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► **To cite this version:**

Dominique Guegan, Jing Zhang. Change analysis of a dynamic copula for measuring dependence in multivariate financial data. *Quantitative Finance*, 2010, 10 (4), pp.421-430. 10.1080/14697680902933041 . halshs-00368334

**HAL Id: halshs-00368334**

**<https://shs.hal.science/halshs-00368334v1>**

Submitted on 15 Apr 2009

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# Change analysis of dynamic copula for measuring dependence in multivariate financial data

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**Abstract.** This paper proposes a new approach to measure dependences in multivariate financial data. Data in finance and insurance often cover a long time period. Therefore, the economic factors may induce some changes inside the dependence structure. Recently, two methods using copulas have been proposed to analyze such changes. The first approach investigates changes inside copula's parameters. The second one determines sequence of copulas using moving windows. In this paper we take into account the non stationarity of the data and analyze the impact of (1) time-varying parameters for a copula family, (2) sequence of copulas, on the computations of the VaR and ES measures. We propose some tests based on conditional copulas and goodness-of-fit (GOF) tests to decide the type of change, and further give the corresponding change analysis. We illustrate our approach using Standard & Poor 500 and Nasdaq indices in order to compute risk measures, using the two previous methods.

**Keywords:** Dynamic copula; Goodness-of-Fit test; Change-point; Time-varying parameter; VaR; ES.

**JEL:** C51 - G12

## 1 Introduction

Determining the dependences between assets is an important domain of research. It is useful for portfolio management, risk assessment, option pricing and hedging. The correlation matrices have been a lot considered to quantify the dependence structure between assets,

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but it is now well known that this kind of approach is only satisfactory when we work inside a Gaussian or an Elliptical framework. The recent work of [Embrechts et al. \(2001\)](#) proposing the concept of copulas to measure dependence between financial data has opened the routes to a very interesting research domain, which has shown its ability to improve the domain of quantitative finance.

Assuming stationarity of a data set  $X_1, \dots, X_d$ , its joint distribution function  $H(X_1, \dots, X_d)$  is totally determined using copulas. Indeed, the idea of separating  $H$  into two parts, one describing the dependence structure and the other one describing the marginal behavior only, has led to the well known concept of copulas, [Joe \(1997\)](#) and [Nelsen \(1999\)](#). To determine such copula under stationary has already been studied using different criteria such as AIC criterion, [Akaike \(1974\)](#),  $D_2$  diagnostic, [Caillault and Guégan \(2005\)](#).

Furthermore, most of data sets cover a reasonably long time period and economic factors may induce some changes in the dependence structure: we can observe tranquil periods and turmoil periods for instance, and then the notion of strict stationarity fails, [Guégan \(2007\)](#). To take into account this phenomenon, the notion of dynamic copula has also been introduced in risk management by [Dias and Embrechts \(2004\)](#), [Jondeau and Rockinger \(2006\)](#), and [Granger et al. \(2006\)](#). The dynamics are introduced inside the copula's parameters using some time-varying function of predetermined variables. In all these cases the family of the copula remains changeless. Recently, [Caillault and Guégan \(2009\)](#) proposed a new method to take into account the possibility of changes of the copula's family and changes inside the parameters, using moving windows. On a sequence of subsamples, a sequence of copulas (adjusted with respect to the AIC criterion) is selected. However, some problems remain opened such as the choice of the width of the moving window or the detection of the change points. These choices influence the accuracy of the results for the copula's adjustment, and by the way for a risk management strategy.

In this paper, we develop a new approach to use the concept of "dynamic copula", which contains copulas with time-varying parameters

or sequence of copulas, in order to compute risk measures like the Value at Risk (VaR) or the Expected Shortfall (ES). Looking at a data set, in order to compute its distribution function using the concept of copula, we proceed in two steps. First we adjust a copula on the whole sample and test if it changes. If not we try to adjust some dynamics on the parameters of the copula. If the copula changes all along the sample, we adjust a set of copulas to model the dynamics of the data set. In order to detect the change type of the copula robustly, we propose a series of nested tests based on conditional copulas, [Anderson \(1969\)](#), [Fermanian \(2005\)](#). Our procedure is as follows. At first, we test whether the copula changes during a given time period. If the copula appears changeless, we keep the copula and we deal with the changes of copula's parameters. Now, if we detect some changes in the copulas, then we apply the so called binary segmentation procedure to detect the change time and to build a sequence of copulas. If only the copula parameters change, we apply the change-point analysis as in [Csörgő and Horváth \(1997\)](#), [Gombay and Horváth \(1999\)](#) and [Dias and Embrechts \(2004\)](#). In this latter case considering that the change-point tests have less power in the case of "small" changes, we assume that the parameters change according to the time-varying functions of some predetermined variables. We summarize our procedure in [Figure 1](#).

In order to illustrate this new approach, we apply it to Standard & Poor 500 and Nasdaq indices. We study their dynamic dependence and use it for risk management, providing risk measures such as the VaR and the ES measures whose values can be totally different with respect the method used.

The paper is organized as follows. In Section 2, we review some useful notions and specify the notations. Section 3 presents a series of tests for detecting the copulas' change. Section 4 analyzes the details for every change type, including the change time, the copulas and the change value of the parameter, *etc.* In section 5, we provide some empirical research applying the previous method on two real data sets and we associate their dynamic risk measures. Section 6 concludes.

