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Lanouar Charfeddine, Dominique Guegan

IS IT POSSIBLE TO DISCRIMINATE BETWEEN DIFFERENT SWITCHING REGRESSIONS MODELS: AN EMPIRICAL INVESTIGATION?

Lanouar CHARFEDDINE *† Dominique GUEGAN ‡

ABSTRACT

In this paper we study, using the sup LR test, the possibility of discrimination between two classes of models: the Markov switching models of Hamilton (1989) and the Threshold Auto-Regressive Models (TAR) of Lim and Tong (1980). This work is motivated by the fact that generally practicians use, in applications, switching models without any statistical justification. Using experiment simulations, we show that it is very difficult to discriminate between the MSAR and the SETAR models specially using large samples. This means that when the null hypothesis is rejected, it appears that different switching models are significant. Moreover, the results show that the power of the sup LR test is sensitive to the mean, the noise variance and the delay parameter.

Finally, we apply this methodology to two time series: the US GNP growth rate and the US/UK exchange rate. We shall retain a Markov switching process for the US GNP growth rate and the US/UK exchange rate (monthly data). For the US/UK exchange rate (quarterly data), we accept the null hypothesis of a random walk.

JEL classification: C12;C15;F31

Keywords: Switching Models, Sup LR test, Empirical power, Exchange rate

1 INTRODUCTION:

In the last two decades, a huge literature on non linear models have been developed. Researchers have largely used this class of models to explain many macroeconomic and financial specific phenomena like sudden changes, irreversibility and asymmetry... From the available non linear models, a specific attention have been attributed to the Markov Switching Auto-Regressive models (MSAR afterhere) of Hamilton (1989) and the SETAR model of Lim and Tong (1980).

Since these two previous seminal papers, applications of the MSAR and the SETAR models have been extensively increased. Their use in empirical applications is motivated by many economics and internationals event like the collapse of the Bretton Wood's system, the two oil crisis, the Plaza Accord, the collapse of the European Monetary System (EMS) and the intervention of Banks in foreign exchanges market. In empirical applications, the Markov switching models have been employed to study the aggregate outputs (Hamilton, 1990), the annual growth rate of consumption in an asset-pricing (Cecchetti et *al.*, 1990), the behavior of foreign exchanges rates (Engle and Hamilton, 1989), the effect of federal reserve actions on interest rates (Hamilton 1988, Garcia and Perron 1989), for instance. In the other hand, many others researchers have used the SETAR models in empirical applications, we refer to Potter (1995), Hansen (1997), Proietti (1998) and Ferrara and Guégan (2005) for instance.

These two models have the advantage of being able to model and to capture asymmetry, sudden changes and irreversibility time observed in many economic and financial time series. Despite these similarities, these models have been involved, in the literature, largely independently. Moreover, testing between these two types of switching regression models are seldom explored by researchers. In the literature only few papers have dealt with this problem, see for example Carrasco (2002). Thus, it appears very important to analyze the behavioral similarities created by these two models. Moreover, one should investigates and proposes a way to discriminate between them.

Generally, before applying a regime switching model, one should test the null hypothesis of a single state against the alternative of several states. Building

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†University Marne-la-Vallée. Tel: (0033)627144742, lanouar_charf@yahoo.fr.

‡Professor at University Paris 1 Sorbonne, MSE-laboratoire CES-CERMSEM, 106 bd de l'hopital 75013 Paris.

such tests is problematic because of the presence of nuisance parameters which are not identified under the null hypothesis. For instance, these parameters are: the probabilities of transition p_{00} and p_{11} for the MSAR models and the threshold parameter r and the delay parameter d for the SETAR models. Moreover, in the case of the Markov switching models another problem occurs through the score function which is identically zero under the null. As a result, the Likelihood Ratio (LR), the Lagrange Multiplicateur (LM) and the Wald tests (W) have not their standard asymptotic distribution, see Davies (1977, 1987), Hansen (1992,1996), Gong and Mariano (1997), Garcia (1998), and Carrasco (2004).

This paper has two mains objectives. First, we analyze the behavior of the sup LR test against the particular alternative of a SETAR model. Precisely, we investigate the behavior of this test with respect to the delay parameter d and different values of the mean and the noise variance. Then, we applied the previous results to two real data sets, largely used in the empirical literature. The results show that the sup LR test rejects the null hypothesis of linearity whatever the switching models under the alternative (a SETAR or a MSAR models). This means that when the null hypothesis is rejected, it appears that different switching models are significant[1]. So, one should be careful to interpret this result because the model under the alternative is a possible candidate but other nonlinear models can be also candidates. Moreover, we show that discrimination between the two models is impossible as soon as the data set's sample size is large.

The outline of the paper is as follows. In Section 2, we discuss the sup LR test proposed by Garcia (1998) under the null hypothesis of an AR process and the alternative of a MSAR model. In Section 3, we report some new results concerning the possibility of differentiation between the SETAR and MSAR models. In Section 4, we investigate the US/UK exchange rates series and the US GNP growth rate. Section 5 concludes.

[1] Hamiton (1989) applied a Markov switching model, with autoregressive order equal to four, to the growth rates of U.S GNP but Hansen (1992) and Garcia (1998) show, based on Sup LR test, that the null hypothesis of $AR(4)$ cannot be rejected. Potter (1995) applied SETAR model to the U.S GNP but Hansen (1996) doubts about the evidence for the SETAR model.

2 Models and Testing procedure:

This section introduces the Markov switching model, the SETAR model and the sup LR test developed by Garcia (1998).

2.1 The Models

Consider the stationary Markov switching model (y_t) defined by:

$$y_t = \phi_1 + (\phi_2 - \phi_1)s_t + z_t \tag{1}$$
$$\text{with } z_t = \theta z_{t-1} + u_t,$$

where u_t is a Gaussian strong white noise N(0,σ_u^2). We assume that the state s_t is independent of u_t. The parameters ϕ_1, ϕ_2 and θ take values in $\Re$. The state s_t is an unobserved Markov chain whose transition probability is defined by:

$$p(s_t = j | s_{t-1} = i) = p_{ij}, \qquad i, j = 0, 1, \tag{2}$$

with $0 \leq p_{ij} \leq 1$ and $\sum_{j=0}^{1} p_{ij} = 1, i = 0, 1$, and the transition matrix is

$$\mathbf{P} = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix}.$$

We denote π_i the unconditional probabilities for the process (s_t) to be in each regime : $\pi_i = P(s_t = i), i = 0, 1$. These unconditional probabilities are equal to :

$$\pi_0 = P(s_t = 0) = \frac{1-p_{11}}{2-p_{00}-p_{11}} \text{ and}$$
$$\pi_1 = P(s_t = 1) = \frac{1-p_{00}}{2-p_{00}-p_{11}}.$$

In the Markov switching model (1) the 'state' or 'regime' plays an important role. Indeed, Hamilton (1989) suggests that the existence of discrete 'regimes' explain the nonlinearity in GNP growth rates. The first 'state' will correspond to a fast growth and the second one to a slow growth. Here we assume that the MSAR model is strictly stationary and β-mixing. Guégan and Rioublanc (2005) show that a sufficient condition for strict stationarity for Markov switching model is $(1 - p_{00} - p_{11}) \log |\theta| \leq 0$, see also Yao and Attali (2000) for geometric ergodicity of MSAR Models.

