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Rose-Anne Dana, Cuong Le Van. No-arbitrage, overlapping sets of priors and the existence of efficient allocations and equilibria in the presence of risk and ambiguity. 2008. halshs-00281582v1

**HAL Id: halshs-00281582**

**<https://shs.hal.science/halshs-00281582v1>**

Submitted on 23 May 2008 (v1), last revised 23 Nov 2009 (v2)

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**2008.39**



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# No-arbitrage, Overlapping sets of priors and the existence of efficient allocations and equilibria in the presence of risk and ambiguity

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May 21, 2008

## Abstract

The theory of existence of equilibrium with short-selling is reconsidered under risk and ambiguity modelled by risk averse variational preferences. A sufficient condition for existence of efficient allocations is that the relative interiors of the risk adjusted sets of expectations overlap. This condition is necessary if agents are not risk neutral at extreme levels of wealths either positive or negative. It is equivalent to the condition that there does not exist mutually compatible trades, with non negative expected value with respect to any risk adjusted prior, strictly positive for some agent and some prior. It is shown that the more uncertainty averse and the more risk averse, the more likely are efficient allocations and equilibria to exist.

**Keywords:** Uncertainty, risk, common prior, equilibria with short-selling. Variational preferences.

**JEL Classification:** C62, D50, D81,D84,G1.

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# 1 Introduction

The issue of the relationship between agents beliefs and risk tolerances and the existence of efficient allocations and equilibria has first been considered, in the early seventies, by Grandmont [9], Green [10] ) and Hart [13] for markets with short-selling in the context of temporary equilibrium models and equilibrium models of assets models and reconsidered later by Hammond [11] and Page [16], [17]. In these early models, investors were assumed to have a *single* homogeneous or heterogeneous probabilistic belief and be von Neumann-Morgenstern (vNM), risk averse utility maximizers. Two sufficient conditions for existence of an equilibrium were given:

- the *overlapping expectations* condition which expresses that investors are sufficiently similar in their beliefs and risk tolerances so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded vNM utility nondecreasing purchases (see Hammond [11], Page [16], [17]),

- *the no unbounded utility arbitrage* condition, a condition of absence of collective arbitrage, which requires that investors do not engage in mutually compatible, utility nondecreasing trades (see Hart [13], Page [17], Nielsen [15]).

These conditions were shown to be equivalent under adequate assumptions and necessary for existence of equilibrium (see e.g. Page [17], Page and Wooders [19]) under further assumptions.

The problem of existence of equilibria with consumption sets unbounded below has been extended to abstract economies and the assumptions mentioned above generalized (see Werner [22] and Nielsen [15]). For a review of the subject in finite dimension, see Allouch et al [1], Dana et al [5], Page [16],[18]. The theory has also been developed in infinite dimension (see Brown and Werner [4] and Dana et al [6] for some of the difficulties encountered).

This paper generalizes the original theory of existence of equilibrium with short-selling, by relaxing the assumption that agents hold a single probabilistic belief and introducing risk and ambiguity. It uses the variational preferences axiomatized by Maccheroni, Marinacci and Rustichini [14] (denoted MMR from now on). Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literatures, in particular, the maxmin expected utility of Gilboa and Schmeidler [8], the convex Choquet expected utility of Schmeidler and the  $\varepsilon$ -contaminated model introduced in statistics, the penalty preference functionals of Hansen and Sargent [12] and finally the convex measures of risk introduced in mathematical finance. A risk averse variational preference is characterized by a convex cost (penalty) function defined on the probability simplex and a concave utility index that

models risk-aversion. Without loss of generality, attention may be restricted to the probabilities with finite cost that we call the *priors*. To simplify as much as possible the analysis, we assume complete markets and consider a standard Arrow-Debreu model of state contingent claims.

We reconsider for MMR preferences all the basic concepts of the theory of equilibrium with short-selling, the useful and useless trading directions, the no-arbitrage prices (weak no-arbitrage prices) and the concepts of collective absence of arbitrage. We first show that for a risk neutral MMR agent, a useful trade for an agent is a trade with non negative expectation with respect to any of her prior and the set of normalized no-arbitrage prices (weak no-arbitrage prices) is the interior (relative interior) of the set of priors. A sufficient condition for existence of an equilibrium is that the intersection of the relative interiors of the sets of priors of the agents be non empty. It is equivalent to the inexistence of mutually compatible trades with non negative expectations with respect to any prior, strictly positive for some agent and some prior. These conditions are necessary when the utility has no half-line. When this last condition is not fulfilled, a necessary condition for existence of an equilibrium is that the intersection of the sets of priors of the agents be non empty. This in particular implies that for risk neutral MMR agents, the more ambiguity and the more likely is an equilibrium to exist.

We next consider the case of risk averse MMR agents and show that, as far as arbitrage theory is concerned, agents behave as if they were risk neutral MMR with a larger set of priors which, in the financial tradition, we call the *risk adjusted set of priors*. The more ambiguous and the more risk averse the agent, the larger is the risk adjusted set of priors. As a benchmark, for an expected utility maximizer whose belief give positive weight to each state of the world, the risk adjusted set of priors contains all the probabilities which give positive weight to any state of the world when the ratio of the asymptotic marginal utilities is zero (the agent is *infinitely risk averse*). We show that for a risk averse MMR agent, a useful trade for an agent is a trade with non negative expectation with respect to any of her risk adjusted set of priors and that the set of no-arbitrage prices (weak no-arbitrage prices) is the interior (relative interior) of the risk adjusted set of priors. A sufficient condition for existence of efficient allocations or of equilibria is that the intersection of the relative interiors of the risk adjusted set of priors be non empty. It is equivalent to the inexistence of mutually compatible trades with non negative expectations with respect to any risk adjusted prior, strictly positive for some agent and some prior. This condition is also sufficient when agents marginal utilities are not constant at extreme levels of wealths either positive or negative, a condition that we now

on call *absence of risk neutrality at infinity*. The more ambiguous and the more risk averse the agents, the larger are the risk adjusted sets of priors and the more likely is an equilibrium to exist. These conditions generalize the conditions given in the early seventies for single beliefs. An equilibrium does not exist if agents disagree "very much". This happens if for example some agents give no weight whatever prior they use to disjoint subsets of the states of the world. When agents agree on the states of the world which are relevant, there may not be an equilibrium if their priors are too different. However, when the agents are very risk averse, strong disagreement on expectations may be compatible with the existence of an equilibrium.

The paper is organized as follows. Section 2 introduces variational preferences and standard concepts in equilibrium theory. Section 3 recalls and characterizes for MMR agents, the concepts of useful and useless trading directions, that of a no-arbitrage price (weak no-arbitrage price) and of collective absence of arbitrage. Section 4 deals with existence of efficient allocations and equilibria. Necessary and sufficient conditions are provided in the case of strictly concave utilities, sufficient conditions in the case of concave utilities as well as necessary conditions. Some examples are given to show that the existence of an equilibrium is unrelated to the intersection of agents' priors being non empty. A last section provides all the proofs that are not given in the main part of the paper.

## 2 Variational preferences

### 2.1 M.M.R. preferences

We consider a standard Arrow-Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state  $s$  from a state space  $\Omega = \{1, \dots, k\}$  will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from  $\Omega \rightarrow \mathbb{R}$ . The random variable  $X$  which equals  $x_1$  in state 1,  $x_2$  in state 2 and  $x_k$  in state  $k$ , is identified with the vector in  $X \in \mathbb{R}^k$ ,  $X = (x_1, \dots, x_k)$ . Let  $\Delta = \{\pi \in \mathbb{R}_+^k : \sum_{s=1}^k \pi_s = 1\}$  be the probability simplex in  $\mathbb{R}^k$ . For a given  $\pi \in \Delta$ , we denote by  $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$  the expectation of  $X$  and for a given price  $p \in \mathbb{R}^k$  by  $p \cdot X := \sum_{l=1}^k p_l x_l$ , the price of  $X$ .

There are  $m$  agents indexed by  $i = 1, \dots, m$ . Agent  $i$  has an endowment  $E^i \in \mathbb{R}^k$  of contingent claims. We denote by  $(E^i)_{i=1}^m$  the  $m$ -tuple of endowments and by  $E = \sum_{i=1}^m E^i$  aggregate endowment. We assume that each agent has a preference order  $\succeq$  over  $\mathbb{R}^k$  represented by a utility function  $V$  which verify: there exists a concave, strictly increasing differentiable utility index  $u : \mathbb{R} \rightarrow \mathbb{R}$

and a convex lower semi-continuous function  $c : \Delta \rightarrow [0, \infty]$  such that the utility  $V : \mathbb{R}^k \rightarrow \mathbb{R}$  is given by

$$V(X) = \min_{\pi \in \Delta} E_{\pi}(u(X)) + c(\pi) \quad (1)$$

Utilities of type 1 capture risk and uncertainty. Indeed, let  $\mathcal{M}_b(\mathbb{R})$  be the set of boundedly supported measures on  $\mathbb{R}$  and  $\delta_x$  be the Dirac measure at  $x$ . Regarding  $\mathbb{R}^k$  as a subset of  $(\mathcal{M}_b(\mathbb{R}))^k$  by identifying  $X = (x_1, \dots, x_k)$  with  $(\delta_{x_1}, \dots, \delta_{x_k})$ , we consider that  $\succeq$  is the restriction to  $\mathbb{R}^k$  of a preference relation also denoted  $\succeq$  on  $(\mathcal{M}_b(\mathbb{R}))^k$  which belongs to a subclass of the variational preferences axiomatized by Maccheroni et al [14] and has the following representation:

$$\tilde{V}(\Lambda) = \min_{\pi \in \Delta} \sum_{s=1}^k \pi_s \int u(x) d\lambda_s(x) + c(\pi), \quad \Lambda = (\lambda_1, \dots, \lambda_k), \lambda_s \in \mathcal{M}_b(\mathbb{R})$$

Restricting  $\succeq$  to constant  $\Lambda$  ( $\Lambda_s = \lambda$  for all  $s$ ), we obtain a preference relation on  $\mathcal{M}_b(\mathbb{R})$ , also denoted  $\succeq$  with some abuse of notations, with utility representation  $\int u(x) d\lambda(x)$ . Let  $\int x d\lambda(x) = m(\lambda)$  be the expectation of  $\lambda$  assumed to be finite. Risk aversion is expressed by: for any  $\lambda \in \mathcal{M}_b(\mathbb{R})$  with finite expectation

$$\delta_{m(\lambda)} \succeq \lambda \Leftrightarrow u\left(\int x d\lambda(x)\right) \geq \int u(x) d\lambda(x)$$

and risk aversion is modelled by  $u$  being concave. As usual,  $v$  is more risk averse than  $u$  if  $\int v(x) d\lambda(x) \geq v(a)$  with  $a \in \mathbb{R}$  implies  $\int u(x) d\lambda(x) \geq u(a)$ . From Arrow-Pratt's theorem,  $v$  is more risk averse than  $u$  if and only if  $v = \psi \circ u$  for some  $\psi$  concave increasing.

