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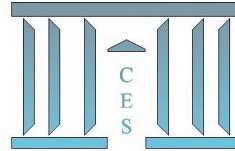
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Abstract

In a pure exchange economy, we define a new concept of local equilibria where the consumers are restricted by sale constraints. So the equilibrium consumptions are close to the initial endowments. We prove that the local equilibrium is always unique when utility functions are smooth and the constraints are sufficiently tight. We also prove that the local equilibria converge to the equilibrium of the linear tangent economy when the sale constraints converge to zero.

JEL classification: C62, D50, D62.

Keywords: General economic equilibrium, local equilibria, uniqueness

1 Introduction

In an exchange economy, under suitable assumptions on the differentiability of the utility functions, it is well known that equilibrium is generically locally unique (See, for example, [5, 6, 8, 1]). This local uniqueness means that there exists only one equilibrium in a sufficiently small neighborhood around an equilibrium price.

In this paper, we focus our attention to another approach with a new concept of local equilibrium. Actually, the consumers do not consider all possible consumptions but only those which are close to the initial endowments. So the net trades on the market remain small. Formally, this means that the consumption set of a consumer is the set of consumptions which are above a given fraction of the initial endowments. We can also interpret this as the fact that the consumers face a restricted market participation where the trades must lie in bounded below subsets depending on endowments with

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a lower bound close to 0. Then, a local equilibrium is a Walras equilibrium of this economy with restricted consumption sets.

We first study the link between local equilibria and equilibria of the tangent linear economy introduced in [4, 3]. The tangent linear economy is a linear economy where the marginal utilities of the agents are the partial derivatives of the utility functions at the initial endowments. The associated equilibrium consumptions are the instantaneous consumptions in the Walras exchange process built by Champsaur and Cornet and the equilibrium price is the supporting price of the instantaneous trades.

Since the trades in a local economy converge to 0 when the trades are more and more restricted, we have to rescale them in order to be comparable with the ones of the tangent linear economy. Actually, we show that we can associate to a local economy an auxiliary economy with the same initial endowments, the positive orthant for the consumption sets and new utility functions derived from the original one by a rescaling of the net trade. Then, we show that the equilibria of this auxiliary economy converge to an equilibrium of the tangent linear economy. This means that an equilibrium price of the local equilibrium is close to the unique equilibrium price of the tangent linear economy and the rescaled equilibrium allocations are also close to the equilibrium allocation of the tangent linear economy.

The main contribution shows that the local equilibrium price is unique for sufficiently restricted trade when the utility functions are strictly quasi-concave. Obviously, we also have the uniqueness of the equilibrium allocation. The proof is based on a concave representation of preferences and a convexity-like property of the indirect utility functions.

This result can be related to the fact that the equilibrium price is unique when the initial endowments are in a neighborhood of the contract curve or of the set of Pareto optimal allocations (See [1]). In that case, without any a priori restrictions, the trades are small. So we remain in the same kind of interpretation: the equilibrium is unique when the trades are small, that is when the gains in terms of utility level are closed to the first order linear approximation.

Consequently, we have two approaches of the uniqueness of equilibria. Either, as in the literature, we consider a neighborhood of the price and we require that the equilibrium price is unique in this neighborhood or we consider restricted trades, that is equilibrium allocations in a neighborhood of the initial endowments and we have a unique equilibrium price.

This paper is a first step toward the definition of a discrete Walrasian exchange process where the exchanges are determined at each period by the unique net trades associated to the unique local equilibrium.

2 Local equilibrium

We consider a family of pure exchange economies with ℓ commodities and m consumers. The consumption sets and the utility functions are kept fixed whereas the initial endowments are taken in a nonempty compact subset \mathbf{E} of $(\mathbb{R}_{++}^\ell)^m$. So, for an element $\mathbf{e} = (e_i)_{i=1}^m \in \mathbf{E}$, the economy $\mathcal{E}(\mathbf{e})$ is defined by $(\mathbb{R}_+^\ell, u_i, e_i)_{i=1}^m$.

For all $\tau \in [0; 1]$, we define a τ -local equilibrium as follows:

Definition 2.1 $(p^*, (x_i^*)) \in \mathbb{R}_+^\ell \times (\mathbb{R}_+^\ell)^m$ is a τ -local equilibrium of the economy $\mathcal{E}(\mathbf{e})$ if

- (i) for all $i = 1, \dots, m$, x_i^* is a solution of
$$\begin{cases} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i \\ x_i \geq (1 - \tau)e_i \end{cases}$$
- (ii) $\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$.

The difference with a standard Walras equilibrium comes from the constraint $x_i \geq (1 - \tau)e_i$ instead of $x_i \geq 0$. This constraint means that the consumption set of the consumer is $\{(1 - \tau)e_i\} + \mathbb{R}_+^\ell$ instead of \mathbb{R}_+^ℓ , so a τ -local equilibrium is a Walras equilibrium of the τ -local economy $\mathcal{E}^\tau(\mathbf{e}) = (\{(1 - \tau)e_i\} + \mathbb{R}_+^\ell, u_i, e_i)_{i=1}^m$. One remarks that a 1-local equilibrium is merely a Walras equilibrium of the economy $\mathcal{E}(\mathbf{e})$. A 0- local equilibrium is $(p^*, (e_i))$, where p^* is any non-zero price vector in \mathbb{R}_+^ℓ .

This constraint $x_i \geq (1 - \tau)e_i$ is equivalent to the constraint on the net trade $z_i = x_i - e_i \geq -\tau e_i$. So a local equilibrium is an equilibrium with restricted trades, where the maximization problem of a consumer is:

$$\begin{cases} \text{maximize } u_i(e_i + z_i) \\ p^* \cdot z_i \leq 0 \\ z_i \geq -\tau e_i \end{cases}$$

When τ is small, the equilibrium allocation of a τ -local equilibrium is close to e_i . Indeed, if we combine this lower constraint with the market clearing condition (ii), one gets $(1 - \tau)e_i \leq x_i^* \leq e_i + \tau \sum_{j \neq i} e_j$.