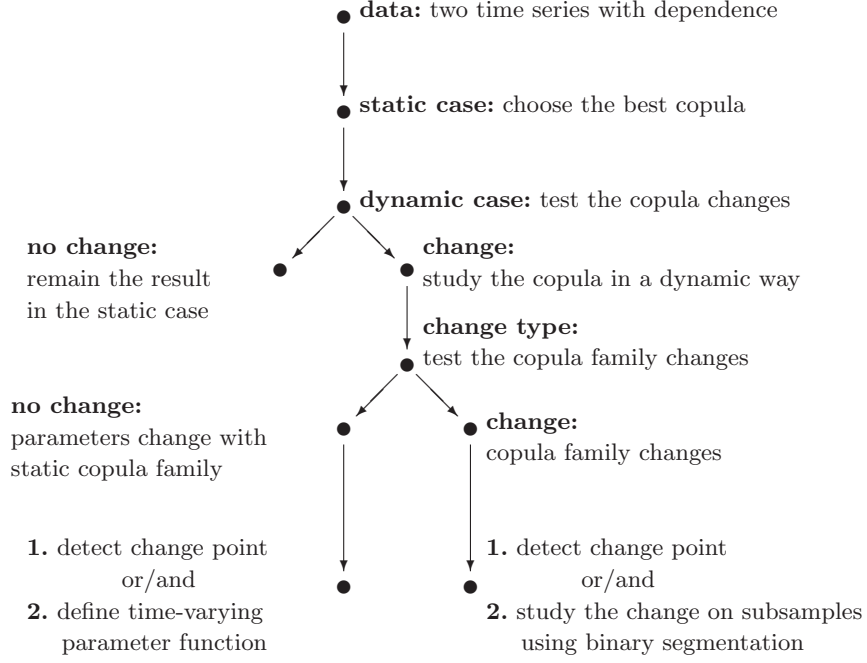


Fig. 1. Change analysis of copula

## 2 Preliminaries and notations

In order to detect the change of dependence structure, we use conditional copulas. We recall some definitions and introduce some notations. We specify some assumptions useful to apply the Goodness-of-Fit tests.

### 2.1 Conditional copulas

Following Patton (2006), the conditional copulas are defined as the following.

**Definition 1.** A  $d$ -dimensional conditional copula is a function  $C: [0, 1]^d \rightarrow [0, 1]$  such that for some conditioning set  $\mathcal{F}$ :

1. For every  $\mathbf{u} = (u_1, u_2, \dots, u_d) \in [0, 1]^d$ ,  $C(\mathbf{u}|\mathcal{F}) = 0$  when at least one coordinate of  $\mathbf{u}$  is zero, and if all coordinates of  $\mathbf{u}$  are 1 except  $u_k$ , then  $C(\mathbf{u}|\mathcal{F}) = u_k$ ,  $k = 1, \dots, d$ .

2.  $C$  is  $d$ -increasing conditioned on  $\mathcal{F}$ ,

The Sklar's theorem (Sklar, 1959) can be extended for conditional distributions and conditional copulas, then we get:

**Theorem 1.** *Let  $H$  be a  $d$ -dimensional conditional distribution function with continuous margins  $F_1, F_2, \dots, F_d$ , and let  $\mathcal{F}$  be some conditioning set, then there exists a unique conditional  $d$ -copula  $C: [0, 1]^d \rightarrow [0, 1]$  such that for all  $\mathbf{x} = (x_1, x_2, \dots, x_d)$  in  $\overline{\mathbb{R}}^d$ ,*

$$H(\mathbf{x}|\mathcal{F}) = C(F_1(x_1|\mathcal{F}), F_2(x_2|\mathcal{F}), \dots, F_d(x_d|\mathcal{F})). \quad (1)$$

*Conversely, if  $C$  is a conditional  $d$ -copula and  $F_1, F_2, \dots, F_d$  are univariate conditional distribution functions, then the function  $H$  defined by Equation (1) is a  $d$ -dimensional conditional distribution function with margins  $F_1, F_2, \dots, F_d$ .*

## 2.2 Assumptions and Goodness-of-Fit (GOF) tests

We specify some assumptions in order to use the GOF tests. For a  $d$ -dimensional stationary process with  $n$  observations  $(\mathbf{X}_n)_{n \in \mathbb{Z}} = \{(X_{i1}, X_{i2}, \dots, X_{in}) : i = 1, 2, \dots, d\}$ , let  $H$  be its cumulative distribution function. A GOF test permits to distinguish between two hypotheses. We denote  $H_0$  a known cumulative distribution function, and  $\mathcal{H} = \{H_\theta | \theta \in \Theta\}$  a known parametric family of cumulative distribution functions, then the GOF test is:

1.  $\mathcal{H}_0 : H = H_0$ , against  $\mathcal{H}_a : H \neq H_0$ , when the null hypothesis is simple; or
2.  $\mathcal{H}_0 : H \in \mathcal{H}$ , against  $\mathcal{H}_a : H \notin \mathcal{H}$  when the null hypothesis is composite.

The assumptions are:

**Assumption 1.** *Let be  $K$ , a probability kernel function on  $\mathbb{R}^d$ , twice continuously differentiable, which is the product of  $d$  univariate kernels  $K_i$  ( $i = 1, 2, \dots, d$ ) with compact supports.*

**Assumption 2.** *Let be  $\mathbf{h}_n = (h_{1n}, h_{2n}, \dots, h_{dn})$  a bandwidth vector, where  $h_n = h_{1n} = h_{2n} = \dots = h_{dn}$  such that  $h_n \rightarrow 0$ ,  $nh_n^d \rightarrow \infty$ ,*

$nh_n^{4+d} \rightarrow 0$  and  $nh_n^{3+d/2}/(\ln(\ln n))^{3/2} \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Assumption 3.** Let be  $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ , and  $\varphi_{n-1} = \sigma((X_{1,s}, X_{2,s}, \dots, X_{d,s}) : s \leq n-1)$  the conditional information set available at  $n-1$  and  $\varphi_{i,n-1} = \sigma(X_{i,s} : s \leq n-1)$  the conditional information set, for the  $i$ -th variable, available at  $n-1$ .

**Assumption 4.** Let be  $C_0$  the true copula associated to  $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ . For  $\forall \mathbf{u} \in [0, 1]^d$ , we denote  $c_0 = c_0(\mathbf{u}, \theta)$  its copula density function, and  $\theta$  the parameter vector. In addition, the first two derivatives of  $c_0$  with respect to  $\mathbf{u}$  are assumed to be uniformly continuous on  $\mathcal{Y}(\mathbf{u}_j) \times \mathcal{Y}(\theta_0)$ , where  $\mathcal{Y}(\mathbf{u}_j)$  represents an open neighborhood of the points  $(\mathbf{u}_j)_{j=1,2,\dots,m} \in [0, 1]^d$ , ( $m \in \mathbb{Z}$ ),  $\mathcal{Y}(\theta_0)$  denotes an open neighborhood of  $\theta_0$ .

