i(k^i, y^i)$:

Lemma 6.2. *The second partial derivatives of $T^i(k^i, y^i)$ are given by:*

$$T^i_{11}(k^i,y^i) \;=\; \tfrac{T^i_1(k^i,y^i)}{k^i-g^i}\left\{\tfrac{[(1-\alpha^i_1)(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)-\alpha^i_2\beta^i_2](k^i-g^i)-(1-\alpha^i_1-\alpha^i_2)\alpha^i_1\beta^i_2k^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}\right\}$$

$$T^i_{12}(k^i,y^i) \;=\; \tfrac{T^i_1(k^i,y^i)}{k^i-g^i}\tfrac{g^i}{y^i\beta^i_1}\left\{\tfrac{-(1-\alpha^i_1)(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)(k^i-g^i)+(1-\alpha^i_1-\alpha^i_2)\alpha^i_1\beta^i_2k^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}\right\}$$

$$\begin{aligned}T^i_{22}(k^i,y^i) \;=\; & -\tfrac{T^i_1(k^i,y^i)}{k^i-g^i}\left(\tfrac{g^i}{y^i\beta^i_1}\right)^2 \\ & \times\left\{\tfrac{(1-\beta^i_1-\beta^i_2)(k^i/g^i)\alpha^i_2\beta^i_1(k^i-g^i)-(\beta^i_1-\alpha^i_1)(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)(k^i-g^i)+(1-\alpha^i_1-\alpha^i_2)\alpha^i_1\beta^i_2k^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}\right\}\end{aligned}$$

with $|H^i(k^i,y^i)| \equiv T^i_{11}(k^i,y^i)T^i_{22}(k^i,y^i)-T^i_{12}(k^i,y^i)^2>0.$

Proof: Recall that by definition of g^i we have the identity:

$$y^i[\alpha^i_2\beta^i_1k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i]^{\beta^i_2}=\mathcal{E}^i_y(\alpha^i_1\beta^i_2)^{\beta^i_2}(g^i)^{\beta^i_1+\beta^i_2} \tag{36}$$

Total differentiation gives after simplifications:

$$g^i\Big\{dy^i\Delta^i+\beta^i_2y^i[\alpha^i_2\beta^i_1dk^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)dg^i]\Big\}=(\beta^i_1+\beta^i_2)y^i\Delta^idg^i$$

We then get

$$\begin{aligned}g^i_1 &= \tfrac{dg^i}{dk^i}=\tfrac{\alpha^i_2\beta^i_2g^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}\\ g^i_2 &= \tfrac{dg^i}{dy^i}=\tfrac{\Delta^ig^i}{y^i\beta^i_1\left[\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i\right]}\end{aligned}$$

The second partial derivatives of $T^i(k^i,y^i)$ are obtained by differentiating (33) and (34):

$$\begin{aligned}T^i_{11}(k^i,y^i) &= -\tfrac{(1-\alpha^i_1-\alpha^i_2)(1-g^i_1)T^i_1(k^i,y^i)}{k^i-g^i}-\tfrac{\alpha^i_2\left[\alpha^i_2\beta^i_1+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i_1\right]T^i_1(k^i,y^i)}{\Delta^i}\\ T^i_{12}(k^i,y^i) &= \tfrac{(1-\alpha^i_1-\alpha^i_2)g^i_2T^i_1(k^i,y^i)}{k^i-g^i}-\tfrac{\alpha^i_2(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i_2T^i_1(k^i,y^i)}{\Delta^i}\\ T^i_{22}(k^i,y^i) &= \tfrac{(1-\alpha^i_1-\alpha^i_2)g^i_2T^i_2(k^i,y^i)}{k^i-g^i}+\tfrac{(1-\beta^i_1-\beta^i_2)g^i_2T^i_2(k^i,y^i)}{g^i}\\ &+\tfrac{(\beta^i_2-\alpha^i_2)(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i_2T^i_2(k^i,y^i)}{\Delta^i}\end{aligned}$$

