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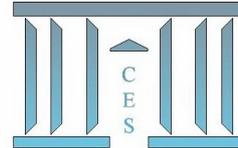
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**The core of games on distributive lattices : how to share
benefits in a hierarchy**

Michel GRABISCH, Lijue XIE

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The core of games on distributive lattices: how to share benefits in a hierarchy

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Abstract

Finding a solution concept is one of the central problems in cooperative game theory, and the notion of core is the most popular solution concept since it is based on some rationality condition. In many real situations, not all possible coalitions can form, so that classical TU-games cannot be used. An interesting case is when possible coalitions are defined through a partial ordering of the players (or hierarchy). Then feasible coalitions correspond to teams of players, that is, one or several players with all their subordinates. In these situations, it is not obvious to define a suitable notion of core, reflecting the team structure, and previous attempts are not satisfactory in this respect. We propose a new notion of core, which imposes efficiency of the allocation at each level of the hierarchy, and answers the problem of sharing benefits in a hierarchy. We show that the core we defined has properties very close to the classical case, with respect to marginal vectors, the Weber set, and balancedness.

Keywords: cooperative game, feasible coalition, core, hierarchy

JEL Classification: C71

1 Introduction

In cooperative game theory, a central topic is to define a rational way for distributing the total outcome among players (solution concept of this game). For transferable utility (TU) games, there exist two well-known solutions: the Shapley value [16], and the core [11]. The first one is defined by a set of rationality axioms: linearity, null player axiom, symmetry, and efficiency. It is applicable to any game. The second one avoids the formation of subcoalitions of the grand coalition, in the sense that any subcoalition will receive at least the amount it can achieve by itself. It may happen that no such solution exist. Classical results show under which conditions the core is nonempty, and give the structure of the core when the game is convex [17, 15].

In the classical setting of TU-games, any coalition $S \subseteq N$ can form, and each player can participate or not participate to the game. Mathematically speaking, this amounts to define the characteristic function of a game as a real-valued function v on the Boolean lattice 2^N , and vanishing at the empty set. More general definitions allowing a better modelling of reality have been proposed. We may distinguish between games having a restricted set of feasible coalitions (which may induce in some cases a hierarchy among players), and games permitting a more complex mechanism of participation. In the first category, we find games with precedence constraints, first proposed by Faigle [7] (see also [8]), games on matroids, convex geometries and other combinatorial structures [1, 2], games on regular set systems [19]; in the second category, we find multichoice games of Hsiao and Raghavan [14], fuzzy games [4], and games on product of distributive lattices [12]. In many cases, the characteristic function of such general games can be considered as a real-valued function defined over a (often distributive) lattice.

In this paper, we propose a definition for the core of TU-games whose characteristic function is $v : L \rightarrow \mathbb{R}$, where L is a distributive lattice. The mathematical motivation clearly appears from the above discussion, since many particular cases are recovered in our framework. On an application point of view, it solves in particular the problem of sharing benefits or costs in a hierarchical structure¹. Suppose that a company, whose set of employees is N , earns each year a total benefit $v(N)$. Employees are structured in a hierarchy, and form teams with other employees. The question is how to distribute $v(N)$ among employees, knowing the part of the total benefit achieved by all teams, so that no team can complain that it receives less than what it achieved by itself. This problem is exactly the problem of defining the core of a game on a distributive lattice, the latter representing the structure of teams in the company. Although our framework can be applied to many other situations, we mainly focus on this example for interpretation. A similar approach has been proposed by the authors for multichoice games [13]. The present work is much more general.

As our study will show, the situation appears to be more complex than for the classical case, although similar results still hold. A first immediate generalization of the classical definition of the core leads to what we call the *precore*, which happens to be a convex polyhedron which may be unbounded. We propose to call *core* a particular closed convex subset of it, satisfying some normalization constraint. Similarly to the classical case, we call *pre-Weber set* the convex hull of the marginal worth vectors, and the Weber set is a particular subset of it. We show that in case of convexity, the Weber set is included in the core. Moreover, the inclusion of the core into the Weber set holds in any case.

We begin by introducing and recall essential definitions about lattices and partially ordered sets (posets) in Section 2. Then Sections 3, 4 present the basic definitions for games on distributive lattices and the core. In the next sections 5, 6, 7, we study their properties. We indicate in Section 8 how to apply our results to the case of product lattices, encompassing the case of multichoice games. In Section 9, we give a brief account on related works in the literature.

¹In this paper, we consider games as *profit games*, hence the core is seen as a rational way to share benefits. We may consider *cost games* as well, reversing inequalities accordingly.

2 Posets, distributive lattices and levels

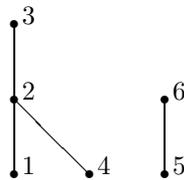
(see, e.g., Davey and Priestley [5]) In all this section, sets are considered to be finite. A set P equipped with a binary relation \leq is a *partially ordered set* (or *poset*) if the binary relation \leq satisfies reflexivity, antisymmetry and transitivity (*partial order*). For any two elements $x, y \in P$, $x < y$ means $x \leq y$ and $x \neq y$. If neither $x \leq y$ nor $y \leq x$, we say that x and y are *incomparable*. If there exists no $y \in P$ such that $y < x$, we call x a *minimal element* of this poset; if there exists no $y \in P$ such that $y > x$, x is a *maximal element* of this poset. We say that x is a *greatest element* of P if $x \geq y$ for all $y \in P$ (and similarly for the *least element*). The greatest and least elements of P are unique whenever they exist, and are denoted by \top and \perp respectively.

Let $x, y \in P$ and $x < y$. If there is no $z \in P$, such that $x < z < y$, we say that x *covers* y , denoted by $x \prec y$. A subset $A \subseteq P$ is called an *antichain* if it is a singleton or if any two elements of A are incomparable. On the other hand, a subset $C \subseteq P$ is called a *chain* if it contains no pair of incomparable elements². For $x, y \in P$ and $x < y$, a *chain* C *from* x *to* y can therefore be considered as a sequence of totally ordered elements from x to y , i.e., $C = \{x =: z_0 < z_1 < \dots < z_{k-1} < z_k := y\}$. The chain is *maximal* if no other chain from x to y contains it (equivalently, if $z_0 \prec z_1 \prec z_2 \prec \dots \prec z_k$). For convenience, a maximal chain from some minimal element to some maximal element of P is called simply a *maximal chain*. The set of all maximal chains of P is denoted by $\mathcal{C}(P)$. The *length* $\ell(C)$ of a chain C is $|C| - 1$. For any element $x \in P$, its *height* $h(x)$ is the length of a longest chain from some minimal element to x :

$$h(x) = \max\{\ell(C) \mid C = \{x_0, x_1, \dots, x\}\}.$$

The height function induces a natural partition $\{Q_1, \dots, Q_q\}$ of P as follows: Q_i is the set of elements of height $i - 1$, $i = 1, \dots, q$. Evidently, Q_1 is the set of all minimal elements of P , and Q_q is a subset of its maximal elements. The set Q_i is called the *i -th level* of P .

Example 1. Let us consider the following poset.



$$P = \{1, 2, 3, 4, 5, 6\}$$

This poset has 3 levels: $Q_1 = \{1, 4, 5\}$, $Q_2 = \{2, 6\}$ and $Q_3 = \{3\} \subseteq \{3, 6\}$, the set of maximal elements.