We consider also the stationary first-order threshold autoregressive process (y_t) introduced by Lim et Tong (1980):

$$y_t = \phi_1 + (\phi_2 - \phi_1)s_t + z_t \tag{3}$$

with $z_t = \theta z_{t-1} + u_t$, and

$$s_t = I\{y_{t-d} \leq r\} = \begin{cases} 0 & \text{if } y_{t-d} \leq r \\ 1 & \text{if } y_{t-d} > r \end{cases}$$

where r and d are the threshold and delay parameters and ϵ_t is a Gaussian strong white noise N(0,σ_ϵ^2). The necessary and sufficient conditions for the geometrical ergodicity of (y_t) in model (3) is simply $\theta \quad \leq 1$, see Chen and Tsay (1991). In simulations, we assume that $r = 0$ and $d = 1, 2$.

The SETAR (3) and the MSAR (1) models are intimately related. In both models, changes in regimes is permanent contrary to structural change model where change occur only one time in the series. Moreover, the SETAR model is a special case of the Markov switching model when $\theta = 0$, although in that case the Markov chain $I\{y_{t-d} \leq r\}$, $d = 1, 2$ is endogenous contrary to the Markov switching model proposed by Hamilton (1989) where the Markov chain is exogenous. Despite these similarities, the main difference between the two models concerns the processus that govern changes in regimes under each model. Contrary to the MSAR models where changes in regimes are governed by an unobserved Markov chain, under the SETAR models changes in regimes occur when an observed variable y_t passes a certain threshold parameter r.

Figures 1-4, in appendix, give some representations of the trajectories of the MSAR and SETAR models. These figures show that the trajectories are similar and one cannot differentiate between them. A same conclusion can be drawn from the scatters plots of y_t versus y_{t-1}. So, this graphical analysis can lead to misspecification which means that if researchers based their choice only in simple linear properties, they will be misleading and a misspecified model will be chosen. This create a problems concerning predictions and forecasting. Thus, it is very important to investigate the possibility of discrimination between different switching regressions models.

2.2 The Sup LR test for Markov Switching Model

The literature on testing, when nuisance parameters are present under the alternative hypothesis, has growing rapidly and a variety of statistical tests have been developed. Most of the tests adopt the approach developed by Davies (1977,1987), which proposes a sup LR test. The weakness of this test lies on the fact that we do not know the asymptotic distribution of this test under the alternative.

Hansen (1992) proposes the likelihood ratio statistic. He considers the likelihood function as a function of unknown parameters, and he gets a bound for the asymptotic distribution of the test. The Hansen's approach (1992) is time consuming and makes it inappropriate in applications. In another hand, this test provides only a bounds and in terms of decision theory, it does not provide critical values. Andrews (1993) and Andrews and Ploberger (1994) proposed the sup LM test and a class of average exponential LM, Wald and LR tests. They showed that they are optimal in terms of weighted average power. Andrews and Ploberger (1995) show that the sup LR test is asymptotically admissible. It is the best test against alternatives that are sufficiently distant from the null hypothesis. Gong and Mariano (1997) developed two statistic tests for Markov switching models: the Difference Test $\mathrm{DT_N}$ (analogous to the LR test) and the LM test in the frequency domain. They derive their exact asymptotic null distributions under the condition of unidentified nuisance parameters. They show that, they only have to face to the problem of unidentified nuisance parameters in nonlinear context because the singularity problem disappears. Recently, Carrasco et *al.* (2004) proposed the sup TS test which is 'asymptotically equivalent to Garcia's test in the sense that both are some kind of likelihood ratio tests and hence they are expected to have similar powers'.

As shown in the literature the likelihood ratio statistics behave relatively smoothly when a nuisance parameter is present under the alternative contrary to the Wald tests. The sup LR test is the most widely used and suggested to be the best test under such irregularity.

Let be the model (1). We test,

$$H_0 : \phi_1 = \phi_2 \quad \text{against the alternative} \quad H_1 : \phi_1 \neq \phi_2. \quad (4)$$

This means that we test a linear AR model under (H_0) against a MSAR model under (H_1). Following the works of Davies (1977, 1987), Hansen (1992, 1996) and Garcia (1998), we use the statistic:

$$LR = 2[L(\hat{\beta}, \hat{\gamma}) - L(\tilde{\beta})], \quad (5)$$

where $L(.)$ is the log-likelihood function, $\beta = (\phi_1, \phi_2, \theta, \sigma_u^2)$ and the vector of nuisance parameter is given by $\gamma = (p_{00}, p_{11})$. For the statistic LR, $\hat{\beta}$ and $\hat{\gamma}$ is the maximum likelihood estimator of β under the alternative of Markov switching model, and $\tilde{\beta}$ is the estimated value of β under the null hypothesis (H_0). Garcia (1998) use the distributional theory of Davies (1977, 1987), Hansen (1991a, 1996) and Andrews and Ploberger (1994) in order to determine the asymptotic distribution theory of usual tests statistics. His method is based on the theorem showed by Hansen (1991). This theorem shows that 'under certain regularity conditions, the asymptotic distribution of standard tests statistics are shown to be functionals of chi-square processes'[2]. This method considers the LR test as the Supremum of a *chi-square* functional,

$$LR_N = Sup_{\gamma \in \Gamma} LR_N(\gamma) \Rightarrow SupC = Sup_{\gamma \in \Gamma} C(\gamma),$$

where $\Gamma \in]0, 1[$ and $C(\gamma)^3$ is a $chi - square$ processus with covariance matrix $\bar{K}(.,.)$ given by,

$$\bar{K}(\gamma_1, \gamma_2) = R_k V(\gamma_1)^{-1} K(\gamma_1, \gamma_2) V(\gamma_2)^{-1} R_k^{'},$$

where R_k is a vector with ones in the position of the parameters constrained and zero elsewhere, k is the dimension of the parameter space under the alternative, and:

$$K(\gamma_1, \gamma_2) = \lim_{n \to \infty} E\left[S_N^c(\beta_0, \gamma_1) S_N^c(\beta_0, \gamma_2)\right],$$

$$S_N^c(\beta_0, \gamma) = \frac{\partial}{\partial \beta} Q_N(\beta, \gamma),$$

$$V(\gamma) = V(\theta_0, \gamma).$$

To derive the asymptotic distribution of LR test under H_0, Garcia (1998) starts by deriving the covariance function of the chi-square processus $C(\gamma)$. Then, the author proceeds as if the distributional theory applied and shows by Monte Carlo simulation that the LR asymptotic distributions give a good approximations of the empirical distributions. We use the critical values given by Garcia (1998). They are equal to 10.89 for the MSAR with non-autoregressive order variable and 8.68 for the MSAR (1) model for the 5% significance level. To avoid the possibility of local maxima we use a lot of starting values in order to be sure that the maximum obtained is a global one.

3 Simulation Experiment

In this section, we investigate the size and the empirical power of the test introduced in (5) under the assumptions (H_0) and (H_1). This permits to calibrate the test and to specify its sensitivity to different values of the mean and noise's variance parameters. Then, using 1000 simulated series of the SETAR processes defined in (3), we assess the capability of Garcia's test to reject this last model. In all the empirical study we use the 5% significant level.

[2]see Hansen (1991) for the proof of this theorem and the conditions to be checked.

[3]see Garcia (1998) for the method for simulating Chi-square processes, Appendix.4, 786-7.

3.1 The size and the empirical power of the Sup LR test

To explore the size and the empirical power of this test, we generate 1000 series of sample size 100, 300 and 1000. First, to assess the size of the test we simulate a linear Gaussian AR(1) model which corresponds to the hypothesis (H_0). Rejection's percentages of the $AR(1)$ processes are reported in table 1. Three combinations of the mean parameters and the noise's variances are considered, with $\theta = 0.5$. Table 1's results suggest that

Table 1: Size of the sup LR test when the DGP is an AR(1) model for the nominal size of 5%

mean parameters	σ^2	100	300	1000
$\phi_2 = -\phi_1 = 0.5$	0.36	0.028	0.029	0.025
	1	0.027	0.020	0.021
	4	0.019	0.018	0.018
$\phi_2 = -\phi_1 = 1$	0.36	0.024	0.017	0.020
	1	0.026	0.020	0.020
	4	0.023	0.021	0.022
$\phi_2 = -\phi_1 = 2$	0.36	0.021	0.018	0.021
	1	0.023	0.019	0.020
	4	0.021	0.019	0.019

the size of the test (5) is not sensitive neither to the mean nor to the noise's variance. The test provides a size around 2% which means that the test underreject (H_1), whatever the sample size.