We now introduce a family of auxiliary economies $(\hat{\mathcal{E}}^\tau(\mathbf{e}))_{(\tau, \mathbf{e}) \in]0, 1[\times \mathbf{E}}$ and we show the link between a τ -local equilibrium of $\mathcal{E}(\mathbf{e})$ and a Walras equilibrium of the auxiliary economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$. The difference between $\mathcal{E}^\tau(\mathbf{e})$ and $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is that we modify the preferences instead of the consumption sets.

Definition 2.2 For all $\tau \in]0, 1[$ and $\mathbf{e} \in \mathbf{E}$, the pure exchange economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is the economy where the consumption sets are equal to \mathbb{R}_+^ℓ , the initial endowments are $\mathbf{e} = (e_i)_{i=1}^m$ and the preferences are represented by the utility function $u_{ie_i}^\tau$ defined by:

$$u_{ie_i}^\tau(x_i) = \frac{1}{\tau} (u_i(e_i + \tau(x_i - e_i)) - u_i(e_i))$$

Note that the coefficient $\frac{1}{\tau}$ and the additional term $-u_i(e_i)$ are useless in the definition since the preferences are the same with or without them. Nevertheless we introduce them to obtain a simpler formulation later on the convergence towards the tangent linear economy. We can interpret the preferences of the agent as follows. For two consumptions x_i and x'_i , far from e_i , the consumer compares them by considering the two allocations $e_i + \tau(x_i - e_i)$ and $e_i + \tau(x'_i - e_i)$ which are close to the current allocation e_i . So, it is as if the consumer has well known preferences around her initial endowments and extrapolates them for the farther allocations. Once again, if $\tau = 1$, then $\hat{\mathcal{E}}^1(\mathbf{e})$ is merely the initial economy $\mathcal{E}(\mathbf{e})$.

Remark 2.1 We can remark that the definition of $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is independent of the choice of the representations of preferences. That is, if \tilde{u}_i represents the same preferences as u_i then $\tilde{u}_{ie_i}^\tau$ represents the same preferences as $u_{ie_i}^\tau$ since we have:

$$\begin{aligned} u_i^\tau(x_i) \geq u_i^\tau(x'_i) &\Leftrightarrow u_i(e_i + \tau(x_i - e_i)) \geq u_i(e_i + \tau(x'_i - e_i)) \\ &\Leftrightarrow \tilde{u}_i(e_i + \tau(x_i - e_i)) \geq \tilde{u}_i(e_i + \tau(x'_i - e_i)) \\ &\Leftrightarrow \tilde{u}_{ie_i}^\tau(x_i) \geq \tilde{u}_{ie_i}^\tau(x'_i). \end{aligned}$$

So, the economies with the utility functions $u_{ie_i}^\tau$ or with the utility functions $\tilde{u}_{ie_i}^\tau$ are identical.

The next proposition gives the link between τ -local equilibrium of $\mathcal{E}(\mathbf{e})$ and Walras equilibrium of $\hat{\mathcal{E}}^\tau(\mathbf{e})$ for each $\tau \in]0, 1[$ and $\mathbf{e} \in \mathbf{E}$.

Proposition 2.1 *For all $\tau \in]0, 1[$ and $\mathbf{e} \in \mathbf{E}$, if $(p^*, (x_i^*))$ is a τ -local equilibrium of $\mathcal{E}(\mathbf{e})$, then $\left(p^*, \left(e_i + \frac{x_i^* - e_i}{\tau}\right)\right)$ is an equilibrium of the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$. Conversely, if $(p^*, (\xi_i^*))$ is a Walras equilibrium of $\hat{\mathcal{E}}^\tau(\mathbf{e})$, then $(p^*, (e_i + \tau(\xi_i^* - e_i)))$ is a τ -local equilibrium of the economy $\mathcal{E}(\mathbf{e})$.*

The proof is obvious starting from the definitions of Walras equilibrium and τ -local equilibrium.

We now end this section by considering the tangent linear economy associated to $\mathcal{E}(\mathbf{e})$ as defined in [4]. We denote it by $\hat{\mathcal{E}}^0$ since we shall prove later that it is the limit in a precise sense of the economies $\hat{\mathcal{E}}^\tau(\mathbf{e})$ when τ converges to 0.

Definition 2.3 If the utility functions are differentiable at the initial endowments $\mathbf{e} = (e_i)$, the tangent linear economy of $\hat{\mathcal{E}}^0(\mathbf{e})$ is the economy with the consumption sets \mathbb{R}_+^ℓ , the initial endowments e_i and the linear preferences represented by the utility function $u_{ie_i}^0$ defined by:

$$u_{ie_i}^0(x_i) = \nabla u_i(e_i) \cdot (x_i - e_i)$$

Once again, we remark that the preferences are represented by the simpler utility function $\nabla u_i(e_i) \cdot x_i$ but we choose such formulation to get simpler statements later.

3 Convergence of τ -local equilibria

From now on, we consider $\tau \in]0, \frac{1}{2}]$. The upper bound $\frac{1}{2}$ is arbitrary chosen in $]0, 1[$ to guarantee the compactness of the sets we are considering in the proofs. Recall that \mathbf{E} is a compact of $(\mathbb{R}_{++}^\ell)^m$.

We posit the following classical assumption on the preferences in the theory of general economic equilibrium from a differentiable approach.

Assumption C. For all i , u_i is twice continuously differentiable on \mathbb{R}_{++}^ℓ and $\nabla u_i(x_i) \in \mathbb{R}_{++}^\ell$ for all $x_i \in \mathbb{R}_{++}^\ell$.

The following proposition explains in what sense the economy $\hat{\mathcal{E}}^0(\mathbf{e})$ is the limit of economies $\hat{\mathcal{E}}^\tau(\mathbf{e})$.

Proposition 3.1 *Let \mathbf{K} be a compact subset of $(\mathbb{R}_+^\ell)^m$. Under Assumption C, for all $\varepsilon > 0$, there exists $\bar{\tau} > 0$ such that for all $\tau \in]0, \bar{\tau}]$, for all $(\mathbf{e}, x) \in \mathbf{E} \times \mathbf{K}$, for all i ,*

$$|u_{ie_i}^\tau(x_i) - u_{ie_i}^0(x_i)| \leq \varepsilon$$

The proof of this proposition is a direct consequence of Lemma 5.1 in Appendix. We deduce from this uniform convergence on compacta a result of continuity (see the proof in Appendix).