According to Maccheroni et al [14],  $\succeq_1$  is more ambiguity averse than  $\succeq_2$  if

$$\Lambda \succeq_1 (\lambda, \dots, \lambda) \Rightarrow \Lambda \succeq_2 (\lambda, \dots, \lambda)$$

As shown by Maccheroni et al [14]  $\succeq_1$  is more ambiguity averse than  $\succeq_2$  if and only if  $u_1 = au_2 + b$  for some  $a > 0$ ,  $b \in \mathbb{R}$  and  $c_1 \leq c_2$  provided  $u_1 = u_2$ . Hence  $c$  is an index of ambiguity aversion.

Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literatures, in particular:

- the maxmin expected utility of Gilboa and Schmeidler [8]

$$V(X) = \min_{\pi \in P} E_{\pi}(u(X)) \quad (2)$$

which is obtained for  $c = \delta_P$ , an indicator function of a convex compact subset  $P$  of  $\Delta$  ( $c(\pi) = 0$  if  $\pi \in P$  and  $c(\pi) = \infty$  otherwise),

- the convex Choquet expected utility of Schmeidler where  $P$  above is the core of a convex capacity and in particular, the  $\varepsilon$ -contaminated model introduced in statistics, where  $P = \varepsilon\pi + (1 - \varepsilon)\Delta$  for some  $\pi \in \Delta$ ,
- the multiplier utility used by Hansen and Sargent [12] where

$$c(q | p) = \begin{cases} \theta E\left(\frac{dq}{dp} \ln\left(\frac{dq}{dp}\right)\right) & \text{if } q \ll p \\ = \infty & \text{otherwise} \end{cases}$$

where  $\theta > 0$  is a parameter of ambiguity aversion.

Utilities of type (1) also include the *monetary utility functions* where we recall that a function  $V : \mathbb{R}^k \rightarrow \mathbb{R}$  is a monetary utility function if it is concave monotone increasing and has the cash invariance property  $V(X + M) = V(X) + M$ , for any  $X \in \mathbb{R}^k$ ,  $C$  constant. From Fenchel's duality theory, monetary utility functions have the following representation

$$V(X) = \min_{\pi \in \Delta} E_{\pi}(X) + c(\pi) \quad (3)$$

where  $c(\pi) = \sup_{X \in \mathbb{R}^k} V(X) - E_{\pi}(X) \in \mathbb{R} \cup \{\infty\}$  which is convex, lower semi-continuous, is the conjugate function of  $V$ . A monetary utility is obtained from (1) by assuming risk-neutrality and cash invariance. Monetary utility functions can be identified with *convex measures of risk*. We recall that a map  $\rho : \mathbb{R}^k \rightarrow \mathbb{R}$  is a monetary convex measures of risk if it is convex, monotone decreasing and  $\rho(X + M) = \rho(X) - M$ , for any  $X \in \mathbb{R}^k$ ,  $C$  constant (see Föllmer and Schied [7]) by defining  $\rho = -V$ . Monetary utilities with cost function  $c = \delta_P$ , the indicator function of a convex compact subset  $P$  of  $\Delta$

$$V(X) = \min_{\pi \in P} E_{\pi}(X)$$

correspond to coherent measures of risk and will be called coherent monetary utilities.

Let  $P = \text{dom } c$  be the set of effective priors. Clearly we have

$$V(X) = \min_P E_{\pi}(u(X)) + c(\pi) \quad (4)$$

For a fixed  $u$ , the higher the ambiguity aversion, the smaller  $c$  and the larger is  $P$ .

## 2.2 Concepts in equilibrium theory

We next recall standard concepts in equilibrium theory.



Given  $(E^i)_{i=1}^m$ , an allocation  $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$  is attainable if  $\sum_{i=1}^m X^i = E$ . The set of individually rational attainable allocations  $A((E^i)_{i=1}^m)$  is defined by

$$A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \mid \sum_{i=1}^m X^i = E \text{ and } V^i(X^i) \geq V^i(E^i), \forall i \right\}.$$

The individually rational utility set  $U((E^i)_{i=1}^m)$  is defined by

$$U((E^i)_{i=1}^m) = \{(v_1, v_2, \dots, v_m) \in \mathbb{R}^m \mid \exists X \in A((E^i)_{i=1}^m) \text{ s. t. } V^i(E^i) \leq v_i \leq V^i(X^i), \forall i\}.$$

**Definition 1** Given  $(E^i)_{i=1}^m$ , an attainable allocation  $(X^i)_{i=1}^m$  is efficient (or Pareto optimal) if there does not exist  $(X^i)_{i=1}^m$  attainable such that  $V_i(X_i^h) \geq V_i(X_i)$  for all  $i$  with a strict inequality for some  $i$ . It is individually rational efficient if it is efficient and  $V^i(X^i) \geq V^i(E^i)$  for all  $i$ .

**Definition 2** A pair  $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \setminus \{0\}$  is a contingent Arrow-Debreu equilibrium if

1. for each agent  $i$  and  $X^i \in \mathbb{R}^k$ ,  $V^i(X^i) > V^i(X^{i*})$  implies  $p^* \cdot X^i > p^* \cdot X^{i*}$ ,
2. for each agent  $i$ ,  $p^* \cdot X^{i*} = p^* \cdot E^i$ .

### 3 Useful vectors and no-arbitrage concepts

In this section, we recall the concepts of useful and useless trading directions and characterize the useful (useless) trading directions for a utility of type (4). We then turn to the concepts of no-arbitrage prices and weak no-arbitrage prices. We finally recall concepts of collective absence of arbitrage.

#### 3.1 Useful vectors

Let  $V$  be a utility of type (4). For  $X \in \mathbb{R}^k$ , let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid V(Y) \geq V(X)\}$$

be the set of contingent claims preferred to  $X$  and let  $R(X)$  be its asymptotic cone (see Rockafellar [20], section 8). Since  $V$  is concave, by Rockafellar's theorem 8.7 in [20],  $R(X)$  is independent of  $X$  and called the set of *useful vectors* for  $V$ . It will be denoted by  $R$ . We recall that

$$R = \{W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0\}.$$

The lineality space of  $V$  or set of useless vectors is defined by

$$L = \{W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \in \mathbb{R}\} = R \cap (-R).$$

Using the concavity of  $V$ , we also have:

$$L = \{W \in \mathbb{R}^k \mid V(\lambda W) = V(0), \text{ for all } \lambda \in \mathbb{R}\}.$$

Let us first consider the risk neutral case:

$$V(X) = \min_P aE_\pi(X) + c(\pi), \quad a > 0 \quad (5)$$

with  $c : \Delta \rightarrow [0, \infty]$ .

**Proposition 1** *Let  $V$  fulfill (5). We then have*

$$R = \{W \in \mathbb{R}^k \mid E_\pi(W) \geq 0, \text{ for all } \pi \in P\}$$

$$L = \{W \in \mathbb{R}^k \mid E_\pi(W) = 0, \text{ for all } \pi \in P\}$$

$L = \{0\}$  if and only there exists  $\pi \in \text{int } P$ .

It follows from proposition 1 that if  $\bar{P} = \Delta$ , then  $R = \mathbb{R}_+^k$ . This is the case when the cost function is the entropy or the Gini index with respect to a strictly positive probability measure.

Let us next consider the risk averse case. We first show that  $V$  is the minimum of a family of affine combinations of linear expectations over a set of priors  $\tilde{P}$  which is larger than  $P$ . Indeed, since  $u$  is concave and differentiable,

$$u(x) = \min_{z \in \mathbb{R}} \{u'(z)x + u(z) - u'(z)z\}. \quad (6)$$

We may therefore characterize  $V$  as follows.

**Lemma 1** *Let  $V$  fulfill (4) and  $u$  be non linear. For any  $X \in \mathbb{R}^k$ , we have*

$$V(X) = \min_\eta \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\} \quad (7)$$

where  $\eta = (\pi, Z) \in P \times \mathbb{R}^k$  and  $\gamma(\eta) = E_\pi u(Z) - E_\pi(u'(Z)Z) + c(\pi)$ .

The above representation leads us to introduce a new set of priors which, in the financial tradition, we call the *risk adjusted set of priors*,

$$\tilde{P} = \left\{ p \in \Delta \mid \exists \pi \in P, Z \in \mathbb{R}^k \text{ s. t. } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}, \forall s = 1, \dots, k \right\} \quad (8)$$

The next proposition states some of the properties  $\tilde{P}$ . We use the following notations.

Two equivalent probabilities  $p$  and  $\pi$  will be denoted by  $p \simeq \pi$ . For  $\pi \in P$ , let  $I_\pi = \{s \mid \pi_s > 0\}$ . Let  $\text{int } \Delta = \{p \in \Delta \mid p_s > 0 \text{ for all } s\}$  and for  $A \subseteq \Delta$ , let

$\text{int}A = \{p \in A \mid \exists \text{ a ball } B(p, \varepsilon) \text{ s.t. } B(p, \varepsilon) \cap \text{int } \Delta \subseteq A\}$ .

Let  $a = u'(+\infty)$  and  $b = u'(-\infty)$  be the asymptotic slopes of  $u$  and  $t = \frac{a}{b}$  be their ratio. Note that  $t = 0$  if and only if  $a = 0$  or  $b = +\infty$  while  $t = 1$  if and only if the agent is risk neutral. For an expected utility maximizer, it follows from Arrow-Pratt's theorem that  $t$  is a measure of risk tolerance: the more risk averse the agent and the smaller is  $t$ .

**Proposition 2** 1.  $P \subseteq \tilde{P}$ .  $\tilde{P} = P$  when the agent is risk neutral ( $t = 1$ ), when  $P = \text{int}\Delta$  or  $P = \Delta$ .

2. The set  $\tilde{P}$  is convex.

3. If  $t = 0$ , then  $\tilde{P} = \{p \in \Delta \mid \exists \pi \in P, \pi \simeq p\}$ . If moreover,  $P \cap \text{int } \Delta \neq \emptyset$ , then  $\text{int } \Delta \subseteq \tilde{P}$ .