Now, to assess the power of the test we generate a two states Markov switching model (1) with $\theta = 0.5$, $p_{00} = 0.95$, $p_{11} = 0.95$ and several means and noise's variances. The results are reported in table 2. For small samples, the power depends on the noise's variance. The empirical power is higher than 0.8 except in three cases. When we use a samples size greater than 300, the test has an empirical power close to 1. Table 2 shows also that the power of this test depends on the ratio,

$$|(\phi_2 - \phi_1)| / \sigma^2. \tag{6}$$

For instance, if we take $\phi_2 = -\phi_1 = 0.5$ with $\sigma = 1$ or $\phi_2 = -\phi_1 = 1$ with $\sigma = 2$ and $\phi_2 = -\phi_1 = 2$ with $\sigma = 0.6$ we observe that when the value of the ratio (6) is large, the empirical power of the test (5) is close to 1 and small when the value of (6) is very small. An interesting case is when $\phi_2 = -\phi_1 = 0.5$ and $\sigma = 2$, for N=100 and N=300, the power is smaller. This result can be explained by the high volatility of the data compared to the small value of the mean.

Table 3: Power of sup LR test when the DGP is SETAR (0) with d=1

(ϕ_1, ϕ_2)	σ^2	100	200	300	400	500	1000	p_{00} and p_{11}
	0.36	99.3	100	100	100	100	100	$p_{00} = 0.86$, $p_{11} = 0.86$
(-0.5, 0.5)	1	74.3	97.5	100	100	100	100	$p_{00} = 0.80$, $p_{11} = 0.80$
	4	13.8	38.5	59.1	74.6	99.6	100	$p_{00} = 0.74$, $p_{11} = 0.72$
	0.36	100	100	100	100	100	100	$p_{00} = 0.95$, $p_{11} = 0.95$
(-1, 1)	1	98.3	100	100	100	100	100	$p_{00} = 0.88$,$p_{11} = 0.88$
	4	66.8	97.7	100	100	100	100	$p_{00} = 0.80$, $p_{11} = 0.80$
	0.36	45.5	69.2	82	90.5	94.5	100	$p_{00} = 0.99$, $p_{11} = 0.99$
(-1.5, 1.5)	1	99.4	100	100	100	100	100	$p_{00} = 0.94$, $p_{11} = 0.94$
	4	97.8	100	100	100	100	100	$p_{00} = 0.84$, $p_{11} = 0.84$
	0.36	3.4	9	10.2	15.7	18.4	26.4	$p_{00} = 0.999$, $p_{11} = 0.77$
(-2, 2)	1	88.7	98.5	100	100	100	100	$p_{00} = 0.98$, $p_{11} = 0.97$
	4	34.7	64.3	84.7	93.8	96.3	100	$p_{00} = 0.88$, $p_{11} = 0.88$
	0.36	99.5	100	100	100	100	100	$p_{00} = 0.93$, $p_{11} = 0.85$
(-0.8, 0.5)	1	89.4	100	100	100	100	100	$p_{00} = 0.89$, $p_{11} = 0.84$
	4	29.4	65.5	88.6	96.5	99.3	100	$p_{00} = 0.88$, $p_{11} = 0.84$
	0.36	13.8	32.4	43.3	54	63.2	82.5	$p_{00} = 0.68$, $p_{11} = 0.95$
(-1.5, 0)	1	52.6	83.5	96.8	99.3	99.7	100	$p_{00} = 0.69$, $p_{11} = 0.94$
	4	28.4	68	89.6	96.5	99.6	100	$p_{00} = 0.70$, $p_{11} = 0.82$

Table 2: Empirical power of the sup LR test when the DGP is a MSAR(1) process

(ϕ_1, ϕ_2)	σ^2	100	300	1000
	0.36	97	100	100
(-0.5, 0.5)	1	79.3	100	100
	4	15.5	60.1	99.3
	0.36	98.4	100	100
(-1, 1)	1	98.3	100	100
	4	78	100	100
	0.36	87.6	98.2	100
(-2, 2)	1	95.2	99.8	100
	4	99	100	100

This empirical study shows that the test (5) is able to recognize a Markov switching model even using small samples sizes. Nevertheless, we observe through the articles of Garcia (1998), Gong and Mariano (1997) and Coe (2002), that using real data, the results are no so evident.

In a simulation study, not reported here, our results concerning the size and the empirical power of the sup LR test are close to those of Carrasco (2002) when under (H_0) we use a strong white noise.

3.2 Capability of the Sup LR test to reject SETAR model

Now, to study the ability of the sup LR test to identify the alternative as a non Markov switching model when the DGP is a SETAR model, we simulate 1000 time series from model (3) with different samples sizes N=100, 200, 300, 400, 500 and 1000. We set the threshold parameter and the initial value y_0 equal to zero.

Six combinations for the parameters (ϕ_1, ϕ_2) are considered. The values of the noise's variance are setted equal to $\sigma = 0.6, 1$ and 2. All combinations are chosen in order to be sure that we have enough points in each regime except for one combination parameters. To minimize the influence of starting values we have discarded the first 200 observations. We do not investigate the influence of the autoregressive parameter θ, which is setted equal to 0.5. Garcia (1998) shows that the distribution of the test is not sensitive to the value of the autoregressive parameters.

In the following paragraph, we analyze the results reported in tables 3-6. These tables report the empirical power of the sup LR test, percentage of rejections of the null assumption, with respect to different values of the delay parameter d.

3.2.1 d=1 in (6)

1- *Results for the SETAR(0)*[4]

The results of simulation, when the DGP follows the SETAR process (3) with $r = 0$, $d = 1$ and $\theta = 0$, are reported in table 3.

For a sample size larger than 200, the sup LR test correctly rejects the null hypothesis of linear model. Under the alternative, the sup LR test is unable to recognize this SETAR process. It accepts nearly always the SETAR process although we use a statistic built to recognize the MSAR process (1). Garcia (1998, appendix.3, p.785) shows that the LR test has the same asymptotic distribution under (H_0) whatever the process that we consider under the alternative: MSAR(0) or SETAR(0). Our empirical results confirm their theoretical finding.