Corollary 3.1 *Under Assumption C, the function U from $[0, \frac{1}{2}] \times \mathbf{E} \times (\mathbb{R}_+^\ell)^m$ to \mathbb{R}^m defined by*

$$U(\tau, e, x) = (u_{ie_i}^\tau(x_i))_{i=1}^m$$

is continuous.

We now deduce the convergence of equilibrium prices and equilibrium allocations.

We normalize the equilibrium prices in the simplex S of \mathbb{R}^ℓ . Under Assumption C, we know that the normalized price equilibrium correspondence $e \in \mathbf{E} \mapsto P^0(e)$ associated to the linear tangent economy $\hat{\mathcal{E}}^0(\mathbf{e})$ reduces to a continuous mapping (see [4, 7]). For all $\tau > 0$ we denote by P^τ the normalized price equilibrium correspondence associated to $\hat{\mathcal{E}}^\tau$.

Proposition 3.2 *Under Assumption C, for all $\varepsilon > 0$, there exists $\hat{\tau} > 0$ such that for all $\tau \in]0, \hat{\tau}]$, for all $\mathbf{e} \in \mathbf{E}$,*

$$P^\tau(\mathbf{e}) \subset \bar{B}(P^0(\mathbf{e}), \varepsilon).$$

Remark 3.1 Note that the above proposition implies that for all sequences $(\mathbf{e}^\nu, \tau^\nu, p^\nu) \in \mathbf{E} \times]0, 1/2] \times S$ converging to $(\mathbf{e}, 0, p)$ and satisfying $p^\nu \in P^{\tau^\nu}(\mathbf{e}^\nu)$ for all integer ν , then $p \in P^0(\mathbf{e})$.

Proof. Assume on the contrary that there exists $\varepsilon > 0$ such that for all integer $\nu \geq 1$, there exists $\tau^\nu \leq (1/\nu)$, $\mathbf{e}^\nu \in \mathbf{E}$ such that $P^{\tau^\nu}(\mathbf{e}^\nu) \not\subset \bar{B}(P^0(\mathbf{e}^\nu), \varepsilon)$. Then, for all integer ν , there exists $\bar{\mathbf{x}}^\nu \in (\mathbb{R}_+^\ell)^m$ and $p^\nu \in P^{\tau^\nu}(\mathbf{e}^\nu)$ such that $(p^\nu, \bar{\mathbf{x}}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$.

The sequence $(\bar{\mathbf{x}}^\nu)$ is bounded since \mathbf{E} is compact and $\bar{x}_i^\nu \leq \sum_{i=1}^m e_i^\nu$ for all ν . Since \mathbf{E} and S are compact subsets, there exists a subsequence of $(\tau^\nu, \mathbf{e}^\nu, p^\nu, \bar{\mathbf{x}}^\nu)$ (denoted identically for the sake of simpler notations) which converges to $(0, \mathbf{e}, p, \bar{\mathbf{x}})$. From the fact that $(p^\nu, \bar{\mathbf{x}}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one deduces that $\sum_{i=1}^m \bar{x}_i^\nu = \sum_{i=1}^m e_i^\nu$ and $p^\nu \cdot x_i \leq p^\nu \cdot e_i$ for all i . Furthermore, $p \notin B(P^0(\mathbf{e}), \varepsilon)$. Hence p is not an equilibrium price vector of $\hat{\mathcal{E}}^0(\mathbf{e})$, so there exists i such that \bar{x}_i does not belong to the demand for $u_{ie_i}^0$ at $(p, p \cdot e_i)$.

Consequently there exists a vector $\xi_i \in \mathbb{R}_+^\ell$ such that $p \cdot \xi_i \leq p \cdot e_i$ and $u_{ie_i}^0(\xi_i) > u_{ie_i}^0(\bar{x}_i)$. Since according to Assumption C, $\nabla u_i(e_i) \gg 0$ and $p \cdot e_i > 0$, we can choose ξ_i in such a way that $p \cdot \xi_i < p \cdot e_i$.

For ν large enough, $p^\nu \cdot \xi_i < p^\nu \cdot e_i^\nu$. Since $(p^\nu, \bar{\mathbf{x}}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one deduces that $u_{ie_i^\nu}^{\tau^\nu}(\xi_i) < u_{ie_i^\nu}^{\tau^\nu}(\bar{x}_i^\nu)$. From Corollary 3.1, one deduces that $u_{ie_i}^0(\xi_i) \leq u_{ie_i}^0(\bar{x}_i)$, which is in contradiction with $u_{ie_i}^0(\xi_i) > u_{ie_i}^0(\bar{x}_i)$. \square

We now state a result on the convergence of equilibrium allocations. Since there is no uniqueness of the equilibrium allocation for the linear tangent economy, we do not really have a convergence property but a closed graph property.

Proposition 3.3 *Under Assumption C, for all sequences $(\tau^\nu, \mathbf{e}^\nu, \mathbf{x}^\nu)$ of $]0, \frac{1}{2}] \times \mathbf{E} \times (\mathbb{R}_+^\ell)^m$ which converges to $(0, \mathbf{e}, \bar{\mathbf{x}})$, if \mathbf{x}^ν is a an equilibrium allocations of the economy $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$ for all ν then $\bar{\mathbf{x}}$ is an equilibrium allocation of the tangent linear economy $\hat{\mathcal{E}}^0(\mathbf{e})$.*

Proof. Let (p^ν) be the sequence of normalized equilibrium price associated to (\mathbf{x}^ν) and $p^0 = P^0(\mathbf{e})$. From Proposition 3.2, (p^ν) converges to p^0 . Consequently, one easily checks that $p^0 \cdot \bar{x}_i = p^0 \cdot e_i$ and $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$.

