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Document de Travail
n°2006-44

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September 2006

DT-GREQAM

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Abstract: *We present a survey of the main conditions for the occurrence of indeterminacy in discrete-time infinite-horizon models with technological external effects. One-sector models are characterized by global external effects coupled with increasing social returns. We will show that indeterminacy of equilibria is fundamentally based on the consideration of endogenous labor demand and externalities coming both from capital and labor. Most of the two-sector models are characterized by constant returns to scale at the social level. We will show that depending on whether external effects are sector-specific or intersectoral, some simple but different conditions on capital intensity differences across sectors generate indeterminate equilibria.*

Keywords: *Two-sector models, externalities, depreciation of capital, CES technologies, indeterminacy.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

1 Introduction

Recently there has been an increasing interest in sunspot equilibria as a possible explanation of business cycle fluctuations. In a macroeconomic context, sunspot fluctuations is a topic that dates back to the early work of Shell [25], Azariadis [1] and Cass and Shell [10]. This renewed interest is explained by the fact that during the last decade a variety of economic models that incorporate some degree of market imperfections have been shown to exhibit multiple equilibria and local indeterminacy.[1] As shown by Woodford [29], the existence of sunspot equilibria is closely related to the indeterminacy of perfect foresight equilibrium.

Indeterminacy, or multiple equilibria, is known to occur in dynamic models with small market distortions and generates some coordination problems. Basically, the occurrence of indeterminacy needs a mechanism such that, starting from an equilibrium, if all agents were simultaneously to increase their investment in, say, the capital good, the rate of return on this good would tend to increase, and in turn set off relative price changes that would drive the economy back towards the steady state. In one-sector models, such a mechanism may be associated with external effects in production and increasing returns. However, in a two sector model, the rate of return and marginal product of capital depend not only on factor inputs, but also on the composition of output and thus on the relative factor intensities. An increase of the production and the stock of capital following an increase in its price may well increase its rate of return. Therefore constant aggregate returns at the social level are compatible with indeterminacy if there are minor external effects in some of the sectors.

In this paper we will present the main conditions for the occurrence of indeterminacy in one and two-sector optimal growth models extended to include market imperfections based on technological external effects. We will focus almost exclusively on discrete-time models. We will distinguish between different formulations for externalities which will be in general associated with different assumptions concerning the returns to scale at the

[1] See Benhabib and Farmer [5] for an extensive bibliography.

social level. Following Romer [24], one-sector models are characterized by global external effects coupled with increasing social returns. We will show that indeterminacy of equilibria is fundamentally based on the consideration of endogenous labor demand and externalities coming both from capital and labor. In two-sector models, Benhabib and Farmer [4] have introduced sector-specific external effects. While their initial formulation assumed increasing social returns, most of the papers that followed the contribution of Benhabib and Nishimura [7] are based on constant returns to scale at the social level. We will show that some simple conditions on capital intensity differences across sectors generate some amplification mechanisms that produce the existence of indeterminate equilibria.

The paper is organized as follows. Section 2 presents one-sector models. Two-sector models with Cobb-Douglas technologies, complete depreciation of capital and sector-specific externalities are analyzed in Section 3. Section 4 is devoted to the presentation of similar two-sector models but with CES production functions. The cases with symmetric and asymmetric elasticities of capital-labor substitution are consecutively considered. In Section 5 we discuss extensions of the two-sector Cobb-Douglas formulation. Firstly, we present how the conditions for local indeterminacy are modified when partial depreciation of capital is assumed. Secondly, we introduce a formulation for intersectoral externalities that is compatible with both sector-specific and global externalities specifications. We will then show how additional intersectoral mechanisms provide new room for local indeterminacy. Finally, in Section 6, other formulations of infinite-horizon models are explored. We first deal with the consideration of models with capacity utilization in which the speed of capital depreciation is endogenously determined. Then we present two-sector models derived from general technologies.

