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**A coalition formation value for games with externalities**

Michel GRABISCH, Yukihiro FUNAKI

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# A coalition formation value for games with externalities

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## Abstract

The coalition formation problem in an economy with externalities can be adequately modeled by using games in partition function form (PFF games), proposed by Thrall and Lucas. If we suppose that forming the grand coalition generates the largest total surplus, a central question is how to allocate the worth of the grand coalition to each player, i.e., how to find an adequate solution concept, taking into account the whole process of coalition formation. We propose in this paper the original concepts of scenario-value, process-value and coalition formation value, which represent the average contribution of players in a scenario (a particular sequence of coalitions within a given coalition formation process), in a process (a sequence of partitions of the society), and in the whole (all processes being taken into account), respectively. We give an application to Cournot oligopoly, and two axiomatizations of the scenario-value.

**Keywords:** coalition formation, games in partition function form, solution concept, Cournot oligopoly

**JEL code:** C71

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# 1 Introduction

The coalition formation problem is one of the important issues of economics and game theory, both in cooperative and non-cooperative games. There are several attempts to analyze this problem. Many papers tried to find stable coalition structures in a cooperative game theoretic fashion (see, e.g., Ray and Vohra [17], Diamantoudi and Xue [6], and Funaki and Yamato [9], or a dynamic process using a non-cooperative approach, as Bloch [3]).

If we suppose that forming the grand coalition generates the largest total surplus, it is natural to assume that the grand coalition structure will eventually form after some negotiations. Then, the worth of the grand coalition has to be allocated to the individual players, according to the contribution of each player. The question is how to do that, taking into account the whole process of coalition formation.

In a coalition formation problem, it is important to consider situations with externalities. Typically, a coalition formation problem in an economy with externalities is related to public goods, public bads and common pool resource games. One of the best way to analyze it in game theory is to use games in partition function form (PFF games for short) introduced in Thrall and Lucas [18] (see also Funaki and Yamato [9]). A partition function assigns a worth to each pair consisting of a coalition and a coalition structure which contains that coalition. Such pairs are called *embedded coalitions*.<sup>1</sup> Games in partition function form are considered as a useful extension of classical TU games, since they well capture the externalities in an economy [14].

Then the above problem of allocation of the worth of the grand coalition amounts to defining a suitable solution concept or value for PFF games. For TU games, one of the most well-known solutions is the Shapley value. This solution concept is based on the marginal contribution of players when they enter the game one by one, considering all possible orders. There are already many attempts to define a modification of the Shapley value for PFF games, e.g., by Myerson [16], Bolger [4], Pham Do and Norde [7], de Clippel and Serrano [5], Albizuri et al. [1], Macho-Stadler et al. [15], etc. They proposed several new kinds of null player or dummy player axioms, and carrier axioms, which are extensions of the original axioms in TU games. Then the resulting formulas are averaging of marginal contributions of players when the players enter the game one by one. However, these approaches do not reflect the process of coalition formation from singletons coalition structure to the grand coalition structure. Since in TU games, the process of

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<sup>1</sup>This term is used by de Clippel and Serrano [5], and Macho-Stadler et al. [15] among others, and seems to have become the standard term in this field.

entering of the players is expressed by some order on coalitions, in PFF games the process of coalition formation should concern not only coalitions but also the corresponding coalitions structures, so that it should be expressed by an order on embedded coalitions.

Mathematically, the set of embedded coalitions has a structure much more complicated than the lattice structure of coalitions in a TU game, however, they share similar properties. One of the most interesting property is that all the maximal chains (paths from the minimal element to the maximal element) are of the same length. This means that a coalition formation process from the singleton coalition structure to the grand coalition structure in a PFF game has always the same number of steps. They clearly correspond to the marginal contribution of a player in this coalition formation process, which is a key concept to define the original Shapley value. Hence, roughly speaking, a permutation of players in a TU game corresponds to a maximal chain in the set of embedded coalitions for a PFF game.

Our idea is to take advantage of the structure of embedded coalitions endowed with a suitable ordering, and to follow the original idea of Shapley based on maximal chains. In this way we can propose a new value for PFF games, which is rooted in the process of coalition formation.

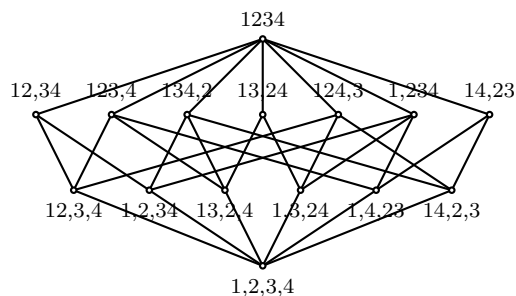
An original feature of our approach is that we define three notions of value, which are in some sense embedded one into the others. The first one, called scenario-value, consider only one given maximal chain in the set of embedded coalitions, i.e., a given scenario of coalition formation, starting from a single player and arriving to the grand coalition. The second one, called process-value, consider all scenarios which follow the same fixed sequence of partitions of the society of players, starting from a society of individual players and arriving to the grand coalition. The third one, which corresponds to the usual notion of value, considers all possible processes (sequences of scenarios). According to the applicative context, the one which best makes sense can be used: in Section 5, we give an application to Cournot oligopoly, where the process-value is the most appropriate.

The rest of the paper is organized as follows. Section 2 introduces embedded coalitions and their structure, and Section 3 gives the definition of scenarios and processes. Section 4 introduces our coalition formation value. First, we introduce scenario-values and process-values, then we show how the classical Shapley value can be recovered and axiomatized by the use of scenario-values. Next, we introduce the axioms giving rise to what we call the egalitarian scenario-value, and we give also a second axiomatization of it. Then we give an explicit expression of the coalition formation value, and finally we give the relation with the Shapley value when the game has no externalities. After the application to Cournot oligopoly given in Section 5,

we end the paper with a comparison of our value with other values of PFF games.

## 2 Partitions and embedded coalitions

Let  $N := \{1, 2, \dots, n\}$  be the set of players. We denote by  $S, T, \dots$ , the subsets of  $N$ , and their cardinality by  $s, t, \dots$ . A subset of  $N$  is called a *coalition*. A partition  $\pi := \{S_1, \dots, S_k\}$  is a collection of disjoint nonempty coalitions  $S_1, \dots, S_k$  with  $\cup_{l=1}^k S_l = N$ . Subsets  $S_1, \dots, S_k$  are called *blocks* of  $\pi$ . We denote by  $\Pi(N)$  or  $\Pi(n)$  the set of all possible partitions of  $N$ . A partition shows the structure of sub-groups in the player set, then it is usually called a *coalition structure*. A partition with  $k$  blocks is called a *k-partition*. A natural ordering of partitions is given by the notion of “coarsening” or “refinement”, like an ordering of subsets. Taking  $\pi, \pi'$  partitions in  $\Pi(N)$ , we say that  $\pi$  is a *refinement* of  $\pi'$  or  $\pi'$  is a *coarsening* of  $\pi$ , denoted by  $\pi \leq \pi'$ , if any block of  $\pi$  is contained in a block of  $\pi'$  (or equivalently, every block of  $\pi'$  fully decomposes into blocks of  $\pi$ ). Then  $(\Pi(N), \leq)$  is a lattice, called the *partition lattice*. With this ordering, the bottom element of the lattice is the finest partition  $\pi^\perp := \{\{1\}, \dots, \{n\}\}$  called *the singletons coalition structure*, while the top element is the coarsest partition  $\pi^\top := \{N\}$  called *the grand coalition structure*. An example with  $n = 4$  is given below.



An *embedded coalition* is a pair  $(S, \pi)$ , where  $S$  is a coalition such that  $S \in 2^N \setminus \{\emptyset\}$ , and  $\pi$  is a coalition structure such that  $S \in \pi$ . We also call  $S$  the *base coalition* of  $(S, \pi)$ . We denote by  $\mathfrak{C}(N)$  (or by  $\mathfrak{C}(n)$ ) the set of embedded coalition on  $N$ . For the sake of concision, we often denote by  $S\pi$  the embedded coalition  $(S, \pi)$ , and omit braces and commas for subsets (example for  $n = 3$ :  $12\{12, 3\}$  instead of  $(\{1, 2\}, \{\{1, 2\}, \{3\}\})$ ). Remark that  $\mathfrak{C}(N)$  is a proper subset of  $2^N \times \Pi(N)$ . We propose the following order relation on embedded coalition, which is merely the product order on  $2^N \times \Pi(N)$ :

$$(S, \pi) \sqsubseteq (S', \pi') \Leftrightarrow S \subseteq S' \text{ and } \pi \leq \pi'$$

Evidently, the top element of this ordered set is  $(N, \pi^\top)$  (denoted more simply by  $N\{N\}$  according to our conventions). However, due to the fact that the empty set is not allowed in  $(S, \pi)$ , there is no bottom element in the ordered structure  $(\mathfrak{C}(N), \sqsubseteq)$ . Indeed, all elements of the form  $(\{i\}, \pi^\perp)$  are minimal elements (i.e., there is no smaller element than them). For mathematical convenience, we introduce an artificial bottom element  $\perp$  to  $\mathfrak{C}(N)$  (it could be considered as  $(\emptyset, \pi^\perp)$ ), and denote  $\mathfrak{C}(N)_\perp := \mathfrak{C}(N) \cup \{\perp\}$ . We give as illustration the partially ordered set  $(\mathfrak{C}(N)_\perp, \sqsubseteq)$  with  $n = 3$  (Fig. 1).

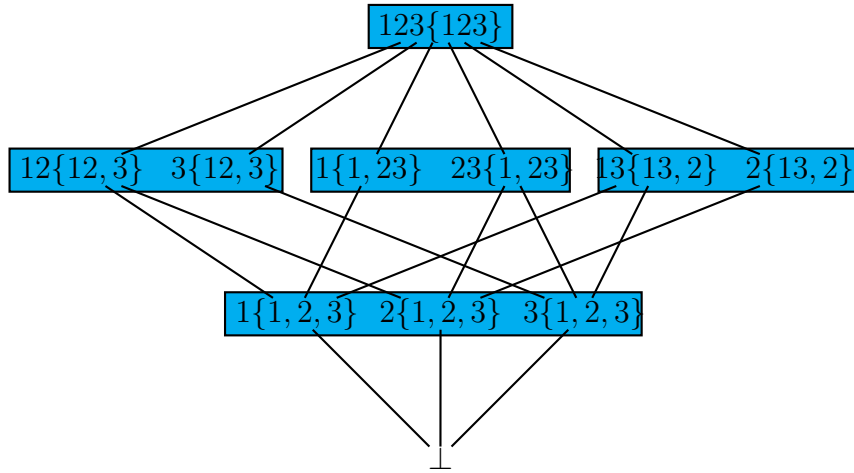


Figure 1: Diagram of  $(\mathfrak{C}(N)_\perp, \sqsubseteq)$  for  $n = 3$ . Elements with the same partition are framed in grey.

**Definition 1.** A *game in partition function form* (PFF-game in short) on  $N$  is a mapping  $v : \mathfrak{C}(N)_\perp \rightarrow \mathbb{R}$ , such that  $v(\perp) = 0$ . The set of all PFF-games on  $N$  is denoted by  $\mathcal{PG}(N)$ .

Following the usual interpretation, the payoff for coalition  $S$  depends on the situation of the outsiders, depicted by the partition  $\pi$ , which represents the externalities in an economy.

To be meaningful, we first assume that forming the grand coalition generates the largest total surplus, i.e.,  $v(N\{N\}) \geq \sum_{S \in \pi} v(S, \pi)$ , for all  $\pi \in \Pi(N)$ . Hence, we consider economic environments where doing so is the best for the society and the total surplus  $v(N\{N\})$  is distributed among the players in the society. In the same spirit, we assume that for a given coalition, the more the society is organized, the better the payoff, i.e., for every coalition

$S$ ,  $v(S, \pi) \leq v(S, \pi')$  if  $\pi$  is finer than  $\pi'$ . We call this condition *monotonicity w.r.t. the partition*<sup>2,3</sup>.

We recall that in a partially ordered set  $(P, \leq)$  with a bottom element  $\perp$  and a top element  $\top$ , a *chain from  $\perp$  to  $\top$*  is a totally ordered sequence of elements of  $P$  including  $\perp, \top$ . The chain is *maximal* if no other chain can contain it (equivalently, if between two consecutive elements  $x_i, x_{i+1}$  of the sequence, there is no element  $x \in P$  such that  $x_i < x < x_{i+1}$ ). If no ambiguity occurs, we say *maximal chain* instead of *maximal chain from  $\perp$  to  $\top$* . The set of maximal chains in  $P$  is denoted by  $\mathcal{C}(P)$ . The *length* of a maximal chain is the number of elements of the sequence minus 1. If all maximal chains have the same length, this length is the *height* of the partially ordered set. In Fig. 1, the sequence  $\perp, 1\{1, 2, 3\}, 1\{1, 23\}, 123\{123\}$  is a maximal chain, and there are 9 maximal chains in total, all of length 3, hence the height of  $(\mathfrak{C}(123)_\perp, \sqsubseteq)$  is 3. Concerning the partition lattice  $\Pi(N)$ , it is easy to see that its height is  $n - 1$ . In [12], it is proved that  $(\mathfrak{C}(N)_\perp, \sqsubseteq)$  is a lattice, whose maximal chains have all the same length  $n$ , hence the height of this lattice is  $n$ . The combinatorial complexity of  $(\mathfrak{C}(N)_\perp, \sqsubseteq)$  is far beyond the complexity of the Boolean lattice of coalitions in a TU game (for  $n$  players, there are  $2^n$  coalitions and  $n!$  maximal chains), as illustrated by the following facts [12]:

- The total number of elements is  $\sum_{k=1}^n kS_{n,k} + 1$ , where  $S_{n,k}$  is the Stirling number of second kind.