### 3 Tests for copula's change

In this section we introduce a series of specified GOF tests to apply on the copulas.

#### 3.1 Test to detect the change of copula

Using the previous notations and the notion of the conditional copula, we test the null hypothesis,

$$\mathcal{H}_0^{(1)} : \text{For every } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) = C_0(\cdot),$$

against

$$\mathcal{H}_a^{(1)} : \text{For some } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) \neq C_0(\cdot),$$

where  $C_0$  has been introduced before.

In order to apply this test, we need to build an estimate of the conditional density  $c_0(\mathbf{u}_j|\varphi_{n-1})$  at point  $\mathbf{u}_j$ . Assuming that we observe an  $n$ -sample, the estimate is equal to:

$$\hat{c}(\mathbf{u}_j|\varphi_{n-1}) = \frac{1}{nh_n^d} \sum_{i=1}^n K\left(\frac{\mathbf{u}_j - \mathbf{U}_i}{\mathbf{h}_n}\right), \quad (2)$$

where  $\mathbf{h}_n$  and  $h_n$  are claimed in Assumption 2 and the kernel function  $K$  is claimed in Assumption 1. The vector  $\mathbf{U}_i$  is such that

$$\mathbf{U}_i = (\hat{F}_1(X_{1,i}), \hat{F}_2(X_{2,i}), \dots, \hat{F}_d(X_{d,i})),$$

$i = 1, 2, \dots, n$ , and  $\hat{F}_l$  is the empirical  $l$ -th marginal cumulative distribution function of  $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ , for  $l = 1, 2, \dots, d$ :

$$\hat{F}_l(X_{l,i}) = \frac{1}{n+1} \sum_{p=1}^n \mathbf{1}_{\{X_{lp} < X_{li}\}}.$$

Now we introduce the test statistics:

$$\mathcal{T} = (nh_n^d) \sum_{j=1}^m \frac{\{\hat{c}(\mathbf{u}_j | \varphi_{n-1}) - c_0(\mathbf{u}_j | \varphi_{n-1})\}^2}{\sigma^2(\mathbf{u}_j)}, \quad (3)$$

where  $\sigma(\mathbf{u}_j)$  satisfies:

$$\sigma^2(\mathbf{u}_j) = c_0^2(\mathbf{u}_j | \varphi_{n-1}) \cdot \int K^2.$$

Under the null hypothesis  $\mathcal{H}_0^{(1)}$ , the statistics  $\mathcal{T}$  defined in equation (3) tends to a Chi-square distribution with  $m$  degrees of freedom when  $n \rightarrow \infty$ , [Fermanian \(2005\)](#). Through this test based on  $\mathcal{T}$ , we can detect whether or not the copula changes during a considered time period.

Note that the points  $(\mathbf{u}_j)_{j=1,2,\dots,m} \in [0,1]^d$  are chosen arbitrarily. Clearly, the power of the test  $\mathcal{T}$  depends on the choice of the points  $(\mathbf{u}_j)_{j=1,2,\dots,m}$ , which is a drawback as also the choice of cells in the usual GOF Chi-square test. Without a priori, given an integer  $N$ , it is always possible to choose an uniform grid of the type  $(i_1/N, i_2/N, \dots, i_k/N)$ , for every integers  $1 \leq i_1, i_2, \dots, i_k \leq N - 1$ .

### 3.2 Test to detect the change type of the copula

If we reject  $\mathcal{H}_0^{(1)}$ , then we should study the dependence structure inside the  $d$ -dimensional vector, in a dynamic way. Thus, we test the change type of the copula. Let be  $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$  a family of



copulas and  $\theta_{n-1}$  the parameter depending on the past information set of the process.

Let be the null hypothesis,

$$\mathcal{H}_0^{(2)} : \text{For every } n \in \mathbb{N}, \theta_{n-1} = \theta(\varphi_{n-1}), C(\cdot|\varphi_{n-1}) = C_{\theta_{n-1}} \in \mathcal{C},$$

and the alternative,

$$\mathcal{H}_a^{(2)} : \text{For some } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) \notin \mathcal{C}.$$

We use the same notations as before and we introduce the statistics associated to this test:

$$\mathcal{R} = (nh_n^d) \sum_{j=1}^m \frac{\{\hat{c}(\mathbf{u}_j|\varphi_{n-1}) - c_{\hat{\theta}_{n-1}}(\mathbf{u}_j|\varphi_{n-1})\}^2}{\hat{\sigma}^2(\mathbf{u}_j)}, \quad (4)$$

where  $\mathbf{u}_j$  ( $j = 1, 2, \dots, m$ ) is described in Assumption 4, the  $\sigma$ -algebra  $\varphi_{n-1}$  is introduced in Assumption 3, and  $\hat{\theta}_{n-1}$  is the consistent estimator of  $\theta_{n-1}$ .  $c_{\hat{\theta}_{n-1}}(\mathbf{u}_j|\varphi_{n-1})$  denotes the density of the conditional copula  $C_{\hat{\theta}_{n-1}}$ , and  $\hat{c}(\mathbf{u}_j|\varphi_{n-1})$  is the empirical copula density given in Equation (2). Moreover,

$$\hat{\sigma}^2(\mathbf{u}_j) = c_{\hat{\theta}_{n-1}}^2(\mathbf{u}_j|\varphi_{n-1}) \cdot \int K^2.$$

Under the null hypothesis  $\mathcal{H}_0^{(2)}$ , the statistics  $\mathcal{R}$  defined in Equation (4) tends to a Chi-square distribution with  $m$  degrees of freedom, when  $n \rightarrow \infty$ . If we reject  $\mathcal{H}_0^{(2)}$ , the copula family changes. On the other hand, if we do not reject  $\mathcal{H}_0^{(2)}$ , the copula family remains static, then we say that only the copula's parameters change. After determining the change type of the copula by testing  $\mathcal{H}_0^{(2)}$ , we analyze in details the copula's changes.