2 One-sector models

One-sector discrete-time models with Romer-type [24] global externality and increasing returns at the social level have been considered initally by Kehoe [14] and Boldrin and Rustichini [9]. The aggregate production function is augmented to include a new factor which represents the effect of knowledge on production and productivity:

$$Y_t = F(K_t, L_t, A_t)$$

with A_t the externality at time t which will be equal at the equilibrium to K_t/L_t. For any given A, $F(.,.,A)$ is increasing, concave and homogeneous of degree 1, and labor is inelastic. Under constant population, the intensive formulation for the capital accumulation equation is:

$$k_{t+1} = f(k_t, A_t) - c_t$$

with $f(k, A) = F(k, 1, A) + (1 - \mu)k_t$ and $\mu \in [0, 1]$ the rate of depreciation of capital.

Assumption 1 . *$f(k, A)$ is C^2 and such that for any $k, A > 0$, $f_1(k, A) > 0$, $f_{11}(k, A) < 0$ while for any $A > 0$, $f(0, A) = 0$.*

From a standard utility function $u(c)$ which satisfies:

Assumption 2 . *$u(c)$ is C^2 and such that for any $c > 0$, $u'(c) > 0$, $u''(c) < 0$, $u(0) = 0$, $u'(0) = +\infty$ and $u'(+\infty) = 0$.*

we define the parameterized maximisation program of a representative consumer as

$$\begin{array}{cl} \max\limits_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t u(c_t) \\ s.t. & k_{t+1} = f(k_t, A_t) - c_t \\ & k(0), \{A_t\}_{t=0}^{+\infty} \text{ given} \end{array}$$

with $\delta \in (0, 1]$ the discount factor. Along an equilibrium path, $A_t = k_t$ and the Euler equation easily writes as

$$\delta u'(c_{t+1}) f_1(k_{t+1}, k_{t+1}) - u'(c_t) = 0$$

A steady state k^* is obtained considering $k_{t+1} = k_t$ and $c_{t+1} = c_t$ in the Euler equation, i.e. k^* is a solution of

$$f_1(k, k) = 1/\delta$$

It follows that $c^* = f(k^*, k^*)$. Contrary to the optimal growth framework, existence and uniqueness are no longer ensured under Assumption 1. We will however assume that there exists one locally unique steady state k^*. Linearizing the Euler equation around this steady state easily shows that the sum and product of the characteristic roots satisfy:

$$\begin{aligned}\mathcal{T} &= 1+\delta^{-1}+f_2(k^*,k^*)+\frac{u'(c^*)}{u''(c^*)}[f_{11}(k^*,k^*)+f_{12}(k^*,k^*)] \\ \mathcal{D} &= \delta^{-1}+f_2(k^*,k^*)\end{aligned}$$

Definition 1 . *A steady state k^* is called locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^*-\epsilon, k^*+\epsilon)$ there are infinitely many equilibrium paths converging to the steady state.*

If both characteristic roots have modulus less than one then the steady state is locally indeterminate. If a steady state is not locally indeterminate, then we call it locally determinate.

As shown by Kehoe [14], it follows easily from the above expressions that necessary conditions for local indeterminacy are:

$$f_2(k^*,k^*) < 0 \text{ and } f_{12}(k^*,k^*) > 0 \text{ such that } f_{11}(k^*,k^*)+f_{12}(k^*,k^*) > 0$$

Such conditions imply very strong negative externalities which improves enough the private marginal productivity of capital to destroy concavity at the social level. Obviously they cannot be met by usual Cobb-Douglas or CES technologies. When standard positive externalities are considered, Boldrin and Rustichini [9] then show that the steady state is either saddle-point stable (if $f_{11}(k^*,k^*)+f_{12}(k^*,k^*) < 0$) or totally unstable (if $f_{11}(k^*,k^*)+f_{12}(k^*,k^*) > 0$).[2]