Let P be a poset and its partition in levels $\mathcal{Q} = \{Q_1, \dots, Q_q\}$. Clearly, for any $x \in Q_i$, $y \in Q_j$ such that $x < y$, we have $i < j$. But the converse is not always true: even if $x \in Q_i$, $y \in Q_j$ and $i < j$, x and y may be incomparable.

For any two elements $x, y \in P$, the *supremum* $x \vee y$ of x and y is the least element of all those greater than x and y (least upper bound), whenever it exists. Similarly, the

²Note that a singleton is both an antichain and a chain.

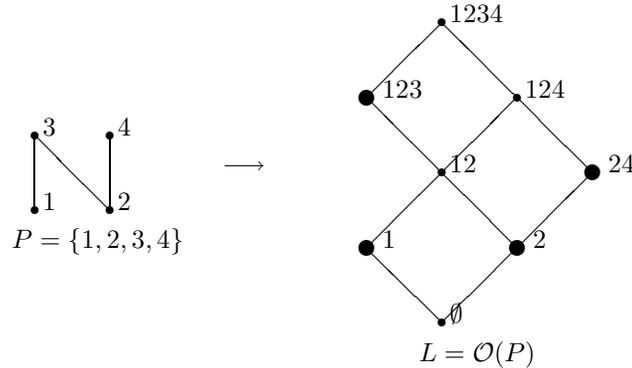
infimum $x \wedge y$ of x and y is the greatest lower bound of x and y . A *lattice* L is a poset such that for any $x, y \in L$, $x \vee y$ and $x \wedge y$ exist. Clearly, in a finite lattice, \top, \perp always exist. In addition, L is *distributive* if \vee, \wedge satisfy the distributive law: for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{or equivalently} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let L be a lattice and $x \in L$. If $x \neq \perp$ cannot be written as a supremum, i.e., $x = y \vee z$ implies $y = x$ or $z = x$, then x is said to be *join-irreducible*. Equivalently, a join-irreducible element covers only one element. Denote the set of all join-irreducible elements of L by $\mathcal{J}(L)$, and the set of join-irreducible elements less than or equal to an element $x \in L$ by $\eta(x)$. In a distributive lattice, any maximal chain has length $|\mathcal{J}(L)|$.

Let P be a poset and consider $x \in P$. The *principal ideal* of x is defined by $\downarrow x := \{y \in P \mid y \leq x\}$. Similarly, the *principal filter* of x is $\uparrow x := \{y \in P \mid y \geq x\}$. Let $Q \subset P$. The subset Q is a *downset* of P if $x \in Q, y \leq x$ imply $y \in Q$. Any downset is a union of principal filters. We denote the set of all downsets of P by $\mathcal{O}(P)$. Birkhoff proved that, if L is a distributive lattice, L is isomorphic to $\mathcal{O}(\mathcal{J}(L))$ by the isomorphism η [3]. Put otherwise, any poset P generates a distributive lattice $\mathcal{O}(P)$, whose set of join-irreducible elements is isomorphic to P . This well-known result, fundamental in this paper, is illustrated in the next example.

Example 2. We consider the poset (P, \leq) given in the left. The set $\mathcal{O}(P)$ of all its downsets is given in the right (for ease of notation, $\{i, j\}$ is denoted by ij and so on.). It is a distributive lattice, and its join-irreducible elements are 1, 2, 24 and 123 (figured with larger circles on the figure). Observe that the sub-poset $\mathcal{J}(\mathcal{O}(P))$ of its join-irreducible elements is isomorphic to P .



Let P be a poset. The partition of P into levels Q_1, \dots, Q_q induces a partition $\{S_1, \dots, S_q\}$ of its corresponding distributive lattice $\mathcal{O}(P)$ in the following way:

$$S_1 := \mathcal{O}(Q_1), \quad S_2 := \mathcal{O}(Q_1 \cup Q_2) \setminus S_1, \dots, \quad S_q := \mathcal{O}(P) \setminus (S_1 \cup \dots \cup S_{q-1}).$$

The following proposition shows some properties of the partition $\{S_1, \dots, S_q\}$.

Proposition 1. Let P be a poset and $\{Q_1, \dots, Q_q\}$ be its partition into levels. Then the following holds.

- (i) $\top_i := \cup_{j=1}^i Q_j$ is the greatest element of S_i for all $i = 1, \dots, q$;

(ii) Denoting respectively by \perp, \top the bottom and top elements of $\mathcal{O}(P)$, we have $\perp < \top_1 < \dots < \top_q = \top$;

(iii) $S_1 = \downarrow \top_1$, and $S_i = (\downarrow \top_i) \setminus (\downarrow \top_{i-1})$ for $i = 2, \dots, q$.

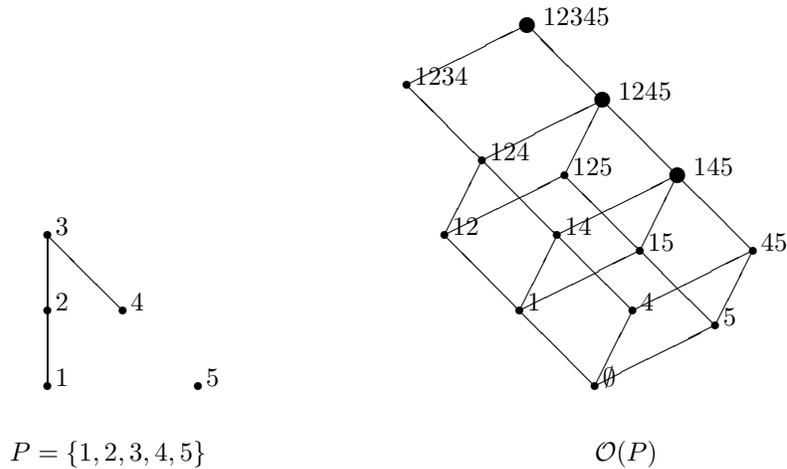
Proof. (i) We first show that $\cup_{j=1}^i Q_j \in \mathcal{O}(P)$. Consider $x \in \cup_{j=1}^i Q_j$ and $y < x$. Suppose $y \in Q_j$ for some $j > i$. Then there exists a longest maximal chain of length at least i from some minimal element y_0 to y . But this chain could be prolonged till x and would have a length greater than i , a contradiction with the definition of x . Hence $y \in \cup_{j=1}^i Q_j$, and $\cup_{j=1}^i Q_j$ is a downset of P .

Moreover, $\cup_{j=1}^i Q_j \in \mathcal{O}(\cup_{j=1}^i Q_j)$ and does not belong to S_1, \dots, S_{i-1} , hence $\cup_{j=1}^i Q_j \in S_i$. Since any $x \in S_i$ is such that $x \subseteq \cup_{j=1}^i Q_j$, this proves that $\cup_{j=1}^i Q_j$ is the greatest element of S_i .

(ii) and (iii) are straightforward. □

The collection of all \top_i 's is denoted by \top_P . A maximal chain of $L := \mathcal{O}(P)$ passing through all \top_i 's in \top_P is called a *restricted maximal chain*. We denote the set of restricted maximal chains by $\mathcal{C}_r(L)$. From Proposition 1, $\mathcal{C}_r(L)$ is never empty.