Note that if we consider the Archimedean copula family  $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$ , the parameter  $\theta$  can be estimated using the Kendall's tau.

## 4 Detail analysis for the copula change

According to the test results for the hypotheses  $\mathcal{H}_0^{(1)}$  and  $\mathcal{H}_0^{(2)}$ , we determine the change type of the copula during the time period. Here, we analyze two kinds of changes.

#### 4.1 Detail analysis for the change of copula's family

If we reject  $\mathcal{H}_0^{(2)}$ , then the copula's family may change. We apply the so called binary segmentation procedure to detect the change point. This procedure proposed by [Vostrikova \(1981\)](#) enables to simultaneously detect the number and the location of change-points. The procedure can be described as follows. Firstly, we choose the best copula according to the AIC criterion on the whole sample. Then the sample is divided into two subsamples, we choose the best copulas on these two subsamples respectively. If the two best copulas are different from the copula on the whole period, we continue this segmentation procedure, i.e., we again divide each subsample into two parts, and do the same work as in the previous step. Finally, the procedure stops when all the best copulas on each subsample have been adjusted. Therefore, we get all the change points for the family changes.

#### 4.2 Detail analysis for the change of copula's parameters

If  $\mathcal{H}_0^{(2)}$  is not rejected, the copula's family remains changeless. Therefore, we say that only the copula parameters change. Then, we need to deal with the analysis of the time-varying parameters.

To find the change time, we apply the change point technique introduced by [Dias and Embrechts \(2004\)](#). Let  $\mathbf{u}_1, \dots, \mathbf{u}_n$  be a sequence of independent random vectors in  $[0, 1]^d$  with univariate uniformly distributed margins and copulas  $C(\mathbf{u}; \theta_1, \eta_1), \dots, C(\mathbf{u}; \theta_n, \eta_n)$ , respectively, where  $\theta_i$  and  $\eta_i$  represent the dynamic and the static copula parameters satisfying  $\theta_i \in \Theta^{(1)} \subseteq \mathbb{R}^p$  and  $\eta_i \in \Theta^{(2)} \subseteq \mathbb{R}^q$ . We test the null hypothesis

$$\mathcal{H}_0^{(3)} : \theta_1 = \theta_2 = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n$$

against

$$\mathcal{H}_a^{(3)} : \theta_1 = \dots = \theta_{k^*} \neq \theta_{k^*+1} = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n.$$

Here  $k^*$  is the location or time of the change-point if we reject the null hypothesis. The hypotheses are tested through the generalized

likelihood ratio, that is, the null hypothesis would be rejected for small values of the likelihood ratio:

$$\Lambda_k = \frac{\sup_{(\theta, \eta) \in \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq n} c(\mathbf{u}_i; \theta, \eta)}{\sup_{(\theta, \theta', \eta) \in \Theta^{(1)} \times \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq k} c(\mathbf{u}_i; \theta, \eta) \prod_{k < i \leq n} c(\mathbf{u}_i; \theta', \eta)},$$

where  $c$  is the density of copula  $C$ . The statistic  $\Lambda_k$  is carried out through maximum likelihood method, all the necessary conditions of regularity and efficiency have to be assumed, [Lehmann and Casella \(1998\)](#).

If  $L_k(\theta, \eta) = \sum_{1 \leq i \leq k} \log c(\mathbf{u}_i; \theta, \eta)$ , and  $L_k^*(\theta, \eta) = \sum_{k < i \leq n} \log c(\mathbf{u}_i; \theta, \eta)$ , then, the likelihood ratio equation can be written as

$$-2 \log(\Lambda_k) = 2(L_k(\hat{\theta}_k, \hat{\eta}_k) + L_k^*(\hat{\theta}_k^*, \hat{\eta}_k) - L_n(\hat{\theta}_n, \hat{\eta}_n)).$$

The hypothesis  $\mathcal{H}_0^{(3)}$  is rejected for large values of

$$Z_n = \max_{1 \leq k < n} (-2 \log(\Lambda_k)).$$

Pursuing [Gombay and Horváth \(1996\)](#), the following approximation holds:

$$\mathbb{P}(Z_n^{1/2} \geq x) \approx \frac{x^p \exp(-x^2/2)}{2^{p/2} \Gamma(p/2)} \cdot (HL - \frac{p}{x^2} HL + \frac{4}{x^2} + O(\frac{1}{x^4})),$$

as  $x \rightarrow \infty$ , where  $HL = \log \frac{(1 - g_n)(1 - l_n)}{g_n l_n}$ ,  $g_n = l_n = (\log n)^{3/2}/n$ , [Dias and Embrechts \(2004\)](#).

If we assume that there is exactly one change point, then the estimate for the change time is given by  $\hat{k}_n = \min\{1 \leq k < n : Z_n = -2 \log(\Lambda_k)\}$ .

Considering that the change-point test has less power for small changes, we analyze the dependence more specifically by assuming a time-varying behavior for the corresponding parameter. In order to show how it works, we provide now the dynamics of the parameters for the copulas that we use in the applications. The definitions of the

copulas are recalled in an Annex.

Using the dynamic Gaussian copula, we define the time-varying correlation as :

$$\rho_t = h^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 h(\rho_{t-1})), \quad (5)$$

where  $(x_{1,t})_t$  and  $(x_{2,t})_t$  are the samples,  $r_0, r_1, s_1$  the parameters estimated by maximum likelihood, and  $h(\cdot)$  the Fisher's transformation such that  $h(\rho) = \log\left(\frac{1+\rho}{1-\rho}\right)$ , to ensure that  $-1 < \rho < 1$ .