4. The more ambiguous and more risk averse the agent and the larger is  $\tilde{P}$ .

We may now characterize the useful vectors of an agent with a utility of type (4). Assertion 1 of the next proposition implies that the set of useful vectors of a risk averse agent with a variational utility and set of priors  $P$  is the set of useful vectors of a risk-neutral agent with a variational utility and set of priors  $\tilde{P}$ . In particular, the set of useful vectors of a risk averse expected utility maximizer is the set of useful vectors of a risk neutral ambiguous agent with set of priors, the risk adjusted priors of a given probability. This characterization will turn out to be very important from a technical point of view since it will enable us to fully characterize all concepts of absence of arbitrage for MMR utilities.

**Proposition 3** Let  $V$  fulfill (4) with  $t < 1$ . Then

1.  $R = \{W \in \mathbb{R}^k \mid E_p(W) \geq 0, \text{ for all } p \in \tilde{P}\}$

2. If  $t = 0$  and  $P \cap \text{int}\Delta \neq \emptyset$  or if  $\bar{P} = \Delta$ , then  $R = \mathbb{R}_+^k$ .

3.  $L = \{W \in \mathbb{R}^k \mid E_p(W) = 0, \text{ for all } p \in \tilde{P}\}$ .  $L = \{0\}$  iff  $\text{int } \tilde{P} \neq \emptyset$ .

It follows from propositions 2 and 3 that for a utility of type (4) the more ambiguous and the more risk averse the agent, the larger is  $\tilde{P}$ , the smaller are the sets of useful and useless vectors.

**Remark 1** Let us show that the condition

$$E_p(W) \geq 0, \text{ for all } p \in \tilde{P} \tag{9}$$

is equivalent to

$$\sum_{\{l \mid w_l \geq 0\}} a\pi_l w_l + \sum_{\{l \mid w_l < 0\}} b\pi_l w_l \geq 0, \text{ for all } \pi \in P \tag{10}$$

or to

$$tE_\pi(W_+) - E_\pi(W_-) \geq 0, \text{ for all } \pi \in P. \quad (11)$$

where  $(W_+)_l = w_l$  if  $x_l \geq 0$  and  $(W_-)_l = -w_l$  if  $w_l \leq 0$ . When  $P$  is a singleton, (11) is the incomplete mean condition given by Bertsekas [3] and Hart [13]. Clearly (10) is equivalent to (11). Let us show that (9) and (10) are equivalent. (9) is equivalent to  $\sum_l \pi_l u'(x_l) w_l \geq 0$  for every  $X \in \mathbb{R}^k$  and  $\pi \in P$ . Letting  $x_l$  go to  $+\infty$  for any  $l$  such that  $w_l \geq 0$  and  $x_l$  go to  $-\infty$  for any  $l$  such that  $w_l < 0$ , we obtain (10). Conversely, since, for any  $X \in \mathbb{R}^k$  and  $\pi \in P$ ,

$$\frac{1}{E(u'(X))} \sum_l \pi_l u'(x_l) w_l \geq \frac{1}{E(u'(X))} \left( \sum_{\{l \mid w_l \geq 0\}} \pi_l a w_l + \sum_{\{l \mid w_l < 0\}} \pi_l b w_l \right)$$

(10) implies (9).

### 3.2 Concepts of absence of arbitrage

The first concept that we recall is that of a no-arbitrage price, a price for which no agent can make costless unbounded utility nondecreasing purchases.

**Definition 3** A price vector  $p \in \mathbb{R}^k$  is a "no-arbitrage price" for agent  $i$  if  $p \cdot W > 0$ , for all  $W \in R^i \setminus \{0\}$ . A price vector  $p \in \mathbb{R}^k$  is a "no-arbitrage price" for the economy if it is a no-arbitrage price for each agent.

For  $A \subseteq \mathbb{R}^k$ , we note  $A^0$  the polar of  $A$  where we recall that  $A^0 = \{p \in \mathbb{R}^p \mid p \cdot X \leq 0, \text{ for all } X \in A\}$ .

Let  $S^i$  denote the set of non arbitrage prices for  $i$ . Then  $S^i = \text{int} - (R^i)^0$ . A price vector  $p \in \mathbb{R}^k$  is a "no-arbitrage price" for the economy if and only if  $p \in \cap_i S^i = \cap_i \text{int} - (R^i)^0$ . From Proposition 3, we may characterize the set of no-arbitrage prices for agent  $i$  and for the economy. A no-arbitrage normalized price for  $i$  is a strictly positive risk adjusted probability in  $\tilde{P}^i$  that fulfills (12) below. A no-arbitrage normalized price for the economy is a strictly positive common risk adjusted probability that fulfills (12) for each  $i$ .

**Proposition 4** Let  $V^i$  fulfill (4) for each  $i$ . Then

1. the set of no-arbitrage prices for agent  $i$  is  $S^i = \text{cone int } \tilde{P}^i$ .

2. If  $t^i < 1$ ,  $p \in \text{int } \tilde{P}^i$  if and only if

$$\exists \pi \in P^i \cap \text{int } \Delta, Z \in \mathbb{R}^k, \forall s, a < u'(z_s) < b \text{ and } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}. \quad (12)$$

Hence  $S^i \neq \emptyset$  if and only if  $P^i \cap \text{int } \Delta \neq \emptyset$ .

If  $t^i = 1$ , then  $S^i \neq \emptyset$  if and only if,  $\text{int } P^i \neq \emptyset$

3. The set of no-arbitrage prices for the economy is  $\cap_i S^i = \text{cone} \cap_i \text{int} \tilde{P}^i$ .
4. Let  $I_1 = \{i \mid t^i < 1\}$  and  $I_2 = \{i \mid t^i = 1\}$ . Then  $\cap_i S^i \neq \emptyset$  if and only if, for any  $i \in I_1$ , there exists  $\pi^i \in P^i \cap \text{int} \Delta$ ,  $Z^i \in \mathbb{R}^k$  with  $u^i(+\infty) < u^i(z_s^i) < u^i(-\infty)$  for all  $s$  and  $\pi \in \cap_{i \in I_2} \text{int} P^i$  such that, for all  $i \in I_1$ ,  $j \in I_1$ ,  $s = 1, \dots, k$ ,

$$\frac{\pi_s^i u^i(z_s^i)}{E_{\pi^i} u^i(Z^i)} = \frac{\pi_s^j u^j(z_s^j)}{E_{\pi^j} u^j(Z^j)} = \pi_s$$

Let us give a few simple sufficient conditions that insure the non-emptiness of the set of no-arbitrage prices for the economy.

**Corollary 1** 1. If  $\cap_i \text{int} P^i \neq \emptyset$ , then  $\cap_i S^i \neq \emptyset$ .

2. If  $t^i = 0$  and  $P^i \cap \text{int} \Delta \neq \emptyset$  for all  $i$ , then  $\cap_i S^i = \text{int} \mathbb{R}_+^k$ .

**Proof:** To prove assertion 1 that allows for risk neutral agents, if  $t^i < 1$ , let  $Z^i$  be constant in (12) with  $a^i < u^i(z_s^i) < b^i$ . We obtain that  $\text{int} P^i \subseteq \text{int} \tilde{P}^i$ . Hence  $\cap_i \text{int} P^i \neq \emptyset$  imply  $\cap_i \text{int} \tilde{P}^i \neq \emptyset$ . From proposition 4, assertion 3,  $\cap_i S^i \neq \emptyset$ . To prove the second assertion, from proposition 2 assertion 4,  $\text{int} \tilde{P}^i = \text{int} \Delta$ , therefore  $S^i = \text{int} \mathbb{R}_+^k$  for all  $i$  and  $\cap_i S^i = \text{int} \mathbb{R}_+^k$ . ■

From proposition 4,  $S^i \neq \emptyset$  if and only if  $\text{int} \tilde{P}^i \neq \emptyset$ . If  $t^i < 1$ , from proposition 4,  $P^i \cap \text{int} \Delta \neq \emptyset$  is a necessary and sufficient for the non-emptiness of  $\text{int} \tilde{P}^i$ . When this condition is not satisfied, there are states of the world that agent  $i$  considers as totally unlikely :  $P^i$  and  $\tilde{P}^i$  are in a facet of  $\Delta$  that we next define.

**Lemma 2** Let  $P \cap \text{int} \Delta = \emptyset$ , then  $\{l \mid \pi_l = 0, \text{ for all } \pi \in P\} \neq \emptyset$ .

**Proof:** Assume on the contrary that for any  $l$ , there exists  $\pi(l) \in P$  such that  $\pi(l)_l > 0$ . Let  $\lambda \in \text{int} \Delta$ . Then  $\nu = \sum_s \lambda_s \pi(s) \in P \cap \text{int} \Delta$ , a contradiction. ■

Let  $G_P$  be the complement of the set  $\{l \mid \pi_l = 0, \text{ for all } \pi \in P\}$ ,  $|G_P|$  be its cardinal and

$$\Delta_{G_P} = \{\pi \in \Delta \mid \sum_{s \in G_P} \pi_s = 1\} \quad (13)$$

be the set of probabilities with support in  $G_P$ . If  $P \cap \text{int} \Delta = \emptyset$ , by definition of  $G_P$ ,  $P \subseteq \Delta_{G_P}$  and  $\tilde{P}$  being absolutely continuous with respect to  $P$ ,  $\tilde{P} \subseteq \Delta_{G_P}$ .

From proposition 4,  $S^i \neq \emptyset$  if and only if  $\text{int} \tilde{P}^i \neq \emptyset$  or equivalently if and only if  $L^i = \{0\}$ . If  $P^i \subseteq \Delta_{G_{P^i}}$ , then  $L^i = \{X \in \mathbb{R}^k, \mid x_l = 0, \text{ for all } l \in G_{P^i}\}$ . This leads us to introduce a weaker no-arbitrage price concept due to Werner [22].

**Definition 4** A price vector  $p \in \mathbb{R}^k$  is a "weak no-arbitrage price" for agent  $i$  if  $p \cdot W > 0$  for all  $W \in R^i \setminus L^i$ . A price vector  $p \in \mathbb{R}^k$  is a "weak no-arbitrage price" for the economy if it is a weak no-arbitrage price for each agent.