Example 3. We consider the following poset P and its corresponding distributive lattice $\mathcal{O}(P)$.



Then

$$Q_1 = \{1, 4, 5\}, Q_2 = \{2\}, Q_3 = \{3\}.$$

$$S_1 = \{1, 4, 5, 14, 15, 45, 145\}, S_2 = \{12, 124, 125, 1245\}, S_3 = \{1234, 12345\}.$$

$$\top_1 = 145, \top_2 = 1245, \top_3 = 12345.$$

3 Distributive games

As said in the introduction, there are two main applications of games defined on distributive lattices, namely to model restriction on the set of feasible coalitions, and to allow for each player several possible (partially ordered) actions for participation to the game. Our development will follow the first stream, and so is close to the framework of Faigle and

Kern [8]. We will comment briefly the second one, which is developed in [12], in Section 8, where we will indicate how our results can be straightforwardly applied to this case.

In the rest of the paper, $N = \{1, \dots, n\}$ denotes the set of players, which we suppose to be endowed with a partial order \leq . The relation $i \leq j$, with $i, j \in N$, indicates that player i is below player j , or a subordinate of j (this is called *precedence constraint* by Faigle [7]). Hence, the relation \leq describes a *hierarchy* among players. Practically, this means that, if j participates to the game, all subordinates of j must also participate to it. Therefore, a coalition $S \subseteq N$ is *feasible* if $j \in S$ and $i \leq j$ implies that $i \in S$. This has two important consequences, which can be drawn from Section 2:

- (i) The set of feasible coalitions is precisely the set of all downsets of (N, \leq) , denoted by $\mathcal{O}(N)$.
- (ii) The set of feasible coalitions is a distributive lattice.

Definition 1. Let $L := \mathcal{O}(N)$ be the collection of all feasible coalitions (all downsets of (N, \leq)). A *game on the distributive lattice* L is a real-valued function $v : L \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. More simply, we call it a *distributive game*.

Note that the classical definition of a TU-game is recovered when (N, \leq) is an antichain, that is, when there is no hierarchy and all players are “on the same level”. Then clearly no restriction on coalitions exist, and any $S \in 2^N$ is feasible.

We introduce the following properties.

Definition 2. Let v be a distributive game on $\mathcal{O}(N)$.

- (i) v is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \in \mathcal{O}(N)$.
- (ii) v is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T, S, T \in \mathcal{O}(N)$.

4 Precore and core

We take the classical point of view for defining the core, that is, it is a set of pre-imputations satisfying some rationality condition, which prevent players to form sub-coalitions. A *pre-imputation* is a vector $\phi \in \mathbb{R}^n$ such that $\sum_{i=1}^n \phi_i = v(N)$, where ϕ_i is the amount of money given to player i . We use the usual shorthand $\phi(S) := \sum_{i \in S} \phi_i$ for any subset $S \subseteq N$.

4.1 The precore

In the classical case, the rationality condition is $\phi(S) \geq v(S)$ for all coalitions S . Adapting it to our framework leads to the following notion we call “precore”, for reasons which will become clear after.

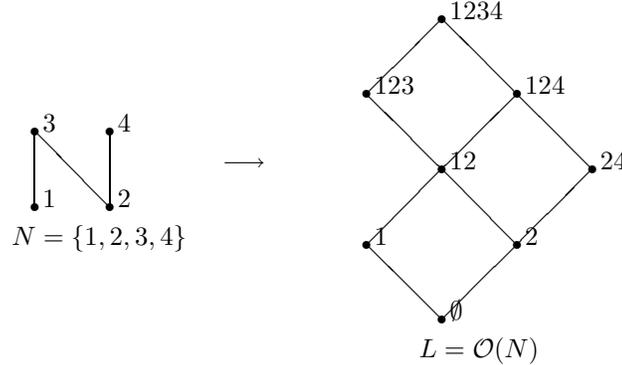
Definition 3. The *precore* of a distributive game v on $\mathcal{O}(N)$ is defined by the following set.

$$\mathcal{PC}(v) := \{\phi \in \mathbb{R}^n \mid \phi(N) = v(N) \text{ and } \phi(S) \geq v(S), \forall S \in \mathcal{O}(N)\}.$$

It is equivalent to definitions of cores defined by Faigle [7] for games under precedence constraints, and by Van den Nouweland *et al.* [18] for multichoice games (see Section 8).

Clearly, the precore is a closed convex polyhedron. In the classical case (TU-games), the conditions $\phi(S) \geq v(S)$ for singletons suffice to ensure the boundedness of ϕ ³. However, in our framework, it may happen that some singletons are not feasible (because they are not subordinates). If i is such a singleton, there is no lower bound for ϕ_i . The consequence of this is that in general the precore is unbounded, and hence its definition cannot be operational (see next example).

Example 4. We consider the poset (N, \leq) of Example 2 (left), and its corresponding distributive lattice $\mathcal{O}(N)$ (right).



Let v be a distributive game on $\mathcal{O}(N)$. By definition of the precore, any element ϕ of the precore must satisfy:

$$\begin{aligned} \phi_1 + \phi_2 + \phi_3 + \phi_4 &= v(\top) = v(1234) \\ \phi_1 &\geq v(1) \\ \phi_2 &\geq v(2) \\ \phi_1 + \phi_2 &\geq v(12) \\ \phi_2 + \phi_4 &\geq v(24) \\ \phi_1 + \phi_2 + \phi_3 &\geq v(123) \\ \phi_1 + \phi_2 + \phi_4 &\geq v(124). \end{aligned}$$

Whenever ϕ_1, ϕ_2 are large enough, we can always find out some ϕ_3, ϕ_4 to satisfy all conditions, i.e., ϕ_1, ϕ_2 can be arbitrarily large. Hence the precore of this game has four rays (infinite directions): two positive rays for ϕ_1, ϕ_2 and two negative rays for ϕ_3, ϕ_4 .

Since a sharing cannot attain infinite values, the precore is of no practical use, and we have to refine our definition so that we obtain a bounded set. We denote the set of vertices of some convex set by $\text{Ext}(\cdot)$, and the convex hull of some set by $\text{co}(\cdot)$. We define the *finite part of the precore* by $\mathcal{PC}^F(v) := \text{co}(\text{Ext}(\mathcal{PC}(v)))$. It is a polytope, and by the theory of polyhedra (see, e.g., Ziegler [20]), we know that $\mathcal{PC}(v)$ is the Minkovski sum of its finite part and its rays.

A simple remedy to the above described drawback would be to impose that ϕ should be bounded from below by some quantity. In the sequel, we will provide a much less

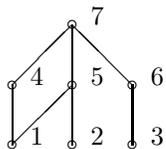
³In most cases, $v(\{i\}) \geq 0$ for all $i \in N$, which entails the nonnegativity of ϕ .

arbitrary and much better answer to this problem, both for mathematical properties (since we will see that we are able to keep many of the classical results on the core), and for the practical side, illustrated hereafter with an example of benefit sharing in a hierarchical structure, one of our main motivation.