Table 4: Power of Sup LR test when the DGP is a SETAR (1) process with d=1

(ϕ_1, ϕ_2)	σ^2	100	200	300	400	500	1000	p_{00} and p_{11}
(-0.5, 0.5)	0.36	22.4	36.6	50	55.6	60.4	64.2	$p_{00} = 0.79$, $p_{11} = 0.70$
	1	5.5	10.1	11.9	12.2	15.5	21.4	$p_{00} = 0.80$, $p_{11} = 0.65$
	4	2.3	3.7	4	4.9	5.7	6.7	$p_{00} = 0.82$, $p_{11} = 0.54$
(-1, 1)	0.36	96.1	100	100	100	100	100	$p_{00} = 0.85$, $p_{11} = 0.85$
	1	33.4	66.6	84.7	90.6	94.5	100	$p_{00} = 0.73$, $p_{11} = 0.73$
	4	6.2	9.3	14.7	16.3	21	25.8	$p_{00} = 0.79$, $p_{11} = 0.50$
(-1.5, 1.5)	0.36	99.9	100	100	100	100	100	$p_{00} = 0.92$, $p_{11} = 0.93$
	1	87	98.5	100	100	100	100	$p_{00} = 0.82$, $p_{11} = 0.82$
	4	16.6	33.3	45.3	57.6	70.7	93.2	$p_{00} = 0.67$, $p_{11} = 0.66$
(-2, 2)	0.36	100	100	100	100	100	100	$p_{00} = 0.97$, $p_{11} = 0.97$
	1	93.8	99.5	100	100	100	100	$p_{00} = 0.88$, $p_{11} = 0.88$
	4	34.7	64.3	84.7	93.8	96.3	100	$p_{00} = 0.73$, $p_{11} = 0.73$
(-0.8, 0.5)	0.36	35	52	57.2	58.7	59.6	65.9	$p_{00} = 0.78$, $p_{11} = 0.78$
	1	11.9	15.8	18.3	22.4	26.9	30.5	$p_{00} = 0.80$, $p_{11} = 0.70$
	4	3.6	4	4.2	5.6	8.6	9.6	$p_{00} = 0.84$, $p_{11} = 0.55$
(-1.5, 0)	0.36	54	79.7	88.8	94.6	96.1	100	$p_{00} = 0.52$, $p_{11} = 0.93$
	1	17.2	30.8	43.1	57	66.9	87.9	$p_{00} = 0.52$, $p_{11} = 0.85$
	4	6.6	7.6	8.9	11.6	15.6	20.5	$p_{00} = 0.60$, $p_{11} = 0.64$

Now, by analyzing the probabilities to remain in the same regime, we observe that $\hat{p}_{00}$ and $\hat{p}_{11}$ are equals when the means (ϕ_1 and ϕ_2) are symmetrical with respect to the threshold r. These value $\hat{p}_{00}$ and $\hat{p}_{11}$ increase with (ϕ_1, ϕ_2) and decrease with σ. Their behaviors depend on the expressions $p_{00} = \Phi(r - \phi_1/\sigma)$ and $p_{11} = \Phi(r - \phi_2/\sigma)$, where $\Phi(.)$ is the c.d.f (cumulative distributive function) of the standard Gaussian distribution, see Appendix for more details. When $\phi_2 = -\phi_1 = 2$ and $\sigma = 0.6$, we get a very small empirical power for the sup LR test. In this later case, among the 1000 generated series only one hundred series present changes from one regime to another one[5]. This is due to the high value of the mean and the small value of the noise's variance. For these series, $\hat{p}_{00} = 0.999$ which indicates that the probability to change from one regime to another one is very small.

2- *Results for SETAR(1)*

Table 4 gives the results of the empirical power of the sup LR test for different sample sizes when the data are generated under model (3) using $d = 1$, $r = 0$ and $\theta = 0.5$. For large sample sizes, the test has no ability to differentiate between SETAR(1) and MSAR(1) models. We analyze below in more details the behavior of this test under the alternative.

- Assume that σ is fixed:

When the value of ϕ_2 and ϕ_1 are large, $\phi_2 = -\phi_1 = 1.5$ or $\phi_2 = -\phi_1 = 2$, the sup LR test rejects the null with high empirical power. This means that if the data are generated from SETAR(1) this test builded to recognize a MSAR(1) model accepts, with a very high probability, the SETAR(1) as a MSAR(1) model.

Again, we observe that the test cannot discriminate between the two models, in particular when the data are less noisy ($\sigma = 0.6$ or $\sigma = 1$). In another hand, when the difference between the two means decreases, for instance when ϕ_1 and ϕ_2 are small, the test can distinguish between the two models (1) and (3). The empirical power to reject the SETAR(1), for $\phi_2 = -\phi_1 = 1.5$ and $\phi_2 = -\phi_1 = 2$ with $\sigma = 2$ is equal to 83.4% and 65.3% respectively, using N=100 observations.

- Assume that ϕ_1 and ϕ_2 are fixed:

By varying the noise variance σ, we observe that the ability of the sup LR test to discriminate between the two models (1) and (3) increases specially when the data are very noisy. For example, when $\phi_1 = -0.8, \phi_2 = 0.5$ and N=100, the rejection of the alternative of SETAR(1) model increases from 65% ($\sigma = 0.6$) to 88.1% ($\sigma = 1$) and 96.4% ($\sigma = 2$). This means that when the noise variance σ increases the data set becomes very noisy and $p_{00} + p_{11}$ decreases. Thus, the switches between the two states increase and the test rejects the alternative in most of the cases. For example, for N=100 with $\phi_1 = -0.8, \phi_2 = 0.5$, we observe that $p_{00} + p_{11}$ decreases from 1.56 when ($\sigma = 0.6$) to 1.5 when ($\sigma = 1$) and then to 1.39 when ($\sigma = 2$).

3.2.2 d=2 in (6)

Table 5 reports the empirical powers of the sup LR test when the Data Generated Process is a SETAR(0) model with $d = 2$, $r = 0$ and $\theta = 0$ in (6).

The results show that when the sample size is large (N=1000) the test has a reasonable empirical power against the null hypothesis of a linear model, except for a few combinations. When the value of the mean

Table 5:Power of sup LR test when the DGP is a SETAR (0) model with $d = 2$ and $\theta = 0$.

(ϕ_1, ϕ_2)	σ^2	100	200	300	400	500	1000	p_{00} and p_{11}
	0.36	36.3	72.1	87.9	96.7	99.2	100	$p_{00} = 0.94$, $p_{11} = 0.94$
(-0.5, 0.5)	1	14.3	34.3	50.8	67.7	80.4	99	$p_{00} = 0.91$, $p_{11} = 0.91$
	4	4.1	6.7	11.1	17.4	24.8	53.1	$p_{00} = 0.88$, $p_{11} = 0.64$
	0.36	59.7	90.9	97.7	98.4	99.1	100	$p_{00} = 0.94$, $p_{11} = 0.94$
(-1, 1)	1	45.4	79.7	93.4	98.7	99.7	100	$p_{00} = 0.97$, $p_{11} = 0.96$
	4	13.3	34.3	55.2	69.6	79.1	99	$p_{00} = 0.90$, $p_{11} = 0.90$
	0.36	28.7	51	64.6	74.3	92.5	100	$p_{00} = 0.73$, $p_{11} = 0.57$
(-1.5, 1.5)	1	57.7	86	94	98.8	99.5	100	$p_{00} = 0.97$, $p_{11} = 0.96$
	4	34	64.3	83.2	91.6	97.1	99.6	$p_{00} = 0.93$, $p_{11} = 0.93$
	0.36	3.7	8.2	12.8	23.3	29	38.2	$p_{00} = 0.69$, $p_{11} = 0.85$
(-2, 2)	1	57.7	87	97.5	100	100	100	$p_{00} = 0.64$, $p_{11} = 0.53$
	4	45.6	79.9	93	98.1	100	100	$p_{00} = 0.95$, $p_{11} = 0.91$
	0.36	44.4	79.3	95.5	98.1	99.9	100	$p_{00} = 0.93$, $p_{11} = 0.95$
(-0.8, 0.5)	1	20.8	52.7	73.3	87.1	96.2	100	$p_{00} = 0.90$, $p_{11} = 0.93$
	4	5.8	14.6	22.2	30.6	44.2	78	$p_{00} = 0.89$, $p_{11} = 0.88$
	0.36	5.5	12.3	23.4	29.7	30.8	54.6	$p_{00} = 0.73$, $p_{11} = 0.98$
(-1.5, 0)	1	10.6	32.5	47.4	63.8	75.3	99	$p_{00} = 0.84$, $p_{11} = 0.96$
	4	12.6	14	27.3	36.6	43.9	82.1	$p_{00} = 0.86$, $p_{11} = 0.90$

increases, the power of the test increases too. In these latter cases, the ability of the sup LR test to discriminate between the MSAR(0) and the SETAR(0) models is larger in small samples, specially when the data set is very noisy. Two series seem to have a specific behaviors, when $\phi_2 = -\phi_1 = 0.5$ with $\sigma = 2$ and $\phi_2 = -\phi_1 = 2$ with $\sigma = 0.6$. For these two processes the empirical power is very low and the ratio (6) attains respectively its minimum and its maximum values. From the empirical results, it appears that when the ratio (6) is very high or very small, the power of the test is high and small in the other cases.