$n$	1	2	3	4	5	6	7	8
$ \mathfrak{C}(n)_\perp $	2	4	11	38	152	675	3264	17008

- The number of maximal chains from  $\perp$  to  $(N, \{N\})$  is  $|\mathcal{C}(\mathfrak{C}(n)_\perp)| = \frac{(n!)^2}{2^{n-1}}$ .

$n$	1	2	3	4	5	6	7	8
$ \mathcal{C}(\mathfrak{C}(n)_\perp) $	1	2	9	72	900	16 200	396 900	12 700 800

For simplicity, we put  $c := |\mathcal{C}(\mathfrak{C}(n)_\perp)|$  and it is shown in [12] that  $|\mathcal{C}(\Pi(n))| = \frac{c}{n}$ .

<sup>2</sup>In de Clippel and Serrano [5], this condition is called *positive externalities*.

<sup>3</sup>These conditions are here to ensure that our framework of coalition formation, defined through the notions of scenarios, processes, etc., as it will be explained later, is meaningful. Clearly, if the first condition is not fulfilled, the grand coalition will never form, and it is “unlikely” to form if the second one does not hold. However, these conditions are not used *mathematically* in the sequel, i.e., our definitions and results remain valid even if these conditions do not hold.



### 3 Processes and scenarios

Let us describe what we mean by a process of coalition formation. We consider that we start from the society of individual players, and after several steps of negotiation, the grand coalition has formed. This is guaranteed by our first assumption above. Moreover, we consider that at each step, exactly two blocks (coalitions) of the current partition merge to form a new coalition. This means that we first exclude that more than two blocks merge in one or several new coalitions, and secondly, we exclude that a splitting of some block occurs. The first condition is by no means a restriction, since it simply decomposes the process in elementary steps, which is always possible<sup>4</sup>. The second condition is natural if we recall our second assumption, namely that the game is monotonic w.r.t. the partition: coalitions do not have incentive to split, therefore only merging can occur. Summarizing, we can mathematically define a process as follows: A *coalition formation process*  $\mathcal{P}$  is any maximal chain in  $\Pi(n)$ , i.e., a sequence of partitions starting from  $\pi^\perp$  (singleton coalition structure) and ending at  $\{N\}$  (the grand coalition). The set of processes is  $\mathcal{C}(\Pi(n)) =: \mathfrak{P}$ .

Our second fundamental ingredient is the notion of scenario. Let us consider some process  $\mathcal{P}$ . A process is an “external” description of how coalitions form, as it could be seen by some observer outside the game. What could be then an “internal” view? Obviously, this means how the process is seen by some player in  $N$ . Let us consider player  $i \in N$ . At the initial step, player  $i$  is alone, and during the subsequent steps, s/he will join with some other blocks, to become finally included in  $N$ . Hence, the sequence of embedded coalitions  $(S_1, \pi_1), \dots, (S_n, \pi_n)$ , where the sequence  $\pi_1, \dots, \pi_n$  is  $\mathcal{P}$  itself, and  $S_1 = \{i\}$ ,  $S_n = N$  and all  $S_2, \dots, S_{n-1}$  contain  $i$ , is the internal view of the process  $\mathcal{P}$ , seen by player  $i$ . We call this the scenario for

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<sup>4</sup>This step-by-step approach, which allows only bilateral changes in merging at each step, is useful in describing negotiation steps concretely. Actually, mergers of more than two firms or organizations have been rarely observed. This is because the costs of merging more than two organizations are much larger than those for two organizations. For example, most of all mergers of major banks in Japan were bilateral and large banks have been formed sequentially in bilateral mergers: Tokay bank and Sanwa bank merged and became UFJ bank; Mitsubishi bank and Tokyo bank merged and became Tokyo-Mitsubishi bank; Tokyo-Mitsubishi bank and UFJ bank merged and became Tokyo-Mitsubishi-UFJ bank, which is Japan’s largest bank (this corresponds to process  $\mathcal{P}'$  in Section 5). Another example is Mitsui-Sumitomo bank, the third largest bank: Taiyo bank and Kobe bank merged and became Taiyo-Kobe bank; Taiyo-Kobe bank and Mitsui bank merged and became Sakura bank; Sakura bank and Sumitomo bank merged and became Mitsui-Sumitomo bank (this corresponds to process  $\mathcal{P}$  in Section 5). See also Houston et al. [13] and Macho-Stadler et al. [15] for various examples of bilateral mergers in economic environment.

player  $i$  (in the process  $\mathcal{P}$ )<sup>5</sup>. Summarizing, we get the following definition: a *scenario*  $\mathcal{S}$  in a process  $\mathcal{P}$  is any maximal chain in  $\mathfrak{C}(n)_\perp$  so that the sequence of partitions corresponds to  $\mathcal{P}$  (notation:  $\mathcal{S} \leftarrow \mathcal{P}$ ). The set of all scenarios, considering all processes, is therefore  $\mathcal{C}(\mathfrak{C}(n)_\perp)$ , denoted by  $\mathfrak{S}$  for simplicity.

For a given process  $\mathcal{P}$ , there are  $n$  scenarios  $\mathcal{S}_i$ ,  $i \in N$ . Note that a given scenario belongs to a unique process. For example, in a 3 persons game with  $N = \{1, 2, 3\}$ , we have three processes:

$$\begin{aligned}\mathcal{P}_1 &: \{1, 2, 3\} \rightarrow \{12, 3\} \rightarrow \{123\} \\ \mathcal{P}_2 &: \{1, 2, 3\} \rightarrow \{13, 2\} \rightarrow \{123\} \\ \mathcal{P}_3 &: \{1, 2, 3\} \rightarrow \{1, 23\} \rightarrow \{123\}\end{aligned}$$

In process  $\mathcal{P}_2$ , first players 1 and 3 merge, then coalitions 13 and 2 merge and form the grand coalition. A process describes how exactly the coalition structure evolves step by step, by a process of coalition formation.

Each coalition process has three scenarios. In the above example, process  $\mathcal{P}_2$  has the following three scenarios:

$$\begin{aligned}\mathcal{S}_1 &: \perp \rightarrow 1\{1, 2, 3\} \rightarrow 13\{13, 2\} \rightarrow 123\{123\} \\ \mathcal{S}_2 &: \perp \rightarrow 2\{1, 2, 3\} \rightarrow 2\{13, 2\} \rightarrow 123\{123\} \\ \mathcal{S}_3 &: \perp \rightarrow 3\{1, 2, 3\} \rightarrow 13\{13, 2\} \rightarrow 123\{123\}\end{aligned}$$

In  $\mathcal{S}_1$ , player 1 first merges with player 3 at the singletons coalition structure, then this base coalition 13 containing player 1 merges with player 2 and becomes the grand coalition structure. Different views of players correspond to different scenarios.

In a scenario  $\mathcal{S}$ , some elements play a special role. We consider those elements  $S\pi$  such that in the sequence of elements of  $\mathcal{S}$  from bottom to top,  $S\pi$  is the last element with base coalition  $S$ . They are called *terminal elements*. Specifically, let us denote  $\mathcal{S}$  by

$$\mathcal{S} = \{\perp, S_1\pi_{1,1}, \dots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \dots, S_2\pi_{2,m_2}, \dots, S_k\pi_{k,1}, \dots, S_k\pi_{k,m_k}, N\{N\}\}, \quad (1)$$

with  $S_1 \subsetneq \dots \subsetneq S_k \neq N$ . Then the terminal elements are  $S_i\pi_{i,m_i}$ ,  $i = 1, \dots, k$ . We denote by  $\mathcal{F}(\mathcal{S})$  this family of elements. A terminal element is an embedded coalition such that the base coalition of the embedded coalition changes at the next step in the scenario. We will motivate this definition in the next section.

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<sup>5</sup>We may also say that the scenario tracks the history of player  $i$  in the coalition formation process.

**Example 1.** We consider 4 players and the following process  $\mathcal{P}$ :

$$\{1, 2, 3, 4\} \rightarrow \{13, 2, 4\} \rightarrow \{13, 24\} \rightarrow \{1234\}.$$

and the four different scenarios in  $\mathcal{P}$  where terminal elements are in bold:

$$\mathcal{S}_1 : \perp \rightarrow \mathbf{1}\{1, 2, 3, 4\} \rightarrow 13\{13, 2, 4\} \rightarrow \mathbf{13}\{13, 24\} \rightarrow \mathbf{N}\{N\}$$

$$\mathcal{S}_2 : \perp \rightarrow 2\{1, 2, 3, 4\} \rightarrow \mathbf{2}\{13, 2, 4\} \rightarrow \mathbf{24}\{13, 24\} \rightarrow \mathbf{N}\{N\}$$

$$\mathcal{S}_3 : \perp \rightarrow \mathbf{3}\{1, 2, 3, 4\} \rightarrow 13\{13, 2, 4\} \rightarrow \mathbf{13}\{13, 24\} \rightarrow \mathbf{N}\{N\}$$

$$\mathcal{S}_4 : \perp \rightarrow 4\{1, 2, 3, 4\} \rightarrow \mathbf{4}\{13, 2, 4\} \rightarrow \mathbf{24}\{13, 24\} \rightarrow \mathbf{N}\{N\}$$

In scenario  $\mathcal{S}_1$ , player 1 first merges with player 3, then this coalition remains unchanged in the next step, and finally base coalition 13 merges with coalition 24, and becomes the grand coalition. Note that the second embedded coalition is not a terminal element because at the next step the base coalition does not change (player 1 is not concerned by the move).

## 4 Values for coalition formation processes

In this section, we introduce our concept of value for PFF games. Since it is based on an interpretation in terms of coalition formation, we call it a *coalition formation value*. To be more exact, we will define a whole family of such values.

### 4.1 Scenario-values, process-values and values

Consider again the initial state of the society where all players are individual, and no coalition has formed. We know from the previous section that under our assumptions finally the grand coalition will form, following some process  $\mathcal{P}$ , and our aim is to share the total surplus of the game among the players, taking into account the contribution of each player during the coalition formation process. Here, two situations can arise: either we know precisely the process  $\mathcal{P}$  (i.e., the grand coalition has formed and we observed how it was formed), or we do not know it, either because we were not able to observe the process, or because the process has not yet realized. In the first situation, it seems natural to make the calculation of the sharing by using only those embedded coalitions  $(S, \pi)$  which are related to the process  $\mathcal{P}$ , in other words, all  $(S, \pi)$  in the  $n$  scenarios of  $\mathcal{P}$ , the other ones being irrelevant. We call the sharing obtained in this way a *process-value*. In the second situation, our ignorance obliges us to use a priori all embedded coalitions, since any process may realize or have realized. If we have no knowledge at all about

the process, the principle of insufficient reason tells us that we should consider equally all processes; therefore, our value will be the arithmetic mean of the process-values, for all possible processes  $\mathcal{P} \in \mathfrak{P}$ . Otherwise, we may define some weight vector on each process representing, e.g., a probability of occurrence of each process, and compute a weighted arithmetic mean. Hence, depending on the practical situation under consideration, either the process-value or the (weighted) value is better suited (see an example in Section 5).

Let us now consider a given process  $\mathcal{P}$  and try to define the process-value. Unlike global games [11] which assigns worths to partitions —and therefore processes fully make sense for them, PFF games assign worths to coalitions *given* a partition, and therefore we must consider the  $n$  scenarios induced by a process, i.e., all possible sequences of embedded coalitions pertaining to the same process. In each scenario, we should compute the contribution of each player, which we call the scenario-value.

Summarizing the above discussion, we are led to the following definition.

**Definition 2.** (i) A *scenario-value* is a mapping  $\psi : \mathcal{PG} \rightarrow \mathbb{R}^{n \times |\mathfrak{S}|}$ . Components of  $\psi(v)$  are denoted by  $\psi_i^{\mathfrak{S}}(v)$  for scenario  $\mathfrak{S}$  and player  $i$ . If there is no confusion, we also call each component  $\psi_i^{\mathfrak{S}}(v)$  a scenario-value.