4.2 How to share benefits in a hierarchical structure

The example we develop in this section will lead naturally to a new definition of the core.

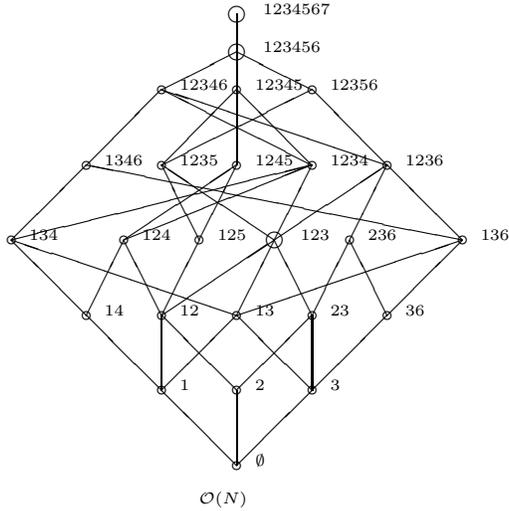
We consider for illustration purpose a company with 7 employees $N = \{1, 2, 3, 4, 5, 6, 7\}$, and we represent the hierarchy among employees by the partial order \leq on N . To be enough general, we may even consider that one employee may have more than one direct superior (it could be the case if the employee participates to several projects or belongs to several teams). Hence the partial order is not necessarily a tree. The poset below depicts the hierarchy in N .



$$N = \{1, 2, 3, 4, 5, 6, 7 \mid 1 < 4 < 7, 2 < 5 < 7, 3 < 6 < 7 \text{ and } 1 < 5\}$$

We see that employee 1 has two direct superiors, namely 4 and 5.

As explained in Section 3, feasible coalitions are downsets of (N, \leq) . In this context, feasible coalitions correspond to *feasible teams* of the company, in the sense that the presence of an employee in a feasible team implies the presence of all employees below. It must be remarked that in general a feasible team in the above sense may be formed of several teams in the usual sense, which we may call *elementary teams* (that is, a boss and all employees below): in terms of poset terminology, this amounts to say that a downset is the union of principal ideals (see Section 2). For example, the feasible team 12356 is formed of the two elementary teams 125 and 36, with bosses 5 and 6. Note also that 3 itself is a team reduced to a singleton. We give below the distributive lattice of all teams ordered by inclusion.



Computing the levels Q_k and top elements \top_k , we get

$$Q_1 = \{1, 2, 3\}, Q_2 = \{4, 5, 6\}, Q_3 = \{7\},$$

$$\top_1 = 123, \top_2 = 123456, \top_3 = 1234567 = N.$$

Level Q_k corresponds to employees having the same rank⁴ k in the company, and \top_k is the smallest feasible team containing all employees up to rank k . We call it the *principal team of rank k* . All feasible teams in S_k are called *feasible teams of rank k* . From Proposition 1 (iii), we know that $S_k = \downarrow \top_k \setminus \downarrow \top_{k-1}$.

At the end of each year, the total benefit (or a fixed proportion of it) has to be distributed among all employees as a bonus. We denote it by $v(N)$. For a given feasible team S , we denote by $v(S)$ the benefit achieved by S (and only by S) which is brought to the company, and we denote by $\phi(S)$ the bonus or reward given to S . To achieve the sharing, we propose to perform a local sharing at each hierarchical level Q_k . More precisely:

- For hierarchical level Q_k , the amount to be shared among the employees of this level is $v(\top_k) - v(\top_{k-1})$, that is, roughly speaking, the difference between the benefit achieved by all employees up to level k , and the benefit achieved by all employees of level strictly lower than k . In a sense, this is the genuine contribution of level k .
- Inside a given level Q_k , the sharing is done freely, up to the condition that for each feasible team $S \in S_k$, $\phi(S) \geq v(S)$. Otherwise, if for some S , $\phi(S) < v(S)$, then the team S may split from N and build a new independent company, because the benefit achieved by S alone is more than that S will receive.

⁴Mathematically speaking the same height, see Section 2.

Assuming there are l hierarchical levels, this gives the linear system in ϕ

$$\begin{aligned}\phi(Q_l) &= v(N) - v(\top_{l-1}) \\ \phi(Q_{l-1}) &= v(\top_{l-1}) - v(\top_{l-2}) \\ &\vdots \\ \phi(Q_1) &= v(\top_1)\end{aligned}$$

and since $\top_k = \cup_{i=1}^k Q_i$, and the Q_k 's are pairwise disjoint, we deduce that $\phi(\top_k) = \sum_{i=1}^k \phi(Q_i) = v(\top_k)$ for $k = 1, \dots, l$. Conversely, imposing $\phi(\top_k) = v(\top_k)$ for $k = 1, \dots, l$ leads to the above system.

Applying this procedure to our example, we get

$$\begin{aligned}v(N) - v(123456) &\text{ is given to the group } \mathcal{Q}_3 = \{7\}, \\ v(123456) - v(123) &\text{ is given to the group } \mathcal{Q}_2 = \{4, 5, 6\}, \\ v(123) &\text{ is given to the group } \mathcal{Q}_1 = \{1, 2, 3\}.\end{aligned}$$

4.3 The core

From the previous development, we are led to the following definition.

Definition 4. *The core of a distributive game v on $\mathcal{O}(N)$ is defined by*

$$\mathcal{C}(v) := \{\phi \in \mathcal{PC}(v) \mid \phi(\top_i) = v(\top_i), \forall \top_i \in \top_N\}.$$

Hence, the normalization condition is imposed at each level of the hierarchy.

Evidently, the core is a closed convex polyhedron.

Theorem 1. The core of a distributive game v on $\mathcal{O}(N)$ is compact, hence it is a polytope.

Proof. We know that, in the space \mathbb{R}^n , a subspace is compact if and only if it is closed and bounded. Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be the collection of levels of the poset (N, \leq) . We show by induction on the level number that $\phi_i, \forall i \in N$ is lower bounded.

In the first level, any element $i_1 \in Q_1$ corresponds to the singleton $\{i_1\}$ of $\mathcal{O}(N)$. Hence $\phi_{i_1} \geq v(i_1)$ for all $\phi \in \mathcal{C}(v)$.

Suppose that the property holds till the k -th level.

In the $(k+1)$ -th level, by definition of levels, any element $i_{k+1} \in Q_{k+1}$ corresponds to some subsets $L^1 \subseteq Q_1, \dots, L^k \subseteq Q_k$ such that $\{i_{k+1}\} \cup (\cup_{i=1}^k L^i) = \downarrow i_{k+1} \in \mathcal{O}(N)$. Hence $\phi(\downarrow i_{k+1}) \geq v(\downarrow i_{k+1})$ for all $\phi \in \mathcal{C}(v)$. We have

$$\begin{aligned}\phi_{i_{k+1}} &= \phi(\downarrow i_{k+1}) - \phi(\cup_{i=1}^k L^i) \\ &\geq v(\downarrow i_{k+1}) - \phi(\cup_{i=1}^k L^i) \\ &= v(\downarrow i_{k+1}) - \phi(\top_k) + \phi(\top_k \setminus (\cup_{i=1}^k L^i)) \\ &= v(\downarrow i_{k+1}) - v(\top_k) + \phi(\top_k \setminus (\cup_{i=1}^k L^i))\end{aligned}$$

By induction, $\phi(\top_k \setminus (\cup_{i=1}^k L^i))$ is lower bounded, so $\phi_{i_{k+1}}$ is also lower bounded.