Under standard formulations for the fundamentals, Benhabib and Farmer [3] have shown that local indeterminacy in one-sector models requires the consideration of elastic labor supply and aggregate externalities on capital and labor. They consider a CES separable utility function and a Cobb-Douglas technology such that

$$U(C,L)=\log C-\frac{L^{1-\chi}}{1-\chi},\ F(K,L,\bar{K},\bar{L})=K^{\alpha}L^{1-\alpha}\bar{K}^{\alpha\eta}\bar{L}^{(1-\alpha)\eta}$$

[2] In a continuous-time framework, Spear [26] assumes that a positive externality A_t is given by tomorrow's aggregate capital stock k_{t+1} and gives sufficient conditions for the existence of sunspot equilibria in a neighborhood of the steady state.

with $\chi \leq 0$, $\eta > 0$ and $\bar{K}, \bar{L}$ the economy-wide averages of capital and labor. Denoting $\theta = \delta(1-\mu) \in [0,1]$ the discounted value of capital carried over to the next period per unit of capital used in the current period, standard linearization of the first order conditions around the steady state allows to show that the product and sum of the characteristic roots satisfy[3]

$$\begin{aligned} \mathcal{D} &= \tfrac{1}{\delta}\left[1 - \tfrac{\eta(1-\theta)(1-\chi)}{\theta(1-\alpha)(1+\eta)-(1-\chi)}\right] \\ \mathcal{T} &= 1 + \mathcal{D} - \tfrac{(1-\theta)[1-\alpha(1-\eta)](1-\chi)\left(\frac{1-\theta}{\delta\alpha}-\mu\right)}{\theta(1-\alpha)(1+\eta)-(1-\chi)} \end{aligned}$$

where steady-state conditions imply $(1-\theta)/\delta\alpha - \mu > 0$. In a discrete-time framework, local indeterminacy requires $|\mathcal{D}| < 1$ and $|\mathcal{T}| < 1 + \mathcal{D}$. Assuming that the aggregate share of capital satisfies $\alpha(1+\eta) < 1$, the main conclusion of Benhabib and Farmer is the following: in order to generate multiple equilibria, externalities and thus the degree of increasing returns to scale must be large enough to imply that the aggregate labor demand curve should be upward-sloping and steeper than the aggregate labor supply curve, i.e. $(1-\alpha)(1+\eta) - 1 > -\chi > 0$. This is obviously a non-standard configuration for the labor market. More recently, Pintus [23], by considering a general separable utility function $U(C,L) = u(C) - v(L)$ and a general technology $F(K,L)A(\bar{K},\bar{L})$ with constant returns to scale at the private level, show that the conditions of Benhabib and Farmer are not necessary. Local indeterminacy may indeed arise with a standard decreasing equilibrium labor demand function and small externalities provided the elasticity of capital-labor substitution is significantly greater than one.

3 Two-sector models with Cobb-Douglas technologies

In order to weaken their conditions for local indeterminacy, Benhabib and Farmer [4] consider a two-sector continuous-time model with Cobb-Douglas

[3]Benhabib and Farmer deal with a continuous-time model. We consider here the corresponding discrete-time formulation (see also Farmer and Guo [12]) in order to provide in Section 6.1 comparisons with the Wen's [28] extension to variable capacity utilization.

technologies and sector-specific rather than aggregate externalities. They provide conditions which are compatible with mild externalities and downward sloping labor demand curves. However they assume that each sector is characterised by the same private technology. Benhabib and Nishimura [7] have extended their results to distinct private Cobb-Douglas technologies and provide some nice conditions in terms of capital intensity differences. Even if they still consider an elastic labor supply in order to provide a version of a standard real business cycles model, similar conditions for local indeterminacy may be obtained with inelastic labor.