Table 6 provides the empirical power of the sup LR test when the data are generated with a SETAR(1) model using $d = 2$ and $\theta = 0.5$. In this case, the test has a good empirical power against the null hypothesis of a linear model like in previous cases, even if the sample size is small. This means that the test is unable to differentiate the SETAR model from the MSAR one. From table 6, it appears that for a small sample size, N=100 observations, the power is not sensitive neither to the mean nor to the noise's variance. The empirical power is around 60%, except for the two series that have the lower and higher values of the ratio $|(\phi_2 - \phi_1)| / \sigma^2$. In these two cases, the power is 16% and 47.7% respectively.

After analyzing the behavior of the sup LR test, it is interesting to assess its behavior when we consider a real data. From the economic and financial time series, we are interesting by the series of the US GNP growth and the US/UK exchange rate. For each one, we applies the results obtained from the previous empirical simulations to investigate which model (SETAR or MSAR) describes better their evolutions.

4 Applications

In economic and financial domains, two particular series have been widely examined by researchers to justify the existence of shifts and change between regimes: the US GNP growth rate and the US/UK exchange rate. Many factors induce nonlinearities inside those time series. For the US GNP growth rate, many researchers argue that this variable is characterized by an asymmetric behavior and suggest that shocks are more persistent during great depression. In the other hand, the presence of nonlinearities and the adoption of switching model, for the US/UK exchange rates, is motivated mainly by the heterogeneity of the participants in the foreign exchanges markets, transaction costs, and the differences between domestic and foreign monetary and fiscal policies. Also, the rigidities of certain markets and the presence of chartists and fundamentalists in the foreign exchanges markets induce differences in opinions and in expectations.

For the first time series, Hamilton (1990) proposes a MSAR(4) process to model the growth rate of the US GNP. However, Hansen (1992) doubts in Hamilton specification and proposes a constrained model in which he allows to the intercept, the slope parameters, and the error variance to shift between the 'states'. Garcia (1998) confirms that there is no evidence for a MSAR(4) model for the US GNP growth rate. In another hand, this same series have been modeled by the SETAR model. For instance, Potter (1995) proposes a SETAR(5) model without the third and fourth lags. Hansen (1996) doubts that the SETAR model proposed by Potter (1995) captures changes in the US GNP growth rate series.

Table 6: Power of sup LR test when the DGP is SETAR (1) with $d = 2$ and $\theta = 0.5$.

(ϕ_1, ϕ_2)	σ^2	100	200	300	400	500	1000	p_{00} and p_{11}
(-0.5, 0.5)	0.36	73.4	96.7	99.5	100	100	100	$p_{00} = 0.60$, $p_{11} = 0.60$
	1	50.1	81	96.1	98.6	99.9	100	$p_{00} = 0.67$, $p_{11} = 0.67$
	4	16	36.4	51.9	68.5	79.9	98.2	$p_{00} = 0.66$, $p_{11} = 0.64$
(-1, 1)	0.36	66.8	92.5	97.8	98.1	99.4	100	$p_{00} = 0.65$, $p_{11} = 0.47$
	1	74.4	95.5	99.5	99.9	100	100	$p_{00} = 0.55$, $p_{11} = 0.55$
	4	50	83.9	94.6	99.8	99.9	100	$p_{00} = 0.66$, $p_{11} = 0.6$
(-1.5, 1.5)	0.36	66.3	90.7	95.2	98	99.2	100	$p_{00} = 0.39$, $p_{11} = 0.37$
	1	66.6	94	99.1	99.6	99.8	100	$p_{00} = 0.49$, $p_{11} = 0.46$
	4	66.9	96.1	99.4	99.8	99.9	100	$p_{00} = 0.61$, $p_{11} = 0.61$
(-2, 2)	0.36	47.7	77.6	93.3	97.9	100	100	$p_{00} = 0.39$, $p_{11} = 0.37$
	1	65.3	92.1	96.5	99.5	100	100	$p_{00} = 0.43$, $p_{11} = 0.40$
	4	73	96.8	99.7	100	100	100	$p_{00} = 0.57$, $p_{11} = 0.57$
(-0.8, 0.5)	0.36	74.7	97	98.2	99.9	100	100	$p_{00} = 0.62$, $p_{11} = 0.55$
	1	63.4	93	99.3	99.8	100	100	$p_{00} = 0.68$, $p_{11} = 0.66$
	4	29.2	53.8	74.3	86.4	95	100	$p_{00} = 0.66$, $p_{11} = 0.65$
(-1.5, 0)	0.36	76.7	99.9	100	100	100	100	$p_{00} = 0.94$, $p_{11} = 0.88$
	1	71.5	98.1	99.7	100	100	100	$p_{00} = 0.76$, $p_{11} = 0.70$
	4	33.1	65.6	81.2	93	97.2	100	$p_{00} = 0.68$, $p_{11} = 0.66$

For the US/UK exchange rate, Engle and Hamilton (1990) propose a MSAR model without autoregressive order. Under this specification, the US/UK exchange rate follows a random walk specification under the null and the alternative hypothesis, see also Cheung and Erlandsson (2005).Now, we propose to use the previous test (5) to test the presence of changes in regimes inside these time series. If the test is positive[6], we investigate the two alternatives of a MSAR or a SETAR model using the previous results given in simulations.

4.1 The data

We use a quarterly data set for the U.S real GNP, over the period January 1947 to April 2005. This provides us a sample size of 232 points. For the US/UK exchange rates, we use two frequency data sets, monthly and quarterly data. In the first case we have 176 points, from January 1986 to September 2000, and 58 points in the latter case from January 1986 to October 2000. The GNP data set is provided from the website of Federal Reserve Bank at St Louis in USA and the exchange rate data set comes from Data stream Base. In order to make the data stationary we use the following transformation $100 * [\log(X_t) - \log(X_{t-1})]$, where X_t represents the observed data.

4.2 Results

First, we start by estimating a SETAR process for the GNP growth rate and the US/UK exchange rate data. The results are provided in table 7, using d=1 in model (6). Here we are interested by the value of the ratio given in the last line of the table 7. In all cases, this value is very small. Then, following the results given in paragraph 3, the sup LR test rejects the SETAR model and accepts the null hypothesis of a linear model.

To confirm this result, we simulate 1000 series, using the previous estimated parameters for the SETAR processes, with N=100, 200, 300, 400 and 500 samples sizes. Then, we compute the expected empirical power of the sup LR test for each experiment.

Table 7: Estimates of SETAR model of the U.S GNP growth rate (Quarterly data) and the US/UK exchange rate (Monthly and Quarterly data)

Parameter	$U.S\ \ GNP$	$US/UK(M)$	$US/UK(Q)$
	SETAR(1)	SETAR(0)	SETAR(0)
ϕ_1	0.614	-0.551	0.044
	(0.1046)	(0.362)	(0.801)
ϕ_2	1.172	0.446	-1.384
	(0.119)	(0.300)	(2.163)
θ	0.598	-	-
	(0.049)		
σ^2	0.931	9.447	33.607
	(0.92)	(0.082)	(0.973)
r	0.740	0.049	3.953
d	1	1	1
$\lvert(\phi_2 - \phi_1)\rvert / \sigma^2$	0.599	0.105	0.042

The results are given in table 8. In all cases, we reject the alternative with a high power. This means that the SETAR(1) specification is inappropriate. Thus, we adjust a Markov switching process for these data sets.