Hence, every coordinate has a lower bound. Finally since $\phi(\top) = v(\top)$, the core is bounded.

To show that the core is closed, by the convexity of the core, it suffices to show that all vertices are in the core. By the definition of the core and linear programming, if ϕ is a vertex of the core, then there exist some downsets S of (N, \leq) such that $\phi(S) = v(S)$, and for other downsets T of (N, \leq) , $\phi(T) > v(T)$. Hence ϕ trivially belongs to the core. \square

An important remark is that our definition collapses to the classical one if the set of feasible coalitions is 2^N . Indeed, in this case, (N, \leq) is an antichain, so that there is only one level $Q_1 = N$, and $\top_1 = N$.

To end this section, we come back to Example 4, and compute its core. We have

$$Q_1 = \{1, 2\}, Q_2 = \{3, 4\}, \quad \top_1 = 12, \top_2 = 1234.$$

Hence to the previous system, we add the following equation:

$$\phi(1) + \phi(2) = v(12).$$

Clearly, $\phi(1), \phi(2)$ can no more take infinite values.

5 Balancedness

To find out necessary and sufficient conditions for the nonemptiness of the precore, we introduce the notion of pre-balancedness.

Definition 5. (i) A collection \mathcal{B} of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ is *pre-balanced* if it exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S: S \ni i} \mu(S) = 1$, for all $i \in N$.

(ii) A distributive game v is *pre-balanced* if for every pre-balanced collection \mathcal{B} of elements of $L \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S \in \mathcal{B}} \mu(S) v(S) \leq v(N).$$

Proposition 2. A distributive game has a nonempty precore if and only if it is pre-balanced.

Proof. Nonemptiness of the precore of a distributive game is equivalent to find out a vector $\phi \in \mathbb{R}^n$ satisfying the following conditions:

$$\phi(N) = \sum_{i \in N} \phi_i = v(N) \text{ and } \phi(S) = \sum_{i: i \in S} \phi_i \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Consider the following linear program with the variables $\phi_i \in \mathbb{R}$, $i \in N$:

$$\min z = \sum_{i \in N} \phi_i \text{ under } \sum_{i: i \in S} \phi_i \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Its optimal value is $z = v(N)$ if and only if the precore is nonempty.

Its dual problem is

$$\max w = \sum_{S \in \mathcal{O}(N) \setminus \{\emptyset\}} \mu(S)v(S)$$

under

$$\begin{aligned} \sum_{S: i \in S} \mu(S) &= 1, \quad \forall i \in N, \\ \mu(S) &\geq 0, \quad \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}. \end{aligned}$$

By the duality theorem, it has the same optimal value $w = v(N)$ if we can find out some μ satisfying all conditions. This is the desired result. \square

Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be the collection of levels of N and $\mathbb{T}_N = \{\mathbb{T}_1, \dots, \mathbb{T}_q\}$ be the collection of top elements of every level of $\mathcal{O}(N)$. Similarly, we introduce the notion of balancedness as follows.

Definition 6. (i) A collection \mathcal{B} of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ is *balanced* if it exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S: S \ni i} \mu(S) = q - k + 1$, for all $i \in Q_k$, $k = 1, \dots, q$.

(ii) A distributive game v is *balanced* if for every balanced collection \mathcal{B} of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S \in \mathcal{B}} \mu(S)v(S) \leq v(N).$$

Let us come back to Example 4. The conditions for balancedness read

$$\sum_{S \ni 1} \mu(S) = \sum_{S \ni 2} \mu(S) = 2, \quad \sum_{S \ni 3} \mu(S) = \sum_{S \ni 4} \mu(S) = 1.$$

The sum for elements of lower height have a higher value since the more an element is in the bottom of the hierarchy, the more it is frequent in coalitions. Examples of balanced collections are

$$\begin{aligned} \mathcal{B} &= \{123, 124\} \text{ with weights } 1, 1 \\ \mathcal{B} &= \{1234, 12, 1, 2\} \text{ with weights } 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}. \end{aligned}$$

Proposition 3. A distributive game has a nonempty core if and only if it is balanced.

Proof. Nonemptiness of the core of a distributive game is equivalent to find out a vector $\phi \in \mathbb{R}^n$ satisfying the following conditions:

$$\phi(\mathbb{T}_i) = v(\mathbb{T}_i), \forall \mathbb{T}_i \in \mathbb{T}_N \text{ and } \phi(S) \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Consider the following linear program with the variables $\phi_i \in \mathbb{R}$, $i \in N$:

$$\min z = \sum_{\mathbb{T}_i \in \mathbb{T}_N} \phi(\mathbb{T}_i) = \sum_{i=1}^q \sum_{k=1}^i \phi(Q_k)$$

under

$$\sum_{i:i \in S} \phi_i \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Its optimal value is $z = \sum_{T_i \in \mathcal{T}_N} v(T_i)$ if and only if the core is nonempty.

Its dual problem is

$$\max w = \sum_{S \in L \setminus \{\emptyset\}} \mu(S)v(S)$$

under

$$\sum_{S:S \ni i} \mu(S) = q - k + 1, \quad \forall i \in Q_k, k = 1, \dots, q$$

$$\mu(S) \geq 0, \quad \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

By the duality theorem, it has the same optimal value $w = \sum_{T_i \in \mathcal{T}_N} v(T_i)$ if we can find out some μ satisfying all conditions. This is the desired result. \square

6 Marginal worth vectors

Since $\mathcal{O}(N)$ is a distributive lattice with n join-irreducible elements, we know from Section 2 that any maximal chain has length n . Therefore, let $C = \{S_0 := \emptyset \prec S_1 \prec \dots \prec S_n := N\}$ be a maximal chain in $L := \mathcal{O}(N)$. To each maximal chain we associate a permutation on N , $\pi : N \rightarrow N$, such that the additional element between any two consecutive coalitions S_{i-1}, S_i of C is $\pi(i)$. So we have $S_i = \{\pi(1), \pi(2), \dots, \pi(i)\}$.

It is easy to see that π defines a linear extension of \leq on N , and moreover, any linear extension of \leq corresponds to such a permutation π . Indeed, $i < j$ implies that $\pi(i) > \pi(j)$ will never happen, for any permutation. Conversely, if i_1, \dots, i_n is a linear extension, then $k < l$ implies that $i_k > i_l$ cannot happen. Hence $\{\{i_1\}, \{i_1, i_2\}, \dots, \{i_1, \dots, i_n\}\}$ is a chain of downsets, defining a permutation π on N .

Definition 7. The *marginal worth vector* $\psi^C \in \mathbb{R}^n$ associated to C and v is defined by

$$\psi_j^C := v(S_i) - v(S_{i-1}), \quad \forall i \in N,$$

with $j = S_i \setminus S_{i-1}$.