We then extend to a framework with externalities the contribution of Nishimura and Yano [22] which study an optimal growth model.[4] We consider a discrete-time two-sector economy having an infinitely-lived representative agent with single period linear utility function, i.e. $u(c) = c$. We assume that the labor supply is inelastic. There are two goods: the pure consumption good, c, and the pure capital good, k. Each good is assumed to be produced with a Cobb-Douglas technology which contains some positive sector specific externalities. We denote by c and y the outputs of sectors c and k, and by e_c and e_y the corresponding external effects:

$$c = K_c^{\alpha_1} L_c^{\alpha_2} e_c(\bar{K}_c, \bar{L}_c), \;\; y = K_y^{\beta_1} L_y^{\beta_2} e_y(\bar{K}_y, \bar{L}_y)$$

The externalities $e_c(\bar{K}_c, \bar{L}_c)$ and $e_y(\bar{K}_y, \bar{L}_y)$ depend on $\bar{K}_i$, $\bar{L}_i$ which denote the average use of capital and labor in sector $i = c, y$ and will be equal to

$$e_c(\bar{K}_c, \bar{L}_c) = \bar{K}_c^{a_1} \bar{L}_c^{a_2}, \;\; e_y(\bar{K}_y, \bar{L}_y) = \bar{K}_y^{b_1} \bar{L}_y^{b_2} \tag{1}$$

with $a_i, b_i \geq 0$, $i = 1, 2$. We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. Denoting $\hat{\alpha}_i = \alpha_i + a_i$, $\hat{\beta}_i = \beta_i + b_i$, the *social production functions* are defined as

$$c = K_c^{\hat{\alpha}_1} L_c^{\hat{\alpha}_2}, \;\; y = K_y^{\hat{\beta}_1} L_y^{\hat{\beta}_2}$$

We assume $\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\beta}_1 + \hat{\beta}_2 = 1$. The returns to scale are therefore constant

[4] The proof of the results presented in this section can be found in Benhabib, Nishimura and Venditti [8].

at the social level, and decreasing at the private level.[5] Factor intensities may be determined by the coefficients of the Cobb-Douglas functions. The investment (consumption) good sector is capital intensive from the private perspective if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 < (>)0$. The investment (consumption) good sector is capital intensive from the social perspective if and only if $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < (>)0$.[6]

Labor is normalized to one, $L_c + L_y = 1$, and the total stock of capital is given by $K_c + K_y = k$. We assume complete depreciation of capital in one period so that the capital accumulation equation is $y_t = k_{t+1}$. The consumer's optimization program will be given by:

$$\begin{aligned} \max_{\{K_{ct},L_{ct},K_{yt},L_{yt},k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \delta^t K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} e_{ct} \\ s.t. \quad & y_t = K_{yt}^{\beta_1} L_{yt}^{\beta_2} e_{yt} \\ & 1 = L_{ct} + L_{yt} \\ & k_t = K_{ct} + K_{yt} \\ & y_t = k_{t+1} \\ & k_0, \{e_c(\bar{K}_{ct}, \bar{L}_{ct})\}_{t=0}^{+\infty}, \{e_y(\bar{K}_{ct}, \bar{L}_{ct})\}_{t=0}^{+\infty} \quad given \end{aligned}$$

Denote by p_t, w_{0t} and w_t respectively the price of the capital good, the wage rate of labor and the rental rate of the capital good at time $t \geq 0$, all in terms of the price of the consumption good. Let $e_{it} = e_i(\bar{K}_{it}, \bar{L}_{it})$, $i = c, y$. For any given sequences $\{e_{ct}\}_{t=0}^{\infty}$ and $\{e_{yt}\}_{t=0}^{\infty}$ of external effects, the Lagrangian at time $t \geq 0$ is:

$$\begin{aligned} \mathcal{L}_t &= K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} e_{ct} + w_{0t}(1 - L_{ct} - L_{yt}) + w_t(k_t - K_{ct} - K_{yt}) \\ &+ p_t \left[K_{yt}^{\beta_1} L_{yt}^{\beta_2} e_{yt} - k_{t+1} \right] \end{aligned} \tag{2}$$