(ii) A *process-value* is a mapping  $\psi : \mathcal{PG} \rightarrow \mathbb{R}^{n \times |\mathfrak{P}|}$ . Components of  $\psi(v)$  are denoted by  $\psi_i^{\mathcal{P}}(v)$  for process  $\mathcal{P}$  and player  $i$ . If there is no confusion, we also call each component  $\psi_i^{\mathcal{P}}(v)$  a process-value. Any scenario-value  $\psi$  induces a process-value (denoted with some abuse by the same letter  $\psi$ ) by:

$$\psi_i^{\mathcal{P}}(v) := \frac{1}{n} \sum_{\mathfrak{S} \leftarrow \mathcal{P}} \psi_i^{\mathfrak{S}}(v), \quad i \in N, \mathcal{P} \in \mathfrak{P}.$$

(iii) A *value* is a mapping  $\psi : \mathcal{PG} \rightarrow \mathbb{R}^n$ . Components of  $\psi(v)$  are denoted by  $\psi_i(v)$  for player  $i$ . Any scenario-value or process-value  $\psi$  induces a value by:

$$\psi_i(v) := \frac{n}{c} \sum_{\mathcal{P} \in \mathfrak{P}} \psi_i^{\mathcal{P}}(v) = \frac{1}{c} \sum_{\mathfrak{S} \in \mathfrak{S}} \psi_i^{\mathfrak{S}}(v). \quad (2)$$

## 4.2 The classical Shapley value revisited

We now revisit the classical Shapley value in our setting. In the classical case, processes and scenarios collapse to simply sequences of coalitions induced by the different orderings of the players: this is obvious from the mathematical

structures of the two frameworks, since maximal chains in  $\mathfrak{C}(N)_\perp$  are scenarios and maximal chains in  $2^N$  correspond to orderings (see in Section 4.6 for the exact correspondance between processes, scenarios and orderings).

Let us consider a permutation  $\sigma$  and its associated scenario (sequence of coalitions)  $\mathfrak{S}(\sigma) = \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \dots, N$ . Shapley considers that the contribution of player  $\sigma(i)$  is  $\Delta_i(v, \sigma) := v(\{\sigma(1), \dots, \sigma(i)\}) - v(\{\sigma(1), \dots, \sigma(i-1)\})$ . In our language, this defines a scenario-value

$$\phi_i^{\mathfrak{S}(\sigma)}(v) = \Delta_i(v, \sigma), \quad (3)$$

for any  $i \in N$  and any permutation  $\sigma$  on  $N$ . Then the Shapley value reads  $\phi_i(v) = \frac{1}{n!} \sum_{\sigma} \phi_i^{\mathfrak{S}(\sigma)}(v)$  as it is well known. Observe that this is exactly the value induced by (3) through the mechanism (2).

Let us show briefly that it is easy to axiomatize, using axioms very close to the classical ones. Our first axiom is linearity (L): the scenario-value  $\psi$  should be linear on the set of games:  $\psi_i^{\mathfrak{S}(\sigma)}(v+w) = \psi_i^{\mathfrak{S}(\sigma)}(v) + \psi_i^{\mathfrak{S}(\sigma)}(w)$  for all permutations  $\sigma$  and players  $i$ . The second axiom is the null axiom. We say that player  $i$  is null in scenario  $\mathfrak{S}(\sigma)$  if  $v(\{\sigma(1), \dots, \sigma(i)\}) = v(\{\sigma(1), \dots, \sigma(i-1)\})$ . Then the null axiom says that  $\psi_i^{\mathfrak{S}(\sigma)}(v) = 0$  if  $i$  is null in  $\sigma$ . The third axiom is symmetry (S): for every permutation  $\rho$  on  $N$ , the value is insensitive to  $\rho$ :  $\psi_i^{\mathfrak{S}(\sigma)}(v) = \psi_{\rho(i)}^{\mathfrak{S}(\sigma)}(v \circ \rho^{-1})$ . The last axiom is scenario-efficiency (SE):

$$\sum_{i \in N} \psi_i^{\mathfrak{S}(\sigma)}(v) = v(N).$$

Then, the following holds.

**Proposition 1.** The Shapley scenario-value (3) is the only scenario-value satisfying (L), (N), (S) and (SE).

*Proof.* It is easy to see that (3) satisfies all axioms. Conversely, from (L) we immediately deduce that  $\psi^{\mathfrak{S}(\sigma)}$  reads

$$\psi_i^{\mathfrak{S}(\sigma)}(v) = \sum_{S \subseteq N} \gamma_{\mathfrak{S}(\sigma), S}^i v(S)$$

with real coefficients  $\gamma_{\mathfrak{S}(\sigma), S}^i$ . Let  $S$  be the smallest set in  $\mathfrak{S}(\sigma)$  containing  $i$ , and consider any game  $v$  such that  $v(S) := 1$  and  $v(S \setminus i) = 0$ . Then  $i$  is null in  $\mathfrak{S}(\sigma)$  for any such  $v$ , which by (N) entails

$$\gamma_{\mathfrak{S}(\sigma), S}^i = -\gamma_{\mathfrak{S}(\sigma), S \setminus i}^i, \quad \gamma_{\mathfrak{S}(\sigma), T}^i = 0, \forall T \neq S, S \setminus i.$$

It follows that

$$\psi_i^{\mathfrak{S}(\sigma)}(v) = \gamma_{\mathfrak{S}(\sigma), S}^i (v(S) - v(S \setminus i)).$$

Let us apply now symmetry. We have for any permutation  $\rho$  on  $N$ :

$$\psi_{\rho(i)}^{\rho(\mathfrak{S}(\sigma))}(v \circ \rho^{-1}) = \gamma_{\rho(\mathfrak{S}(\sigma)), \rho(S)}^{\rho(i)}(v(S) - v(S \setminus i)),$$

therefore by (S)  $\gamma_{\rho(\mathfrak{S}(\sigma)), \rho(S)}^{\rho(i)} = \gamma_{\mathfrak{S}(\sigma), S}^i$  holds for every permutation  $\rho$ , every  $i \in N, S \in \mathfrak{S}(\sigma)$ . Clearly  $\rho(\mathfrak{S}(\sigma)) = \mathfrak{S}(\rho \circ \sigma)$ , therefore  $\rho(S) \in \mathfrak{S}(\rho \circ \sigma)$  and this set is the smallest containing  $\rho(i)$ . Hence, we may consider that  $\sigma = Id$ , and the set of coefficients reduce to  $\gamma_{\mathfrak{S}, \{1\}}^1, \gamma_{\mathfrak{S}, \{1,2\}}^2, \dots, \gamma_{\mathfrak{S}, N}^n$ , where  $\mathfrak{S}$  is the scenario  $\{1\}, \{1, 2\}, \dots, N$ . Observe then that the coefficients  $\gamma_{\mathfrak{S}, S}^i$  depends only on the size of the set  $S$ . Therefore we obtain

$$\psi_i^{\mathfrak{S}(\sigma)}(v) = \gamma_{|S|}(v(S) - v(S \setminus i)).$$

Finally we apply efficiency. We obtain for any game  $v$ :

$$\begin{aligned} \sum_{i \in N} \psi_i^{\mathfrak{S}(\sigma)}(v) &= \gamma_1 v(\{\sigma(1)\}) + \gamma_2 (v(\{\sigma(1), \sigma(2)\}) - v(\{\sigma(1)\})) + \dots \\ &\quad + \gamma_n (v(N) - v(\{\sigma(1), \dots, \sigma(n-1)\})) \\ &= v(\{\sigma(1)\})(\gamma_1 - \gamma_2) + v(\{\sigma(1), \sigma(2)\})(\gamma_2 - \gamma_3) + \dots + v(N)\gamma_n. \end{aligned}$$

By (SE), we deduce the linear system

$$\begin{aligned} \gamma_1 - \gamma_2 &= 0 \\ \gamma_2 - \gamma_3 &= 0 \\ &\vdots = 0 \\ \gamma_{n-1} - \gamma_n &= 0 \\ \gamma_n &= 1 \end{aligned}$$

whose unique solution is  $\gamma_1 = \gamma_2 = \dots = \gamma_n = 1$ . Therefore,  $\psi_i^{\mathfrak{S}(\sigma)} = \phi_i^{\mathfrak{S}(\sigma)}$ , as desired.  $\square$

### 4.3 Axioms

Our task is now to translate the above axiomatization to PFF games. Due to the much higher complexity of PFF games compared to coalitional games, our task will be more difficult and we will need one more axiom, however the principles remain the same.

A scenario-value satisfies **linearity (L)** if it is linear on the set of PFF-games. We define similarly linearity for process-values and values.

**Proposition 2.** If  $\psi$  is a linear scenario-value, then for all  $v \in \mathcal{PG}(N)$

$$\psi_i^{\mathcal{S}}(v) = \sum_{S\pi \in \mathcal{C}(n)} \gamma_{\mathcal{S}, S\pi}^i v(S\pi), \quad \forall i \in N, \forall \mathcal{S} \in \mathfrak{S}$$

for some real constants  $\gamma_{\mathcal{S}, S\pi}^i$  (and similarly for a linear process-value, with constants  $\gamma_{\mathcal{P}, S\pi}^i$ , and for a linear value, with constants  $\gamma_{S\pi}^i$ ).

*Proof.* As usual, consider the elementary PFF-games  $e_{S\pi}(S'\pi') := 1$  iff  $S\pi = S'\pi'$  and 0 otherwise, for all  $S\pi \in \mathcal{C}(n)$ . Then linearity for scenario-values implies:

$$\psi(v) = \psi\left(\sum_{S\pi \in \mathcal{C}(n)} v(S\pi)e_{S\pi}\right) = \sum_{S\pi \in \mathcal{C}(n)} v(S\pi)\psi(e_{S\pi}),$$

hence the result, letting  $\psi_i^{\mathcal{S}}(e_{S\pi}) =: \gamma_{\mathcal{S}, S\pi}^i$  for  $i \in N, \mathcal{S} \in \mathfrak{S}$ .  $\square$

*Remark 1.* (i) Each linear value  $\psi$  is representable by at least one linear scenario-value  $\psi'$ . Indeed, by linearity we have:

$$\psi_i(v) = \sum_{S\pi \in \mathcal{C}(n)} \gamma_{S\pi}^i v(S\pi) = \sum_{S\pi \in \mathcal{C}(n)} v(S\pi) \sum_{\mathcal{S} \ni S\pi} \gamma_{\mathcal{S}, S\pi}^i = \sum_{\mathcal{S} \in \mathfrak{S}} \sum_{S\pi \in \mathcal{S}} \gamma_{\mathcal{S}, S\pi}^i v(S\pi),$$

taking any set of coefficients  $\gamma_{\mathcal{S}, S\pi}^i$  solution of the system  $\sum_{\mathcal{S} \ni S\pi} \gamma_{\mathcal{S}, S\pi}^i = \gamma_{S\pi}^i, S\pi \in \mathcal{C}(n)$ . Evidently, this system has in general infinitely many solutions. Then it suffices to take the scenario-value defined by  $\psi_i^{\mathcal{S}}(v) := \sum_{S\pi \in \mathcal{S}} \gamma_{\mathcal{S}, S\pi}^i v(S\pi), i \in N, \mathcal{S} \in \mathfrak{S}$ , which is linear. The same remark applies to process-values as well.

(ii) If a scenario-value is linear, then clearly its induced value is linear too. The converse is not true in general. Take simply  $n = 2$  and call  $\mathcal{S}_1, \mathcal{S}_2$  the two scenarios. Define for  $i = 1, 2$ :

$$\begin{aligned} \psi_i^{\mathcal{S}_1}(v) &:= v(1\{1, 2\}) + (v(12\{12\}))^2 \\ \psi_i^{\mathcal{S}_2}(v) &:= v(2\{1, 2\}) - (v(12\{12\}))^2. \end{aligned}$$

Then clearly the scenario-value is not linear but the induced value is.

**Definition 3.** Let us consider  $i \in N$ , a scenario  $\mathcal{S}$ , and denote by  $S^-\pi^-$  the last element in  $\mathcal{S}$  not containing  $i$ , and  $S\pi$  its successor in  $\mathcal{F}(\mathcal{S})$ , as in Figure 2. Player  $i$  is *null in scenario  $\mathcal{S}$*  for  $v$  if  $v(S\pi) = v(S^-\pi^-)$ . Player  $i$  is *null for  $v$*  if  $i$  is null for every scenario  $\mathcal{S}$ .

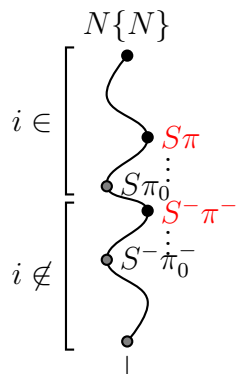


Figure 2: Notation used for scenario  $\mathcal{S}$  (terminal elements are in black, others are in grey)

We make some comments on this definition. The intuition behind the null axiom is that a player bringing no contribution should not be rewarded. The difficulty here is to distinguish between the effect of one player entering the base coalition  $S^-$  and any move in the coalition structure  $\pi^-$  (the externalities). This principle prevents us to impose equality everywhere between  $S\pi_0$  and  $S\pi$ , because this would simply mean that externalities have no effect on  $S$ . Therefore, there are two remarkable embedded coalitions containing  $S$ , which are  $S\pi_0$  and  $S\pi$ , and player  $i$  should be said to be null if  $v(S^-\pi^-)$  is equal to either  $v(S\pi_0)$  or  $v(S\pi)$ . We choose the latter for two reasons: under the hypothesis of positive externalities,  $v(S\pi)$  is the most favorable outcome for  $S$  inside scenario  $\mathcal{S}$  (and doing so will impose in fact  $v(S^-\pi^-) = v(S\pi_0) = \dots = v(S\pi)$ ), and more surprisingly, choosing the former makes the null axiom incompatible with efficiency, as it will be shown in Remark 2. Therefore, both economically and mathematically we are compelled to choose  $v(S^-\pi^-) = v(S\pi)$ <sup>6</sup>.

**Scenario Null axiom (SN):** If  $i$  is null in scenario  $\mathcal{S}$  for  $v$ , then  $\psi_i^{\mathcal{S}}(v) = 0$ .