The set of all marginal worth vectors ψ^C for all maximal chains is denoted by $\mathcal{M}(v)$. We can easily get

$$\psi^C(S_i) := \sum_{k=1}^i \psi_{\pi(k)}^C = \sum_{k=1}^i (v(S_k) - v(S_{k-1})) = v(S_i), \forall S_i \in C.$$

Definition 8. The *pre-Weber set* $\mathcal{PW}(v)$ of v is defined as the convex hull of all vectors in $\mathcal{M}(v)$:

$$\mathcal{PW}(v) := \text{co}(\mathcal{M}(v)).$$

Theorem 2. For any distributive game v , the polytope of the precore is included in the pre-Weber set, i.e, $\mathcal{PC}^F(v) \subseteq \mathcal{PW}(v)$.

Proof. We show that all vertices of the precore are included in the convex hull of the set of marginal worth vectors by induction on the number of players $|N|$.

- (i) If $N = \{1\}$, then $\mathcal{PC}(v) = \mathcal{PW}(v) = \mathcal{M}(v)$. The statement is true.
- (ii) Suppose that the statement is true whenever $|N| < n$.
- (iii) Let $N = \{1, \dots, n\}$ and $\phi \in \text{Ext}(\mathcal{PC}(v))$. Then $\exists S \in \mathcal{O}(N) \setminus \{N, \emptyset\}$ such that $\phi(S) = v(S)$.

Let $u(T) := v(T)$, $\forall T \subseteq S, T \in \mathcal{O}(N)$. We have clearly $\phi|_S \in \mathcal{PC}(u)$, and by induction,

$$(\phi|_S)_i = \sum_{\psi^k \in \mathcal{M}(u)} \alpha_k \psi_i^k \text{ with } \sum_{k: \psi^k \in \mathcal{M}(u)} \alpha_k = 1, \alpha_k \in [0, 1], \quad \forall i \in S.$$

Let $w(T) := v(S \cup T) - v(S)$, $\forall T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$. Evidently, w is a game on the distributive lattice $\mathcal{O}(N \setminus S)$, the latter being isomorphic to the distributive sublattice $\uparrow S = \{T \supseteq S \mid T \in \mathcal{O}(N)\}$ by the mapping $\theta : T \rightarrow T \cup S, \forall T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$. We have, for all $T \subseteq N \setminus S, S \cup T \in \mathcal{O}(N)$,

$$\phi|_{N \setminus S}(T) = \phi(T) = \phi(S \cup T) - \phi(S) \geq v(S \cup T) - v(S) = w(T)$$

and

$$\phi|_{N \setminus S}(N \setminus S) = \phi(N) - \phi(S) = v(N) - v(S) = w(N \setminus S).$$

Hence $\phi|_{N \setminus S} \in \mathcal{PC}(w)$, i.e., $(\phi|_{N \setminus S})_i = \sum_{\psi^k \in \mathcal{M}(w)} \beta_k \psi_i^k$ where $\sum_{k: \psi^k \in \mathcal{M}(w)} \beta_k = 1, \beta_k \in [0, 1] \quad \forall i \in N \setminus S$.

Any $\psi^i \in \mathcal{M}(u)$ corresponds to a maximal chain C^i from \emptyset to S . Any $\psi^j \in \mathcal{M}(w)$ corresponds to a maximal chain C^j from \emptyset to $N \setminus S$ in $\mathcal{O}(N \setminus S)$. By the mapping θ , each element $T \in C^j$ corresponds to an element $\theta(T) = T \cup S \in \mathcal{O}(N)$, i.e., the maximal chain C^j corresponds to a maximal chain $C'^j := \theta(C^j)$ of $\mathcal{O}(N)$ from S to N . Let

$$\psi_k^{(i,j)} := \begin{cases} \psi_k^i, & \text{if } k \in S \\ \psi_k^j, & \text{if } k \in N \setminus S. \end{cases}$$

Then $\psi^{(i,j)}$ corresponds to the maximal chain $C = (C^i, C'^j)$ from \emptyset to N , i.e., $\psi^{(i,j)} \in \mathcal{M}(v)$. We can show that, for all i such that $\psi^i \in \mathcal{M}(u)$ and all j such that $\psi^j \in \mathcal{M}(w)$,

$$\begin{aligned} \phi_k &= \sum_i \alpha_i \psi_k^i = \sum_i \alpha_i \psi_k^{(i,j)} \\ &= \sum_j \beta_j \left(\sum_i \alpha_i \psi_k^{(i,j)} \right) = \sum_j \sum_i \alpha_i \beta_j \psi_k^{(i,j)}, \quad \forall k \in S, \end{aligned}$$

and

$$\begin{aligned} \phi_k &= \sum_j \beta_j \psi_k^j = \sum_j \beta_j \psi_k^{(i,j)} \\ &= \sum_i \alpha_i \left(\sum_j \beta_j \psi_k^{(i,j)} \right) = \sum_i \sum_j \alpha_i \beta_j \psi_k^{(i,j)}, \quad \forall k \in N \setminus S, \end{aligned}$$

i.e., $\phi = \sum_i \sum_j \alpha_i \beta_j \psi^{(i,j)}$ where $\sum_i \sum_j \alpha_i \beta_j = \sum_i \alpha_i (\sum_j \beta_j) = 1$. Hence $\phi \in \mathcal{PW}(v)$.

By induction, all vertices of the precore belong to the pre-Weber set. Therefore the polytope of the precore is included in the pre-Weber set. \square

We consider marginal worth vectors ψ^{C_r} associated to restricted chains C_r . The set of all such marginal worth vectors is denoted by $\mathcal{M}^r(v)$.

Definition 9. The *Weber set* is defined as the convex hull of all marginal worth vectors associated to restricted maximal chains:

$$\mathcal{W}(v) := \text{co}(\mathcal{M}^r(v)).$$

Theorem 3. For any distributive game v , the core is included in the Weber set, i.e., $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.

Proof. We prove it by induction on the level number. If a poset N has only one level, then the proof is the same as the one of Theorem 2.

Suppose that the statement is true for all posets having at most k levels. We assume now that the poset has $k+1$ levels. Let $v'(T) := v_{|\top_k}(T) = v(T), \forall T \subseteq \top_k, T \in \mathcal{O}(N)$, and $\phi \in \text{Ext}(\mathcal{C}(v)) \subseteq \mathcal{C}(v)$. Clearly, $\phi_{|\top_k} \in \mathcal{C}(v_{|\top_k}) = \mathcal{C}(v')$. Then $\phi_{|\top_k} \in \text{Ext}(\mathcal{C}(v'))$. Indeed, if $\phi_{|\top_k} \notin \text{Ext}(\mathcal{C}(v'))$, then $\exists \phi^1 \neq \phi^2 \in \mathcal{C}(v'), \exists \lambda \in (0, 1)$ such that $\phi_{|\top_k} = \lambda \phi^1 + (1 - \lambda) \phi^2$. Let

$$\phi_i^{1j} := \begin{cases} \phi_i^j & \forall i \in \top_k, \\ \phi_i & \forall i \in \top \setminus \top_k \end{cases}$$

for $j = 1, 2$. Then $\phi = \lambda \phi^1 + (1 - \lambda) \phi^2$, which contradicts $\phi \in \text{Ext}(\mathcal{C}(v))$.