Table 8: Power of the Sup LR test when the DGP is the estimates parameter of real data ($U.S\ \ GNP$, $US/UK(M)$ and $US/UK(Q)$)

	100	200	300	400	500
$U.S\ \ GNP$	3.4	3.6	4.5	5.4	6.4
$US/UK(M)$	6.3	14.4	22.3	35.3	44.7
$US/UK(Q))$	1.8	3.4	4.3	7.4	8.6

In table 9, columns 2 and 3, we provide the estimation of the real growth rate GNP data using an AR(1) linear model and a MSAR(1) model. We provide also

Table 9: Estimates of MSAR model for the U.S GNP growth rate (Quarterly data) and the US/UK exchange rate (Monthly and Quarterly data)

Parameter	$U.S\ \ GNP$		$US/UK(M)$		$US/UK(Q)$	
	AR(1)	MSAR(1)	AR(0)	MSAR(0)	AR(0)	MSAR(0)
ϕ_1	0.563	0.952	-0.018	0.320	-0.0743	0.766
	(0.095)	(0.103)	(0.231)	(0.228)	(0.739)	(0.863)
ϕ_2	-	-1.213	-	-8.295	-	-13.855
		(0.467)		(1.536)		(5.583)
θ	0.332	0.391	-	-	-	-
	(0.073)	(0.071)				
p_{00}	-	0.9618	-	0.967	-	0.950
		(0.023)		(0.020)		(0.048)
p_{11}	-	0.262	-	0.205	-	0.227
		(0.223)		(0.181)		(0.425)
σ_u^2	0.938	0.451	9.42	6.687	31.764	20.813
	(0.092)	(0.083)	(0.054)	(0.214)	(1.231)	(0.973)
LL	-101.205	-96.736	-284.002	-274.483	-129.267	-125.823

the value of the log-likelihood function under each model. In the case of the US GNP series, the value of the likelihood ratio statistic is 8.938. This value is greater than Garcia's 95% asymptotic critical value of 8.68. Thus, we accept the alternative of MSAR(1) model against the null of linear AR(1) model. The value of the ratio in (7) is between 1 and 10 for this series. This means that, if we use the estimate value of the U.S real GNP to generate artificial data under an MSAR(1) model, the empirical power of the sup LR test will be close to 1. Simulation experiments results confirm this intuition. Their power is equal to 80.2%, 99.8% and 100% for a sample size equal to N=100, N=200 and N=300 respectively, when the DGP use the estimated parameters for the U.S real GNP provided in table 9. These results induce to retain a MSAR(1) model for the U.S real GNP data.

In the table 9, colums 4-7, we provide an estimate of the monthly and quarterly US/UK exchange rate data. Under the null, the series are modeled by a random walk and by a Markov switching model under the alternative. The likelihood ratio statistic for the monthly data is equal to 19.38. This value is largely greater than the asymptotic critical value, 10.89. Thus, for this US/UK series we accept a Markov switching model against a random walk process. Using quarterly data (58 observations) the likelihood ratio statistic is equal to 6.9. This value does not excess the Garcia's 95% asymptotic critical value of 10.89. Thus, for the quarterly data, we accept the random walk specification (linear model) against the Markov switching model. We are not surprised by this result because the sample size and the frequency of the data plays an important role for the choice of the best model. Now, we propose a simulation using the estimated parameters of the MSAR(0) for US/UK exchange rate given in table 7. In order to study the influence of the sample size on the empirical power of the Sup LR test, we simulate 1000 series for different samples sizes, N=100, 200, 300, 400, 500.

The simulations results show that the power of the test is equal to 85% when N=100, and close to 1 when the sample size is $N \geq 200$. We remark also that there is small difference between the empirical power for the monthly and the quarterly data set. This comes from the values obtained for the ratio (6).

In fine, we retain a MSAR(1) model for the US GNP growth rate, a MSAR(0) model for the US/UK monthly data and a random walk when we use a quarterly data set for the US/UK.

5 Conclusion

This paper study the power of sup LR test (5), constructed for a MSAR model, when the underlying simulated process is a SETAR model. We explore the sensitivity of the empirical power with respect to the mean, the variance and the delay parameter d of the model (6). The results show that it is very difficult to discriminate between the SETAR model and the MSAR model, in particular using large sample sizes. For small sample sizes the Sup LR test allows to doubt about the nature of the changes in regimes. It seems that the higher and the lower value of the ratio $|(\phi_2 - \phi_1)| / \sigma^2$ reduce the power of the sup LR test for small samples, but in large samples this test still has good power.

Tow specific cases are of interest, when the ratio is very lower and very greater. For these two extreme cases the test has a correct power to discriminate between different types of switching models. We apply this approach to the US GNP growth rate and the US/UK exchange rate and we show that a Markov switching specification is more appropriate for the US GNP growth rate and for the US/UK exchange rate (Monthly data). For the US/UK Quarterly data we accept a random walk specification because we have only a few observations, 58 points.

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Appendix

Consider the simple first-autoregressive SETAR model:

$$y_t = \phi_1 + (\phi_2 - \phi_1)s_t + z_t \tag{7}$$

with $z_t = \theta z_{t-1} + u_t$, and

$$s_t = I\{y_{t-1} \le r\} = \begin{cases} 0 & \text{if } y_{t-1} \le r \\ 1 & \text{if } y_{t-1} > r \end{cases}$$

This model can be rewriting as:

$$y_t = \begin{cases} \mu_0 + \phi_1(y_{t-1} - \mu_0) + \epsilon_t & \text{if } y_{t-1} \le r \text{ and } y_{t-2} \le r \\ \mu_0 + \phi_1(y_{t-1} - \mu_1) + \epsilon_t & \text{if } y_{t-1} \le r \text{ and } y_{t-2} \ge r \\ \mu_1 + \phi_1(y_{t-1} - \mu_0) + \epsilon_t & \text{if } y_{t-1} \ge r \text{ and } y_{t-2} \le r \\ \mu_1 + \phi_1(y_{t-1} - \mu_1) + \epsilon_t & \text{if } y_{t-1} \ge r \text{ and } y_{t-2} \ge r, \end{cases} \tag{8}$$

Let be

$Pr(s_t = 0|s_{t-1}, Y) = Pr(y_{t-1} \le r|s_{t-1}, Y)$

$= Pr(\phi_1 + (\phi_2 - \phi_1)I\{s_{t-1} = 0\} + \theta(y_{t-2} - \phi_1 - (\phi_2 - \phi_1)I\{s_{t-2} = 0\}) + \epsilon_{t-1} \le r)$

$= Pr(\epsilon_{t-1} \le r - \phi_1 - (\phi_2 - \phi_1)I\{s_{t-1} = 0\} - \theta(y_{t-2} - \phi_1 - (\phi_2 - \phi_1)I\{s_{t-2} = 0\})$

$= \Phi(\frac{r - \phi_1 - (\phi_2 - \phi_1)I\{s_{t-1} = 0\} - \theta(y_{t-2} - \phi_1 - (\phi_2 - \phi_1)I\{s_{t-2} = 0\})}{\sigma})$

$= Pr(s_t = 0|s_{t-1}, s_{t-2}, y_{t-2}) \ne Pr(s_t = 0|s_{t-1})$

Therefore for the SETAR(1) the probability $Pr(s_t = 0|s_{t-1}, s_{t-2}, Y)$ depends on y_{t-2} then the process $I\{y_{t-d} \le r\}$ is not an a exogenous Markov chain like an Hamilton (1989).