If  $p$  is a weak no-arbitrage price for  $i$ , then for every  $W \in R^i \cap (L^i)^\perp \setminus \{0\}$  and  $W' \in L^i$ ,  $\alpha W + \beta W' \in R^i \setminus L^i$  for every  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ . Hence  $p \cdot (\alpha W + \beta W') = \alpha p \cdot W + \beta p \cdot W' > 0$  for every  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ . Therefore  $p \cdot W' = 0$  for any  $W' \in L^i$ . In other words a "weak no-arbitrage price" for  $i$  gives 0 value to any useless trade for  $i$ . Let  $S_w^i$  denote the set of weak no arbitrage prices for  $i$ . We have the following characterization of  $S_w^i$  where for a convex subset  $A \subseteq \mathbb{R}^p$ , the relative interior of  $A$ ,  $\text{ri } A$ , is the interior which results when  $A$  is regarded as a subset of its affine hull  $\text{aff}A$ . Let  $I_1 = \{i \mid t^i < 1\}$ ,  $I_2 = \{i \mid t^i = 1\}$  be respectively the set of risk averse and risk neutral agents.

**Proposition 5** Let  $V^i$  fulfill (4) for each  $i$ . Then

1.  $S_w^i = \text{ri} - (R^i)^0 = \text{cone ri } \tilde{P}^i$ .
2. If  $t^i < 1$ ,  $p \in \text{ri} \tilde{P}^i$  if and only if there exists  $\pi^i \in P^i$  with  $\pi_s^i > 0$  for  $s \in G_{P^i}$ ,  $\pi_s^i = 0$  for  $s \notin G_{P^i}$

$$Z^i \in \mathbb{R}^{|G_{P^i}|}, \forall s \in G_{P^i}, a^i < u^{i'}(z_s^i) < b^i, p_s = \frac{\pi_s u^{i'}(z_s^i)}{E_{\pi} u^{i'}(Z^i)} \quad (14)$$

3. The set of weak no arbitrage prices for the economy is  $\cap_i S_w^i = \cap_i \text{ri} - (R^i)^0 = \text{cone } \cap_i \text{ri } \tilde{P}^i$ .
4.  $\cap_i S_w^i \neq \emptyset$  if and only if  $G_{P^i}$  is independent of  $i$  ( $G_{P^i} := G$ ) and for  $i \in I_1$ , there exists  $\pi^i \in P^i$  with  $\pi_s^i > 0$  for  $s \in G$ ,  $\pi_s^i = 0$ , for  $s \notin G$ ,  $Z^i \in \mathbb{R}^{|G|}$  with  $u^{i'}(+\infty) < u^{i'}(z_s^i) < u^{i'}(-\infty)$ , for all  $s \in G$  and  $\pi \in \cap_{i \in I_2} \text{ri} P^i$ ,  $\pi_s > 0$  for  $s \in G$  such that, for all  $(i, j) \in (I_1)^2$ ,  $s \in G$

$$\frac{\pi_s^i u^{i'}(z_s^i)}{E_{\pi^i} u^{i'}(Z^i)} = \frac{\pi_s^j u^{j'}(z_s^j)}{E_{\pi^j} u^{j'}(Z^j)} = \pi_s$$

From proposition 5, a necessary condition for existence of weak no-arbitrage prices is that agents agree on the irrelevant states of the world, those which have no weight whatever prior they use.

Let us give simple sufficient conditions that insure the non-emptiness of the set of weak no-arbitrage prices for the economy.

**Corollary 2** 1. If  $\cap_i \text{ri} P^i \neq \emptyset$ , then  $\cap_i S_w^i \neq \emptyset$ .

2. Let  $G_{P^i} = G$  for all  $i$ . If  $t^i = 0$  and  $P^i \cap \text{ri} \Delta_G \neq \emptyset$  for all  $i$ , then  $\cap_i S_w^i = \text{cone } \text{ri} \Delta_G$ .

**Proof:** From assertion 2 of proposition 5 with  $p^i = \pi^i$  and  $Z$  constant, we obtain that  $\text{ri}P_i \subseteq \text{ri}\tilde{P}^i$  which proves the first assertion. To prove the second,  $\text{ri}\tilde{P}^i = \text{ri}\Delta_G$ . Hence  $S_w^i = \text{cone ri}\Delta_G$  for all  $i$  and  $\cap_i S_w^i = \text{cone ri}\Delta_G$ . ■

It follows from propositions 4 and 5 that when agents have utilities of type (4), the more risk averse and the more uncertainty averse are the agents, the larger are the sets of no-arbitrage and weak no-arbitrage prices.

We now turn to concepts of *collective absence of arbitrage*. From now on, a *feasible trade* is an  $m$ -tuple  $W^1, \dots, W^m$  with  $W^i \in \mathbb{R}^k$  for all  $i$  and  $\sum_i W^i = 0$ .

Let us recall the *no-unbounded-arbitrage* condition (NUBA) introduced by Page [17] which requires inexistence of unbounded feasible trades which are utility nondecreasing and the Weak No-Market-Arbitrage condition (WNMA), introduced by Hart [13] which requires that feasible trades which are utility nondecreasing be useless.

**Definition 5** 1. *The economy satisfies the NUBA condition if  $\sum_i W^i = 0$  and  $W^i \in R^i$  for all  $i$ , implies  $W^i = 0$  for all  $i$ .*

2. *The economy satisfies WNMA if  $\sum_i W^i = 0$  and  $W^i \in R^i$  for all  $i$  implies  $W^i \in L^i$ , for all  $i$ .*

From proposition 3, we may now characterize the NUBA and WNMA conditions.

**Proposition 6** 1. *NUBA is equivalent to: there exists no feasible trade  $W^1, \dots, W^m$  with  $W^i \neq 0$  for some  $i$  that fulfills  $E_\pi(W^i) \geq 0$  for all  $\pi \in \tilde{P}^i$  and for all  $i$ .*

2. *WNMA condition is equivalent to: there exists no feasible trade  $W^1, \dots, W^m$  that fulfills  $E_\pi(W^i) \geq 0$  for all  $\pi \in \tilde{P}^i$  and for all  $i$  with a strict inequality for some  $i$  and  $\pi \in \tilde{P}^i$ .*

In the risk neutral case, since  $P^i = \tilde{P}^i$  for all  $i$ , the concepts introduced in this section take a simpler form that we next summarize.

**Corollary 3** *The risk neutral case. Let  $V^i$  fulfill (5) for each  $i$ . Then*

1. *the set of non arbitrage prices for agent  $i$  is  $S^i = \text{cone int } P^i$ ,*
2. *the set of non arbitrage prices for the economy is:  $\cap_i S^i = \text{cone } \cap_i \text{int } P^i$ ,*
3. *the set of weak non arbitrage prices for agent  $i$  is  $S_w^i = \text{cone ri } P^i$ ,*
4. *The set of weak no arbitrage prices for the economy is  $\cap_i S_w^i = \text{cone } \cap_i \text{ri } P^i$ ,*

5. *NUBA* : there exists no feasible trade  $W^1, \dots, W^n$ ,  $W^i \neq 0$  for some  $i$  with  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$ ,
6. *WNMA*: there exists no feasible trade  $W^1, \dots, W^n$ , with  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .

## 4 Existence of efficient allocations

### 4.1 Necessary and sufficient conditions for existence of efficient allocations

We first characterize the existence of efficient allocations and of equilibria under the condition that the utilities do not contain half-lines (or weak half-lines). They follow from theorem 3 and 4 in the appendix and propositions 3, 4 and 5. The no half-line condition implies in particular that  $P^i \cap \text{int}\Delta \neq \emptyset$  for all  $i$ .

**Definition 6** *Let  $V$  fulfill (4). A trade  $W \in \mathbb{R}^k \setminus \{0\}$  is a half-line if there exists  $X \in \mathbb{R}^k$  such that  $V(X + \lambda W) = V(X)$  for all  $\lambda \geq 0$ .*

We shall use the following notation: for a given  $X \in \mathbb{R}^k$ , let

$$P(X) = \{\pi \in P \mid V(X) = E_\pi(u(X)) + c(\pi)\}$$

be the set of minimizing probabilities at  $X$ .

**Lemma 3** 1. *Let  $V$  fulfill (5). Then  $V$  has no-half line if and only if  $P(X) \subseteq \text{int}P$  for any  $X \in \mathbb{R}^k$ .*

2. *Let  $V$  fulfill (4). Assume that  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$  and that  $a < u'(x)$  for all  $x$  or  $u'(x) < b$  for all  $x$  (no risk neutrality at infinity). Then  $V$  has no-half line.*

3. *If  $V$  has no-half line, then  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$ . If  $V$  fulfills (2) and has no-half line, then  $P \subseteq \text{int}\Delta$ .*

4. *Let  $V(X) = E_\pi(u(X))$ . Then  $V$  has no-half line if and only if  $\pi \in \text{int}\Delta$  and  $a < u'(x)$  for all  $x$  or  $u'(x) < b$  for all  $x$ .*

When there is no risk aversion, the no half-line condition is fulfilled for example in the case of multiplier utilities but it is not fulfilled for utilities of the type  $V(X) = \min_{\pi \in P} E_\pi(X)$  since minimizing probabilities are at the boundary of  $P$ . When there is risk aversion, the no half-line condition is fulfilled for Gilboa-Schmeidler's utilities  $V(X) = \min_{\pi \in P} E((u(X)))$  if  $P \subseteq \text{int}\Delta$  and the agent is not risk neutral at infinity.

Strictly concave utilities have no half-lines. Let us therefore give necessary and sufficient conditions for the strict concavity of  $V$ .



**Lemma 4** *Let  $V$  fulfill (4). Then  $V$  is strictly concave if and only if  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$  and  $u$  is strictly concave. If  $V$  fulfills(2), then  $V$  is strictly concave if and only if  $u$  is strictly concave and  $P \subseteq \text{int}\Delta$ .*

**Theorem 1** *Let  $V^i$  fulfill (4) or (5) and have no-half line for all  $i$ . Let  $I_1 = \{i \mid t^i < 1\}$  be the set of risk averse agents and  $I_2 = \{i \mid t^i = 1\}$  the set of risk neutral agents. Then the following are equivalent:*

1.  $\bigcap_i \text{int} \tilde{P}^i \neq \emptyset$  with  $\tilde{P}^i = P^i$  for any  $i \in I_2$ ,
2. For any  $i \in I_1$ , there exist  $\pi^i \in P^i \cap \text{int} \Delta$ , and  $Z^i \in \mathbb{R}^k$  with  $a^i < u^{i'}(z_s^i) < b^i$  for all  $s$  and  $\pi \in \bigcap_{i \in I_2} \text{int} P^i$  such that for all  $i \in I_1, j \in I_2, s = 1, \dots, k$ , we have

$$\frac{\pi_s^i u^{i'}(z_s^i)}{E_{\pi^i} u^{i'}(Z^i)} = \frac{\pi_s^j u^{j'}(z_s^j)}{E_{\pi^j} u^{j'}(Z^j)} = \pi_s$$

3. there exists no feasible trade  $W^1, \dots, W^m$  with  $W^i \neq 0$  for some  $i$  and  $E_{\pi}(W^i) \geq 0$  for all  $\pi \in \tilde{P}^i$  and for all  $i$ ,
4. there exists an individually rational efficient allocation for any distribution of initial endowments,
5. there exists an equilibrium for any distribution of initial endowments.