Similarly, let $v''(T) = v(\top_k \cup T) - v(\top_k), \forall T \subseteq \top \setminus \top_k$, then

$$\phi_{|\top \setminus \top_k}(T) = \phi(T \cup \top_k) - \phi(\top_k) \geq v(T \cup \top_k) - v(\top_k) = v''(T), \forall T \subseteq \top \setminus \top_k.$$

So $\phi_{|\top \setminus \top_k} \in \mathcal{C}(v'')$. Then

$$\phi_k = \begin{cases} \sum_{\psi^i \in \mathcal{PM}(v')} \alpha^i \psi_k^i & \forall k \in \top_k \\ \sum_{\psi^j \in \mathcal{PM}(v'')} \beta^j \psi_k^j & \forall k \in \top \setminus \top_k. \end{cases}$$

By the proof of Theorem 2, let $\psi^{(i,j)} = (\psi^i, \psi^j)$ where $\psi^i \in \mathcal{M}^r(v'), \psi^j \in \mathcal{M}^r(v'')$, then $\phi = \sum_i \sum_j \alpha_i \beta_j \psi^{(i,j)}$, i.e., $\phi \in \mathcal{W}(v)$. This is the desired result. \square

7 Core of convex distributive games

We give our main results as follows.

Theorem 4. Let v be any distributive game on N . Then v is convex if and only if $\text{Ext}(\mathcal{PC}(v)) = \mathcal{M}(v)$, i.e., $\mathcal{PC}^F(v) = \mathcal{PW}(v)$.

To prove this theorem, we must show some lemmas.

Lemma 1. If a distributive game v is convex, then the pre-Weber set is a subset of the precore:

$$\mathcal{PW}(v) \subseteq \mathcal{PC}(v).$$

Proof. Because $\mathcal{PW}(v) = \text{co}(\mathcal{M}(v))$, if all vectors of $\mathcal{M}(v)$ belongs to the precore, by the convexity of the precore, all elements of the Weber set must be contained in the precore. Now we show $\mathcal{M}(v) \subseteq \mathcal{PC}(v)$.

Let $C = \{S_0 := \emptyset \prec S_1 \prec \dots \prec S_n := N\}$ be a maximal chain in $\mathcal{O}(N)$. Because $\psi^C(S_i) = v(S_i), \forall S_i \in C$, it remains to show that $\psi^C(S) \geq v(S), \forall S \in \mathcal{O}(N) \setminus C$. We prove it by induction on $|S|$.

If $S = \{i\}$, then $\exists j$ such that $S_{j+1} = S_j \cup \{i\}$. By the convexity of v , we have $\psi_i^C = v(S_{j+1}) - v(S_j) \geq v(i) - v(\emptyset) = v(i)$.

Assume that $\psi^C(S) \geq v(S)$ for any $S \in \mathcal{O}(N) \setminus C$ and $|S| < s$. Let $S \in \mathcal{O}(N) \setminus C$ and $|S| = s$. Denote by π the permutation associated to C , such that $S_i = \{\pi(1), \dots, \pi(i)\}$ and $j := \pi(i)$. Then for S , we can get a sequence $i_1 < \dots < i_s$ such that $i_k = \pi^{-1}(j_k)$ for all $j_k \in S$. Hence by the convexity of v , we have

$$v(S_{i_s}) - v(S_{i_s-1}) \geq v(S) - v(S \setminus \{\pi(i_s)\}),$$

then

$$\psi_{\pi(i_s)}^C = v(S_{i_s}) - v(S_{i_s-1}) \geq v(S) - v(S \setminus \{\pi(i_s)\}).$$

By induction, for $S \setminus \{\pi(i_s)\} = \{\pi(i_1), \dots, \pi(i_{s-1})\}$, we have

$$\psi^C(S \setminus \{\pi(i_s)\}) \geq v(S \setminus \{\pi(i_s)\}).$$

Finally

$$\psi^C(S) = \psi_{\pi(i_s)}^C + \psi^C(S \setminus \{\pi(i_s)\}) \geq v(S).$$

Hence ψ^C belongs to the precore. \square

Lemma 2. If a distributive game v is convex, then any marginal worth vector in $\mathcal{M}(v)$ is a vertex of the precore:

$$\mathcal{M}(v) \subseteq \text{Ext}(\mathcal{PC}(v)).$$

Moreover, $\mathcal{M}(v) = \text{Ext}(\mathcal{PW}(v))$.

Proof. By Lemma 1, we have $\mathcal{M}(v) \subseteq \mathcal{PC}(v)$, it remains to show that every ψ^C is a vertex of the precore. Suppose there exist vectors $\phi^1, \phi^2 \neq \psi^C \in \mathcal{PC}(v)$, and $\lambda \in (0, 1)$ such that $\psi^C = \lambda\phi^1 + (1 - \lambda)\phi^2$. Because we have $\psi^C(S_i) = v(S_i)$ for any $S_i \in C$, we have $v(S_i) = \lambda\phi^1(S_i) + (1 - \lambda)\phi^2(S_i)$. But $\phi^k(S_i) \geq v(S_i)$ for all $S_i \in C, k = 1, 2$, hence necessarily $\phi^1(S_i) = \phi^2(S_i) = v(S_i)$, i.e., $\phi^1 = \phi^2 = \psi^C$, a contradiction. Hence, ψ^C is a vertex of the precore.

To prove $\mathcal{M}(v) = \text{Ext}(\mathcal{PW}(v))$, we have to prove only that $\mathcal{M}(v) \subseteq \text{Ext}(\mathcal{PW}(v))$. But this follows from the fact that $\mathcal{PW}(v) \subseteq \mathcal{PC}(v)$ (Lemma 1) and that any marginal vector is a vertex of the precore. \square

Lemma 3. If a distributive game v is convex, then $\text{Ext}(\mathcal{PC}(v)) = \mathcal{M}(v)$, or equivalently $\mathcal{PC}^F(v) = \mathcal{PW}(v)$.

Proof. By Lemma 2, we know that for a convex game v , any vertex of the Weber set is a vertex of the precore, also of the finite part of the precore. Since the finite part of the precore is included in the Weber set by Theorem 2, it follows that the vertices of the two sets coincide. \square

Now let us prove Theorem 4.

Proof. We have already shown in Lemma 3 that, if v is convex, then $\text{Ext}(\mathcal{PC}(v)) = \mathcal{M}(v)$. Conversely, suppose $\text{Ext}(\mathcal{PC}(v)) = \mathcal{M}(v)$. For any $S = \{s_1, \dots, s_k, p_1, \dots, p_s\}$, $T = \{s_1, \dots, s_k, q_1, \dots, q_t\} \in \mathcal{O}(N)$ and $S \cap T, S \cup T \in \mathcal{O}(N)$, we can always find out a maximal chain C passing through the points $S \cap T, S, S \cup T$. Hence $v(S \cup T) - v(S) = \psi^C(S \cup T) - \psi^C(S) = \psi^C(q_1, \dots, q_t) = \psi^C(T) - \psi^C(S \cap T) \geq v(T) - v(S \cap T)$. It implies the convexity of v . \square

For the core, we have a similar result.