The final expressions of these derivatives are obtained after simplifications built on (35) and the fact that

$$\begin{aligned}g^i-k^ig^i_1 &= \tfrac{\Delta^ig^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}=y^i\beta^i_1g^i_2\\ 1-g^i_1 &= \tfrac{\alpha^i_2\beta^i_2(k^i-g^i)+\Delta^i}{\alpha^i_2(\beta^i_1+\beta^i_2)k^i+(\alpha^i_1\beta^i_2-\alpha^i_2\beta^i_1)g^i}\end{aligned}$$

Strict concavity of the production functions implies that the determinant of the Hessian matrix of $T^i(k^i,y^i)$ satisfies $|H^i(k^i,y^i)|\equiv T^i_{11}(k^i,y^i)T^i_{22}(k^i,y^i)-$

$T^i_{12}(k^i, y^i)^2 > 0$. $\square$

We may now prove Proposition 2. If the investment good is capital intensive we derive from Lemma 6.2 that $T^i_{12}(k^i, y^i) > 0$ for any (k^i, y^i). The result then follows from Theorem 3 (p. 296) in Benhabib and Nishimura [3]. $\square$

6.3 Proof of Proposition 3

Consider the second partial derivatives of $T^i(k^i, y^i)$ evaluated at the autarky steady state. Straightforward computations give after simplifications:

$$\begin{aligned}
\frac{\mathcal{P}^i_a(0)}{\Phi^i} &= -\left[(1-\alpha^i_1)(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)(1-\rho\beta^i_1) - (1-\alpha^i_1-\alpha^i_2)\alpha^i_1\beta^i_2\right] \\
&\equiv -\tilde{\mathcal{P}}^i_a(0) \\
\frac{\mathcal{P}^i_a(1)}{\Phi^i} &= -(1-\beta^i_1)(1-\rho\beta^i_1)\left[\alpha^i_2 + \rho(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)\right] < 0 \\
\frac{\mathcal{P}^i_a(-1)}{\Phi^i} &= -\Big\{2(1+\rho)\Big[(1-\alpha^i_1)(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)(1-\rho\beta^i_1) \\
&- (1-\alpha^i_1-\alpha^i_2)\alpha^i_1\beta^i_2\Big] - (1-\rho\beta^i_1)(1-\beta^i_1)\left[\alpha^i_2 + \rho(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)\right]\Big\} \\
&\equiv -\tilde{\mathcal{P}}^i_a(-1)
\end{aligned} \tag{37}$$

with

$$\Phi^i = \frac{T^i_1(\bar{k}^i, \bar{k}^i)\rho}{\bar{k}^i(1-\rho\beta^i_1)[\alpha^i_2(\beta^i_1+\beta^i_2) + (\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)\rho\beta^i_1]} > 0 \tag{38}$$

We derive that $\tilde{\mathcal{P}}^i_a(0) > 0$ if and only if the consumption good is capital intensive with

$$\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1 > \frac{1-\alpha^i_1-\alpha^i_2}{1-\alpha^i_1}\frac{\alpha^i_1\beta^i_2}{1-\rho\beta^i_1} \equiv \mathcal{Z}^i_1 \tag{39}$$

Notice that the right-hand-side is an increasing function of ρ. Therefore, if

$$\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1 > \frac{\alpha^i_1\beta^i_2(1-\alpha^i_1-\alpha^i_2)}{(1-\alpha^i_1)(1-\beta^i_1)}$$

then $\mathcal{P}^i_a(0) < 0$ for any $\rho \in (0, 1]$. Consider now $\mathcal{P}^i_a(-1)$ when $\rho = 1$. By definition we have

$$\begin{aligned}
\mathcal{P}^i_a(-1) &= 2T^i_{12}(\bar{k}^i, \bar{k}^i) - T^i_{11}(\bar{k}^i, \bar{k}^i) - T^i_{22}(\bar{k}^i, \bar{k}^i) \\
&= -\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} T^i_{11}(\bar{k}^i, \bar{k}^i) & T^i_{12}(\bar{k}^i, \bar{k}^i) \\ T^i_{12}(\bar{k}^i, \bar{k}^i) & T^i_{22}(\bar{k}^i, \bar{k}^i) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} > 0
\end{aligned}$$