Similarly as for linearity, if  $i$  is null for every scenario  $\mathcal{S}$ , then  $\psi_i(v) = 0$  (null axiom for the induced value), but the converse does not hold, that is, if  $\psi_i(v) = 0$  holds, then  $i$  is not necessary null in every scenario.

<sup>6</sup>A last possibility would be to impose  $v(S^-\pi_0^-) = v(S\pi_0)$ . But this clearly takes into account the change in externalities from  $\pi_0^-$  to  $\pi^-$ .



**Proposition 3.** Under (L) and (SN), for every scenario  $\mathcal{S}$ , every player  $i$ , the scenario-value reads

$$\psi_i^{\mathcal{S}}(v) = \gamma_{\mathcal{S}, S\pi}^i(v(S\pi) - v(S^-\pi^-)) \quad (4)$$

with notation of Fig 2, and with real constants  $\gamma_{\mathcal{S}, S\pi}^i$ ,  $i \in N$ ,  $\mathcal{S} \in \mathfrak{S}$ .

*Proof.* Take any scenario  $\mathcal{S}$ , and define  $v(S\pi) = v(S^-\pi^-) = 1$  and  $v(S'\pi') = 0$  otherwise. Then  $i$  is null for  $v$  in  $\mathcal{S}$ . Then by (SN) we get  $\gamma_{\mathcal{S}, S^-\pi^-}^i + \gamma_{\mathcal{S}, S\pi}^i = 0$ . Now take  $v' = v$ , except on a single element  $S'\pi' \in \mathfrak{C}(n)$  different from  $S^-\pi^-, S\pi$ . Since  $i$  is still null for  $v'$  in  $\mathcal{S}$ , we get by (SN) that  $\gamma_{\mathcal{S}, S'\pi'}^i = 0$ .  $\square$

The following is the usual symmetry axiom, but unlike in the classical case, its effect on the simplification of the gamma coefficients will be much weaker.

**Symmetry axiom for the scenario-value (SS):** For any  $i \in N$ , any  $\mathcal{S} \in \mathfrak{S}$ , and any permutation  $\sigma$  on  $N$ , it holds

$$\psi_i^{\mathcal{S}}(v) = \psi_{\sigma(i)}^{\sigma(\mathcal{S})}(v \circ \sigma^{-1})$$

with  $\sigma(\mathcal{S}), \sigma(S, \pi)$  defined naturally as follows:  $\sigma(S, \pi) = (\sigma(S), \sigma(\pi))$ , where  $\sigma(S) = \{\sigma(i) \mid i \in S\}$ ,  $\sigma(\pi) = \{\sigma(S) \mid S \in \pi\}$ , and  $\sigma(\mathcal{S}) = \{\sigma(S, \pi) \mid (S, \pi) \in \mathcal{S}\}$ .

For any scenario  $\mathcal{S} := (\perp, S_1\pi_1, \dots, S_n\pi_n = N\{N\})$ , we can define what we call its signature, which is the part of the scenario being invariant under permutations. For this, we need an unambiguous way to arrange blocks in the embedded coalitions of  $\mathcal{S}$ . First, for  $\pi_1 = \pi^\perp$ , an order on the blocks (singletons) is fixed. Then, for any  $\pi_k$ ,  $k = 2, \dots, n-1$ , the blocks are arranged in decreasing order of their size, and blocks of same size are arranged in the lexicographic order induced by the order on singletons. We call this the *natural ordering*. Now, the *signature* of  $\mathcal{S}$ , assuming that blocks are arranged with the natural ordering, is the sequence  $\tau(\mathcal{S}) := ((s_1, \rho(\pi_1), (i_1, j_1)), \dots, (s_{n-1}, \rho(\pi_{n-1}), (i_{n-1}, j_{n-1})), (n, (n)))$ , where  $\rho(\pi_k)$  is the sequence of cardinalities of blocks of  $\pi_k$ , and  $(i_k, j_k)$  denotes the index numbers of blocks in  $\pi_k$  which are merged to form  $\pi_{k+1}$ <sup>7</sup>. Note that there are at most  $n!$  different scenarios with the same signature, generated by all permutations on  $N$ .

<sup>7</sup>This is the long version of the signature, which contains many redundant information. An equivalent shortest (and therefore irredundant) version is  $\tau(\mathcal{S}) = (i; (i_1, j_1), (i_2, j_2), \dots, (i_{n-2}, j_{n-2}))$ , where  $i$  is the index of  $S_1$  in  $\pi_1$ .

**Example 2.** Consider the scenario  $\mathcal{S}$  given by

$$1\{1, 2, 3, 4, 5\} \rightarrow 12\{12, 3, 4, 5\} \rightarrow 12\{12, 34, 5\} \rightarrow 125\{125, 34\} \rightarrow 12345\{12345\}.$$

Its signature's long and shortest versions are

$$\begin{aligned} \tau(\mathcal{S}) = & ((1, (1, 1, 1, 1, 1), (1, 2)), (2, (2, 1, 1, 1), (2, 3)), (2, (2, 2, 1), (1, 3)), \\ & (3, (3, 2), (1, 2)), (5, (5))) \\ \tau(\mathcal{S}) = & (1; (1, 2), (2, 3), (1, 3)). \end{aligned}$$

The scenario  $\mathcal{S}'$  given by

$$4\{1, 2, 3, 4, 5\} \rightarrow 14\{14, 2, 3, 5\} \rightarrow 14\{14, 2, 35\} \rightarrow 124\{124, 35\} \rightarrow 12345\{12345\}$$

has the same signature and is obtained from  $\mathcal{S}$  by the permutation 41532.

**Proposition 4.** Under (L), (SN) and (SS), the scenario-value takes the form

$$\psi_i^{\mathcal{S}}(v) = \gamma_{\tau(\mathcal{S}), |\pi|}^i(v(S\pi) - v(S^-\pi^-)), \quad (5)$$

for any scenario  $\mathcal{S}$  and player  $i$ , with same notation as in Figure 2,  $\tau(\mathcal{S})$  is the signature of the scenario, and  $|\pi|$  is the number of blocks of  $\pi$ .

*Proof.* We know from (L) and (SN) that  $\psi_i^{\mathcal{S}}(v)$  takes the form (4). Hence, for any permutation  $\sigma$  on  $N$  we have for any game  $v$

$$\psi_{\sigma(i)}^{\sigma(\mathcal{S})}(v \circ \sigma^{-1}) = \gamma_{\sigma(\mathcal{S}), \sigma(S\pi)}^{\sigma(i)}(v(S\pi) - v(S^-\pi^-)).$$

The (SS) axiom entails  $\gamma_{\mathcal{S}, S\pi}^i = \gamma_{\sigma(\mathcal{S}), \sigma(S\pi)}^{\sigma(i)}$ , and this holds for any permutation  $\sigma$ .

Considering the coefficient  $\gamma_{\mathcal{S}, S\pi}^i$ , we note that  $S\pi \in \mathcal{S}$  and  $i \in S \setminus S^-$ . Taking a permutation  $\sigma$ , it is plain that  $\tau(\sigma(\mathcal{S})) = \tau(\mathcal{S})$ ,  $|\sigma(\pi)| = |\pi|$ , and that  $\sigma(S\pi) \in \sigma(\mathcal{S})$ ,  $\sigma(i) \in \sigma(S) \setminus \sigma(S^-)$ . Conversely, take any scenario  $\mathcal{S}'$  of same signature,  $S'\pi' \in \mathcal{S}'$  such that  $|\pi'| = |\pi|$  and  $i' \in S' \setminus S'^-$ , where  $S'^-$  precedes  $S'$  in  $\mathcal{S}'$ . Then there exists some permutation  $\sigma$  such that  $\sigma(\mathcal{S}) = \mathcal{S}'$  and since  $|\pi'| = |\pi|$ , we have  $\sigma(S\pi) = S'\pi'$ . Now observe that it is not necessarily true that  $i' = \sigma(i)$ , as shown by the following example:

$$\begin{aligned} \mathcal{S} = & 3\{1, 2, 3, 4, 5\} \rightarrow 3\{12, 3, 4, 5\} \rightarrow 34\{12, 34, 5\} \rightarrow 34\{125, 34\} \rightarrow N\{N\} \\ \mathcal{S}' = & 3\{5, 2, 3, 4, 1\} \rightarrow 3\{25, 3, 4, 1\} \rightarrow 34\{25, 34, 1\} \rightarrow 34\{125, 34\} \rightarrow N\{N\}, \end{aligned}$$

and consider  $S\pi = S'\pi' = N\{N\}$ , and  $i = i' = 1$ . Then  $\sigma$  must necessarily change 1 into 5, i.e.,  $\sigma(i) \neq i'$ . Therefore, the superindex  $i$  is necessary.  $\square$

By analogy with coalitional cooperative games, the *unanimity game*  $u_{S\pi}$  centered on  $S\pi$  is defined by  $u_{S\pi}(S'\pi') = 1$  if  $S'\pi' \sqsupset S\pi$  and 0 otherwise. The last axiom expresses the fact that for the unanimity game  $u_{S\pi}$  and any scenario where  $S\pi$  is a terminal element, there is equal contribution for all players entering  $S$  in the scenario.

**Egalitarian axiom (EG):** consider a scenario  $\mathfrak{S}$  and  $S\pi, S^-\pi^- \in \mathcal{F}(\mathfrak{S})$  where  $S^-\pi^-$  is the predecessor of  $S\pi$  in  $\mathcal{F}(\mathfrak{S})$ , and  $|S \setminus S^-| > 1$ . Then  $\forall i, j \in |S \setminus S^-|, \psi_i^{\mathfrak{S}}(u_{S\pi}) = \psi_j^{\mathfrak{S}}(u_{S\pi})$ .

**Proposition 5.** Under (L), (SN), (SS) and (EG), the scenario-value takes the form

$$\psi_i^{\mathfrak{S}}(v) = \gamma_{\tau(\mathfrak{S}), |\pi|}(v(S\pi) - v(S^-\pi^-)) \quad (6)$$

for any scenario  $\mathfrak{S}$  and player  $i \in N$  (same notation as in Figure 2).

Proof is immediate from (5).

Our last axiom is efficiency.

**Scenario-efficiency axiom (SE):** for every  $\mathfrak{S} \in \mathfrak{G}$ ,  $\sum_{i \in N} \psi_i^{\mathfrak{S}}(v) = v(N\{N\})$ . Similarly, we define **process-efficiency (PE)** as  $\sum_{i \in N} \psi_i^{\mathcal{P}}(v) = v(N\{N\})$  for all  $\mathcal{P} \in \mathfrak{P}$ , and **efficiency (E)** as  $\sum_{i \in N} \psi_i(v) = v(N\{N\})$ .

With some abuse, we say that a scenario-value is (process-)efficient if its induced (process-)value is. Obviously, scenario-efficiency implies process-efficiency and efficiency.

**Theorem 1.** The unique scenario-value satisfying (L), (SN), (SS), (EG) and (SE) is given by

$$\phi_i^{\mathfrak{S}}(v) = \frac{1}{|S \setminus S^-|}(v(S\pi) - v(S^-\pi^-)),$$

with notation of Figure 2). We call it the *egalitarian scenario-value*.

*Proof.* The fact that the egalitarian scenario-value satisfies (L), (SN), (SS), (EG) and (SE) is easy to check and left to the reader.

Conversely, let  $\mathfrak{S} = \{\emptyset\pi^\perp, S_1\pi_{1,1}, \dots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \dots, S_2\pi_{2,m_2}, \dots, S_k\pi_{k,1}, \dots, S_k\pi_{k,m_k}, N\{N\}\}$  be fixed.

From Proposition 5 we get:

$$\begin{aligned}
\sum_{i \in N} \psi_i^S(v) &= \gamma_{\tau(S), |\pi_{1,m_1}|} v(S_1 \pi_{1,m_1}) + |S_2 \setminus S_1| \gamma_{\tau(S), |\pi_{2,m_2}|} (v(S_2 \pi_{2,m_2}) - v(S_1 \pi_{1,m_1})) + \\
&\quad \cdots + |S_j \setminus S_{j-1}| \gamma_{\tau(S), |\pi_{j,m_j}|} (v(S_j \pi_{j,m_j}) - v(S_{j-1} \pi_{j-1,m_{j-1}})) + \\
&\quad \cdots + |N \setminus S_k| \gamma_{\tau(S), 1} (v(N \{N\}) - v(S_k \pi_{k,m_k})) \\
&= (\gamma_{\tau(S), |\pi_{1,m_1}|} - |S_2 \setminus S_1| \gamma_{\tau(S), |\pi_{2,m_2}|}) v(S_1 \pi_{1,m_1}) + \\
&\quad \cdots + (|S_j \setminus S_{j-1}| \gamma_{\tau(S), |\pi_{j,m_j}|} - |S_{j+1} \setminus S_j| \gamma_{\tau(S), |\pi_{j+1,m_{j+1}}|}) v(S_j \pi_{j,m_j}) + \\
&\quad \cdots + |N \setminus S_k| \gamma_{\tau(S), 1} v(N \{N\}).
\end{aligned}$$

From the (SE) axiom, we deduce the following linear system of  $k+1$  equations and  $k+1$  unknowns:

$$\begin{aligned}
\gamma_{\tau(S), |\pi_{1,m_1}|} - |S_2 \setminus S_1| \gamma_{\tau(S), |\pi_{2,m_2}|} &= 0 \\
&\vdots = 0 \\
|S_j \setminus S_{j-1}| \gamma_{\tau(S), |\pi_{j,m_j}|} - |S_{j+1} \setminus S_j| \gamma_{\tau(S), |\pi_{j+1,m_{j+1}}|} &= 0 \\
&\vdots = 0 \\
|N \setminus S_k| \gamma_{\tau(S), 1} &= 1.
\end{aligned}$$

Evidently the system is nonsingular, since from the last equation  $\gamma_{\tau(S), 1}$  is obtained, then substituting it into the last but one, we get  $\gamma_{\tau(S), |\pi_{k,m_k}|}$  and so on. Knowing that the coefficients of the egalitarian scenario-value are solutions of the system, it is the unique solution.  $\square$

We call *coalition formation value* the value induced by the egalitarian scenario-value.