- When $\theta = 0$, we see that the model (7) becomes a particular case of Markov Switching model because $Pr(s_t = 0|s_{t-1}, Y) = Pr(s_t = 0|s_{t-1})$. For model (7) we get:

$$\mathbf{P} = \begin{bmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{bmatrix} = \begin{bmatrix} \Phi(\frac{\phi_2}{\sigma}) & \Phi(\frac{-\phi_2}{\sigma}) \\ \Phi(\frac{\phi_1}{\sigma}) & \Phi(\frac{-\phi_1}{\sigma}) \end{bmatrix}.$$

- When $\theta = 0$ and $\phi_1 = -\phi_2$ in (7), we obtain:

$$\mathbf{P} = \begin{bmatrix} \Phi(\frac{-\phi_1}{\sigma}) & \Phi(\frac{\phi_1}{\sigma}) \\ \Phi(\frac{\phi_1}{\sigma}) & \Phi(\frac{-\phi_1}{\sigma}) \end{bmatrix}.$$

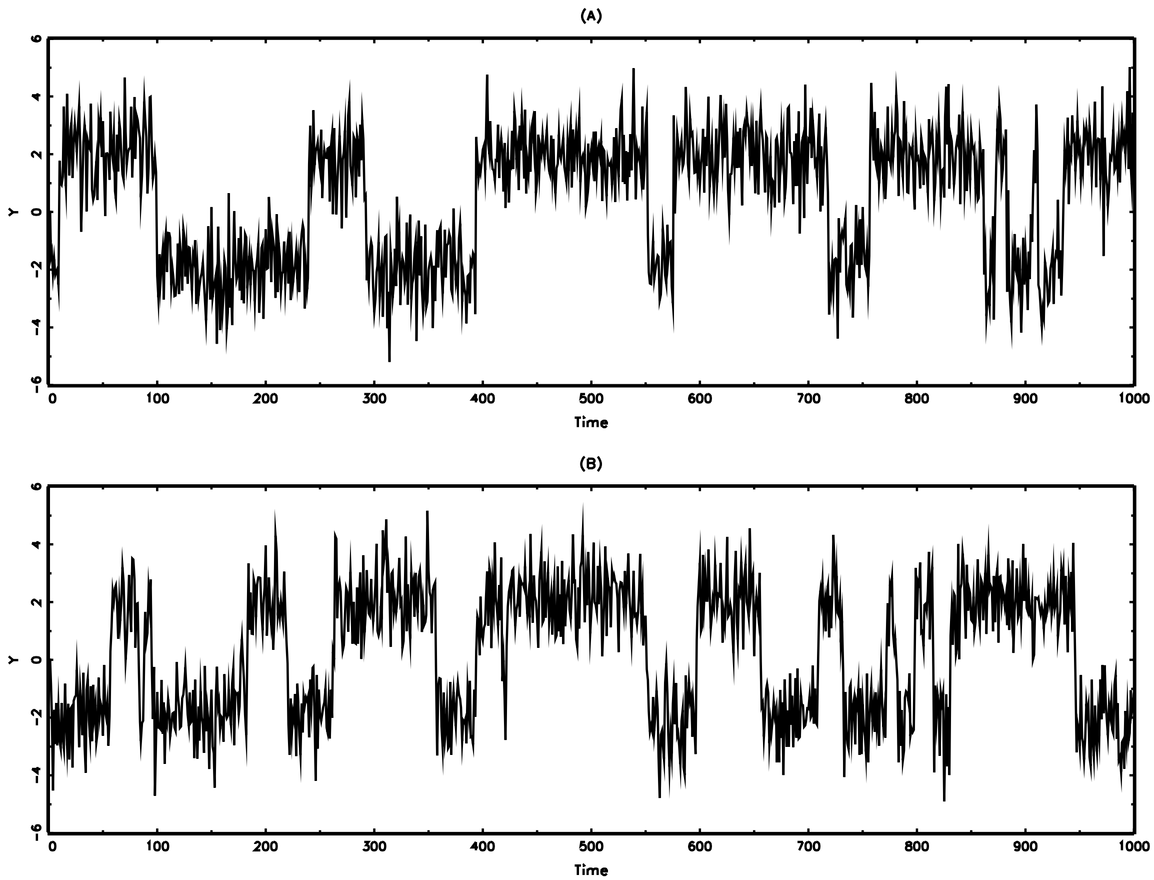

Figure 1: (A) Trajectory of the MSAR model when $p_{00} = 0.98$, $p_{11} = 0.98$ and $\mu_0 = -\mu_1 = 2$, and (B) trajectory of the SETAR model when $r = 0$, $d = 1$ and $\mu_0 = -\mu_1 = 2$.

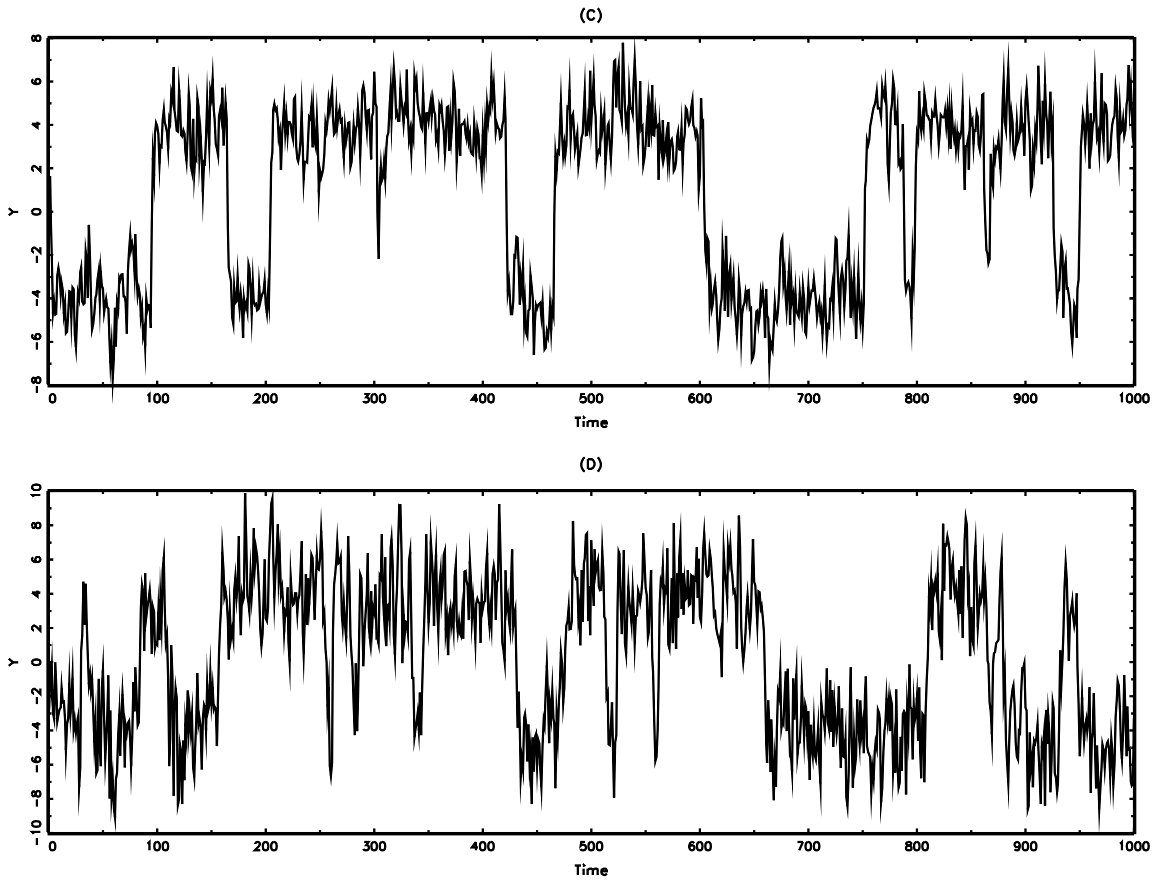

Figure 2: (C) Trajectory of the MSAR model when $p_{00} = 0.98$, $p_{11} = 0.98$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$, and (D) trajectory of the SETAR model when $r = 0$, $d = 1$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$.