On the other side, when $\rho = 0$ we get

$$\tilde{\mathcal{P}}^i_a(-1) = \alpha^i_2\left[2\alpha^i_1(\beta^i_1+\beta^i_2) - 1 - \beta^i_1\right]$$

Therefore, if

$$\alpha^i_1 > \frac{1+\beta^i_1}{2(\beta^i_1+\beta^i_2)} \tag{40}$$

there exists $\bar{\rho} \in (0,1)$ such that $\mathcal{P}_a^i(-1) > 0$ for any $\rho \in (\bar{\rho}, 1]$ and $\mathcal{P}_a^i(-1) < 0$ in a left neighbourhood of $\bar{\rho}$. We then conclude that when $\rho \in (\bar{\rho}, 1]$ the steady state is saddle-point stable with two negative eigenvalues and when ρ crosses $\bar{\rho}$ from above one eigenvalue crosses -1. The result follows from the flip bifurcation theorem.

□

6.4 Proof of Proposition 4

A steady state is obtained as a solution (k^A, k^B, y^A, y^B, k) of the following system

$$
\begin{aligned}
T_2^A(k^A, y^A) + \rho T_1^A(k^A, y^A) &= 0 \quad (41)\\
T_2^B(k^B, y^B) + \rho T_1^B(k^B, y^B) &= 0 \quad (42)\\
T_1^A(k^A, y^A) - T_1^B(k^B, y^B) &= 0 \quad (43)\\
T_2^A(k^A, y^A) - T_2^B(k^B, y^B) &= 0 \quad (44)\\
k^A + k^B = y^A + y^B &= k \quad (45)
\end{aligned}
$$

with $c^A + c^B = V(k,k) = T^A(k^A, k^A) + T^B(k^B, k^B)$. We get from equations (41)-(42) the following property for a steady state under free-trade:

Lemma 6.3. *At a steady state under free-trade, the following relationship holds:*

$$\frac{g^A}{y^A \beta_1^A} = \frac{g^B}{y^B \beta_1^B} = \rho$$

Proof: The first order conditions (43)-(44) show that $T_j^A(k^A, y^A) = T_j^B(k^B, y^B)$, $j = 1, 2$. Since $T_2^i(k^i, y^i) = -T_1^i(k^i, y^i) g^i / y^i \beta_1^i$, we derive that $g^A / y^A \beta_1^A = g^B / y^B \beta_1^B$. Consider now the Euler equation (10) evaluated at a steady state under free-trade. We get $-T_2^i(k^i, y^i) = \rho T_1^i(k^i, y^i)$ and the result follows. □

We may now prove Proposition 4. Using Lemmas 6.1 and 6.3, equations (41) and (42) may be written as

$$
\begin{aligned}
\frac{1}{\beta_1^A} \left(\frac{\alpha_2^A \beta_1^A k^A + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A y^A}{\alpha_1^A \beta_2^A} \right)^{\beta_2^A} &= \rho \\
\frac{1}{\beta_1^B} \left(\frac{\alpha_2^B \beta_1^B k^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_B^i) \rho \beta_1^B y^B}{\alpha_1^B \beta_2^B} \right)^{\beta_2^B} &= \rho
\end{aligned}
$$