*Remark 2.* We now show that the modified null axiom (where  $v(S^- \pi^-) = v(S \pi_0)$ ) and scenario-efficiency are not compatible. It suffices to consider  $n = 3$  and to take a scenario which differs from the classical scenarios *à la Shapley*. First, it is not difficult to see that the modified axiom together with linearity gives:

$$\psi_i^S(v) = \gamma_{S, S \pi_0}^i (v(S \pi_0) - v(S^- \pi^-)).$$

Let us consider  $n = 3$  and the scenario  $1\{1, 2, 3\}, 1\{12, 3\}, 123\{123\}$ . We obtain

$$\begin{aligned}
\psi_1^S(v) &= \gamma_{S, 1\{1,2,3\}}^1 v(1\{1, 2, 3\}) \\
\psi_2^S(v) &= \gamma_{S, 123\{123\}}^2 (v(123\{123\}) - v(1\{12, 3\})) \\
\psi_3^S(v) &= \gamma_{S, 123\{123\}}^3 (v(123\{123\}) - v(1\{12, 3\}))
\end{aligned}$$

Then efficiency yields

$$\begin{aligned}\gamma_{\mathcal{S},123\{123\}}^2 + \gamma_{\mathcal{S},123\{123\}}^3 &= 1 \\ \gamma_{\mathcal{S},1\{1,2,3\}}^1 &= 0 \\ \gamma_{\mathcal{S},123\{123\}}^2 + \gamma_{\mathcal{S},123\{123\}}^3 &= 0\end{aligned}$$

which is clearly an incompatible system.

#### 4.4 A second axiomatization

The egalitarian scenario-value is fully characterized by these axioms, which are very similar to the classical axioms of the Shapley value for TU games. However, every axiom is restricted to a single scenario in a coalition formation process, therefore in some sense we have a local axiomatization. Imposing that each scenario must satisfy these axioms may appear as a strong condition. Our next axiomatization avoids this by using efficiency, the weaker and most global form of efficiency, instead of the strong and local scenario-efficiency.

As a starting point, we come back to the classical Shapley scenario-value, and show that when replacing scenario-efficiency by efficiency (which is weaker), the result still holds. Efficiency reads

$$\sum_{i \in N} \sum_{\sigma} \psi_i^{\mathcal{S}(\sigma)}(v) = n!v(N),$$

which gives

$$\sum_{\sigma} \sum_{i=1}^n \gamma_i(v(\{\sigma(1), \dots, \sigma(i)\}) - v(\{\sigma(1), \dots, \sigma(i-1)\})) = n!v(N)$$

from which we deduce

$$\begin{aligned}\gamma_n &= 1 \\ (n-1)!\gamma_1 - (n-1)!\gamma_2 &= 0 \\ &\vdots = 0 \\ (n-1)!\gamma_{n-2} - (n-1)!\gamma_{n-1} &= 0\end{aligned}$$

from which we get  $\gamma_1 = \dots = \gamma_n = 1$  as before. With PFF games, the situation is more complex and we will need an additional axiom.

Consider the unanimity game  $u_{S\pi_0}$ , where  $S$  is any nonempty coalition, and  $\pi_0$  is the finest partition  $\pi_0$  containing  $S$ , i.e.,  $\pi_0 = \{S, \pi_{N \setminus S}^\perp\}$ , where  $\pi_{N \setminus S}^\perp$

is the partition of  $N \setminus S$  in singletons. Then, any embedded coalition with base coalition  $S$  has worth 1, and any embedded coalition with base coalition  $S^- \subset S$  has worth 0, whatever the externalities are. In this situation, for a fixed  $S^- \subset S$ , it is natural to say that the contribution of a player  $i \in S \setminus S^-$  does not depend on the scenario  $\mathcal{S}$ , provided that in the scenario  $S^-$  precedes  $S$  (more precisely,  $S^- \pi^-$  precedes  $S\pi$  in  $\mathcal{F}(\mathcal{S})$ , for some  $\pi, \pi^-$ ). Put more bluntly, the organisation of the players in  $N \setminus S$  is irrelevant to the contribution of  $i$ .

**Independence of irrelevant players axiom (IIP):** Consider a nonempty coalition  $S \subseteq N$ , and  $\pi_0$  the finest partition containing  $S$ . Then for every scenarios  $\mathcal{S}, \mathcal{S}'$  such that  $S\pi \in \mathcal{F}(\mathcal{S}), S\pi' \in \mathcal{F}(\mathcal{S}')$ , where  $\pi, \pi'$  are arbitrary partitions containing  $S$ , it holds

$$\psi_i^{\mathcal{S}}(u_{S\{S, \pi_{N \setminus S}^-\}}) = \psi_i^{\mathcal{S}'}(u_{S\{S, \pi_{N \setminus S}^-\}}),$$

for every  $i \in S \setminus S^-$ , where  $S^- \pi^-, S^- \pi'^-$  are the predecessors of  $S\pi, S\pi'$  in  $\mathcal{F}(\mathcal{S}), \mathcal{F}(\mathcal{S}')$  respectively.

**Proposition 6.** Under (L), (SN), (SS), (EG) and (IIP), we have, for any scenario  $\mathcal{S}$ , any  $S\pi \in \mathcal{F}(\mathcal{S})$  and its predecessor  $S^- \pi^-$  in  $\mathcal{F}(\mathcal{S})$

$$\psi_i^{\mathcal{S}}(v) = \gamma_{s, s^-}(v(S\pi) - v(S^- \pi^-)) \quad (7)$$

for all  $i \in S \setminus S^-$ . If  $S$  is a singleton,  $\gamma_{s, s^-}$  is denoted by  $\gamma_{1, 0}$ .

*Proof.* Under (L), (SN), (SS), (EG), we already know that  $\psi_i^{\mathcal{S}}(v)$  has the form (6), with coefficients  $\gamma_{\tau(\mathcal{S}), |\pi|}$ . Fix  $S^- \subset S$ , any  $i \in S \setminus S^-$  and any two scenarios  $\mathcal{S}, \mathcal{S}'$  defined as above. Then by (IIP) we get

$$\gamma_{\tau(\mathcal{S}), |\pi|} = \gamma_{\tau(\mathcal{S}'), |\pi'|}.$$

Since  $\pi$  and  $\pi'$  can be any partitions containing  $S$ , the coefficients do not depend on  $\pi$ . Now,  $\tau(\mathcal{S})$  has the form  $\dots, (s^-, \rho(\pi^-), (i^-, j^-)), \dots, (s, \rho(\pi), (i, j)), \dots$ , and any sequence of that type with  $s, s^-$  fixed is produced by some  $\mathcal{S}$  defined as above. Consequently, the dependency to  $\tau(\mathcal{S})$  is reduced to the dependency to  $s, s^-$ .  $\square$

**Theorem 2.** The egalitarian scenario-value is the unique scenario-value satisfying (L), (SN), (SS), (EG), (E) and (IIP).

*Proof.* Again, the egalitarian-scenario-value clearly satisfies these axioms.

Conversely, let us assume that the scenario-value satisfies the five axioms, and let us compute  $\sum_{i \in N} \sum_{\mathcal{S} \in \mathfrak{G}} \psi_i^{\mathcal{S}}(v)$ . From  $\sum_{i \in N} \sum_{\mathcal{S} \in \mathfrak{G}} \psi_i^{\mathcal{S}}(v) = v(N\{N\})$ ,

we get a system of linear equations, one per element  $S\pi$ . We know that there exists at least one solution to this system, since our egalitarian scenario-value satisfies the five axioms. Our task will be to prove that this is the only solution. To this aim, we will prove that there are at least as many equations as variables, and that there exists a subsystem which can be made triangular.

First, we determine the form of the equation for element  $S\pi := S\{S, S_2, \dots, S_k\}$ , assuming  $S\pi$  is any element different from  $N\{N\}$  and  $S$  is not a singleton. From (7), there is a negative contribution for  $v(S\pi)$  with coefficient  $-\gamma_{s^+,s}$  for all  $S^+\pi^+$  such that  $S^+\pi^+ \supset S\pi$ ,  $|S^+| = s^+$ ,  $\pi^+ := \{S^+, S_2^+, \dots, S_{k^+}^+\}$ , and  $S^+ = S \cup S_j$ , for some  $j = 2, \dots, k$ , for all  $i \in S^+ \setminus S$ , and all scenarios  $\mathcal{S}$  passing through  $S^+\pi^+$  and  $S\pi$ , such that  $S^+\pi^+, S\pi \in \mathcal{F}(\mathcal{S})$ . Hence, any scenario of the following form, with  $S^+ = S \cup S_j$ :

$$\perp, \dots, S\pi, \underbrace{S \cup S_j}_{S^+} \pi_{S \cup S_j}, \dots, \underbrace{S \cup S_j}_{S^+} \pi^+, S^+ \cup S_l^+ \pi_{S^+ \cup S_l^+}^+, \dots, N\{N\}, \quad l = 2, \dots, k^+$$

will lead to a negative contribution with coefficient  $-\gamma_{s^+,s}$ . The notation  $\pi_{S \cup S_j}$ , etc., is a shorthand for  $(\pi \setminus \{S, S_j\}) \cup \{S \cup S_j\}$ . The set of all such scenarios (for  $j$  fixed, i.e.,  $S^+$  fixed) is obtained as all possible combinations of:

- (i) all sub-scenarios from  $\perp$  to  $S\pi$ ;
- (ii) all sub-scenarios from  $S \cup S_j \pi_{S \cup S_j}$  to  $S \cup S_j \pi^+$ , for all possible  $\pi^+$  coarser than  $\pi_{S \cup S_j}$  and containing  $S \cup S_j$ ;
- (iii) for a given  $\pi^+$ , all sub-scenarios from  $S^+ \cup S_l^+ \pi_{S^+ \cup S_l^+}^+$  to the top  $N\{N\}$ , for  $l = 2, \dots, k^+$ .

Hence the number of such scenarios is

$$\beta_{s+s_j, s, \pi} = |\mathcal{C}([\perp, S\pi])| \sum_{\substack{\pi^+ \supset \pi \\ \pi^+ \ni S \cup S_j}} \left( |\mathcal{C}([\pi_{S \cup S_j}, \pi^+])| \times \sum_{l=2}^{k^+} |\mathcal{C}([S^+ \cup S_l^+ \pi_{S^+ \cup S_l^+}^+, N\{N\}])| \right),$$

where the notation  $\mathcal{C}([\perp, S\pi])$  stands for the set of maximal chains from  $\perp$  to  $S\pi$ , and so on (for the second term, since the coalition is always  $S^+$ , we can omit it). Although this number seems difficult to compute (!), it depends ultimately only on  $k, s, s_2, \dots, s_k$ , hence on the number of blocks of  $\pi$  and their cardinality. Indeed, the following results are shown in [12].

- Consider two distinct elements  $\pi, \pi' \in \Pi(n)$ , with  $\pi' < \pi$ . Then

$$|\mathcal{C}([\pi', \pi])| = \frac{(k' - k)!}{2^{k' - k}} l_1! l_2! \cdots l_k!$$

with  $\pi := \{S_1, \dots, S_k\}$ ,  $\pi' := \{S_{1l_1}, \dots, S_{1l_1}, S_{2l_2}, \dots, S_{kl_k}\}$ ,  
with  $\{S_{i1}, \dots, S_{il_i}\}$  a partition of  $S_i$ ,  $i = 1, \dots, k$ , and  $k' := \sum_{i=1}^k l_i$ .