**Corollary 4** *Under the hypotheses of theorem 1, any equilibrium price is a no-arbitrage price.*

**Proof:** Indeed if  $(p^*, (X^{*i})_{i=1}^m)$  is an equilibrium, then, from the first order conditions, there exists  $\lambda$  and  $\pi \in \bigcap_i P^i(X^{*i})$  such that  $\lambda p_s^* = \pi_s \frac{u^{i'}(x_s^{*i})}{E_{\pi}(u^{i'}(X^{*i}))}$ . From the proof of lemma 3,  $p^* \cdot W > 0$  for any  $W$  useful. Hence  $p^*$  is a no-arbitrage price. ■

**Remark 2** *Under the hypotheses of theorem 1, if there exists an efficient allocation (or an equilibrium) for some distribution of initial endowments, then there exists an efficient allocation (an equilibrium) for any distribution of initial endowments.*

**Proof:** Indeed if  $(\bar{X}^i)_{i=1}^m$  is an efficient allocation for some distribution of initial endowments, then,  $\bigcap_i \tilde{P}^i(\bar{X}^i) \neq \emptyset$  and is a no-arbitrage price for the economy. Hence assertion 1 of theorem 1 is fulfilled. ■

We now assume that  $P^i \cap \text{int}\Delta = \emptyset$  for some  $i$ . From section 3.2, if  $V$  fulfills (4) and  $P \cap \text{int}\Delta = \emptyset$ , then the set  $G_P^c = \{l \mid \pi_l = 0, \text{ for all } \pi \in P\} \neq \emptyset$  and  $L = \{X \in \mathbb{R}^k \mid x_s = 0, \text{ for all } s \in G_P\}$ .

**Definition 7** Let  $V$  fulfill (4). A trade  $W \in \mathbb{R}^k$  is a weak half-line if  $V(X + \lambda W) = V(X)$  for all  $\lambda \geq 0$  for some  $X \in \mathbb{R}^k$  implies  $W \in L$ .

As in lemma 3, we obtain:

**Lemma 5** 1. Let  $V$  fulfill (5). Then  $V$  has no weak half line if and only if  $P(X) \subseteq riP$  for any  $X \in \mathbb{R}^k$ .

2. Let  $V$  fulfill (4). Assume that  $a < u'(x)$  for all  $x$  or  $u'(x) < b$  for all  $x$  (no risk neutrality at infinity). Then  $V$  has no weak half-line.

The following theorem follows directly from theorem 4 in the appendix. The notations are as in theorem 1.

**Theorem 2** Let  $V^i$  fulfill (4) or (5) and have no weak half-line for all  $i$ . The following are equivalent:

1.  $\cap_i ri(\tilde{P}^i) \neq \emptyset$  with  $\tilde{P}^i = P^i$  for any  $i \in I_2$ ,
2.  $G_{P^i} = G$  and for  $i \in I_1$ , there exists  $\pi^i \in P^i \cap ri\Delta_{G^i}$ ,  $Z^i \in \mathbb{R}^{|G|}$  with  $u^{i'}(+\infty) < u^{i'}(z_s^i) < u^{i'}(-\infty)$ , for all  $s \in G$  and  $\pi \in \cap_{i \in I_2} ri P^i$  such that, for all  $(i, j) \in (I_1)^2$ ,  $s \in G$

$$\frac{\pi_s^i u^{i'}(z_s^i)}{E_{\pi^i} u^{i'}(Z^i)} = \frac{\pi_s^j u^{j'}(z_s^j)}{E_{\pi^j} u^{j'}(Z^j)} = \pi_s$$

3. there exists no feasible trade  $W^1, \dots, W^m$  fulfilling  $E_\pi(W^i) \geq 0$  for all  $\pi \in \tilde{P}^i$  and for all  $i$  with a strict inequality for some  $i$  and  $\pi \in \tilde{P}^i$ .  
Any of the above conditions implies that
4. there exists an individually rational efficient allocation for any distribution of initial endowments,
5. there exists an equilibrium for any distribution of initial endowments.

If furthermore  $V^i$  has no weak half-line for all  $i$ , all of the above assumptions are equivalent and any equilibrium price is a weak no-arbitrage price.

**Remark 3** Theorem 2 can be obtained from Theorem 1. Let us sketch the argument involved. From proposition 5 statement 4, weak no-market arbitrage implies that  $G_{P^i} = G$  for all  $i$ , that  $P^i \subseteq \Delta_G$  for all  $i$  and that  $L^i = \{X \in \mathbb{R}^k \mid x_s = 0, \forall s \in G\}$  for all  $i$ . Consider the economy  $\tilde{E}$  with consumption set equal to  $\mathbb{R}^G$ , utilities

$$\tilde{V}^i(X) = \min_{\pi \in P^i} \left\{ \sum_{s \in G} \pi_s^i u^i(x_s) + c^i(\pi) \right\}$$

and initial endowments  $\tilde{e}_s^i = e_s^i$  for all  $s \in G$ . Note that  $V^i$  has no weak half line if and only if  $\tilde{V}^i$  has no half line. Moreover weak no-market arbitrage for the original economy is equivalent to no-market arbitrage for economy  $\tilde{E}$ . From theorem 1, if  $V^i$  has no weak half line, then  $\tilde{V}^i$  has no half line and weak no-market arbitrage is equivalent to the existence of an equilibrium for  $\tilde{E}$ . Let  $((\tilde{x}^{*i}), \tilde{p}^*)$  be an equilibrium of  $\tilde{E}$ . One can check that  $((x^{*i}), p^*)$  defined by

1.  $x_s^{*i} = \tilde{x}_s^{*i}$ ,  $s \in G$  and  $\sum_i x_s^{*i} = \sum_i e_s^i$ ,  $s \notin G$ ,
2.  $p_s^* = \tilde{p}_s^*$ ,  $s \in G$ , and  $p_s^* = 0$ ,  $s \notin G$ ,

is an equilibrium for the initial economy.

Conversely, if  $((x^{*i}), p^*)$  is an equilibrium for the initial economy, then  $p_s^* = 0$ ,  $s \notin G$  (useless vectors have zero price) and  $((\tilde{x}^{*i}), \tilde{p}^*)$  defined by  $\tilde{x}_s^{*i} = x_s^{*i}$ ,  $s \in G$ ,  $\tilde{p}_s^* = p_s^*$ ,  $s \in G$  is an equilibrium for  $\tilde{E}$ .

Theorem 2 may in particular be applied to the case of risk neutral agents.

**Corollary 5** *The risk neutral case. Let  $V^i$  fulfill (5) for each  $i$ . Then the following are equivalent:*

1.  $\cap_i ri P^i \neq \emptyset$ ,
2. there exists no feasible trade  $W^1, \dots, W^m$  fulfilling  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and for all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .

Any of the above conditions imply that

3. there exists an individually rational efficient allocation for any distribution of initial endowments,
4. there exists an equilibrium for any distribution of initial endowments.

If furthermore  $V^i$  has no weak half-line for all  $i$ , all of the above assumptions are equivalent and any equilibrium price is a weak no-arbitrage price.

It follows from theorems 1 and 2 that if agents have utilities of type (4), the more risk averse and the more uncertainty averse are the agents, the more likely are efficient allocations to exist.

## 4.2 Necessary conditions for existence of efficient allocations

This subsection is of interest only if  $V^i$  has a weak half-line for some  $i$ . We give necessary conditions for existence of an efficient allocations or of an equilibrium for some aggregate endowment  $E$ .

**Proposition 7** *Let  $V^i$  fulfill (4) for each  $i$ . If there exists an efficient allocation for some distributions of endowments  $(E^i)_{i=1}^m$ , then*

1.  $\bigcap_i \tilde{P}^i \neq \emptyset$ ,
2. there exist no feasible trade  $W^1, \dots, W^n$  fulfilling  $E_\pi(W^i) > 0$  for all  $\pi \in \tilde{P}^i$ ,
3. For any distribution of endowments  $(E^i)_{i=1}^m$ , the individually rational utility set  $U((E^i)_{i=1}^m)$  is bounded.

In particular, in the case of risk neutral agents, we obtain the following results:

**Proposition 8** *Let  $V^i$  fulfill (5) for each  $i$ . If there exists an efficient allocation for some distributions of endowments  $(E^i)_{i=1}^m$ , then*

1.  $\bigcap_i P^i \neq \emptyset$ ,
2. there exists no feasible trade  $W^1, \dots, W^n$  fulfilling  $E_\pi(W^i) > 0$  for all  $\pi \in P^i$  and for all  $i$ ,
3. For any distribution of endowments  $(E^i)_{i=1}^m$ , the individually rational utility set  $U((E^i)_{i=1}^m)$  is bounded.

**Corollary 6** *If agents have a single prior and are risk neutral, there exists an efficient allocation for any distribution of initial endowments iff agents have same prior.*

### 4.3 Some examples

The purpose of this subsection is double. Since the necessary and sufficient conditions that we provided for existence of efficient allocations in theorems 1 and 2 were expressed in terms of the risk adjusted sets of priors, our first purpose is to provide sufficient conditions for existence of efficient allocations in terms of priors and utility indices. Our second purpose is to show that the assumption of a common prior is neither sufficient nor necessary for existence of efficient allocations.

The next corollary follows directly from corollary 2.

**Corollary 7** *Assume that  $V^i$  fulfills (4) for all  $i$  and that  $\bigcap_i \text{ri } P^i \neq \emptyset$ . Then there exists efficient allocations. In particular, if  $P^i$  is independent of  $i$ , there exists equilibria for any distribution of endowments  $(E^i)_{i=1}^m$ .*

The first assertion follows from corollary 2, the second from the fact that any convex set has non empty relative interior.