It follows that the autarky steady state, i.e. $k^A = y^A = \bar{k}^A$ and $k^B = y^B = \bar{k}^B$, with $\bar{k}^i$ given in Proposition 1, is a solution of the previous equations and satisfies equation (45). Considering $T_1^i(k^i, y^i)$ in Lemma 6.1 with $\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$, equation (43) with $k^i = y^i = \bar{k}^i$ is satisfied if and only if $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$ with

$$\bar{\mathcal{E}}_c^A = \frac{\left[\alpha_2^A\beta_1^A+(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\rho\beta_1^A\right]^{\alpha_2^A}}{\alpha_1^A(\alpha_2^A\beta_1^A)^{\alpha_2^A}\left[(\bar{k}^A)^{\alpha_1^A-1}(1-\rho\beta_1^A)^{\alpha_1^A+\alpha_2^A-1}\right]}\frac{\alpha_1^B(\alpha_2^B\beta_1^B)^{\alpha_2^B}\left[(\bar{k}^B)^{\alpha_1^B-1}(1-\rho\beta_1^B)^{\alpha_1^B+\alpha_2^B-1}\right]}{\left[\alpha_2^B\beta_1^B+(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\rho\beta_1^B\right]^{\alpha_2^B}}$$

Then, since from (41) and (42) we have $T_1^A(\bar{k}^A, \bar{k}^A) = T_2^A(\bar{k}^A, \bar{k}^A)/\rho$ and $T_1^B(\bar{k}^B, \bar{k}^B) = T_2^B(\bar{k}^B, \bar{k}^B)/\rho$, equation (44) also holds with $k^i = y^i = \bar{k}^i$. □

6.5 Proof of Proposition 5

Consider equations (41)-(45). We know from Lemma 6.3 that equations (41) and (42) imply $g^i = \rho\beta_1^i y^i$, $i = A, B$. Assume then that $k^A = \theta y^A$ and $k^B = y^B/\theta$ with $\theta > 1$ some constant. We will give conditions on the normalization constants $\mathcal{E}_c^i, \mathcal{E}_y^i$ to get these expressions as solutions of equations (41)-(45). Notice first from (45) that these restrictions imply $k^A = \theta k^B$ and $g^A = \rho\beta_1^A k^A/\theta$, $g^B = \rho\beta_1^B k^B\theta$. Substituting these expressions into (31) with $K_y^i = g^i$ and solving for k^i, $i = A, B$, gives

$$\begin{aligned} k^{A*} &= \theta\left(\frac{\alpha_1^A\beta_2^A}{\alpha_2^A\beta_1^A\theta+(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\rho\beta_1^A}\right)^{\frac{\beta_2^A}{1-\beta_1^A}}\left[\mathcal{E}_y^A(\rho\beta_1^A)^{\beta_1^A+\beta_2^A}\right]^{\frac{1}{1-\beta_1^A}} \\ k^{B*} &= \left(\frac{\alpha_1^B\beta_2^B}{\alpha_2^B\beta_1^B+(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^A)\rho\beta_1^B\theta}\right)^{\frac{\beta_2^B}{1-\beta_1^B}}\left[\frac{\mathcal{E}_y^B(\rho\beta_1^B\theta)^{\beta_1^B+\beta_2^B}}{\theta}\right]^{\frac{1}{1-\beta_1^B}} \end{aligned} \tag{46}$$

We may now use the normalization constants $\mathcal{E}_y^A$ and $\mathcal{E}_y^B$ to get $k^{A*} = \theta k^{B*}$. To simplify notation let

$$\Phi_\theta^A = \alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A, \quad \Phi_\theta^B = \alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^A)\rho\beta_1^B\theta$$

Assuming $\mathcal{E}_y^B = 1$, we derive from (46) that $k^{A*} = \theta k^{B*}$ if and only if $\theta \in (1, 1/\rho\beta_1^B)$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ with

$$\mathcal{E}_y^{A*} = \frac{\left(\alpha_1^B\beta_2^B/\Phi_\theta^B\right)^{\frac{\beta_2^B(1-\beta_1^A)}{1-\beta_1^B}}}{(\rho\beta_1^A)^{\beta_1^A+\beta_2^A}\left(\alpha_1^A\beta_2^A/\Phi_\theta^A\right)^{\beta_2^A}}\left[\frac{(\rho\beta_1^B\theta)^{\beta_1^B+\beta_2^B}}{\theta}\right]^{\frac{1-\beta_1^A}{1-\beta_1^B}} \tag{47}$$