If $\bar{\mathbf{x}}$ is not an equilibrium allocation of $\hat{\mathcal{E}}^0(\mathbf{e})$, then there exists i , $\xi_i \in \mathbb{R}_+^\ell$ such that $p^0 \cdot \xi_i \leq p^0 \cdot e_i$ and $u_{ie_i}^0(\xi_i) > u_{ie_i}^0(\bar{x}_i)$. Since $e_i \gg 0$ and $p^0 \in S$, $p^0 \cdot e_i > 0$. Consequently, from the continuity of the utility function $u_{ie_i}^0$, we can assume that $p^0 \cdot \xi_i < p^0 \cdot e_i$.

For ν large enough, $p^\nu \cdot \xi_i \leq p^\nu \cdot e_i^\nu$. Hence, since (p^ν, \mathbf{x}^ν) is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one has $u_{ie_i^\nu}^{\tau^\nu}(\xi_i) \leq u_{ie_i^\nu}^{\tau^\nu}(\bar{x}_i^\nu)$. From Corollary 3.1, one gets at the limit $u_{ie_i}^0(\xi_i) \leq u_{ie_i}^0(\bar{x}_i)$, which is in contradiction with $u_{ie_i}^0(\xi_i) > u_{ie_i}^0(\bar{x}_i)$. \square

4 Uniqueness of the local equilibrium

The main result of the paper is the following one about the uniform uniqueness of the τ -local equilibrium for initial endowments remaining in a compact subset of $(\mathbb{R}_{++}^\ell)^m$ if τ is small enough. For this, we need the utility functions to satisfy, in addition to C, the following standard assumption.

Assumption C'. For all i , for all $x_i \in \mathbb{R}_{++}^\ell$ and for all $z_i \in \mathbb{R}^\ell \setminus \{0\}$, one has :

$$[\nabla u_i(x_i) \cdot z_i = 0] \Rightarrow [z_i \cdot D^2 u_i(x_i)(z_i) < 0].$$

Consider the compact \mathbf{E} of $(\mathbb{R}_{++}^\ell)^m$. For all $\mathbf{e} \in \mathbf{E}$, $\tau \in]0, \frac{1}{2}]$, we denote by $\mathcal{E}(\mathbf{e})$ the economy $(\mathbb{R}_+^\ell, u_i, e_i)_{i=1}^m$ and by $\hat{\mathcal{E}}^\tau(\mathbf{e})$ the auxiliary economy $(\mathbb{R}_+^\ell, u_i^\tau, e_i)_{i=1}^m$, where $(e_i)_{i=1}^m = \mathbf{e}$.

Proposition 4.1 *Under Assumptions C and C', there exists $\tilde{\tau} > 0$ such that, for all $\tau \in]0, \tilde{\tau}]$ and for all $\mathbf{e} \in \mathbf{E}$ the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ has a unique normalized equilibrium. Hence, for all $\tau \in]0, \tilde{\tau}]$ and for all $\mathbf{e} \in \mathbf{E}$, the economy $\mathcal{E}(\mathbf{e})$ has a unique normalized τ -local equilibrium.*

The proof is based on a combination of the concavity of the utility function and a convexity-like property of the indirect utility functions associated to u_i^τ when τ is small enough. This property is known to be satisfied when the utility function is linear (See, [2]). So, as in the proof of the uniqueness of the equilibrium price in a linear exchange economy (see, [2]), we use this property to get the uniqueness of the equilibrium price in the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ for all $\mathbf{e} \in \mathbf{E}$ when τ is small enough.

In order to prepare this proof, we first recall that we can consider concave utility functions without any loss of generality. Let

$$\mathcal{A}(\mathbf{E}) = \{(x_i) \in (\mathbb{R}_+^\ell)^m \mid \exists \mathbf{e} \in \mathbf{E}, \sum_{i=1}^m x_i = \sum_{i=1}^m e_i\}$$

Since \mathbf{E} is compact, $\mathcal{A}(\mathbf{E})$ is a compact subset of $(\mathbb{R}_+^\ell)^m$. Let

$$K = \{\xi \in (\mathbb{R}_+^\ell)^m \mid \exists (\tau, \mathbf{e}, (x_i)) \in [0, (1/2)] \times \mathbf{E} \times \mathcal{A}(\mathbf{E}), \xi_i = (1-\tau)e_i + \tau x_i, \forall i = 1, \dots, m\}$$

Since $[0, (1/2)] \times \mathbf{E} \times \mathcal{A}(\mathbf{E})$ is compact, K is compact and we remark that $K \subset (\mathbb{R}_{++}^\ell)^m$. Finally, for all i , \hat{K}_i is the projection of K on the i -th component of $(\mathbb{R}^\ell)^m$, that is

$$\hat{K}_i = \{x_i \in \mathbb{R}_{++}^\ell \mid \exists x_{-i} \in (\mathbb{R}_{++}^\ell)^{m-1}, (x_i, x_{-i}) \in K\}$$

We now apply the following lemma. For the proof, see [8](p.80).

Lemma 4.1 *Under Assumptions C and C', there exists a C^2 utility function \tilde{u}_i on \mathbb{R}_{++}^ℓ , which represents the same preferences as u_i and which is strictly concave on a convex open neighborhood of \hat{K}_i .*

We are now working on the indirect utility function. For all $\tau \in]0, \frac{1}{2}]$, the indirect utility function $v_{ie_i}^\tau$ associated to $\tilde{u}_{ie_i}^\tau$ is defined by:

$$v_{ie_i}^\tau(p, w) = \max\{\tilde{u}_{ie_i}^\tau(x_i) \mid p \cdot x_i \leq w, x_i \geq 0\}$$