In the previous corollary, we assumed existence of a common prior. The next proposition which follows directly from theorems 1 and 2 and corollaries 1 and 2 shows that if agents are all infinitely risk averse (in other words, if  $t^i = 0$  for any  $i$ ), then the assumption of a common prior is nor sufficient nor necessary for existence of efficient allocations.

**Proposition 9** *Assume that  $V^i$  fulfills (4) with  $t^i = 0$  for all  $i$ .*

1. *Assume that  $P^i \cap \text{int}\Delta \neq \emptyset$  for all  $i$ . Then  $\bigcap_i \text{int} \tilde{P}^i = \text{int}\Delta$  and there exists equilibria for any distribution of endowments  $(E^i)_{i=1}^m$ .*
2. *Let  $G_{P^i}$  be independent of  $i$  and  $G = G_{P^i}$ . Assume that  $P^i \cap \text{ri}\Delta_G \neq \emptyset$  for all  $i$ . Then there exist equilibria for any distribution of endowments  $(E^i)_{i=1}^m$ .*

The previous proposition applies in particular to the case of agents with single heterogeneous beliefs.

**Corollary 8** *Let  $V^i$  fulfill (4) with  $t^i = 0$  and  $P^i = \{\pi^i\}$  with  $\pi^i \in \text{int}\Delta$  for all  $i$  and  $\pi^i \neq \pi^j$  for all  $(i, j)$ . Then  $\bigcap_i \text{int} \tilde{P}^i = \text{int}\Delta$  and there exists equilibria for any distribution of endowments  $(E^i)_{i=1}^m$ .*

#### 4.4 Characterization of efficiency

We end this section by comparing our existence results that hold for any distribution of endowments with the characterization of efficient allocations for given aggregate endowments.

**Proposition 10** *Let  $V^i$  fulfill (4) for all  $i$  and aggregate endowment  $E \in \mathbb{R}^k$  be given. The allocation  $(\bar{X}^i)_{i=1}^m$  is efficient if  $\sum_{i=1}^m \bar{X}^i = E$  and any of the equivalent following conditions are fulfilled:*

1.  $\bigcap_i \tilde{P}^i(\bar{X}^i) \neq \emptyset$ ,
2. *there exists no feasible trade  $(W^i)_{i=1}^m$  such that  $E_\pi W^i > 0$  for all  $\pi \in \tilde{P}^i(\bar{X}^i)$  and all  $i$ .*

Let us compare assertion 1 of proposition 10 with assertion 2 of theorem 1 (or with assertion 2 of theorem 2). In proposition 10,  $(\bar{X}^i)_{i=1}^m$  is an attainable allocation for aggregate endowment  $E$  while in assertion 2 of theorem 1,  $(Z^i)_{i=1}^m$  is not necessarily an attainable allocation. In proposition 10, the common risk adjusted probability is minimizing at  $\bar{X}^i$  for each  $i$  but it need not be in  $\text{int}\Delta$  (it is if for some agent  $V^i$  contains no half-line) while in assertion 2 of theorem 1,  $\pi^i \in \text{int}\Delta$  for all  $i$  but  $\pi^i$  need not be in  $P^i(Z^i)$ . It is also worth noticing that in assertion 2 of proposition 10, we have strict inequalities for all probabilities and for all agents as in the no trade literature while in assertion 3 of theorem 1 or 2 we have strict inequality for some agent and some of her prior.

## 5 Proofs

### 5.1 Proof of proposition 1

Let  $W$  fulfill  $E_\pi(W) \geq 0$  for all  $\pi \in P$ . Then

$$V(\lambda W) = \min_{\pi \in P} aE_\pi(\lambda W) + c(\pi) \geq \min_{\pi \in P} c(\pi) = 0 \text{ for all } \lambda \geq 0$$

which implies that  $W \in R$ . Conversely, let  $W \in R$ . Then

$$V(\lambda W) \geq V(0) = 0, \text{ for all } \lambda \geq 0,$$

hence  $aE_\pi(\lambda W) + c(\pi) \geq 0$ , for all  $\lambda \geq 0$ ,  $\pi \in P$ . For a fixed  $\pi \in P$ , the map  $\lambda \rightarrow \lambda E_\pi(W)$  is bounded below, hence  $E_\pi(W) \geq 0$ . The proofs of the other assertions are straightforward.

### 5.2 Proof of lemma 1

For any  $\eta = (\pi, Z)$ , let  $\gamma(\eta) = E_\pi u(Z) - E_\pi(u'(Z)Z) + c(\pi)$ . We then have

$$\begin{aligned} (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} &= \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) \\ &\geq \min_\pi \left\{ \min_Z \left\{ \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) \right\} \right\} \\ &= \min_\pi \left\{ \sum_s \pi_s u(x_s) + c(\pi) \right\} = V(X) \end{aligned}$$

where the last equality follows from (6). Hence

$$\min_\eta \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\} \geq V(X)$$

Conversely, by definition of  $\gamma$ , we have for any  $\pi \in P$

$$\begin{aligned} \sum_s \pi_s u(x_s) + c(\pi) &= \sum_s \pi_s u'(x_s) x_s + \gamma(\pi, X) \\ &\geq \min_{(\pi, Z)} \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) \end{aligned}$$

hence,

$$V(X) \geq \min_{(\pi, Z)} \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) = \min_\eta \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\}$$

proving lemma 1.

### 5.3 Proof of proposition 2

To prove that  $P \subseteq \tilde{P}$ , it suffices to take  $Z$  constant in (8).

To prove assertion 2, let

$$\tilde{Q} = \left\{ \lambda(\pi_s u'(z_s))_s ; \lambda \geq 0, \pi \in P, Z \in \mathbb{R}^k \right\}$$

be the cone generated by  $\tilde{P}$ . Since  $\tilde{P} = \tilde{Q} \cap \Delta$ , it suffices to prove that  $\tilde{Q}$  is convex. To this end, let  $\lambda(\pi_s u'(z_s))_s \in \tilde{Q}$ ,  $\lambda'(\pi'_s u'(z'_s))_s \in \tilde{Q}$  and  $\alpha \in (0, 1)$ . Then for any  $s$ ,

$$\begin{aligned} (\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(+\infty) &\leq \alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s) \\ &\leq (\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(-\infty). \end{aligned}$$

Hence, there exists  $\zeta_s$  which satisfies

$$(\alpha\lambda\pi_s + (1-\alpha)\lambda'\pi'_s)u'(\zeta_s) = \alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s).$$

Define  $\nu = \frac{\alpha\lambda\pi + (1-\alpha)\lambda'\pi'}{\alpha\lambda + (1-\alpha)\lambda'}$ . Then,  $\nu \in P$  and for any  $s$ ,

$$\alpha\lambda\pi_s u'(z_s) + (1-\alpha)\lambda'\pi'_s u'(z'_s) = (\alpha\lambda + (1-\alpha)\lambda')\nu_s u'(\zeta_s).$$

proving the convexity of  $\tilde{Q}$ .

To prove assertion 3, from its definition  $\tilde{P} \subseteq \{p \in \Delta \mid \exists \pi \in P, \pi \simeq p\}$ . If  $u'(\infty) = 0$  or  $u'(-\infty) = +\infty$ , then for any  $p \simeq \pi$  with  $\pi \in P$ , there exists  $\lambda > 0$  such that

$$u'(\infty) < \lambda \frac{p_s}{\pi_s} < u'(-\infty)$$

and thus, there exists  $Z \in \mathbb{R}^k$  such that, for all  $s \in I_\pi$ ,  $\frac{p_s}{\pi_s} = \frac{u'(z_s)}{E_\pi u'(Z)}$ , hence  $\tilde{P} = \{p \in \Delta \mid \exists \pi \in P, \pi \simeq p\}$ . If there exists  $\pi \in P \cap \text{int } \Delta$ , then  $\tilde{P}$  contains  $\text{int } \Delta$  the set of strictly positive probabilities which are equivalent to  $\pi$ .

To prove the last assertion, clearly the more ambiguous the agent, the larger is  $P$  and hence the larger is  $\tilde{P}$ . Let us show that the more risk averse the agent, the larger is  $\tilde{P}$ . Indeed, if  $v$  is more risk averse than  $u$ , then from Arrow's Pratt theorem,  $v = \psi \circ u$  with  $\psi$  concave. Let  $\tilde{P}_u$  and  $\tilde{P}_v$  be the sets of risk adjusted priors associated to  $u$  and  $v$ . Assume that  $p \in \tilde{P}_u$ . Then there exists  $\pi$  and  $Z \in \mathbb{R}^k$  such that, for all  $s \in I_\pi$ ,

$$u'(+\infty) \leq \frac{p_s}{\pi_s} E_\pi u'(Z) \leq u'(-\infty) \quad (15)$$

If  $v'(+\infty) = 0$  or if  $v'(-\infty) = \infty$ , then from assertion 2,

$\tilde{P}_v = \{p \in \Delta \mid \exists \pi \in P, \pi \simeq p\}$ , hence  $p \in \tilde{P}_v$ . Let us therefore assume that  $0 < v'(+\infty) < v'(-\infty) < \infty$ . We first obtain from (15) that

$$1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} \leq \frac{u'(-\infty)}{u'(+\infty)}$$

Since  $0 < v'(+\infty)$  and  $v'(+\infty) = u'(+\infty)\psi'(u(+\infty))$ , we have  $u'(+\infty) > 0$  and  $\psi'u(+\infty) > 0$ , therefore

$$\frac{v'(-\infty)}{v'(+\infty)} = \frac{\psi'(u(-\infty))u'(-\infty)}{\psi'(u(+\infty))u'(+\infty)} > \frac{u'(-\infty)}{u'(+\infty)}$$

since  $\psi$  is concave but not linear on  $u(\mathbb{R})$ . Hence

$$1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} < \frac{v'(-\infty)}{v'(+\infty)}$$

Let  $\lambda = \frac{v'(+\infty)E_\pi u'(Z)}{u'(+\infty)}$ , we obtain that

$$v'(+\infty) < \lambda \frac{p_s}{\pi_s} < v'(-\infty)$$

and thus, there exists  $Z' \in \mathbb{R}^k$  such that, for all  $s \in I_\pi$ ,  $\frac{p_s}{\pi_s} = \frac{v'(z'_s)}{E_\pi v'(Z')}$  which proves that  $p \in \tilde{P}_v$ .