Considering $T_1^i(k^i, y^i)$ in Lemma 6.1 with $\mathcal{E}_c^B = 1$, equation (43) with $k^A = \theta y^A$, $k^B = y^B/\theta$, and thus $k^A = \theta k^B$, is satisfied if and only if $\theta \in (1, 1/\rho\beta_1^B)$ and $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ with

$$\mathcal{E}_c^{A*} = \frac{(\Phi_\theta^A)^{\alpha_2^A}\alpha_1^B(\alpha_2^B\beta_1^B)^{\alpha_2^B}(1-\rho\beta_1^B\theta)^{\alpha_1^B+\alpha_2^B-1}(k^{B*})^{\alpha_1^B-\alpha_1^A}}{\alpha_1^A(\alpha_2^A\beta_1^A)^{\alpha_2^A}(\theta-\rho\beta_1^A)^{\alpha_1^A+\alpha_2^A-1}(\Phi_\theta^B)^{\alpha_2^B}} \tag{48}$$

Then, since from (41) and (42) we have $T_1^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i) = T_2^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i)/\rho$, $i = A, B$, equation (44) also holds with $k^A = \theta y^A$ and $k^B = y^B/\theta$. □

6.6 Proof of Corollary 1

Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ as given by (47) and (48), and consider $T^i(k^i, y^i)$ as defined by (5) with (27), (29), $k^A = \theta y^A$, $k^B = y^B/\theta$, and thus $k^A = \theta k^B$. We get

$$\begin{aligned} T^{A*} &= \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)}\left(\frac{\alpha_2^B\beta_1^B}{\Phi_\theta^B}\right)^{\alpha_2^B}(1-\rho\beta_1^B\theta)^{\alpha_1^B+\alpha_2^B}(k^{B*})^{\alpha_1^B} \\ &= \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)}T^{B*} \equiv \eta T^{B*} \end{aligned} \tag{49}$$

□

6.7 Proof of Proposition 6

The linearization of the Euler equation around k^* requires the computations of the second partial derivatives of $V(k, y)$. We know from Nishimura and Yano [10] that they are obtained as follows:

Lemma 6.4. *Along a free-trade equilibrium, the second partial derivatives of $V(k, y)$ satisfy the following:*

$$\begin{aligned} V_{11}(k,y) &= \frac{1}{\Theta}\left[T_{11}^A(k^A,y^A)|H^B(k^B,y^B)| + T_{11}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \\ V_{12}(k,y) &= \frac{1}{\Theta}\left[T_{12}^A(k^A,y^A)|H^B(k^B,y^B)| + T_{12}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \\ V_{22}(k,y) &= \frac{1}{\Theta}\left[T_{22}^A(k^A,y^A)|H^B(k^B,y^B)| + T_{22}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \end{aligned}$$

where

$$\begin{aligned} |H^i(k^i,y^i)| &\equiv T_{11}^i(k^i,y^i)T_{22}^i(k^i,y^i) - T_{12}^i(k^i,y^i)^2 > 0 \\ \Theta &\equiv T_{11}^A(k^A,y^A)T_{22}^B(k^B,y^B) + T_{11}^B(k^B,y^B)T_{22}^A(k^A,y^A) \\ &- T_{12}^A(k^A,y^A)T_{21}^B(k^B,y^B) - T_{21}^i(k^A,y^A)T_{12}^B(k^B,y^B) \\ &+ |H^A(k^A,y^A)| + |H^B(k^B,y^B)| > 0 \end{aligned}$$

The stability analysis is made by studying the sign of $\mathcal{P}_f(0)$, $\mathcal{P}_f(1)$ and $\mathcal{P}_f(-1)$. We easily show the following property:

Lemma 6.5. *The characteristic polynomial satisfies* $\mathcal{P}_f(1) < 0$.