- Let  $S\pi$  be an embedded coalition, with  $\pi := \{S, S_2, \dots, S_k\}$ . The number of maximal chains from  $\perp$  to  $S\pi$  is

$$|\mathcal{C}([\perp, S\pi])| = \frac{s(n-k)!}{2^{n-k}} s! s_2! \cdots s_k!$$

- Let  $S\pi$  be an embedded coalition, with  $\pi := \{S, S_2, \dots, S_k\}$ . The number of maximal chains from  $S\pi$  to  $N\{N\}$  is

$$|\mathcal{C}([S\pi, N\{N}])| = \frac{1}{k} |\mathcal{C}(\mathfrak{C}(k)_\perp)| = \frac{k!(k-1)!}{2^{k-1}}.$$

We deduce from this:

$$\begin{aligned} |\mathcal{C}([\pi_{S \cup S_j}, \pi^+])| &= \frac{(k-1-k^+)!}{2^{k-1-k^+}} 1.l_2^+ \cdots l_{k^+}^+ \\ |\mathcal{C}([S^+ \cup S_l^+ \pi_{S^+ \cup S_l^+}^+, N\{N\}])| &= \frac{(k^+-1)!(k^+-2)!}{2^{k^+-2}} \end{aligned}$$

with  $l_2^+, \dots, l_{k^+}^+$  being the numbers of blocks in  $\pi_{S \cup S_j}$  corresponding to  $S_2^+, \dots, S_{k^+}^+$ . Clearly,  $|\mathcal{C}([\perp, S\pi])|$  depends only on  $k$  and the cardinality of blocks of  $\pi$ . For  $|\mathcal{C}([\pi_{S \cup S_j}, \pi^+])|$ , doing the summation over  $\pi^+$ , we have that  $k^+$  will vary from 2 to  $k-1$ . Accordingly, each  $l_j^+$  will vary from 1 (when  $k^+ = k-1$ ) to  $k$  (when  $k^+ = 2$ ). Hence the second term (after summation) depends only on  $k$ . Similarly,  $|\mathcal{C}([S^+ \cup S_l^+ \pi_{S^+ \cup S_l^+}^+, N\{N\}])|$  depends only on  $k$ .

Similarly, there is a positive contribution for  $v(S\pi)$  with coefficient  $\gamma_{s,s^-}$  for all  $S^-\pi^-$  such that  $S^-\pi^- \sqsubset S\pi$ ,  $|S^-| = s^-$ , and  $S = S^- \cup S_1^-$  for some  $S_1^- \in \pi^-$ , all  $i \in S \setminus S^-$ , and all scenarios  $\mathfrak{S}$  passing through elements  $S\pi$  and  $S^-\pi^-$ , so that  $S\pi, S^-\pi^- \in \mathcal{F}(\mathfrak{S})$ . Hence, any scenario of the following form, with  $S^-\pi^-$  defined as above

$$\perp, \dots, S^-\pi^-, \underbrace{S^- \cup S_1^-}_S \pi_{S^- \cup S_1^-}^-, \dots, S\pi, S \cup S_j \pi_{S \cup S_j}, \dots, N\{N\}, \quad j = 2, \dots, k$$

will lead to a positive contribution with coefficient  $\gamma_{s,s^-}$ . The number of such scenarios is, reasoning as above,

$$\alpha_{s,s^-, \pi} = \sum_{\substack{S^- \subset S, |S^-| = s^- \\ \pi^- \subset \pi, \pi^- \ni S^-, S_1^- \\ \text{s.t. } S = S^- \cup S_1^-}} \left( |\mathcal{C}([\perp, S^-\pi^-])| \times |\mathcal{C}([\pi_{S^- \cup S_1^-}^-, \pi])| \right) \times \sum_{j=2}^k |\mathcal{C}([S \cup S_j \pi_{S \cup S_j}, N\{N\}])|$$



Again, this number depends only on  $s^-$ , the number of blocks of  $\pi$  and their cardinality. In summary, the equation for  $S\pi \neq N\{N\}$  is

$$\sum_{s^- < s} (s - s^-) \alpha_{s, s^-, \pi} \gamma_{s, s^-} - \sum_{j=2}^k s_j \beta_{s+s_j, s, \pi} \gamma_{s+s_j, s} = 0.$$

Let us address briefly the case of singletons and  $N$ . If  $S = \{i\}$ , the first term is replaced by  $\alpha_{1,0,\pi} \gamma_{1,0}$ , with

$$\alpha_{1,0,\pi} = \sum_{j=2}^k |\mathcal{C}([S \cup S_j \pi_{S \cup S_j}, N\{N\}])|.$$

If  $S = N$ , the second term does not exist. In summary:

$$\begin{aligned} \alpha_{1,0,\pi} \gamma_{1,0} + \sum_{j=2}^k (s^+ - s) \beta_{s+s_j, s, \pi} \gamma_{s+s_j, s} &= 0, & (S \text{ is a singleton}) \\ \sum_{n^- < n} (n - n^-) \alpha_{n, n^-, \{N\}} \gamma_{n, n^-} &= v(N\{N\}), & (S\pi = N\{N\}). \end{aligned} \quad (8)$$

From the above considerations, equations for  $S\pi$  and  $S'\pi'$  will be identical if and only if  $s = s'$ , and  $\pi$  and  $\pi'$  are of the same type (same number of blocks and same cardinality of blocks). Hence, the number of different equations for  $|S| = s$  is the number of integer partitions of  $n - s$ , denoted by  $p(n - s)$ , for  $s = 1, \dots, n$  (for  $n = s$ , there is only one equation, which is (8)). Hence we put  $p(0) := 1$ ). For example, the numbers of integer partitions of 1, 2, 3, 4 are respectively 1, 2, 3, 5. Hence, for  $n = 2, 3, 4, 5$  we have respectively 2, 4, 7, and 12 different equations.

The number of variables  $\gamma_{s, s'}$  is much easier to compute. For  $s = 1$ , there is only one variable, namely  $\gamma_{1,0}$ . For  $1 < s \leq n$  fixed,  $s'$  varies from 1 to  $s - 1$ . Hence the total number of variables is:

$$1 + \sum_{s=2}^n (s - 1) = \sum_{s=2}^n s - n + 2 = \frac{n^2 - n + 2}{2}.$$

This gives for  $n = 2, 3, 4, 5$  players, 2, 4, 7, 11 variables. It is easy to see that this number is less or equal than the number of equations. Indeed, for large  $n$ , the following formula is known:

$$p(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\left(\frac{2n}{3}\right)^{\frac{1}{2}}\right)$$

which is clearly exponential (see Andrews [2]).

It remains to find a subsystem of equations which can be made triangular. Let us order the variables as follows:  $\gamma_{1,0}, \gamma_{2,1}, \gamma_{3,1}, \gamma_{3,2}, \gamma_{4,1}, \dots$ . For each variable  $\gamma_{s',s}$ , except for  $\gamma_{1,0}$ , let us find an equation using only variables up to  $\gamma_{s',s}$ . It suffices to take the equation for  $S\pi$  such that the largest block of  $\pi \setminus \{S\}$  is of size  $s' - s$ . Doing so for all  $\gamma_{s',s}$ , we form a subsystem. If in this subsystem, we put for each equation  $\gamma_{1,0}$  on the right side, the subsystem becomes triangular. So it has a unique solution in terms of  $\gamma_{1,0}$ , which can be determined by substituting all variables in the equation corresponding to  $S\pi = N\{N\}$ . This proves the uniqueness of the solution.  $\square$

## 4.5 Explicit expression of the coalition formation value

We consider the elementary games  $e_{T\sigma}$  for any  $T\sigma \in \mathfrak{C}(N)$ , defined by  $e_{T\sigma}(T'\sigma') = 1$  iff  $T\sigma = T'\sigma'$ , and 0 otherwise. We have for any game  $v$  in  $\mathcal{PG}(N)$ ,  $v = \sum_{T\sigma \in \mathfrak{C}(N)} v(T\sigma)e_{T\sigma}$ . Since the egalitarian scenario-value is linear, so is the value. Hence, for any game  $v$  in  $\mathcal{PG}(N)$ ,

$$\phi_i(v) = \sum_{T\sigma \in \mathfrak{C}(N)} v(T\sigma)\phi_i(e_{T\sigma}).$$

It remains to compute  $\phi_i(e_{T\sigma})$  for all  $i \in N$  and all  $T\sigma \in \mathfrak{C}(N)$ .

A first important thing is to notice that symmetry (SS) implies  $\phi_i(e_{T\sigma}) = \phi_j(e_{T\sigma})$  for all  $i, j \in T$ . Indeed, consider any permutation  $\tau$  such that  $\tau(T) = T$ , and  $\tau$  is the identity on  $N \setminus T$ . Then, for any  $i \in T$ ,

$$\phi_i(e_{T\sigma}) = \sum_{\mathfrak{S}} \phi_i^{\mathfrak{S}}(e_{T\sigma}) = \sum_{\mathfrak{S}} \phi_{\tau(i)}^{\tau(\mathfrak{S})}(e_{T\sigma \circ \tau^{-1}}) = \sum_{\mathfrak{S}'} \phi_{\tau(i)}^{\mathfrak{S}'}(e_{\tau(T\sigma)}) = \sum_{\mathfrak{S}'} \phi_{\tau(i)}^{\mathfrak{S}'}(e_{T\sigma}) = \phi_{\tau(i)}(e_{T\sigma}).$$

From this and efficiency we immediately have:

$$\phi_i(e_{N\{N\}}) = \frac{1}{n}, \quad \forall i \in N. \quad (9)$$

Let us consider a given  $T\sigma \in \mathfrak{C}(N)$ ,  $T\sigma \neq N\{N\}$ , and consider any  $i \notin T$ . Let us put for convenience  $\sigma = \{T, T_2, \dots, T_k\}$  and consider that  $i \in T_2$ . A scenario  $\mathfrak{S}$  induces  $\phi_i^{\mathfrak{S}}(e_{T\sigma}) \neq 0$  if and only if  $\mathfrak{S}$  contains  $T\sigma$  and the successor of  $T\sigma$  in  $\mathfrak{S}$  is  $T \cup T_2 \sigma_{T \cup T_2}$ , with  $\sigma_{T \cup T_2} := (\sigma \setminus \{T, T_2\}) \cup \{T \cup T_2\}$ , and in this case  $\phi_i^{\mathfrak{S}}(e_{T\sigma}) = -\frac{1}{t_2}$ . The number of such scenarios is (see proof of Th. 2)

$$\frac{t(n-k)!}{2^{n-2}} t! t_2! \cdots t_k! (k-1)! (k-2)!,$$

hence

$$\phi_i(e_{T\sigma}) = -\frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!(t_2-1)!\cdots t_k!, \quad \forall i \in T_2. \quad (10)$$

Establishing the corresponding formulas when  $i \in T_3, \dots, T_k$  successively we deduce

$$\sum_{i \notin T} \phi_i(e_{T\sigma}) = -\frac{2t(k-1)(n-k)!}{n!n!}t!t_2!\cdots t_k!(k-1)!(k-2)!,$$

hence, by efficiency we get:

$$\begin{aligned} \sum_{i \in T} \phi_i(e_{T\sigma}) &= 0 - \sum_{i \notin T} \phi_i(e_{T\sigma}) \\ &= \frac{2t(k-1)(n-k)!}{n!n!}t!t_2!\cdots t_k!(k-1)!(k-2)! \end{aligned}$$

which gives by symmetry:

$$\phi_i(e_{T\sigma}) = \frac{2(n-k)!}{n!n!}(k-1)(k-1)!(k-2)!t!t_2!\cdots t_k!, \quad \forall i \in T. \quad (11)$$

Hence for any game  $v$ , we obtain:

$$\begin{aligned} \phi_i(v) &= \frac{1}{n}v(N\{N\}) + \sum_{\substack{T\sigma \in \mathfrak{C}(N) \setminus \{N\{N\}\} \\ T \ni i}} \frac{2(n-k)!}{n!n!}(k-1)(k-1)!(k-2)!t!t_2!\cdots t_k!v(T\sigma) \\ &\quad - \sum_{\substack{T\sigma \in \mathfrak{C}(N) \\ T \not\ni i}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!(t_2-1)!\cdots t_k!v(T\sigma), \end{aligned} \quad (12)$$

where  $\sigma := \{T, T_2, \dots, T_k\}$ , and it is assumed in the third term that  $i \in T_2$ , the second block of  $\sigma$ . An equivalent expression, although less computationally efficient, is:

$$\begin{aligned} \phi_i(v) &= \frac{1}{n}v(N\{N\}) \\ &\quad + \sum_{\substack{T\sigma \in \mathfrak{C}(N) \\ T \not\ni i, T_2 \supset \{i\}}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!(t_2-1)!\cdots t_k! \left[ \frac{t+1}{t}v(T \cup i\sigma_{T \cup i}) - v(T\sigma) \right] \\ &\quad - \sum_{\substack{T\sigma \in \mathfrak{C}(N) \\ T \not\ni i, T_2 = \{i\}}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!t_3!\cdots t_k!v(T\sigma) \end{aligned}$$

with  $\sigma_{T \cup i}$  the partition obtained from  $\sigma$  by moving  $i \in T_2$  to  $T$ .

## 4.6 Relation with the Shapley value

Using (12), we give the explicit expression of the coalition formation value for a three players game, ( $i, j, k$  denote any three different players):

$$\begin{aligned}\phi_i(v) &= \frac{1}{3}v(N\{N\}) + \frac{1}{9}v(ij\{ij, k\}) + \frac{1}{9}v(ik\{ik, j\}) - \frac{2}{9}v(jk\{i, jk\}) \\ &\quad + \frac{1}{9}v(i\{i, jk\}) - \frac{1}{18}v(j\{j, ik\}) - \frac{1}{18}v(k\{k, ij\}) \\ &\quad + \frac{2}{9}v(i\{i, j, k\}) - \frac{1}{9}v(j\{i, j, k\}) - \frac{1}{9}v(k\{i, j, k\}).\end{aligned}$$

It can be checked that this differs from the other values proposed so far for PFF games (see their definitions in Section 6). One of the most important difference appears when we apply this formula to TU games. If we consider a PFF game with no externality, that is,  $v(S\pi) = v(S\pi')$  for any  $\pi$  and  $\pi'$  with  $\pi \ni S, \pi' \ni S$ , it naturally corresponds to a TU game. We can define the TU game  $\hat{v}$  from the PFF game  $v$  by:

$$\hat{v}(S) := v(S\pi) \text{ for any } S \in 2^N, \text{ any } \pi \ni S.$$

Then we have a value formula for a TU game from the coalition formation value in three person games,

$$\phi_i(\hat{v}) = \frac{1}{3}\hat{v}(N) + \frac{1}{9}\hat{v}(ij) + \frac{1}{9}\hat{v}(ik) - \frac{2}{9}\hat{v}(jk) + \frac{1}{3}\hat{v}(i) - \frac{1}{6}\hat{v}(j) - \frac{1}{6}\hat{v}(k).$$

This differs from the original Shapley value. Player  $i$  gets less share of  $\hat{v}(ij)$ ,  $\hat{v}(ik)$ , and  $-\hat{v}(jk)$  than s/he gets in the original Shapley value. However, most of the proposed values for PFF games coincide with the original Shapley value of TU games. This shows the fundamental difference between our value, which is rooted into the idea of coalition formation process, and the other ones, which are more in the classical coalitional view of games.