#### 5.4 Proof of proposition 3

We first prove assertion 1. From (7), if  $W$  is a useful vector, then for any  $(\pi, Z)$  and any  $\lambda > 0$ , we have:

$$(E_\pi u'(Z)) \left\{ \sum_s \left( \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} (\lambda w_s) \right) + \frac{\gamma(\pi, Z)}{E_\pi u'(Z)} \right\} \geq V(0).$$

Dividing by  $\lambda$  and letting  $\lambda$  go to  $+\infty$ , we obtain:

$$\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0, \forall (\pi, Z)$$

Conversely, if  $\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0, \forall (\pi, Z)$ , then for any  $\lambda > 0$ ,

$$(E_\pi u'(Z)) \left\{ \sum_s \left( \frac{\lambda w_s \pi_s u'(z_s)}{E_\pi u'(Z)} \right) + \frac{\gamma(\pi, Z)}{E_\pi u'(Z)} \right\} \geq \gamma(\pi, Z)$$

and hence  $V(\lambda W) \geq V(0), \forall \lambda \geq 0$  and  $W$  is useful.

Assertion 2 follows from assertion 3 of proposition 2. Assertion 3 is obvious.

#### 5.5 Proof of proposition 4

In order to determine no-arbitrage prices and prove proposition 4, we need to characterize  $\text{int } \tilde{P}$ . In the case  $t = 1$ , we have  $\text{int } \tilde{P} = \text{int } P$ . In the next lemma, we characterize  $\text{int } \tilde{P}$  in the case  $t < 1$ .

**Lemma 6** *Let  $V$  fulfill (4) with  $t < 1$ . Then  $p \in \text{int } \tilde{P}$  if and only if it satisfies*

$$\exists \pi \in P \cap \text{int } \Delta, Z \in \mathbb{R}^k, \text{ s.t. } \forall s, a < u'(z_s) < b, \text{ and } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} \quad (12)$$



**Proof:** Let us first show that if  $p$  satisfies (12), then  $p \in \text{int} \tilde{P}$ . Indeed, we have  $p_s = \frac{\pi_s u'(z_s)}{E_{\pi} u'(Z)}$  for any  $s$ . For any  $\varepsilon \in \mathbb{R}$  close to 0, we can find  $z'_s$  such that  $\frac{p_s + \varepsilon}{1 + k\varepsilon} = \frac{\pi_s u'(z'_s)}{E_{\pi} u'(Z')}$ . Indeed, since  $a < \frac{(p_s + \varepsilon) E_{\pi} u'(Z)}{\pi_s (1 + k\varepsilon)} < b$ , there exists  $z'_s$  such that  $\frac{(p_s + \varepsilon) E_{\pi} u'(Z)}{\pi_s (1 + k\varepsilon)} = u'(z'_s)$ . We then have  $E_{\pi} u'(Z) = E_{\pi} u'(Z')$  which implies that  $\frac{p_s + \varepsilon}{1 + k\varepsilon} = \frac{\pi_s u'(z'_s)}{E_{\pi} u'(Z')}$ . In other words, there exists an open set containing  $p$  which is included in  $\tilde{P}$ . Hence,  $p \in \text{int} \tilde{P}$ .

Since  $\tilde{P}$  is convex, to prove the converse, from Rockafellar's theorem 6.4,  $p \in \text{int} \tilde{P}$  if and only if, for every  $p' \in \tilde{P}$ , there exists  $p'' \in \tilde{P}$  such that  $p = \alpha p'' + (1 - \alpha)p'$  with  $\alpha \in (0, 1)$ . Consider a  $p'$  that verifies (12). Let  $\lambda' = \frac{1}{E_{\pi'} u'(Z')}$  and  $\lambda'' = \frac{1}{E_{\pi''} u'(Z'')}$ . From the proof of Proposition 2 assertion 2, we have that  $p_s = \frac{u'(z_s) \pi_s}{E_{\pi} u'(Z)}$  with

$$\pi_s = \frac{\alpha \lambda'' \pi_s'' + (1 - \alpha) \lambda' \pi_s'}{\alpha \lambda'' + (1 - \alpha) \lambda'}$$

$$a < u'(z_s) = \frac{\alpha \lambda'' \pi_s'' u'(z_s'') + (1 - \alpha) \lambda' \pi_s' u'(z_s')}{\alpha \lambda'' \pi_s'' + (1 - \alpha) \lambda' \pi_s'} < b$$

Since  $\pi' \in P \cap \text{int} \Delta$ ,  $\pi \in P \cap \text{int} \Delta$ . Hence (12) is fulfilled.

■

Let us now prove proposition 4. Given a subset  $A$ ,  $\text{cl } A$  denotes the closure of  $A$

To prove assertion 1, from proposition 3,  $R^i = \{W \in \mathbb{R}^k \mid E_{\pi}(W) \geq 0, \text{ for all } \pi \in \tilde{P}^i\}$ . Hence  $(R^i)^0$  is the closed cone generated by  $\tilde{P}^i$ . Since  $\tilde{P}^i$  is convex,  $S^i = \text{int cl cone } \tilde{P}^i$ . Since cone  $\tilde{P}^i$  is convex,  $\text{int cl cone } \tilde{P}^i = \text{int cone } \tilde{P}^i$  and

$$S^i = \text{int cone } \tilde{P}^i = \text{cone int } \tilde{P}^i.$$

The first part of assertion 2, follows from lemma 6.

To prove that  $\text{int} \tilde{P}^i \neq \emptyset$  if and only if  $P^i \cap \text{int} \Delta \neq \emptyset$ , assume first  $\text{int} \tilde{P}^i \neq \emptyset$ . From (8),  $P^i \cap \text{int} \Delta \neq \emptyset$ . Conversely, if  $P^i \cap \text{int} \Delta \neq \emptyset$ , let  $\pi \in P^i \cap \text{int} \Delta$ , then  $\pi \in \text{int} \tilde{P}^i$ .

If  $t^i = 1$ , then  $\text{int} \tilde{P}^i = \text{int } P^i$ .

To prove assertion 3, the set of no-arbitrage prices for the economy

$$\bigcap_i S^i = \bigcap_i \text{int cone } \tilde{P}^i = \text{cone } \bigcap_i \text{int } \tilde{P}^i$$

The last assertion follows from assertions 2 and 3.

## 5.6 Proof of proposition 5

We first characterize  $\text{ri} \tilde{P}$ .

**Lemma 7** *Let  $V$  fulfill (4) with  $t < 1$ . Then  $p \in \text{ri}\tilde{P}$  if and only*

$$\exists \pi \in P \cap \text{ri}\Delta_{G_P}, \exists Z \in \mathbb{R}^k, \forall s \in G_P, a < u'(z_s) < b, \text{ and } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}$$

*If  $t = 0$ , then  $\text{ri}\tilde{P} = \text{ri}\Delta_{G_P}$ .*

**Proof:** Observe that,  $p \in \text{ri}\tilde{P}$  iff  $p_l > 0$  iff  $l \in G_P$ , hence  $\pi_l > 0$  iff  $l \in G_P$ . Without loss of generality, one can assume that  $G_P = \Omega$  and be reduced to lemma 6. ■

Let us now prove proposition 5.

To prove the first assertion, from Allouch et al [1], lemma 2, we have that  $S_w^i = \text{ri} - (R^i)^0$ . Hence the set of weak no arbitrage prices for the economy is  $\cap_i S_w^i = \cap_i \text{ri}(-(R^i)^0)$ . From Rockafellar's [20] theorem 6.3 and corollary 6.6.1, we further obtain

$$S_w^i = \text{ri} - (R^i)^0 = \text{ri cl cone } \tilde{P}^i = \text{ri cone } P^i = \text{cone ri}\tilde{P}^i$$

The second assertion follows from lemma 7. From assertion 1, the set of weak no arbitrage prices for the economy is

$$\cap_i S_w^i = \cap_i \text{ri} - (R^i)^0 = \cap_i \text{ri}(\text{cone}\tilde{P}^i) = \cap_i \text{cone}(\text{ri}\tilde{P}^i) = \text{cone} \cap_i \text{ri}\tilde{P}^i$$

Let us finally prove that  $\cap_i S_w^i \neq \emptyset$  implies that  $G_{P^i} = G_{P^j}$  for any  $i, j$ . Indeed  $\Delta_{G_{P^i}} = \Delta_{G_{P^j}}$  if and only if  $G_{P^i} = G_{P^j}$ . Furthermore  $\Delta_{G_{P^i}} \neq \Delta_{G_{P^j}}$  if and only if  $\text{ri}\Delta_{G_{P^i}} \cap \text{ri}\Delta_{G_{P^j}} = \emptyset$ . Indeed if  $\pi \in \text{ri}\Delta_{G_{P^i}} \cap \text{ri}\Delta_{G_{P^j}}$ , then  $\pi_s > 0, \forall s \in G_{P^i} \cup G_{P^j}$  and  $\sum_{s \in \Delta} \pi_s \geq \sum_{s \in G_{P^i} \cup G_{P^j}} \pi_s > \sum_{s \in G_{P^i}} \pi_s = 1$ , hence a contradiction. The converse is obvious. Hence  $\cap_i S_w^i \neq \emptyset$  implies  $\Delta_{G_{P^i}} = \Delta_{G_{P^j}}$ . The remaining part of the proposition follows directly from lemma 7.

## 5.7 Existence of equilibrium theorems

### 5.7.1 A review of existence of equilibrium theorems

In order to prove theorems 1 and 2, we start this section by recalling two theorems on existence of equilibrium with short-selling.

**Theorem 3** *Let  $V^i$  fulfill (4) for each  $i$ . Then the following assertions are equivalent:*

1.  $\cap_i S^i \neq \emptyset$
2. NUBA is fulfilled,
3. the set of individually rational attainable allocations  $A$  is compact.

*Any of the previous assertions implies any of the following assertions:*

4. *there exists an individually rational efficient allocation for any distribution of initial endowments,*

5. *there exists an equilibrium for any distribution of initial endowments.*

*If  $V^i$  has no half-line for every  $i$ , then assertions 1-5 are equivalent.*

**Proof:** See e.g. Page and Wooders [19], Dana et al [5]. ■

**Theorem 4** *Let  $V^i$  fulfill (4) for each  $i$ . Then the following equivalent assertions are equivalent.*

1.  $\bigcap_i S_w^i \neq \emptyset$

2. *WNMA is fulfilled.*

*Any of the previous assertions implies any of the following assertions:*

3. *The individually rational utility set  $U$  is compact,*

4. *there exists an individually rational efficient allocation for any distribution of initial endowments,*

5. *there exists an equilibrium for any distribution of initial endowments.*

*If  $V^i$  has no weak half-line for every  $i$ , then assertions 1-5 are equivalent.*

**Proof:** See e.g. Page et al [18], Allouch et al [1]. ■

Theorem 3 obviously make stronger requirements than theorem 4. It is particularly useful when the utilities are strictly concave.