Proof: Consider the second partial derivatives of $V(k, y)$ given in Lemma 6.4 with Lemmas 6.2 and 6.3. Let

$$\Phi^i = \frac{T_1^i(k^i,y^i)\rho}{(k^i-g^i)[\alpha_2^i(\beta_1^i+\beta_2^i)k^i+(\alpha_1^i\beta_2^i-\alpha_2^i\beta_1^i)g^i]} \tag{50}$$

For the autarky distribution with $k^i - g^i = \bar{k}^i(1 - \rho\beta_1^i) > 0$, we get:

$$\begin{aligned}\mathcal{P}_f(1) &= -\frac{|H^B(k^B,y^B)|}{\Theta\Phi^A}(k^A - g^A)(1 - \beta_1^A)\left[\alpha_2^A + \rho(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\right] \\ &- \frac{|H^A(k^A,y^A)|}{\Theta\Phi^B}(k^B - g^B)(1 - \beta_1^B)\left[\alpha_2^B + \rho(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)\right] < 0\end{aligned}$$

For the free-trade distribution with $\theta \in (1, 1/\rho\beta_1^B)$, $k^A - g^A = (\theta - \rho\beta_1^A)k^{A*}/\theta > 0$ and $k^B - g^B = (1 - \rho\beta_1^B\theta)k^{B*} > 0$, we get:

$$\begin{aligned}\mathcal{P}_f(1) &= -\frac{|H^B(k^B,y^B)|}{\Theta\Phi^A}(k^A - g^A)\Big[\rho(1 - \beta_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A) \\ &+ \alpha_2^A[\beta_2^A + (1 - \beta_1^A - \beta_2^A)\theta]\Big] \\ &- \frac{|H^A(k^A,y^A)|}{\Theta\Phi^B}\frac{k^B-g^B}{\theta}\Big[(1 - \beta_1^B)\left[\alpha_2^B + \rho(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)\theta\right] \\ &+ \alpha_2^B\beta_2^B(\theta - 1)\Big]\end{aligned}$$

Let us then denote $h(\theta) = \rho(1-\beta_1^A)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)+\alpha_2^A[\beta_2^A+(1-\beta_1^A-\beta_2^A)\theta]$. Since $h(1) > 0$ and $h'(\theta) > 0$ we conclude that $h(\theta) > 0$ for any $\theta > 1$. Therefore we derive $\mathcal{P}_f(1) < 0$ for any $\theta \in (1, 1/\rho\beta_1^B)$. □

We may now prove Proposition 6. Using again Lemma 6.4 with Lemmas 6.2 and 6.3, we get:

$$\begin{aligned}\mathcal{P}_f(0) &= -\frac{|H^B(k^B,y^B)|}{\Theta\Phi^A}\Big[(1 - \alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(k^A - g^A) \\ &- (1 - \alpha_1^A - \alpha_2^A)\alpha_1^A\beta_2^A k^A\Big] \\ &- \frac{|H^A(k^A,y^A)|}{\Theta\Phi^B}\Big[(1 - \alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(k^B - g^B) \\ &- (1 - \alpha_1^B - \alpha_2^B)\alpha_1^B\beta_2^B k^B\Big] \\ \mathcal{P}_f(-1) &= 2(1 + \rho)\mathcal{P}_f(0) - \mathcal{P}_f(1)\end{aligned} \tag{51}$$

From Lemma 6.5 we know that $\mathcal{P}_f(1) < 0$. If the investment good is capital intensive in each country then we get $\mathcal{P}_f(0) > 0$ and thus $\mathcal{P}_f(-1) > 0$. It follows that the autarky and free-trade steady states are saddle-point stable with positive characteristic roots. The result follows from Theorems 1 and 2 in Nishimura and Yano [10]. □