We now introduce two functions. The first one is denoted by φ and defined from \mathbb{R}^ℓ to \mathbb{R}_{++}^ℓ by:

$$\psi(q) = (\exp(q_1), \exp(q_2), \dots, \exp(q_\ell)).$$

The second one is denoted by $V_{ie_i}^\tau$ and defined from \mathbb{R}^ℓ to \mathbb{R} by:

$$V_{ie_i}^\tau(q) = v_{ie_i}^\tau(\psi(q), \psi(q) \cdot e_i).$$

Before stating the key lemma on the convexity of $V_{ie_i}^\tau$, we first define a compact subset of potential equilibrium prices. As already noticed, the function P^0 , which gives the unique normalized equilibrium price of the tangent linear economy $\hat{\mathcal{E}}^0$ is a continuous mapping. So $\mathbb{P}^0 = P^0(\mathbf{E})$ is a compact subset of \mathbb{R}_{++}^ℓ . Let $\varepsilon > 0$ small enough, such that $\bar{B}(\mathbb{P}^0, \varepsilon) \subset \mathbb{R}_{++}^\ell$. Then, from Proposition 3.2, there exists $\hat{\tau} > 0$ such that for all $(\tau, \mathbf{e}) \in]0, \hat{\tau}] \times \mathbf{E}$, $P^\tau(\mathbf{e}) \subset \bar{B}(\mathbb{P}^0, \varepsilon)$. Let

$$Q = \{(\ln(p_1/p_\ell), \dots, \ln(p_{\ell-1}/p_\ell), 0) \mid p \in \bar{B}(\mathbb{P}^0, \varepsilon)\}$$

Q is a compact subset of \mathbb{R}^ℓ . The proof of the following key lemma is given in Appendix.

Lemma 4.2 *Under Assumption C and C', there exists $\bar{\tau} \in]0, \frac{1}{2}]$ such that for all $(\tau, \mathbf{e}) \in]0, \bar{\tau}] \times \mathbf{E}$, for all i , the function $V_{ie_i}^\tau$ is strictly convex on the convex hull of Q .*

Proof of Proposition 4.1

Let $\tilde{\tau} = \min\{\hat{\tau}, \bar{\tau}\}$ where $\bar{\tau}$ is given by Lemma 4.2 and $\hat{\tau}$ by Proposition 3.2. Let us assume by contraposition that there exists $(\tau, \mathbf{e}) \in]0, \tilde{\tau}] \times \mathbf{E}$ such that the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ has two different normalized equilibrium prices p and p' . Let (x_i) (resp. (y_i)) be the equilibrium allocation associated to p (resp. q).

Let r be the price vectors defined by $r_h = p_h p'_h$ for $h = 1, \dots, \ell$. Let q and q' be the price vector defined as follows:

$$q = (\ln(p_1/p_\ell), \dots, \ln(p_{\ell-1}/p_\ell), 0)$$

$$q' = (\ln(p'_1/p'_\ell), \dots, \ln(p'_{\ell-1}/p'_\ell), 0)$$

Since $\mathbf{e} \in \mathbf{E}$, $\tau \leq \tilde{\tau} \leq \bar{\tau}$, q and q' belongs to Q and $q \neq q'$. We deduce from Lemma 4.2 that for all i ,

$$V_{ie_i}^\tau \left(\frac{1}{2}q + \frac{1}{2}q' \right) < \frac{1}{2}V_{ie_i}^\tau(q) + \frac{1}{2}V_{ie_i}^\tau(q').$$

In addition, one remarks that for all h , $\ln(r_h/r_\ell) = \frac{1}{2}q_h + \frac{1}{2}q'_h$, $V_{ie_i}^\tau(q) = v_{ie_i}^\tau(p)$ and $V_{ie_i}^\tau(q') = v_{ie_i}^\tau(p')$. Finally we have $\frac{1}{2}v_{ie_i}^\tau(p) + \frac{1}{2}v_{ie_i}^\tau(p') > v_{ie_i}^\tau(r)$ for all i . So, since (x_i) (resp. (y_i)) is the equilibrium allocation associated to p (resp. p'), we get, for all i ,

$$\frac{1}{2}\tilde{u}_{ie_i}^\tau(x_i) + \frac{1}{2}\tilde{u}_{ie_i}^\tau(y_i) > v_{ie_i}^\tau(r).$$

From the definition of $\mathcal{A}(\mathbf{E})$ and K , $e_i + \tau(x_i - e_i)$ and $e_i + \tau(y_i - e_i)$ belong to \hat{K}_i . Since $\tilde{u}_{ie_i}^\tau(\xi_i) = (1/\tau)(\tilde{u}_i(e_i + \tau(\xi_i - e_i)) - \tilde{u}_i(e_i))$ and $(1/2)(e_i + \tau(x_i - e_i)) + (1/2)e_i + \tau(y_i - e_i) = e_i + \tau(((1/2)x_i + (1/2)y_i) - e_i)$, one deduces from Lemma 4.1 that

$$\tilde{u}_{ie_i}^\tau \left(\frac{1}{2}x_i + \frac{1}{2}y_i \right) \geq \frac{1}{2}\tilde{u}_{ie_i}^\tau(x_i) + \frac{1}{2}\tilde{u}_{ie_i}^\tau(y_i) > v_i^\tau(r) \quad \forall i.$$

From the definition of the indirect utility function, this implies $r \cdot (\frac{1}{2}x_i + \frac{1}{2}y_i) > r \cdot e_i$ for all i . We get a contradiction since

$$\sum_{i=1}^m x_i = \sum_{i=1}^m y_i = \sum_{i=1}^m e_i.$$

From Assumptions C and C' , u_i is strictly quasi-concave for all i . So is $u_{ie_i}^\tau$ for all e_i and $\tau \in]0, (1/2)]$. Hence, the uniqueness of the normalized equilibrium price implies the uniqueness of the equilibrium allocation since the demand is single valued. \square

For all $(\mathbf{e}, \tau) \in \mathbf{E} \times [0, (1/2)]$, we denote by $X(\mathbf{e}, \tau)$ the set of normalized τ -local equilibrium of the economy $\mathcal{E}(\mathbf{e})$. The previous result shows that X is a mapping on $\mathbf{E} \times [0, \tilde{\tau}]$.

Corollary 4.1 *Under assumption C and C' , the normalized τ -local equilibrium mapping X is continuous on $\mathbf{E} \times]0, \tilde{\tau}]$.*

Proof. Since the range of X is included in $S \times \mathcal{A}(\mathbf{E})$, which is bounded, it suffices to show that for all sequence $(\mathbf{e}^\nu, \tau^\nu)$ in $\mathbf{E} \times]0, \tilde{\tau}]$ converging to $(\mathbf{e}, \tau) \in \mathbf{E} \times]0, \tilde{\tau}]$ such that the sequence $(X(\mathbf{e}^\nu, \tau^\nu))$ converges to $(\bar{p}, (\bar{x}_i))$, then $X(\mathbf{e}, \tau) = (\bar{p}, (\bar{x}_i))$.