### 5.7.2 Proof of lemma 3

Let  $V$  fulfill (5) and have no half-line. Then for every  $X \in \mathbb{R}^k$  and  $W \neq 0$  useful, there exists  $\lambda \geq 0$  such that

$$0 < V(X + \lambda W) - V(X) \leq E_\pi(X + \lambda W - X) = \lambda E_\pi(W)$$

for any  $\pi \in P(X)$ . Hence  $\pi$  is a no-arbitrage price. From corollary 3,  $P(X) \subseteq \text{int}P$  for any  $X \in \mathbb{R}^k$ . Conversely assume that  $P(X) \subseteq \text{int}P$  or equivalently that any  $\pi \in P(X)$  is a no-arbitrage price for any  $X \in \mathbb{R}^k$  and that there is a half-line. Then there exists  $X \in \mathbb{R}^k$  and  $W \neq 0$  useful such that  $V(X + \lambda W) = V(X)$  for all  $\lambda \geq 0$ . Let  $\pi_\lambda \in P(X + \lambda W)$ . We then have

$$0 \geq E_{\pi_\lambda}(X + \lambda W) - X = \lambda E_{\pi_\lambda}(W)$$

contradicting the fact that  $\pi$  is a no-arbitrage price.

Assume that  $V$  fulfill (4) and that  $V$  has a half-line. Then there exists  $X \in \mathbb{R}^k$  and  $W \neq 0$  useful such that  $V(X + \lambda W) = V(X)$  for all  $\lambda \geq 0$ . Let  $\pi_\lambda \in P(X + \lambda W)$ . We then have

$$0 \geq E_{\pi_\lambda}(u(X + \lambda W)) - u(X) \geq E_{\pi_\lambda}(u'(X + \lambda W)\lambda W)$$

Since  $W$  is useful, from proposition 3,  $E_{\pi_\lambda}(u'(X + \lambda W)\lambda W) \geq 0$ , hence

$$E_{\pi_\lambda}(u'(X + \lambda W)W) = 0$$

Assume now that  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$ . Since  $\pi_\lambda \subseteq \text{int}\Delta$  and  $u' > 0$ ,  $W_+ \neq 0$  and  $W_- \neq 0$ . If  $a < u'(x)$  or  $u'(x) < b$  for all  $x$ , then have  $0 > aE_{\pi_\lambda}(W_+) - bE_{\pi_\lambda}(W_-)$  contradicting (11) of remark 1.

Let us now show that if  $V$  has no-half line, then  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$ . Indeed if  $V$  has no-half line, for any  $X \in \mathbb{R}^k$  and any  $W \in \mathbb{R}^k$  useful, there exists  $\lambda > 0$  such that  $0 < V(X + \lambda W) - V(X)$ . Thus for any  $\pi \in P(X)$ , we have

$$0 < V(X + \lambda W) - V(X) \leq E_\pi(u'(X)\lambda W)$$

Hence  $E_\pi(u'(X)W) > 0$  for any  $W$  useful in particular for any  $W \geq 0$ ,  $W \neq 0$ . Hence  $\pi$  is strictly positive. If  $V$  fulfills (2), then any  $\pi \in P$  fulfills  $\pi \in P(a)$ ,  $a \in \mathbb{R}$ , hence  $P \subseteq \text{int}\Delta$ .

Assume that  $V(X) = E_\pi(u(X))$ . From assertion 2,  $\pi \in \text{int}\Delta$  and no risk neutrality is a sufficient condition for no half-line. From assertion 3, if  $V$  has no half-line, then  $\pi \in \text{int}\Delta$  and  $E_\pi(u'(X)W) > 0$  for any non zero useful vector  $W$  and any  $X \in \mathbb{R}^k$ . If there is risk neutrality at infinity, then there exists  $c, d$  such that  $u'(x) = b$  for all  $x \in ]-\infty, d]$  and  $u'(x) = a$  for all  $x \in [c, \infty[$ . Thus we must have

$$aE_\pi(W_+) - bE_\pi(W_+) > 0, \text{ for all } W \neq 0 \text{ useful}$$

However any  $W \neq 0$  such that  $aE_\pi(W_+) - bE_\pi(W_+) = 0$  is useful and violates the strict inequality, hence we obtain a contradiction.

### 5.7.3 Proof of lemma 4

Let us prove that if  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$  and  $u$  is strictly concave, then  $V$  is strictly concave. Indeed, let  $X, Y \in \mathbb{R}^k$ ,  $X \neq Y$ ,  $\lambda \in ]0, 1[$  and  $\pi \in P(\lambda X + (1 - \lambda)Y)$ . We then have

$$\begin{aligned} V(\lambda X + (1 - \lambda)Y) &= E_\pi(u(\lambda X + (1 - \lambda)Y)) + c(\pi) \\ &> \lambda E_\pi(u(X)) + (1 - \lambda)E_\pi(u(Y)) + c(\pi) \\ &\geq \lambda V(X) + (1 - \lambda)V(Y), \end{aligned}$$

proving the desired assertion. Conversely if  $V$  is strictly concave, then restricting attention to constants, we first obtain that  $u$  is strictly concave. As  $V$  has no half-line, from the proof of lemma 3, we obtain that  $P(X) \subseteq \text{int}\Delta$  for any  $X \in \mathbb{R}^k$ . Clearly if  $V$  fulfills (2),  $P \subseteq \text{int}\Delta$ .

## 5.8 Proof of proposition 7

There exists an efficient allocation  $(\bar{X}^i)_{i=1}^m$  for some distribution of endowments if and only if for every  $i$ , there exists  $\lambda^i > 0$ , such that  $\cap_i \lambda^i \delta V^i(\bar{X}^i) \neq \emptyset$ . From Aubin [2] proposition 4.4

$$(\delta V^i(\bar{X}^i))_j = \left\{ u^i(\bar{X}_j^i) \pi_j, \pi \in P^i \mid E_\pi u^i(\bar{X}^i) + c^i(\pi) = \min_{\pi \in P^i} E_\pi u^i(\bar{X}^i) + c^i(\pi) \right\}$$

Therefore for every  $i$ , there exists  $\lambda^i > 0$  and  $\pi^i \in P^i$  such that  $\lambda^i u^i(\bar{X}_j^i) \pi_j^i$  is independent of  $i$  and  $\cap_i \tilde{P}^i(\bar{X}^i) \neq \emptyset$ .

To prove assertion 2, assume on the contrary that there exists an efficient allocation  $(X^i)_{i=1}^m$  for some distribution of endowments and a feasible trade  $W^1, \dots, W^m$  which satisfy  $E_\pi(W^i) > 0$  for all  $i$  and  $\pi \in \tilde{P}^i$ . For any  $i$ , for any  $\pi^i \in P^i$  and  $Z^i \in \mathbb{R}^k$ , we have

$$\sum_s \pi_s^i u^i(z_s^i) w_s^i > 0.$$

In particular, we have, for any  $\pi \in \text{argmin}_{\pi \in P^i} E_\pi(u^i(X^i + W^i)) + c^i(\pi)$ ,

$$V^i(X^i + W^i) - V^i(X^i) \geq E_{\pi^i} u^i(X^i + W^i) W^i > 0$$

contradicting the Pareto optimality of  $(X^i)_{i=1}^m$ .

To prove assertion 3, let  $(E^i)_{i=1}^m$  be fixed. For any  $(z^i) \in U((E^i)_{i=1}^m)$ , there exists  $(X^1, X^2, \dots, X^m) \in A((E^i)_{i=1}^m)$  such that

$$V^i(E^i) \leq z^i \leq V^i(X^i), \text{ for all } i. \quad (16)$$

From assertion 1, if there exist an efficient allocation, there exists  $\tilde{\pi} \in \cap_i \tilde{P}^i$ .

Hence there exists  $(\bar{X}^i, \pi^i)$  with  $\pi^i \in P^i$  such that  $\tilde{\pi}_j := \frac{u^i(\bar{x}_j^i) \pi_j^i}{E_{\pi^i}(u^i(\bar{X}^i))}$  for all  $i$ .

Let us show that if  $(X^1, X^2, \dots, X^m) \in A((E^i)_{i=1}^m)$ , then  $E_{\tilde{\pi}}(X^i)$  is bounded.

We first show that it is bounded below. Indeed

$$\begin{aligned} V^i(X^i) &= \min_{P^i} E_\pi u^i(X^i) + c^i(\pi) \leq E_{\pi^i} u^i(X^i) + c^i(\pi^i) \\ &\leq E_{\pi^i}(u^i(\bar{X}^i)) + E_{\pi^i} \left( u^i(\bar{X}^i)(X^i - \bar{X}^i) \right) + c^i(\pi^i) \\ &= E_{\pi^i}(u^i(\bar{X}^i)) + E_{\pi^i}(u^i(\bar{X}^i)) E_{\tilde{\pi}}(X^i - \bar{X}^i) + c^i(\pi^i) \end{aligned}$$

Thus,

$$m^i = \frac{V^i(E^i) - c^i(\pi^i) - E_{\pi^i}(u^i(\bar{X}^i))}{E_{\pi^i}u^i(\bar{X}^i)} + E_{\bar{\pi}}(\bar{X}^i) \leq E_{\bar{\pi}}(X^i).$$

Since for all  $i$ ,  $E_{\bar{\pi}}(X^i)$  is bounded below by  $m^i$ , it is bounded above by  $M^i = E_{\bar{\pi}}(E) - \sum_{l \neq i} m^l$ . From (16), we thus have

$$z^i \leq V^i(X^i) \leq E_{\pi^i}(u^i(\bar{X}^i)) + (M^i - E_{\bar{\pi}}\bar{X}^i)E_{\pi^i}u^i(\bar{X}^i) + c^i(\pi^i) \text{ for all } i$$

and  $U((E^i)_{i=1}^n)$  is bounded.

## 5.9 Proof of proposition 10

The first assertion is proven as in the proof of proposition 7 assertion 1. The equivalence between the two assertions follows from Samet [21] since  $\tilde{P}^i(\bar{X}^i)$  is compact for every  $i$ .

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