6.8 Proof of Proposition 7

Consider the expressions of $\mathcal{P}_f(0)$ and $\mathcal{P}_f(-1)$ evaluated at the autarky distribution with $k = \bar{k}^A + \bar{k}^B = y$. Using the same arguments as in the proof of Proposition 3 we first derive that if for any $i = A, B$

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \tfrac{\alpha_1^i\beta_2^i(1-\alpha_1^i-\alpha_2^i)}{(1-\alpha_1^i)(1-\beta_1^i)}$$

then $\mathcal{P}_f(0) < 0$ for all $\rho \in (0, 1]$. We also know that when $\rho = 1$, we have by definition

$$\begin{aligned} \mathcal{P}_f(-1) &= 2V_{12}(k,k) - V_{11}(k,k) - V_{22}(k,k) \\ &= -\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} V_{11}(k,k) & V_{12}(k,k) \\ V_{12}(k,k) & V_{22}(k,k) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} > 0 \end{aligned} \tag{52}$$

since under decreasing returns the value function $V(k, y)$ is strictly concave. On the other side, when $\rho = 0$ we get

$$\begin{aligned} \mathcal{P}_f(-1) &= -\tfrac{|H^B(k^B,y^B)|}{\Theta\Phi^A} k^A\alpha_2^A\left[2\alpha_1^A(\beta_1^A + \beta_2^A) - 1 - \beta_1^A\right] \\ &- \tfrac{|H^A(k^A,y^A)|}{\Theta\Phi^B} k^B\alpha_2^B\left[2\alpha_1^B(\beta_1^B + \beta_2^B) - 1 - \beta_1^B\right] \end{aligned}$$

Therefore, if for any $i = A, B$, $\alpha_1^i > (1 + \beta_1^i)/2(\beta_1^i + \beta_2^i)$, there exists $\hat{\rho} \in (0, 1)$ such that $\mathcal{P}_f(-1) > 0$ for any $\rho \in (\hat{\rho}, 1]$ and $\mathcal{P}_f(-1) < 0$ in a left neighbourhood of $\hat{\rho}$. We then conclude that when $\rho \in (\hat{\rho}, 1]$ the steady state is saddle-point stable with two negative eigenvalues and when ρ crosses $\hat{\rho}$ from above one eigenvalue crosses -1. The result follows from the flip bifurcation theorem. $\square$

6.9 Proof of Proposition 8

Consider the free-trade distribution $k = k^A + k^B$ as defined by Proposition 5 with with $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ and $\theta \in (1, 1/\rho\beta_1)$. It follows that $g^A = \rho\beta_1^A k^B$, $k^A - g^A = k^B(\theta - \rho\beta_1^A)$, $g^B = \rho\beta_1^B\theta k^B$, $k^B - g^B = k^B(1 - \rho\beta_1^B\theta)$. From (51) we derive that

$$\begin{aligned} \mathcal{P}_f(0) = &-\tfrac{|H^B(k^B,y^B)|}{\Theta\Phi^A}\tfrac{k^A}{\theta}\Big[(1-\alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(\theta - \rho\beta_1^A) \\ &- (1 - \alpha_1^A - \alpha_2^A)\alpha_1^A\beta_2^A\theta\Big] \\ &- \tfrac{|H^A(k^A,y^A)|}{\Theta\Phi^B} k^B\Big[(1-\alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(1 - \rho\beta_1^B\theta) \\ &- (1 - \alpha_1^B - \alpha_2^B)\alpha_1^B\beta_2^B\Big] \end{aligned}$$

Therefore, $\mathcal{P}_f(0) < 0$ if

$$\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \tfrac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A} \tfrac{\alpha_1^A \beta_2^A \theta}{\theta-\rho\beta_1^A} \equiv \mathcal{Z}_2^A \tag{53}$$

and

$$\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B > \tfrac{1-\alpha_1^B-\alpha_2^B}{1-\alpha_1^B} \tfrac{\alpha_1^B \beta_2^B}{1-\theta\rho\beta_1^B} \equiv \mathcal{Z}_2^B \tag{54}$$