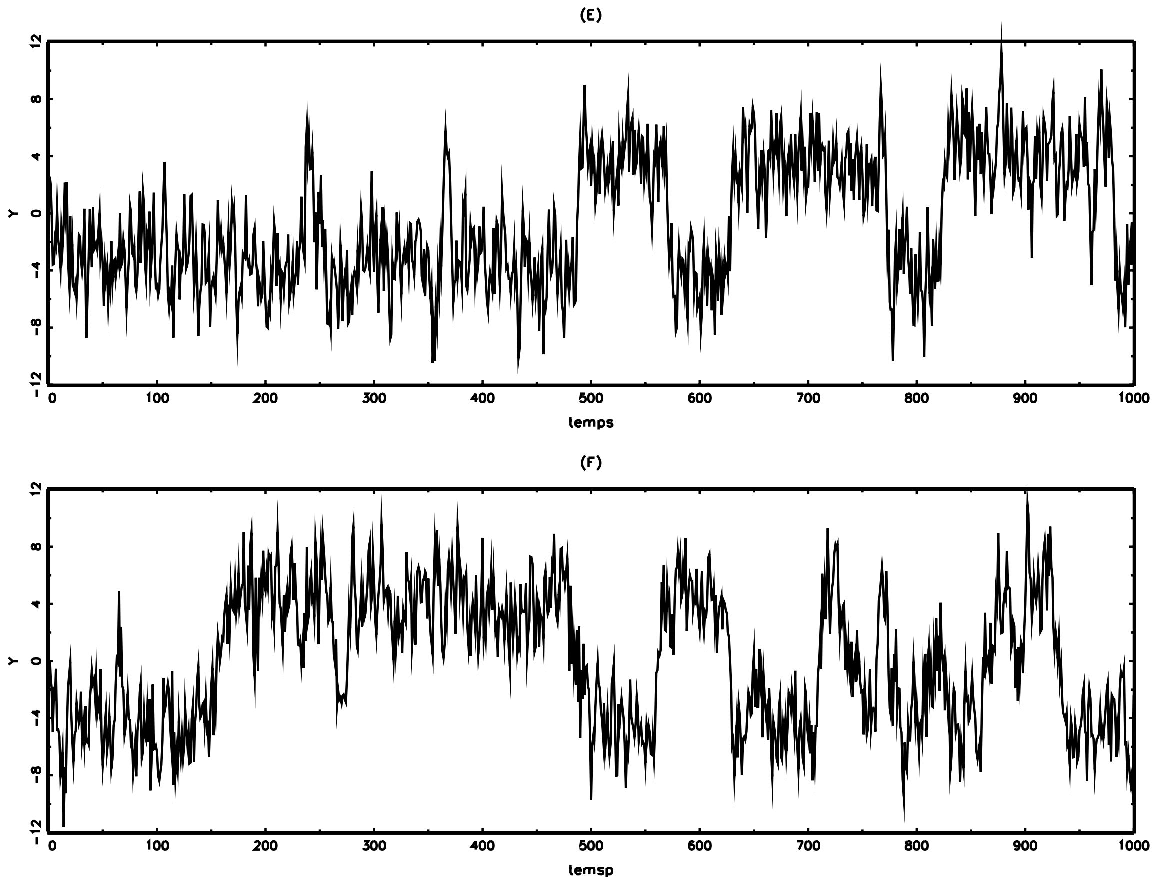

Figure 3: (E) Trajectory of the MSAR model when $p_{00} = 0.95$, $p_{11} = 0.95$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$, and (F) trajectory of the SETAR model when $r = 0$, $d = 2$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$.

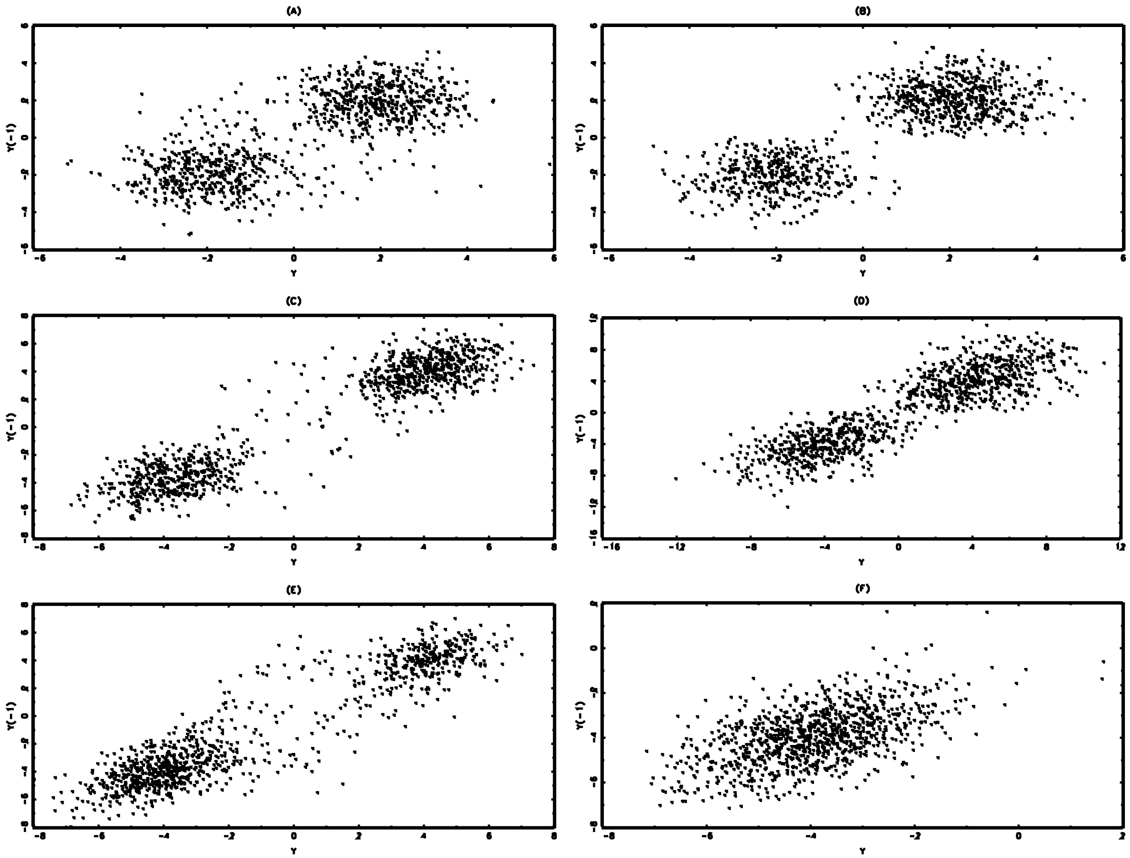

Figure 4: Scatterplots of, (A) MSAR model when $p_{00} = 0.98$, $p_{11} = 0.98$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0$, (B) SETAR model when $r = 0$, $d = 1$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0$, (C) MSAR model when $p_{00} = 0.98$, $p_{11} = 0.98$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$, (D) SETAR model when $r = 0$, $d = 1$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$, (E) MSAR model when $p_{00} = 0.95$, $p_{11} = 0.95$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$ and finally (F) when SETAR model $r = 0$, $d = 2$, $\mu_0 = -\mu_1 = 2$ and $\phi_1 = 0.5$.