If we work with the dynamic Student  $t$ -copula, the time-varying degrees of freedom  $\nu$  can be defined as:

$$\nu_t = l^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 l(\nu_{t-1})), \quad (6)$$

where  $r_0, r_1, s_1$  are parameters estimated by maximum likelihood method, and  $l(\cdot)$  is a function defined as:  $l(\nu) = \log\left(\frac{1}{\nu-2}\right)$ .

For the dynamic Gumbel copula, the time-varying parameter  $\delta$  can be described as:

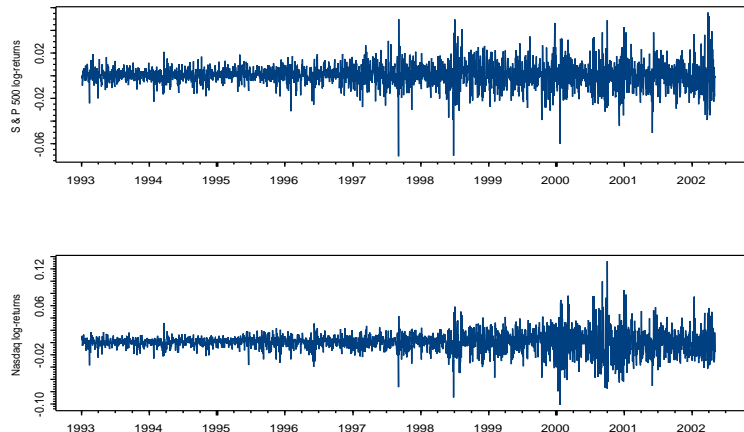
$$\delta_t = w^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 w(\delta_{t-1})), \quad (7)$$

where  $r_0, r_1, s_1$  are parameters estimated by maximum likelihood method, and  $w(\cdot)$  is a function defined as:  $w(\delta) = \log\left(\frac{1}{\delta-1}\right)$ .

## 5 Empirical work

We apply now the above change analysis of dynamic copula to Standard & Poor 500 (S&P500) and Nasdaq indices. The sample data sets contain 2436 daily observations from 4 January, 1993 to 30 August, 2002 for both assets. The log-returns of these two indices are shown in Figure 2.

From Figure 2, it is observed that the outliers of the two underlying log-returns typically occur simultaneously, and almost in the same



**Fig. 2.** Log-returns for S&P500 (up) and Nasdaq (down) Indices

direction. We observe that both assets fluctuate a lot from the middle of 1997 when the Asian financial crisis burst out.

Let  $r_{i,t}$  ( $i = 1, 2$ ) be the daily log-returns for S&P500 and Nasdaq respectively. In order to filter the observed instability, we fit a univariate GARCH(1,1) model to each log-return series, that is:

$$\begin{aligned} r_{i,t} &= \mu_i + \xi_{i,t} \quad \text{with} \quad \xi_{i,t} = \sigma_{i,t} \varepsilon_{i,t}, \\ \sigma_{i,t}^2 &= \alpha_{i,0} + \alpha_{i,1} \varepsilon_{i,t-1}^2 + \beta_{i,1} \sigma_{i,t-1}^2, \\ \varepsilon_{i,t} | \varphi_{i,t-1} &\sim N(0, 1), \end{aligned} \quad (8)$$

where  $\mu_i$  is the drift,  $\alpha_{i,0}$ ,  $\alpha_{i,1}$ ,  $\beta_{i,1}$  are parameters in  $\mathbb{R}$ . The estimation of the parameters using likelihood method are given in Table 1.

### 5.1 Dynamic copula for S&P500 and Nasdaq indices

In order to investigate the dependence between these two data sets, we firstly adjust the best copula for the standard residual-pairs  $(\varepsilon_{1,t}, \varepsilon_{2,t})$  over the whole period using AIC criterion. The set of copulas includes Gaussian, Student  $t$ , Gumbel, Clayton and Frank copulas. The copulas fitting is given in Table 2. Although Student  $t$

**Table 1.** Estimates of GARCH(1,1) parameters

Parameter	S&P500	Nasdaq
$\mu$	6.013e-04 (1.633e-04)	9.395e-04 (2.116e-04)
$\alpha_0$	6.018e-07 (1.579e-07)	1.486e-06 (2.877e-07)
$\alpha_1$	7.947e-02 (6.670e-03)	1.157e-01 (8.902e-03)
$\beta_1$	9.201e-01 (6.761e-03)	8.849e-01 (8.596e-03)

Figures in brackets are standard errors

copula has the smallest AIC value, the estimation, unfortunately, does not converge, therefore, Gaussian copula provides the best copula for the whole sample.

**Table 2.** Copula fitting results

Copula	Parameter	AIC	Convergence
Gaussian	8.116e-01 (2.684e-02)	-2615.196	T
Student $t$	8.143e-01 (3.384e-02); 13.668 (5.078e-01)	-2642.88	F
Gumbel	2.461 (4.090e-02)	-2505.374	T
Clayton	1.659 (5.280e-02)	-1867.982	T
Frank	8.391 (1.878e-01)	-2419.844	T

Figures in brackets are standard errors, for Student  $t$  copula, the first parameter is correlation, the second one is degree of freedom, and “T” means “True”, “F” means “Fault”.

In a first step, we test the stability of this copula. We use the test developed in Section 3.1 and the statistics  $\mathcal{T}$  defined in Equation (3). We assume that the true copula is the Gaussian one specified in Table 2. To apply the test, we choose a kernel function  $K$  given by

$$K(\mathbf{u}) = \left(\frac{15}{16}\right)^2 \prod_{i=1}^2 (1 - u_i^2)^2 \mathbf{1}_{\{u_i \in [0,1]\}},$$

with bandwidth  $\hat{h}_n = \frac{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}{n^{1/6}}$ , and  $\sigma_l^2$  will be the empirical variance of  $\hat{F}_l$  ( $l = 1, 2$ ). Furthermore, for the points  $(\mathbf{u}_j)_{j=1,2,\dots,m}$

in Assumption 4, we choose  $m = 81$  points on the uniform grid  $(1/10, 2/10, \dots, 9/10) \times (1/10, 2/10, \dots, 9/10)$ .