Theorem 5. If a distributive game v is convex, then any marginal worth vector in $\mathcal{M}^r(v)$ is a vertex of the core:

$$\mathcal{M}^r(v) \subseteq \text{Ext}(\mathcal{C}(v)).$$

Moreover, $\mathcal{M}^r(v) = \text{Ext}(\mathcal{W}(v))$.

Proof. Consider a restricted maximal chain C_r and its associated marginal worth vector ψ^{C_r} . We know by Theorem 4 that it is a vertex of the precore, and since ψ^{C_r} coincide with v on C_r , it has the property $\psi^{C_r}(x) = v(x)$, $\forall x \in \top_N$, hence it belongs to the core and is a vertex of it.

Finally, any ψ^{C_r} is a vertex of the Weber set since the Weber set is included in the core. Indeed, the Weber set is the convex hull of all marginal vectors associated to restricted maximal chains, which by the above argument, belong to the core. \square

Corollary 1. If a distributive game v is convex, then $\text{Ext}(\mathcal{C}(v)) = \mathcal{M}^r(v)$, or equivalently $\mathcal{C}(v) = \mathcal{W}(v)$.

Proof. using Theorem 3 and 5, we can similarly prove it like Lemma 3. \square

Remark that $\mathcal{C}(v) = \mathcal{W}(v)$ does not imply that v is convex, i.e., $\mathcal{C}(v) = \mathcal{W}(v)$ is not equivalent to $\mathcal{PC}^F(v) = \mathcal{PW}(v)$. This is shown by the following counterexample.

Example 5. Let v be a distributive game on $\mathcal{O}(N)$ with $N = \{1, 2, 3, 4, 5\} : 1 < 2 < 3, 4 < 5$. Consider v satisfying $v(S) = \sum_{s \in S} s$ for any $S \neq \{12\}$ and $v(12) = 1$. We have $\mathcal{C}(v) = \mathcal{W}(v) = \{(1, 2, 3, 4, 5)\}$ but $v(12345) + v(12) = 16 < v(1245) + v(123) = 18$. Therefore v is not convex.

To end this section, we come back to Example 4 and compute its core. The four restricted maximal chains are

$$\begin{aligned} C_1 &:= \{\emptyset, 1, 12, 123, 1234\}, & C_2 &:= \{\emptyset, 1, 12, 124, 1234\} \\ C_3 &:= \{\emptyset, 2, 12, 123, 1234\}, & C_4 &:= \{\emptyset, 2, 12, 124, 1234\}. \end{aligned}$$

Under convexity of v , the core of v is the convex hull of the four following vectors:

$$\begin{aligned}\phi^1 &:= (v(1), v(12) - v(1), v(123) - v(12), v(N) - v(123)) \\ \phi^2 &:= (v(1), v(12) - v(1), v(N) - v(124), v(124) - v(12)) \\ \phi^3 &:= (v(12) - v(2), v(2), v(123) - v(12), v(N) - v(123)) \\ \phi^4 &:= (v(12) - v(2), v(2), v(N) - v(124), v(124) - v(12)).\end{aligned}$$

In general, it is a 3-dimensional polytope with 4 vertices, hence a 3-dimensional simplex.

8 Games with a partially ordered set of actions

We end this paper by a brief indication about games where each player has at disposal a partially ordered set of (elementary) actions. This notion of game is described in [12]. Consider a set of players N , and for each $i \in N$, define P_i the partially ordered set of possible actions of player i . A trivial example is to take the case of multichoice games. Then the P_i 's are totally ordered sets $P_i := \{0 =: a_0, a_1, \dots, a_m\}$, where $a_0 < a_1 < \dots < a_m$ indicate levels of participation.

We consider the distributive lattices $L_i := \mathcal{O}(P_i)$, $i \in N$. They represent all possible combinations of elementary actions, where if action x is performed and $y \leq x$ in the poset of actions, then y must be performed too. Considering all players together, a given *profile* of actions is an element of the product lattice $L := L_1 \times \dots \times L_n$.

Since L is again distributive, all previous definitions and results can be applied to L . In particular, the core of v is defined as the set of pre-imputations ϕ on L such that ϕ dominates v on L , and coincides with v on each element of L of the form $(\top_1^k, \top_2^k, \dots, \top_n^k)$, where \top_i^k is the top element associated to the k -th level of P_i .

9 Related works

Besides the works we already cited (games with precedence constraints of Faigle, multichoice games of Van den Nouweland et al.), there are other attempts to define the core for non-classical TU-games. We focus here on games with a restricted set of feasible coalitions, since this is our main framework.

As we have seen, a direct transposition of the definition of the classical core to this framework leads to an unbounded set, which we called precore. Our basic idea was to impose further normalization constraints (in short, to impose efficiency at each level of the hierarchy) to make it bounded, and we showed that our definition was particularly suited to the problem of sharing benefits (or costs) in a hierarchy. The other natural way to get a bounded core is to define an extension of the game which is a classical TU-game, and to define the core as the (classical) core of this extended game. Specifically, let us define $\mathcal{F} \subseteq 2^N$ a family of feasible coalitions, containing the empty set, and a game $v : \mathcal{F} \rightarrow \mathbb{R}$. Let us define by some way an extended game $\tilde{v} : 2^N \rightarrow \mathbb{R}$, so that $\tilde{v}|_{\mathcal{F}} = v$. Let us denote by $\mathcal{C}(\tilde{v})$ the classical core of \tilde{v} . Then $\mathcal{C}(\tilde{v}) \subseteq \mathcal{PC}(v)$, and it is a bounded closed polyhedron. Further properties depends on the way the extension of v is defined. We cite here two definitions.

The first one is due to Faigle and Kern [9]. The extension is defined as follows (we adapt definitions to our case of profit games). For every $S \in 2^N \setminus \{\emptyset\}$

$$\tilde{v}(S) := \begin{cases} \max\{\sum_j v(S_j) \mid S_j \in \mathcal{F} \text{ s.t. the } S_j\text{'s form a partition of } S\}, & \text{if a partition of } S \text{ exists} \\ 0, & \text{if no partition of } S \text{ exists,} \end{cases}$$

and $v(\emptyset) = 0$. Then $\tilde{v}|_{\mathcal{F}} = v$ if and only if v is superadditive. The authors give some properties related to the core. In particular, they show that if v is convex and weakly increasing then the core of \tilde{v} is nonempty, where weakly increasing means: for every $S \in \mathcal{F}$, for every $i \in N$ such that i has at least two upper neighbors in \mathcal{F} and $S \cup i \in \mathcal{F}$, we have $v(S \cup i) \geq v(S)$. In more recent papers of Faigle and Peis [10], the definition is slightly modified: instead of a partition, the subsets S_i 's should only be pairwise disjoint and included in S .

The second definition is due to Derks and Peters [6], and uses the Möbius transform (dividends of a game): for any game v on 2^N , its Möbius transform $m^v : 2^N \rightarrow \mathbb{R}$ is given by

$$m^v(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T),$$

and conversely $v(S) = \sum_{T \subseteq S} m(T)$. We define \tilde{v} by its Möbius transform as follows.

$$m^{\tilde{v}}(S) := \begin{cases} m^v(S), & \text{if } S \in \mathcal{F} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{v}|_{\mathcal{F}} = v$. Based on this, the authors defined the Shapley value, but did not investigate the core in their paper. This could be a topic of further research.

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