For all ν , let $X(\mathbf{e}^\nu, \tau^\nu) = (p^\nu, (x_i^\nu))$. Since $x_i^\nu \geq (1 - \tau^\nu)e_i^\nu$, $p^\nu \cdot x_i^\nu \leq p^\nu \cdot e_i^\nu$ for all i , and $\sum_{i=1}^m x_i^\nu = \sum_{i=1}^m e_i^\nu$ for all ν , we have $\bar{x}_i \geq (1 - \tau)e_i$, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot e_i$ for all i , and $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$.

If $(\bar{p}, (\bar{x}_i))$ is not a τ -local equilibrium of $\mathcal{E}(\mathbf{e})$, then there exists i , $x_i \geq (1 - \tau)e_i$ such that $\bar{p} \cdot x_i \leq \bar{p} \cdot e_i$ and $u_i(x_i) > u_i(\bar{x}_i)$. Since $\bar{p} \in S$ and $e_i \gg 0$, $\bar{p} \cdot e_i > 0$, hence $\bar{p} \cdot e_i > \bar{p} \cdot (1 - \tau)e_i$. Consequently, using the continuity of the utility function u_i , we can choose x_i such that $x_i \gg (1 - \tau)e_i$ and $\bar{p} \cdot x_i < \bar{p} \cdot e_i$. Consequently, for ν large enough, $x_i \geq (1 - \tau^\nu)e_i^\nu$, $p^\nu \cdot x_i \leq p^\nu \cdot e_i^\nu$, hence, since $(p^\nu, (x_i^\nu))$ is a τ -local equilibrium of $\mathcal{E}(\mathbf{e}^\nu)$, $u_i(x_i) \leq u_i(x_i^\nu)$. At the limit, one gets $u_i(x_i) \leq u_i(\bar{x}_i)$, which contradicts $u_i(x_i) > u_i(\bar{x}_i)$. Hence, $(\bar{p}, (\bar{x}_i))$ is a τ -local equilibrium of $\mathcal{E}(\mathbf{e})$ and the uniqueness result implies that $X(\mathbf{e}, \tau) = (\bar{p}, (\bar{x}_i))$. \square

5 Appendix

Lemma 5.1 *Let E be a compact subset of \mathbb{R}_{++}^ℓ , K a compact subset of \mathbb{R}_+^ℓ and u a twice continuously differentiable function on \mathbb{R}_{++}^ℓ . Then for all $\varepsilon > 0$,*

there exists $\bar{\tau} \in]0, 1[$ such that for all $\tau \in]0, \bar{\tau}[$, for all $(e, x) \in E \times K$ one has:

$$|u_e^\tau(x) - u_e^0(x)| \leq \varepsilon$$

where $u_e^\tau(x) = \frac{1}{\tau}(u(e + \tau(x - e)) - u(e))$ and $u_e^0(x) = \nabla u(e) \cdot (x - e)$.

Proof. Let $e \in E$ and $\frac{1}{2} \geq \tau > 0$. Applying the Taylor formula to u at e we can write for all $x \in K$,

$$u(e + \tau(x - e)) - u(e) = \tau \nabla u(e) \cdot (x - e) + \frac{1}{2} D^2 u(\xi) [\tau(x - e), \tau(x - e)]$$

where ξ belongs to the segment $[e, e + \tau(x - e)]$.

Hence $u_e^\tau(x) = u_e^0(x) + \frac{\tau}{2} D^2 u(\xi) [(x - e), (x - e)]$. Since K is a compact of \mathbb{R}_+^ℓ and E a compact of \mathbb{R}_{++}^ℓ the set

$$K' = \{e + \tau(x - e) \mid (e, \tau, x) \in E \times [0, 1/2] \times K\}$$

is a compact subset of \mathbb{R}_{++}^ℓ . Let

$$M' = \sup_{\xi \in K'} \|D^2 u(\xi)\| \text{ and } M = \sup_{(e, x) \in E \times K} \|x - e\|.$$

Since for all $e \in E$, $x \in K$ and $\tau \in]0, \frac{1}{2}[$ the segment $[e, e + \tau(x - e)] \subset K'$, we yield the result by taking $\bar{\tau} = \min\{\frac{2\varepsilon}{M'M^2}, \frac{1}{2}\}$. \square

Proof of corollary 3.1

Let U from $[0, \frac{1}{2}] \times \mathbf{E} \times (\mathbb{R}_+^\ell)^m$ defined by $U(\tau, \mathbf{e}, x) = (U_i(\tau, e_i, x_i))_{i=1}^m$ where

$$U_i(\tau, e_i, x_i) = \frac{1}{\tau} (u_i(e_i + \tau(x_i - e_i)) - u_i(e_i))$$

for $\tau \neq 0$ and

$$U_i(0, e_i, x_i) = u_{ie_i}^0(x_i) = \nabla u_i(e_i) \cdot (x_i - e_i).$$

It suffices to show that U_i is continuous for all i . Since u_i is continuous, U_i is continuous on $]0, \frac{1}{2}] \times \hat{E}_i \times \mathbb{R}_+^\ell$, where

$$\hat{E}_i = \{e_i \in \mathbb{R}_{++}^\ell \mid \exists e_{-i} \in (\mathbb{R}_{++}^\ell)^{m-1}, (e_i, e_{-i}) \in \mathbf{E}\}.$$