Using this approach, the  $p$ -value for the null hypothesis  $\mathcal{H}_0^{(1)}$  is equal to 0. Thus the null hypothesis is rejected and the copula for the data set does not remain static.

In a second step, we detect the changes of copula's family using the binary segmentation procedure described in Section 4.1: we are able to detect all of the time changes for the copula's family that we have detected. The results are given in Table 3.

**Table 3.** Changes of copula's family

Period	Copula	Parameter	Change time
04/01/93-24/10/97	Gaussian	7.716e-01 (3.632e-02)	-
24/10/97-11/01/99	Student $t$	8.497e-01 (3.636e-02); 8.355 (2.071)	24 Oct. 1997
11/01/99-18/08/99	Gumbel	3.06275 (2.027e-01)	11 Jan. 1999
18/08/99-06/12/99	Gaussian	8.429e-01 (1.595e-01)	18 Aug. 1999
06/12/99-24/03/00	Student $t$	6.317e-01 (1.462e-01); 14.564 (1.644)	6 Dec. 1999
24/03/00-09/08/00	Gumbel	2.81704 (2.384e-01)	24 Mar. 2000
09/08/00-22/12/00	Gaussian	8.630e-01 (1.481e-01)	9 Aug. 2000
22/12/00-20/02/01	Student $t$	9.115e-01 (2.844e-01); 1.693383 (9.324e-01)	22 Dec. 2000
20/02/01-08/06/01	Gaussian	8.673e-001 (1.709e-01)	20 Feb. 2001
08/06/01-30/08/02	Student $t$	8.948e-01 (1.200e-01); 24.506 (1.134)	8 Jun. 2001

“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance. Figures in brackets are standard errors, and for Student  $t$  copula, the first parameter is correlation, the second one is degree of freedom.

The results in Table 3 provide the change periods for copula's family that coincide with some financial incidents:

- 24 Oct. 1997: copula family changes from Gaussian to Student  $t$ . This date corresponds to 27 October, 1997 when the Asian financial crisis came to a head.
- 11 Jan. 1999: copula family changes from Student  $t$  to Gumbel. This date corresponds to the introduction of Euro as the unit European currency.
- 24 Mar. 2000: copula family changes from Student  $t$  to Gumbel. This date corresponds to the technology-heavy Nasdaq stock market peaked on 10 Mar. 2000 and S&P 500 peaked on 24 Mar. 2000.
- 8 Jun. 2001: copula family changes from Gaussian to Student  $t$ . This date corresponds to the subsequent 9.11 attacks and the recession lasted from March 2001 to November 2001 in the United States.

Thirdly, for each corresponding period within which the copula’s family does not change, we detect the change points for the copula’s parameters in the way introduced in Section 4.2. We provide the results in Table 4.  $z_n^{1/2}$  is the corresponding observation value for the statistics  $Z_n^{1/2}$ .

**Table 4.** Change-point for copula’s parameters

Period	Copula	$z_n^{1/2}$	$P$	$\mathcal{H}_0^{(3)}$	Change time
04/01/93-24/10/97	Gaussian	3.134	7.250e-02	×	15 Oct. 1997
24/10/97-11/01/99	Student $t$	1.240	4.214e-01	✓	-
11/01/99-18/08/99	Gumbel	2.253	3.471e-01	✓	-
18/08/99-06/12/99	Gaussian	2.938	6.331e-02	×	1 Dec. 1999
06/12/99-24/03/00	Student $t$	1.255	6.648e-01	✓	-
24/03/00-09/08/00	Gumbel	2.761	1.054e-01	✓	-
09/08/00-22/12/00	Gaussian	2.298	2.829e-01	✓	-
22/12/00-20/02/01	Student $t$	2.547	1.272e-01	✓	-
20/02/01-08/06/01	Gaussian	3.398	1.702e-02	×	4 Jun. 2001
08/06/01-30/08/02	Student $t$	1.818	7.743e-01	✓	-

“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance.  $P$  denotes the probability  $P(Z_n^{1/2} > z_n^{1/2})$  in Section 4.2, the null hypothesis  $\mathcal{H}_0^{(3)}$  is rejected at a 10% level, we simply denote “✓” as “not reject” and “×” as “reject”.



The change points for the copula’s parameter shown in Table 4 reflect some financial events, which are:

- 15 Oct. 1997: corresponds to the Asian financial crisis beginning from July 1997;
- 1 Dec. 1999: corresponds to the preparation of the unit European currency, euro;
- 4 Jun. 2001: corresponds to the recession beginning from March 2000 to November 2001, when the real gross domestic product in the United States dropped by 0.2% total from the fourth quarter of 2000;

Finally, notice that the above change-point analysis only detects “large” changes in the parameters thus, we further study the dynamic parameters using the appropriate time-varying functions introduced in Equations (5), (6) and (7). The results are given in Table 5.

## 5.2 Risk management strategy

This systematic change analysis permits to detect changes inside the dependence structure of the financial data. Thus, it can be used to improve the computation of the VaR and ES risks measures providing their evolution in time.