Since the right-hand-side of these expressions is an increasing function of ρ, we conclude that $\mathcal{P}_f(0) < 0$ for any $\rho \in (0, 1)$ if

$$\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \tfrac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A} \tfrac{\alpha_1^A \beta_2^A \theta}{\theta-\beta_1^A}$$

and

$$\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B > \tfrac{1-\alpha_1^B-\alpha_2^B}{1-\alpha_1^B} \tfrac{\alpha_1^B \beta_2^B}{1-\theta\beta_1^B}$$

Consider finally $\mathcal{P}_f(-1)$. From (52) we still have $\mathcal{P}_f(-1) > 0$ when $\rho = 1$. When $\rho = 0$, we get

$$\begin{aligned}\mathcal{P}_f(-1) = & -\tfrac{|H^B(k^B,y^B)|}{\Theta\Phi^A} k^B \alpha_2^A \theta \left[2\alpha_1^A(\beta_1^A + \beta_2^A) - \theta(1 - \beta_1^A - \beta_2^A) - 2\beta_1^A - \beta_2^A\right] \\ & - \tfrac{|H^A(k^A,y^A)|}{\Theta\Phi^B} \tfrac{k^B}{\theta} \alpha_2^B \left[2\alpha_1^B \theta(\beta_1^B + \beta_2^B) - (1 - \beta_1^B - \beta_2^B) - \theta(2\beta_1^B + \beta_2^B)\right]\end{aligned}$$

Hence, if the following conditions hold

$$\alpha_1^A > \tfrac{\theta(1-\beta_1^A-\beta_2^A)+2\beta_1^A+\beta_2^A}{2(\beta_1^A+\beta_2^A)} \tag{55}$$

and

$$\alpha_1^B > \tfrac{1-\beta_1^B-\beta_2^B+\theta(2\beta_1^B+\beta_2^B)}{2\theta(\beta_1^B+\beta_2^B)} \tag{56}$$

there exists $\hat{\rho} \in (0, 1)$ such that $\mathcal{P}_f(-1) > 0$ for any $\rho \in (\hat{\rho}, 1]$ and $\mathcal{P}_f(-1) < 0$ in a left neighbourhood of $\hat{\rho}$. We then conclude that when $\rho \in (\hat{\rho}, 1]$ the steady state is saddle-point stable with two negative eigenvalues and when ρ crosses $\hat{\rho}$ from above one eigenvalue crosses -1. The result follows from the flip bifurcation theorem. □

6.10 Proof of Corollary 2

In the proof of Proposition 3, assume that inequality (39) applied to country A is not satisfied, i.e. for any given $\rho \in (0, 1]$

$$\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A < \tfrac{1-\alpha_1^A-\alpha_2^A}{1-\alpha_1^A} \tfrac{\alpha_1^A \beta_2^A}{1-\rho\beta_1^A} \equiv \mathcal{Z}_1^A$$

It follows that the autarky steady state of country A is saddle-point stable with monotone convergence for any given $\rho \in (0, 1]$ since $\mathcal{P}_a^A(0) > 0$ and

$\mathcal{P}_a^A(1) < 0$. Consider then condition (53) in the proof of Proposition 8. Straightforward computations give $\mathcal{Z}_1^A > \mathcal{Z}_2^A$. It follows that all the conditions of Proposition 8 for country A may be satisfied for the free-trade steady state while the steady state under autarky is characterized by monotone convergence for any given $\rho \in (0,1]$. At the same time, we get for country B that for any given $\rho \in (0,1]$, $\mathcal{Z}_1^B < \mathcal{Z}_2^B$. It follows that if all the conditions of Proposition 8 for country B are satisfied along the free-trade steady state then the the optimal path under autarky is also characterized by endogenous fluctuations (period-2 cycles) in the neighborhood of the flip bifurcation value $\hat{\rho}$. As a result endogenous fluctuations arise at the world level while country A is characterized by monotone convergence under autarky. □

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