Note that \hat{E}_i is a compact subset of \mathbb{R}_{++}^ℓ . We have to check the continuity of U_i at $(0, e_i, x_i)$ for all $(e_i, x_i) \in \hat{E}_i \times \mathbb{R}_+^\ell$. Let $(\tau^\nu, e_i^\nu, x_i^\nu)$ be a sequence of $]0, \frac{1}{2}] \times \hat{E}_i \times \mathbb{R}_+^\ell$ which converges to $(0, e_i, x_i)$. Let K_i be any compact subset of \mathbb{R}_+^ℓ containing x_i^ν for all ν and x_i . Let $\varepsilon > 0$. From Proposition 3.1, there exists $\bar{\tau} > 0$ such that for all $\tau \in]0, \bar{\tau}[$ and for all $(e_i, x_i) \in \hat{E}_i \times K_i$,

$$|U_i(\tau, e_i, x_i) - U_i(0, e_i, x_i)| \leq \frac{\varepsilon}{2}.$$

Since $(\tau^\nu) \rightarrow 0$, for all $\bar{\tau} \in]0, \frac{1}{2}]$ there exists ν_0 such that $\tau^\nu \leq \bar{\tau}$ for all $\nu \geq \nu_0$. In addition, $U_i(0, \cdot, \cdot)$ is continuous on $\hat{E}_i \times K_i$, so there exists an integer ν_1 such that for all $\nu \geq \nu_1$,

$$|U_i(0, e_i^\nu, x_i^\nu) - U_i(0, e_i, x_i)| \leq \frac{\varepsilon}{2}.$$

Finally, let $\bar{\nu} = \max\{\nu_0, \nu_1\}$. Then, for all $\nu \geq \bar{\nu}$, one has:

$$\begin{aligned} |U_i(\tau^\nu, e_i^\nu, x_i^\nu) - U_i(0, e_i, x_i)| &\leq |U_i(\tau^\nu, e_i^\nu, x_i^\nu) - U_i(0, e_i^\nu, x_i^\nu)| \\ &\quad + |U_i(0, e_i^\nu, x_i^\nu) - U_i(0, e_i, x_i)| \\ &\leq \varepsilon. \end{aligned}$$

This shows the continuity of U_i at $(0, e_i, x_i)$. \square

Proof of Lemma 4.2

For all $p \in \mathbb{R}_{++}^\ell$ and $w \geq 0$, one has:

$$v_{ie_i}^\tau(p, w) = \max_{\alpha \in S} \tilde{u}_{ie_i}^\tau \left(\left(\frac{\alpha_h w}{p_h} \right)_{h=1}^\ell \right).$$

and there exists $\alpha \in S$ such that

$$v_{ie_i}^\tau(p, w) = \tilde{u}_{ie_i}^\tau \left(\left(\frac{\alpha_h w}{p_h} \right)_{h=1}^\ell \right).$$

Consequently for all $q \in \mathbb{R}^\ell$, $V_{ie_i}^\tau(q) = \max_{\alpha \in S} V_{ie_i\alpha}^\tau(q)$, where

$$V_{ie_i\alpha}^\tau(q) = \tilde{u}_{ie_i}^\tau \left(\left(\alpha_h \sum_{k=1}^\ell e_{ik} \exp(q_k - q_h) \right)_{h=1}^\ell \right).$$

Hence, to prove the result, it suffices to show that for all i , there exists $\bar{\tau}_i$ such that for all $\tau \leq \bar{\tau}_i$ and for all $\mathbf{e} \in \mathbf{E}$, for all $\alpha \in S$, $V_{ie_i\alpha}^\tau$ is strictly convex on the convex hull of Q , $\text{co}Q$. One then gets the result by taking the minimum of $\bar{\tau}_i$, $i = 1, \dots, m$. For all i , $\hat{E} - i$ denotes the projection of \mathbf{E} on the i -th component of $(\mathbb{R}^\ell)^m$. It is a compact subset included in \mathbb{R}_{++}^ℓ .

For the sake of simpler notation, we omit the subscript i in the remaining of the proof. Let us compute the Hessian matrix of $V_{e\alpha}^\tau$. For all $\chi \in \mathbb{R}^\ell$, $\chi \cdot D^2 V_{e\alpha}^\tau(q)(\chi)$ is equal to¹

$$\sum_{h=1}^\ell \alpha_h u_{eh}^{\tau'}(\xi) \left(\sum_{k \neq h} e_k \exp(q_k - q_h) (\chi_k - \chi_h)^2 \right) + \sum_{h,k=1}^\ell \alpha_h \alpha_k u_{ekh}^{\tau''}(\xi) \zeta_h \zeta_k \quad (1)$$

¹See below for the details of computations.

where

$$\xi = \left(\alpha_h \sum_{k=1}^{\ell} e_k \exp(q_k - q_h) \right)_{h=1}^{\ell}, \quad \zeta = \left(\sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) (\chi_{k'} - \chi_h) \right)_{h=1}^{\ell},$$

$u_{eh}^{\tau'}$ is the first order partial derivative of u_e^{τ} with respect to the h th component and $u_{ekh}^{\tau''}$ is the first order partial derivative of $u_{eh}^{\tau'}$ with respect to the k th component.

From the definition of u_e^{τ} , one remarks that the Hessian matrix of u_e^{τ} at ξ is equal to $\tau^2 D^2 u(e + \tau(\xi - e))$. Note that for all $\tau \in]0, \frac{1}{2}]$ and for all $(e, q) \in \hat{E} \times \text{co}Q$, $e + \tau(\xi - e)$ remains in a compact set A of \mathbb{R}_{++}^{ℓ} . Hence, the norm of the Hessian matrix of u is bounded above on A and we denote by $\bar{\theta}$ an upper bound. Consequently, for all $(e, q, \tau) \in \hat{E} \times \text{co}Q \times]0, \frac{1}{2}]$,

$$\sum_{h,k=1}^{\ell} \alpha_h \alpha_k u_{ekh}^{\tau''}(\xi) \zeta_h \zeta_k \geq -\tau^2 \bar{\theta} \sum_{h=1}^{\ell} \alpha_h^2 \zeta_h^2 \geq -\tau^2 \bar{\theta} \sum_{h=1}^{\ell} \alpha_h \zeta_h^2.$$

The second inequality comes from the fact that $0 \leq \alpha_h \leq 1$ for all h . We remark that

$$\zeta_h = \sum_{k' \neq h} \sqrt{e_{k'} \exp(q_{k'} - q_h)} \sqrt{e_{k'} \exp(q_{k'} - q_h)} (\chi_{k'} - \chi_h).$$

Using the Cauchy-Schwartz inequality, one has

$$\zeta_h^2 \leq \left(\sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) \right) \left(\sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) (\chi_{k'} - \chi_h)^2 \right).$$

Hence, one obtains the following lower bound for $\chi \cdot D^2 V_{e\alpha}^{\tau}(q)(\chi)$

$$\sum_{h=1}^{\ell} \alpha_h \left[u_{eh}^{\tau'}(\xi) - \tau^2 \bar{\theta} \left(\sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) \right) \right] \left(\sum_{k \neq h} e_k \exp(q_k - q_h) (\chi_k - \chi_h)^2 \right)$$

Once again, from the definition of u^{τ} , since $e + \tau(\xi - e)$ remains in A , there exists a real number $\varepsilon > 0$ such that for all h , for all $x \in A$, $\varepsilon \leq u_h'(x)$. Consequently, for all j , for all $(e, q, \tau) \in \hat{E} \times \text{co}Q \times]0, \frac{1}{2}]$, $\tau \varepsilon \leq u_{eh}^{\tau'}(\xi)$.