For any (k_t, y_t), solving the first order conditions w.r.t. $(K_{ct}, L_{ct}, K_{yt}, L_{yt})$ and using $y_t = k_{t+1}$ gives input demand functions such that $\tilde{K}_c = K_c(k_t, k_{t+1}, e_{ct}, e_{yt})$, $\tilde{L}_c = L_c(k_t, k_{t+1}, e_{ct}, e_{yt})$, $\tilde{K}_y = K_y(k_t, k_{t+1}, e_{ct}, e_{yt})$, $\tilde{L}_y = L_y(k_t, k_{t+1}, e_{ct}, e_{yt})$. We then define the production frontier as

[5] Our formulation is however compatible with constant returns at the private level if we assume that there exists a factor in fixed supply such as land in the technologies. In this case, the income of the representative consumer will be increased by the rental of land.

[6] Notice that under constant social returns $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 = \hat{\alpha}_1 - \hat{\beta}_1 = \hat{\beta}_2 - \hat{\alpha}_2$.

$$T(k_t, k_{t+1}, e_{ct}, e_{yt}) = \tilde{K}_{ct}^{\alpha_1} \tilde{L}_{ct}^{\alpha_2} e_{ct}$$

Using the envelope theorem we derive:

$$p_t = -T_2(k_t, k_{t+1}, e_{ct}, e_{yt}), \quad w_t = T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) \tag{3}$$

where $T_1 = \frac{\partial T}{\partial k_t}$ and $T_2 = \frac{\partial T}{\partial k_{t+1}}$. The first order conditions w.r.t. k_t give the Euler equation

$$-p_t + \delta w_{t+1} = 0$$

From the optimal demand functions defined above together with the external effects (1) considered at the equilibrium we may define the equilibrium factors demand fonctions $\hat{K}_i = \hat{K}_i(k_t, k_{t+1})$, $\hat{L}_i = \hat{L}_i(k_t, k_{t+1})$ so that $\hat{e}_c = \hat{e}_c(k_t, k_{t+1}) = \hat{K}_c^{a_1} \hat{L}_c^{a_2}$ and $\hat{e}_y = \hat{e}_y(k_t, k_{t+1}) = \hat{K}_y^{b_1} \hat{L}_y^{b_2}$.[7] From (3) prices now satisfy

$$\begin{aligned} p_t(k_t, k_{t+1}) &= -T_2(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1})) \\ w_t(k_t, k_{t+1}) &= T_1(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1})) \end{aligned}$$

and we get the Euler equation evaluated at $\hat{e}_c$ and $\hat{e}_y$:

$$-p(k_t, k_{t+1}) + \delta w(k_{t+1}, k_{t+2}) = 0 \tag{4}$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ of (4) which also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) = 0$$

is called an equilibrium path.

A steady state is defined by $k_t = k^*$, $y_t = y^* = k^*$ and is given by the solving of $\delta\omega(k^*, k^*) - p(k^*, k^*) = 0$. The methodology consists first in approximating the Euler equation (4), i.e. the first partial derivatives of $T(k_t, k_{t+1}, e_{ct}, e_{yt})$ for any given (e_{ct}, e_{yt}), using the first order conditions derived from the maximization of the Lagrangian (2). Then considering the externalities evaluated at the equilibrium $(\hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))$, we compute the steady state and then the partial derivatives of $T_i(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))$, $i = 1, 2$, in order to get the characteristic polynomial.[8] The first step gives

[7] Since we deal with an example we can show the existence of an equilibrium path together with the local indeterminacy. However if utility and production functions are not specified, then the existence of equilibrium paths is not obvious. For existence proofs in some general cases, see Le Van, Morhaim and Dimaria [15] and Mitra [16].

[8] See Benhabib, Nishimura and Venditti [8] for details.