Recall that, for a given probability level  $\alpha$ ,  $0 < \alpha < 1$ ,  $\text{VaR}_\alpha$  is simply the maximum loss that is exceeded over a specified period with a level of confidence  $1 - \alpha$ . If  $X$  is a random return with distribution function  $F_X$ , then

$$F_X(\text{VaR}_\alpha) = P\{X \leq \text{VaR}_\alpha\} = \alpha.$$

Thus, losses lower than  $\text{VaR}_\alpha$  occur with probability  $\alpha$ . The Expected Shortfall represents the expectation of loss knowing that a threshold is exceeded, for instance  $\text{VaR}_\alpha$ , and we define it as:

$$\text{ES}_\alpha(X) = E\{X|X \leq \text{VaR}_\alpha\}.$$

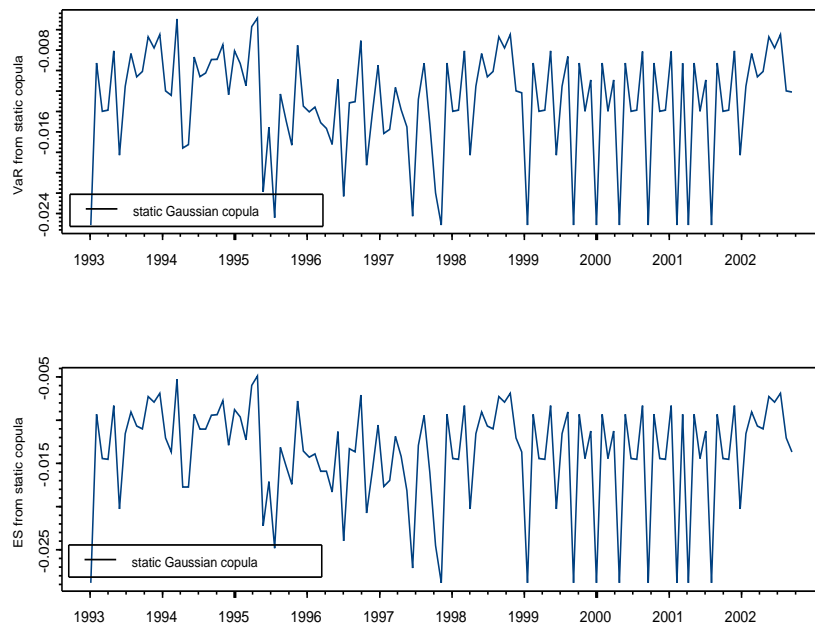
For the portfolio of S&P500 and Nasdaq with equal weight, we compare the VaR and ES values using the static copula and the dynamic

**Table 5.** Estimates for time-varying parameters

Period	Copula	Parameter	$r_0$	$r_1$	$s_1$
04/01/93-24/10/97	Gaussian	dynamic $\rho$	2.620e-02 (4.961e-02)	4.160e-02 (5.347e-02)	9.735e-01 (2.689e-01)
24/10/97-11/01/99	Student $t$	$\rho = 8.249e-01$ ; (2.264e-02)	-	-	-
		dynamic $\nu$	8.915e-01 (8.238e-01)	-1.632e-01 (1.389e-01)	3.313e-01 (1.226e-01)
11/01/99-18/08/99	Gumbel	dynamic $\delta$	-1.263 (2.194e-01)	-5.236e-03 (1.129e-01)	-7.700e-01 (3.890e-02)
18/08/99-06/12/99	Gaussian	dynamic $\rho$	3.266 (1.495)	3.081e-02 (3.636e-03)	-3.557e-01 (3.291e-02)
06/12/99-24/03/00	Student $t$	$\rho = 5.433e-01$ ; (2.938e-02)	-	-	-
		dynamic $\nu$	7.228e-01 (5.977e-01)	4.033e-01 (1.428e-01)	-6.784e-01 (2.584e-01)
24/03/00-09/08/00	Gumbel	dynamic $\delta$	-8.134e-01 (5.823e-01)	-3.892e-02 (3.937e-01)	-4.400e-01 (5.333e-02)
09/08/00-22/12/00	Gaussian	dynamic $\rho$	3.317 (3.055e-02)	1.104e-01 (2.343e-03)	-3.147e-01 (7.314e-01)
22/12/00-20/02/01	Student $t$	$\rho = 9.387e-01$ ; (5.236e-01)	-	-	-
		dynamic $\nu$	-1.922 (1.485)	1.276 (1.893)	-5.806e-01 (5.530e-01)
20/02/01-08/06/01	Gaussian	dynamic $\rho$	4.236e-02 (6.874e-01)	3.808e-02 (5.258e-02)	9.792e-01 (2.180e-01)
08/06/01-30/08/02	Student $t$	$\rho = 8.747e-01$ ; (2.711e-02)	-	-	-
		dynamic $\nu$	-5.764e-01 (5.540e-01)	-3.595e-01 (8.683e-01)	-1.025 (3.303e-01)

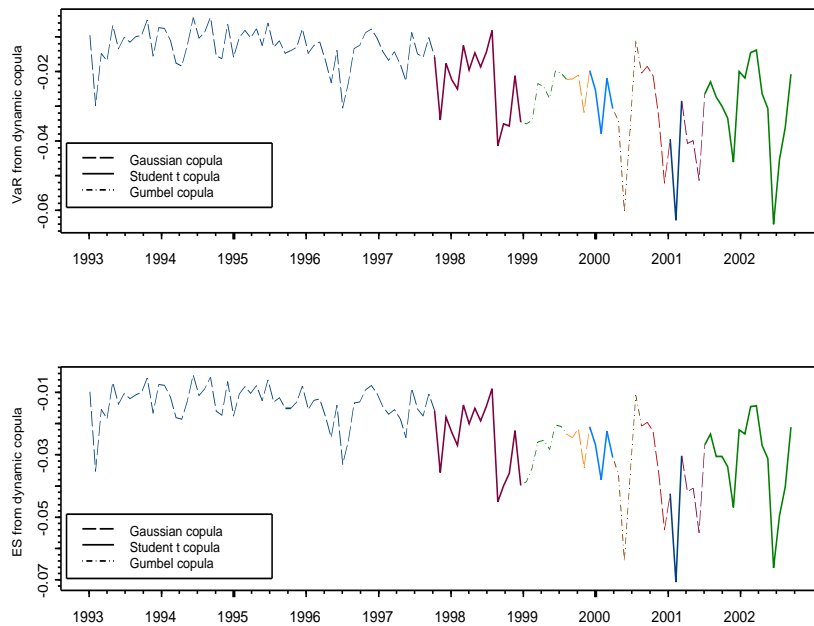
“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance. Figures in brackets are standard errors,  $r_0$ ,  $r_1$  and  $s_1$  have been estimated using maximum likelihood method.

sequence of copulas. In the static case, we use the Gaussian copula obtained in Table 2, estimated on the whole period. In the dynamical case, we use the time-varying parameters provided in table 5, assuming that the copula's family does not change in each subsample (the families of copulas are provided in Table 3). We calculate the VaR and ES values per 20 days in order to take into account the time evolution. The results obtained from the static and time-varying parameters copulas are shown in Figure 3 and Figure 4.