Let $\delta_h = \max_{(e,q) \in \hat{E} \times \text{co}Q} \sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h)$ and $\delta = \max_{h=1, \dots, \ell} \{\delta_h\}$. Then for all $(e, q, \tau) \in \hat{E} \times \text{co}Q \times]0, \frac{1}{2}]$, we get:

$$\chi \cdot D^2 V_{e\alpha}^{\tau}(q)(\chi) \geq \tau [\varepsilon - \tau \bar{\theta} \delta] \sum_{h=1}^{\ell} \alpha_h \left(\sum_{k \neq h} e_k \exp(q_k - q_h) (\chi_k - \chi_h)^2 \right)$$

Hence for $\tau < (\varepsilon / \bar{\theta} \delta)$, for all $(e, q) \in \hat{E} \times \text{co}Q$, $\chi \cdot D^2 V_{e\alpha}^{\tau}(q)(\chi) \geq 0$ for all $(e, q) \in \hat{E} \times \text{co}Q$ and $\chi \cdot D^2 V_{e\alpha}^{\tau}(q)(\chi) > 0$ if $\chi \neq 0$ and $\chi_{\ell} = 0$. Since for all $q \in \text{co}Q$, $q_{\ell} = 0$, one concludes that $V_{e\alpha}^{\tau}$ is strictly convex on $\text{co}Q$ if $\tau < (\varepsilon / \bar{\theta} \delta)$ for all $e \in \hat{E}$. \square

Computation of formula 1

Recall that α belongs to the simplex S , q to a compact subset $\text{co}Q$ of \mathbb{R}_+^ℓ and χ to \mathbb{R}^ℓ . $V_{e\alpha}^\tau(q)$ and $\varphi_j(q)$ are defined as follow:

$$V_\alpha^\tau(q) = (u^\tau \circ \varphi)(q) = u^\tau(\varphi_1(q), \dots, \varphi_\ell(q)); \quad \varphi_j(q) = \alpha_j \sum_{k'=1}^{\ell} e_{k'} \exp(q_{k'} - q_j)$$

$$\text{Let } \xi = \varphi(q) = \left(\alpha_h \sum_{k=1}^{\ell} e_k \exp(q_k - q_h) \right)_{h=1}^{\ell}.$$

From the chain rule for the derivatives, one has for all $(\chi, \chi') \in (\mathbb{R}^\ell)^2$,

$$\chi' \cdot D^2 V_\alpha^\tau(q)(\chi) = D\varphi(q)(\chi') \cdot D^2 u_e^\tau(\xi) D\varphi(q)(\chi) + Du_e^\tau(\xi) D^2 \varphi(q)(\chi, \chi'). \quad (2)$$

Note that

$$D\varphi(q)(\chi) = (\alpha_h \zeta_h)_{h=1}^{\ell} = \left(\alpha_h \sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) (\chi_{k'} - \chi_h) \right)_{h=1}^{\ell} \quad (3)$$

and

$$D^2 \varphi(q)(\chi, \chi') = \left(\alpha_h \sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) (\chi_{k'} - \chi_h) (\chi'_{k'} - \chi'_h) \right)_{h=1}^{\ell} \quad (4)$$

Consequently,

$$\begin{aligned} \chi \cdot D^2 V_\alpha^\tau(q)(\chi) &= (\alpha_h \zeta_h)_{h=1}^{\ell} \cdot D^2 u_e^\tau(\xi) \left((\alpha_h \zeta_h)_{h=1}^{\ell} \right) + \\ &\quad Du_e^\tau(\xi) \left(\alpha_h \sum_{k' \neq h} e_{k'} \exp(q_{k'} - q_h) (\chi_{k'} - \chi_h)^2 \right)_{h=1}^{\ell} \\ &= \sum_{h,k=1}^{\ell} \alpha_h \alpha_k u_{ekh}^{\tau''}(\xi) \zeta_h \zeta_k + \\ &\quad \sum_{h=1}^{\ell} \alpha_h u_{eh}^{\tau'}(\xi) \left(\sum_{k \neq h} e_k \exp(q_k - q_h) (\chi_k - \chi_h)^2 \right) \end{aligned}$$

References

- [1] BALASKO, Y., *Fondements de la Théorie de l'équilibre général*, Economica, 1988.
- [2] CORNET, B. , *Linear Exchange Economies*, Cahier Eco-Math, CERMSEM, Université Paris 1, 1989.
- [3] BOTTAZZI, J-M. *Accessibility of Pareto optima by Walrasian Exchange processes*. Journal of Mathematical Economics, Elsevier, vol. 23(6), pp. 585-603, 1994.

- [4] CHAMPSAUR, P. et CORNET, B.: *Walrasian Exchange Processes*. Economic Decision Making : Games, Econometrics, and Optimisation, Elsevier Sciences Publishers B.V.,1990.
- [5] DEBREU, G., *Economies with a Finite Set of Equilibria*, Econometrica, Vol. 38, pp. 387-392, 1970.
- [6] DIERKER, E., *Regular Economies* , Handbook of Mathematical Economics, Vol. 1, 1981, PP. 795-830.
- [7] GALE, D. *The Theory of Linear Economic Models*. The University of Chicago Press, Chicago 60637, 1960.
- [8] MAS-COLELL, A., *The theory of general economic equilibrium. A differentiable approach*, Cambridge University Press, 1985.