We elaborate more on this surprising result, and claim that we are able to recover the classical Shapley value, provided we use the weighted version of our value. In addition, we will get an interesting interpretation of the Shapley value in terms of coalition formation. Let us consider as above a game without externalities. Recall that the Shapley value can be interpreted as the average marginal contribution of players when considering all possible ways for the players to enter one by one the game, i.e., all permutations on  $N$ . Now note that this can be represented as particular processes of coalition formation. Indeed, suppose that players enter the game in the order  $1, 2, \dots, n$ . This corresponds to the process

$$\{1, 2, 3, \dots, n\} \rightarrow \{12, 3, 4, \dots, n\} \rightarrow \{123, 4, \dots, n\} \rightarrow \dots \rightarrow \{123 \dots n\}, \quad (13)$$

with the following convention: in each partition of the process, the first block is the block of players who have already entered the game (they already form a coalition), the other ones are the remaining players (they are still outside of the game). Note also that each process corresponds to two different orders: in the above example,  $2, 1, \dots, n$  corresponds to the same process. Therefore, if we want to recover the Shapley value, *only* those processes should be taken (there are  $2/n!$  such processes), i.e., their weight is  $\frac{2}{n!}$ , and the weight of the other ones is 0. Now, consider the above process and the scenarios  $\mathcal{S}_1$  of player 1 and  $\mathcal{S}_2$  of player 2:

$$\begin{aligned} 1\{1, 2, 3, \dots, n\} &\rightarrow 12\{12, 3, 4, \dots, n\} \rightarrow 123\{123, 4, \dots, n\} \rightarrow \dots \rightarrow N\{N\} \\ 2\{1, 2, 3, \dots, n\} &\rightarrow 12\{12, 3, 4, \dots, n\} \rightarrow 123\{123, 4, \dots, n\} \rightarrow \dots \rightarrow N\{N\}. \end{aligned}$$

Computing the marginal contribution of any player in these scenarios gives exactly the contribution used in the formula of the Shapley value for a given order, e.g., the contribution of player 3 in both is  $v(123\{123, 4, \dots, n\}) - v(12\{12, 3, 4, \dots, n\})$ , which is equal to  $\hat{v}(123) - \hat{v}(12)$ , while the contribution of player 2 is  $\hat{v}(12) - \hat{v}(1)$  in  $\mathcal{S}_1$  and  $\hat{v}(2)$  in  $\mathcal{S}_2$ . Therefore, putting a weight of  $1/2$  to  $\mathcal{S}_1$  and  $\mathcal{S}_2$  and 0 to the other scenarios  $\mathcal{S}_3, \dots, \mathcal{S}_n$ , we recover the classical Shapley value by averaging all these processes. In summary

$$\phi_i^{\mathcal{S}}(\hat{v}) = \frac{2}{n!} \sum_{\mathcal{P}_\sigma} \left( \frac{1}{2} \phi_i^{\mathcal{S}_{\sigma(1)} \leftarrow \mathcal{P}_\sigma}(v) + \frac{1}{2} \phi_i^{\mathcal{S}_{\sigma(2)} \leftarrow \mathcal{P}_\sigma}(v) \right) = \frac{1}{n!} \sum_{\mathcal{P}_\sigma} \left( \phi_i^{\mathcal{S}_{\sigma(1)} \leftarrow \mathcal{P}_\sigma}(v) + \phi_i^{\mathcal{S}_{\sigma(2)} \leftarrow \mathcal{P}_\sigma}(v) \right),$$

for  $i \in N$ , where  $\phi^{\mathcal{S}}$  denotes the classical Shapley value of a TU game, and  $\mathcal{S}_{\sigma(1)} \leftarrow \mathcal{P}_\sigma$  denotes the scenario of player  $\sigma(1)$  in the process of type (13) induced by  $\sigma$ , that is

$$\{\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(n)\} \rightarrow \{\sigma(1)\sigma(2), \sigma(3), \sigma(4), \dots, \sigma(n)\} \rightarrow \dots \rightarrow \{N\}.$$

Hence, the difference between the classical Shapley value and our value lies in the discarded scenarios  $\mathcal{S}_3, \dots, \mathcal{S}_n$  in each process  $\mathcal{P}_\sigma$  and the discarded processes. While it is clear why processes other than the  $\mathcal{P}_\sigma$ 's should be discarded, let us explain why those scenarios are discarded. In Shapley's view, there is a distinction between players already in the game (those in the room) and those still outside the game. Note that in process (13), only player 1 or player 2 can be first in the room and be present during all the process of formation of the grand coalition, and these correspond precisely to scenarios  $\mathcal{S}_1, \mathcal{S}_2$ . By contrast, in a coalition formation process, *all* players are always present in the game, only the structure of the society evolves. Therefore, all scenarios have to be taken into account. In summary, a well-defined coalition formation value should *never* collapse to the classical Shapley value.

Similarly, it is possible to recover the value  $\phi^{\text{CS}}$  of de Clippel and Serrano (and therefore of Pham Do and Norde, since they coincide, see Section 6). Its formula is

$$\phi_i^{\text{CS}}(v) = \phi_i^{\text{S}}(\tilde{v}), \quad i \in N,$$

where  $\tilde{v}(S) := v(S\{S, \{k\}_{k \in N \setminus S}\})$ . Due to the definition of  $\tilde{v}$ , we see that the only processes involved are, as in the case of the Shapley value, of the form (13), for all permutations on  $N$ . Due to the previous result on the Shapley value, in each such process, we take the scenario corresponding to the first player entering the game. Therefore,

$$\phi_i^{\text{SC}}(v) = \frac{1}{n!} \sum_{\mathcal{P}_\sigma} \left( \phi_i^{\text{S}_{\sigma(1)} \leftarrow \mathcal{P}_\sigma}(v) + \phi_i^{\text{S}_{\sigma(2)} \leftarrow \mathcal{P}_\sigma}(v) \right), \quad i \in N.$$

## 5 Coalition formation process in Cournot oligopoly

In this section, we give an application of the values we defined to a coalition formation process in Cournot oligopoly.

We consider a symmetric case where the worth of the embedded coalitions depends only on the number of blocks of the partition, i.e.,  $v(S\pi) = v_k$ , where  $\pi = \{S, S_2, \dots, S_k\}$ . Symmetric Cournot oligopoly games and symmetric common pool resource games satisfy this property (see Funaki and Yamato [9]). Under this condition, the Shapley coalition formation value is always the equal division of  $v(N\{N\})$  because of the symmetry of  $v$ . Hence this value is of no interest. However if processes matter, the value should also reflect the coalition formation process, hence we consider the Shapley process-value instead. In other words, to see the effect of a coalition process, a symmetric game is more adequate, because the difference of evaluations of the players arises from the asymmetry of the coalition formation process.

Consider a 4-person game with  $N = \{1, 2, 3, 4\}$ , and two typical coalition formation processes  $\mathcal{P}$  and  $\mathcal{P}'$ , defined as follows.

$$\begin{aligned} \mathcal{P} : \{1, 2, 3, 4\} &\rightarrow \{12, 3, 4\} \rightarrow \{123, 4\} \rightarrow \{1234\} \\ \mathcal{P}' : \{1, 2, 3, 4\} &\rightarrow \{12, 3, 4\} \rightarrow \{12, 34\} \rightarrow \{1234\} \end{aligned}$$

Coalition formation process  $\mathcal{P}$  represents a situation where all players enter the largest coalition one by one, while coalition formation process  $\mathcal{P}'$  represents somehow a symmetric coalition formation process. Here we suppose that in the process only bilateral mergers of two separate coalitions are feasible because of high negotiation costs. As mergers of major banks after 1960 in Japan, bilateral mergers are widely observed in economic environments (see Funaki and Yamato [10]).

The Shapley process-values of the processes  $\mathcal{P}$  and  $\mathcal{P}'$  are given by

$$\begin{aligned}\phi^{\mathcal{P}} &= \left( \frac{2v_1 + v_2 + 3v_3}{24}, \frac{2v_1 + v_2 + 3v_3}{24}, \frac{v_1 + 5v_2 - 3v_3}{12}, \frac{3v_1 - 2v_2}{4} \right) \\ \phi^{\mathcal{P}'} &= \left( \frac{v_1}{4}, \frac{v_1}{4}, \frac{v_1}{4}, \frac{v_1}{4} \right)\end{aligned}$$

These are the averages of four Shapley scenario-values induced from the corresponding processes  $\mathcal{P}$  and  $\mathcal{P}'$ . These values are interpreted as average contributions of the players in the processes. They are symmetric for  $\mathcal{P}'$ , but not for  $\mathcal{P}$ . However  $\phi^{\mathcal{P}}$  does not depend on  $v_4$  and  $\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}}$ . This is because in the first step of coalition formation process  $\mathcal{P}$ , the roles of players 1 and 2 are very similar since both are first players to enter, and  $v_4$  has no influence in that case. Here we could compare the contributions of the same firm in the two different processes.

This result can be extended to the general case. First we give the Shapley process-value for the following coalition formation process  $\mathcal{P}$  in an  $n$ -person game.

$$\mathcal{P} : \{1, 2, 3, \dots, n\} \rightarrow \{12, 3, \dots, n\} \rightarrow \{123, \dots, n\} \rightarrow \dots \rightarrow \{N\}$$

The value becomes

$$\phi_k^{\mathcal{P}} = \frac{1}{n} \left[ v_{n-k+2} + (k-1)(v_{n-k+1} - v_{n-k+2}) + \sum_{j=k}^{n-1} \frac{1}{j} (v_{n-j} - v_{n-j+1}) \right]$$

for  $k = 2, \dots, n-1$ , and

$$\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}}, \quad \phi_n^{\mathcal{P}} = \frac{1}{n} [v_2 + (n-1)(v_1 - v_2)].$$

Consider next the case of  $n = 2^m$  and a process  $\mathcal{P}'$  defined by

$$\begin{aligned}\mathcal{P}' : \{1, 2, 3, 4, \dots, n\} &\rightarrow \{12, 3, 4, \dots, n\} \rightarrow \{12, 34, \dots, n\} \rightarrow \dots \rightarrow \{12, 34, \dots, (n-1)n\} \\ &\rightarrow \{1234, \dots, (n-1)n\} \rightarrow \dots \rightarrow \{1234, \dots, (n-3)(n-2)(n-1)n\} \rightarrow \dots \\ &\rightarrow \{12345678, \dots, (n-3)(n-2)(n-1)n\} \rightarrow \dots \rightarrow \{123 \dots (2^{m-1}), (2^{m-1} + 1) \dots n\} \rightarrow \{N\}\end{aligned}$$

The Shapley process-value for process  $\mathcal{P}'$  is given by  $\phi_k^{\mathcal{P}'} = \frac{v_1}{n}$  for  $k = 1, \dots, n$ , which shows that everyone has a perfectly symmetric role in this process. This can be explained as follows. Every single player makes a two-person coalition when making the first coalition. Next, two two-person coalitions merge into 4-person coalitions, and so on. In each step, both coalitions have the same power. Then it is natural that finally each player has an

equal contribution. We remark that this is just an evaluation of a players' contributions in the processes, not the result of payoff negotiation among the players.

Next we apply this to a Cournot oligopoly with linear demand and constant average costs. We assume that the efficient coalition  $\{N\}$  is formed. Then we evaluate each firm's contribution depending on the coalition formation process.