**Fig. 3.** VaR and ES using static copula for the portfolio of S&P500 and Nasdaq Indices for a confidence level  $\alpha = 0.05\%$

On Figure 3 and Figure 4, we observed that the VaR and ES values fluctuate a lot. Through comparison, some conclusions are summarized below:



**Fig. 4.** VaR and ES using dynamic copulas for the portfolio of S&P500 and Nasdaq Indices for a confidence level  $\alpha = 0.05\%$

1. The dynamics of the VaR and ES using the static copula only come from the volatilities of the GARCH model, while using the dynamic copulas, the dynamics of VaR and ES still depend on the dynamic dependence structure;
2. The VaR and ES from the static copula have generally smaller absolute values than those from the dynamic copulas, which means that the dynamic copula model takes into account more risk information than the static one.
3. After the middle of 1997 when the Asian financial crisis broke out, the VaR and ES values calculated from the dynamic copula vary a lot, while this phenomenon does not distinctly appear when we use the static copula. Thus, the dynamic copula permits to

include the shocks when they appear and as frequently as they appear, which is not the case in the static setting.

From the above remarks, it appears that the dynamic changes inside the dependence structure of a portfolio can play an important role in risk management. To model dynamic changes inside copulas has already shown its interest in multivariate option pricing, [Guégan and Zhang \(2009\)](#).

## 6 Conclusion

In this paper, we introduce a new approach to detect the best dynamic copula which characterizes the evolution of several data sets. It is based on a series of nested tests using conditional copula and GOF tests. This approach permits to determine the change type of the copula using the binary segmentation procedure, the change-point analysis and the time-varying parameter functions.

The method is illustrated using S&P500 and Nasdaq indices. The computations of risks measures using the method developed in this paper show its importance for risk management strategy: indeed, it appears that the values of the risk measures can be drastically different with respect of the copula used.

## 7 Annex

### 7.1 Gaussian copula

The copula of the  $d$ -variate normal distribution with linear correlation matrix  $R$  is

$$C_R^{\text{Ga}}(\mathbf{u}) = \Phi_R^d(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \dots, \Phi^{-1}(u_d)),$$

where  $\Phi_R^d$  denotes the joint distribution function of the  $d$ -variate standard normal distribution function with linear correlation matrix  $R$ , and  $\Phi^{-1}$  denotes the inverse of the distribution function of the univariate standard Gaussian distribution. Copulas of the above form are called Gaussian copulas. In the bivariate case, we denote  $\rho$

as the linear correlation coefficient, then the copula's expression can be written as

$$C^{\text{Ga}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right\} ds dt.$$

The Gaussian copula  $C^{\text{Ga}}$  with  $\rho < 1$  has neither upper tail dependence nor lower tail dependence.

## 7.2 Student-t copula

If  $\mathbf{X}$  has the stochastic representation

$$\mathbf{X} \stackrel{d}{=} \mu + \frac{\sqrt{\nu}}{\sqrt{S}} \mathbf{Z}, \quad (9)$$

where  $\stackrel{d}{=}$  represents the equality in distribution or stochastic equality,  $\mu \in \mathbb{R}^d$ ,  $S \sim \chi_\nu^2$  and  $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$  are independent, then  $\mathbf{X}$  has a  $d$ -variate  $t_\nu$  distribution with mean  $\mu$  (for  $\nu > 1$ ) and covariance matrix  $\frac{\nu}{\nu-2}\Sigma$  (for  $\nu > 2$ ). If  $\nu \leq 2$  then  $\text{Cov}(\mathbf{X})$  is not defined. In this case we just interpret  $\Sigma$  as the shape parameter of the distribution of  $\mathbf{X}$ . The copula of  $\mathbf{X}$  given by Equation (9) can be written as

$$C_{\nu,R}^t(\mathbf{u}) = t_{\nu,R}^d(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \dots, t_\nu^{-1}(u_d)),$$

where  $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$  for  $i, j \in \{1, 2, \dots, d\}$ ,  $t_{\nu,R}^d$  denotes the distribution function of  $\sqrt{\nu}\mathbf{Y}/\sqrt{S}$ ,  $S \sim \chi_\nu^2$  and  $\mathbf{Y} \sim \mathcal{N}_d(\mathbf{0}, R)$  are independent. Here  $t_\nu$  denotes the margins of  $t_{\nu,R}^d$ , i.e., the distribution function of  $\sqrt{\nu}Y_i/\sqrt{S}$  for  $i = 1, 2, \dots, d$ . In the bivariate case with the linear correlation coefficient  $\rho$ , the copula's expression can be written as

$$C_{\nu,R}^t(u, v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{s^2 - 2\rho st + t^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2} ds dt.$$

Note that  $\nu > 2$ . And the upper tail dependence and the lower tail dependence for Student  $t$  copula have the equal value.

### 7.3 Gumbel copula

The Gumbel copula is defined as

$$C_{Gu}(u, v; \delta) = \exp\{-[(-\ln u)^\delta + (-\ln v)^\delta]^{1/\delta}\}, \quad \delta \in [1, \infty).$$

It has the properties:

1.  $\delta = 1$  implies  $C_{Gu}(u, v; 1) = uv$ ;
2. As  $\delta \rightarrow \infty$ ,  $C_{Gu}(u, v; \delta) \rightarrow \min(u, v)$ ;
3. Gumbel copula has upper tail dependence:  $2 - 2^{1/\delta}$ ;
4. Gumbel copula has no lower tail dependence.

The Gumbel copula belongs to the Archimedean copula, [Joe \(1997\)](#) and [Nelsen \(1999\)](#).

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