Let us consider the following Cournot model with  $n$  identical firms. Let  $x_i$  be firm  $i$ 's output ( $i = 1, \dots, n$ ). The inverse demand function is given by  $p = a - \sum_{i=1}^n x_i$ , and the total cost function of firm  $i$  is  $cx_i$ , where  $a > c > 0$ . Given a coalition structure  $\pi = \{S_1, S_2, \dots, S_k\}$ , we assume that each coalition  $S_j$  is a player who chooses the total output level of its firms to maximize the sum of their profits, given the output levels of other coalitions. Then it is easy to check that the total profit of each coalition  $S_j$  at a unique Nash equilibrium is given by  $\frac{(a-c)^2}{(k+1)^2}$ . Without loss of generality, we assume that  $a - c = 1$ . This gives

$$v_k = \frac{1}{(k+1)^2}$$

and implies

$$\phi_k^{\mathcal{P}} = \frac{1}{n(n-k+3)^2} + \frac{(k-1)(2n-2k+5)}{n(n-k+2)^2(n-k+3)^2} + \sum_{j=k}^{n-1} \frac{2n-2j+3}{nj(n-j+1)^2(n-j+2)^2}$$

for  $k = 2, \dots, n-1$ ,

$$\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}}, \quad \phi_n^{\mathcal{P}} = \frac{5n-1}{36n}$$

$$\phi_k^{\mathcal{P}'} = \frac{1}{4n} \quad \text{for } k = 1, 2, \dots, n,$$

Moreover it holds that for  $k = 2, \dots, n-1$ ,

$$\phi_{k+1}^{\mathcal{P}} - \phi_k^{\mathcal{P}} = \frac{1}{n} \left[ v_{n-k+1} - v_{n-k+2} + \left(k - \frac{1}{k}\right)(v_{n-k} - v_{n-k+1}) - (k-1)(v_{n-k+1} - v_{n-k+2}) \right].$$

Since  $v_k = \frac{1}{(k+1)^2}$  is a concave decreasing function, the Shapley process-value  $\phi_k^{\mathcal{P}}$  of player  $k$  satisfies  $\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}} < \phi_3^{\mathcal{P}} \dots < \phi_n^{\mathcal{P}}$ , because  $v_{n-k+1} - v_{n-k+2} > 0$ ,  $k - \frac{1}{k} > k-1$  and  $v_{n-k} - v_{n-k+1} > v_{n-k+1} - v_{n-k+2}$ .

This shows that when players enter the game one by one, the later a player joins the current coalition, the more the contribution of this player in Cournot oligopoly. This is natural because  $v_k = \frac{1}{(k+1)^2}$  implies

$$v_n - v_{n-1} < v_{n-1} - v_{n-2} < \dots < v_3 - v_2 < v_2 - v_1.$$



However the original Shapley value of TU games cannot be used to analyze this situation because of the existence of externalities in Cournot oligopoly. Moreover, no value for PFF games can be used, because they will all lead to an equal division of  $v_1$  among players, and give no information on particular processes.

Turning to the symmetric coalition formation process  $\mathcal{P}'$ , we see that it induces a symmetric contribution of the players.

For the sake of completeness, we give numerical results for the 4-person case.

$$\phi^{\mathcal{P}} \approx \frac{1}{4}(.138, .138, .206, .528), \quad \phi^{\mathcal{P}'} = \frac{1}{4}(.25, .25, .25, .25).$$

The last player gets more profit in process  $\mathcal{P}$ , which is more than one half.

Symmetric values for PFF games give the equal division  $\frac{1}{4}(.25, .25, .25, .25)$  and do not say anything about a coalition formation process. However our Shapley process-value shows the difference between two processes  $\mathcal{P}$  and  $\mathcal{P}'$ . Process  $\mathcal{P}'$  is more egalitarian than process  $\mathcal{P}$ . Even in a symmetric oligopoly, the coalition formation process matters a lot.

Again we remark that this can induce the comparison of the two typical bilateral coalition formation processes, but cannot imply the results of the negotiations among firms. One might analyze each firm's contribution in historically given coalition formation process, like bank mergers in Japan, by this way.

## 6 Related works and concluding remarks.

As we discussed in the three person example (Section 4.6), the coalition formation value differs from other values of PFF games, since our value reflects the whole (dynamic) process of formation of coalitions, while the others one are more related to the coalitional (static) view of games. Here we make a comparison of the different axiom systems.

Myerson's value of a PFF game (Myerson [16]) is uniquely determined by the following three axioms: (S: symmetry) for any permutation  $\sigma$ ,  $\psi_i(v) = \psi_{\sigma i}(\sigma v)$ , (ADD: additivity) for any two games  $v, w$ ,  $\psi(v + w) = \psi(v) + \psi(w)$ , and (CAR: carrier) if  $T$  is a carrier, that is,  $v(T\pi) = v(T \cap S, \pi \wedge \{T, N \setminus T\}) \forall S\pi \in \mathfrak{C}(N)$ , then  $\sum_{k \in T} \psi_k(v) = v(N, \{N\})$ . (ADD) is a bit weaker but almost the same as (L). (CAR) is an extension of that in TU games, which is equivalent to the null axiom and the efficiency for the original Shapley value. In this sense it is a direct generalization of an axiom system for the original Shapley value. The most different axiom from our axiomatization of our scenario-value is (CAR). It is important to mention that the Myerson's

value of a PFF game in three person game is not monotonic: if  $v(i\{i, j, k\})$  increases, the Myerson's value of player  $i$  decreases.

Bolger's value (Bolger [4]) of a PFF game is uniquely determined by the following five axioms: (S), (L), (E: efficiency), (B-null) if  $j$  is a null player, that is,  $v(S\pi) = v(S \setminus \{j\}\pi')$  for any  $S, \pi, \pi'$  such that  $j \in S \in \pi, \pi' = \{T \cup j, S_1 \setminus j, \dots, S_k \setminus j\}$  with  $\pi = \{T, S_1, \dots, S_k\}, j \in N \setminus T$ , then  $\psi_j(v) = 0$ , and (MM: Modified Marginality) for any two games  $v, w$ , and  $i \in N$ , if for any  $S \ni i$ , and  $\pi \ni S, \sum_{T \in \pi, T \neq S} [v(S, \pi) - v(S \setminus i, \alpha_{iT})] = \sum_{T \in \pi, T \neq S} [w(S, \pi) - w(S \setminus i, \alpha_{iT})]$ , then  $\psi_i(v) = \psi_i(w)$ , where for  $i \in S, \pi = \{S, T, S_1, S_2, \dots, S_k\}, \alpha_{iT} = \{S \setminus \{i\}, T \cup i, S_1, \dots, S_k\}$ . Axioms (MM) as well as (B-null) are needed, which are both very strong. Again (B-null) is very different from our (SN).

Pham Do and Norde ([7]) proposed a value of a PFF game which is uniquely determined by the following four axioms: (S), (ADD), (E) and (DN-null): if  $j$  is a null player, that is,  $v(S, \pi \cup \{\{j\}\}) = v(S \cup \{j\}, (\pi \setminus \{S\}) \cup \{S \cup j\})$  for any  $S \subseteq N \setminus \{j\}$ , for any  $\pi \in \Pi(N \setminus \{j\})$ , then  $\phi_j(v) = 0$ . Fujinaka [8] gives a different axiomatization of the same value using (M: Marginality) for any two games  $v, w$ , if  $v(S \cup \{j\}, (\pi \setminus \{S\}) \cup \{S \cup j\}) - v(S, \pi \cup \{\{j\}\}) = w(S \cup \{j\}, (\pi \setminus \{S\}) \cup \{S \cup j\}) - w(S, \pi \cup \{\{j\}\})$  for any  $S \subseteq N \setminus \{j\}$ , for any  $\pi \in \Pi(N \setminus \{j\})$ , then  $\phi_j(v) = \phi_j(w)$ . He characterizes it by (S), (ADD), and (M). de Clippel and Serrano ([5]) also give another axiomatization of the same value using a balanced contribution axiom. Again the key difference is based on the null axiom (DN-null). The important remark is that the formula of their value of a PFF game  $v$  is given by  $\phi(v) = Shapley(\tilde{v})$ , where  $\tilde{v}(S) = v(S, \{S, \{k\}_{k \in N \setminus S}\})$ , and  $Shapley(\tilde{v})$  is the Shapley value of TU game  $\tilde{v}$ . This means their value does not utilize the full information of the PFF game.

Macho-Stadler et al. ([15]) proposed a collection of values for PFF games, called average values. They are characterized by (SS: strong symmetry), which is a stronger version of symmetry, (L), (E), and (B-null). The typical representative of this family is given by this formula:  $\phi(v) = Shapley(\bar{v})$ , where  $\bar{v}(S) = \frac{1}{\#\{\pi \in \Pi | \pi \ni S\}} \sum_{\{\pi \in \Pi | \pi \ni S\}} v(S\pi)$ . It satisfies in addition (SI: similar influence): if players  $i$  and  $j$  have similar influence in games  $v, v'$ , that is,  $v = v'$  except for 2 elements  $S\{S, i, j, S_2, \dots, S_k\}$  and  $S\{S, \{i, j\}, S_2, \dots, S_k\}$ , and  $v(S\{S, i, j, S_2, \dots, S_k\}) = v'(S\{S, \{i, j\}, S_2, \dots, S_k\})$ ,  $v(S\{S, \{i, j\}, S_2, \dots, S_k\}) = v'(S\{S, i, j, S_2, \dots, S_k\})$ , then  $\psi_i(v) = \psi_i(w)$ , and  $\psi_j(v) = \psi_j(w)$ . This value is also characterized by Albizuri et al. [1]. Clearly it utilizes the full information given by the PFF game.

Our scenario-value is different from any of the values above, mainly because it is not the (classical) Shapley value of some TU game induced by

the PFF game. Therefore, the null player axiom of the Shapley value of TU games naturally induces the above special null axioms for PFF games, which all differ from our null axiom. Moreover, we claim that the underlying structure of embedded coalitions (which is not explicitly mentioned in the above cited works) is implicitly suggested by the null axioms which are employed. Indeed, in the case of TU games, the null axiom is based on the difference between the worths of  $S$  and  $S \setminus i$ , assuming  $S \ni i$ . These elements are “neighbors” in the lattice  $(2^N, \subseteq)$ . In the PFF case, our scenario null axiom is defined along a maximal chain of  $\mathfrak{C}(N)_\perp$ . The B-null axiom takes the difference of worth between  $S \setminus i$  and  $S \setminus i \cup i$ , for  $S \ni i$ . In  $\mathfrak{C}(N)_\perp$ , these elements are not neighbors because they are on the same level. To recover them as neighbors, one possibility is the following: take the Boolean lattice  $(2^N, \subseteq)$ . Duplicate each element  $S$  as many times there are different possible coalition structures containing  $S$ , and indicate these coalition structures. Put all possible links between duplicates of an element  $S$  and duplicates of an element  $T$  if and only if these elements are linked in the Boolean lattice. Doing so, the B-nullity condition appears for neighbors elements. This structure also explains well the average approach: it can almost be seen on the picture. For illustration, we give  $\mathfrak{C}(N)$  and the structure induced by B-nullity for  $n = 3$ .

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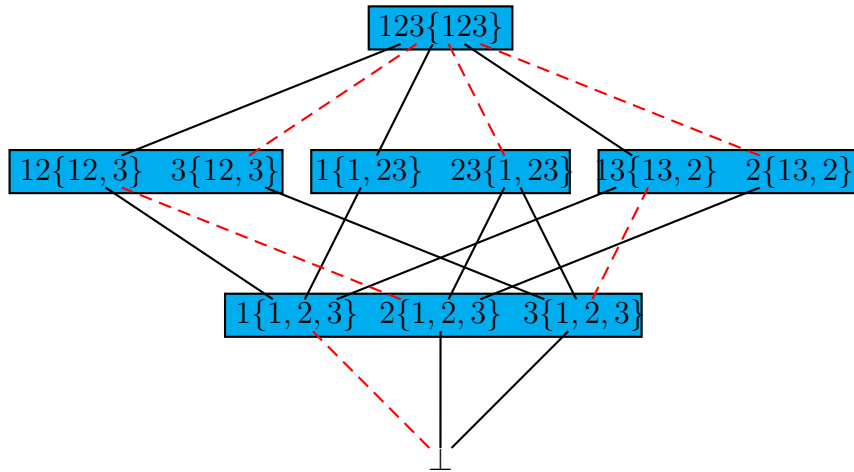


Figure 3: Diagram of  $(\mathfrak{C}(3)_{\perp}, \leq)$ . Elements with the same partition are framed in blue. Elements in nullity condition of player 1 are linked in dashed line

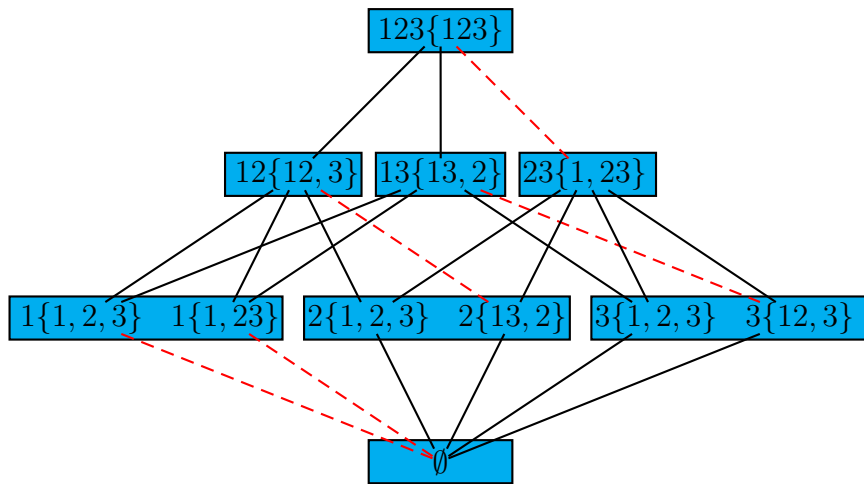


Figure 4: Diagram of the structure induced by the B-null axiom, used by Bolger, and Macho-Stadler et al.  $n = 3$ . Elements with same coalition (duplicates) are framed in blue. Elements in B-nullity condition of player 1 are linked in dashed line

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