Proposition 1 . *There exists a unique stationary capital stock k^* such that:*

$$k^* = \frac{\alpha_1\beta_2}{\alpha_2\beta_1+(\alpha_1\beta_2-\alpha_2\beta_1)\delta\beta_1}\left(\delta\beta_1\right)^{1/\hat{\beta}_2}$$

In the second step, the characteristic polynomial gives the following characteristic roots

Theorem 1 . *The characteristic roots are given by*

$$x_1 = \frac{\alpha_2}{\delta(\alpha_2\beta_1 - \alpha_1\beta_2)}, \ x_2 = \frac{\hat{\alpha}_2\hat{\beta}_1 - \hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2}$$

Remark: Note that x_1 does not depend on external effects while x_2 does. Moreover the sign of x_1 is determined by factor intensity differences at the private level, while the sign of x_2 is determined by factor intensity differences at the social level.

As this was shown in a continuous-time framework by Benhabib and Nishimura [7], a necessary condition for the steady to be locally indeterminate is a capital intensive consumption good from the private perspective. This result also holds in a discrete-time framework.[9] We thus introduce the following restriction

Assumption 3 . *The consumption good is capital intensive at the private level.*

Under this assumption notice that x_1 is negative. Local indeterminacy of the steady state may be obtained under slightly stronger conditions.

Theorem 2 . *Under Assumption 3, let $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$. Then the steady state is locally indeterminate if and only if one of the following sets of conditions is satisfied;*
i) the consumption good is labor intensive from the social perspective;
ii) the consumption good is capital intensive from the social perspective and $\hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$.

[9] If the investment good is capital intensive at the private level, it is easy to show that $x_1 > 1$ and the steady state is locally determinate.

Benhabib and Nishimura [7] have conducted a similar analysis with a two-sector Cobb-Douglas economy in continuous time. They prove that local indeterminacy occurs when there is a capital intensity reversal between the private and social levels: the consumption good needs to be capital intensive from the private perspective, but labor intensive from the social perspective. This corresponds to condition *i*) of Theorem 2 above. In a discrete-time framework however, such a reversal is not necessary. Even in the case the consumption good is capital intensive from both private and social perspectives, indeterminacy can take place as in *ii*) of Theorem 2. These results are based on the fact that the characteristic root x_2 may be either less than 1 when positive or greater than -1 when negative for any sign of the capital intensity difference at the social level.[10]

4 Two-sector models with CES technologies

Until now we have assumed Cobb-Douglas technologies and thus unitary elasticities of capital-labor substitution. In order to question the robustness of indeterminacy with respect to that parameter, we now extend the previous formulation to technologies with constant but non-unitary elasticities of substitution.[11] Consider indeed that each good is produced with a CES technology such that

$$\begin{aligned} c &= \left(\alpha_1 K_c^{-\rho_c} + \alpha_2 L_c^{-\rho_c} + e_c(\bar{K}_c, \bar{L}_c)\right)^{-1/\rho_c} \\ y &= \left(\beta_1 K_y^{-\rho_y} + \beta_2 L_y^{-\rho_y} + e_y(\bar{K}_y, \bar{L}_y)\right)^{-1/\rho_y} \end{aligned}$$

with $\rho_c, \rho_y > -1$ and $\sigma_c = 1/(1+\rho_c) \geq 0$, $\sigma_y = 1/(1+\rho_y) \geq 0$ the elasticities of capital/labor substitution in each sector. As previously, the externalities, $e_c(\bar{K}_c, \bar{L}_c)$ and $e_y(\bar{K}_y, \bar{L}_y)$ depend on $\bar{K}_i$, $\bar{L}_i$ which denote the average use of capital and labor in sector $i = c, y$ and will now be equal to

[10]When the discount factor δ crosses from above the critical value $\delta^* = \alpha_2/(\alpha_1\beta_2 - \alpha_2\beta_1) < 1$, the steady state becomes saddle-point stable, a flip bifurcation occurs and there exist equilibrium period-two cycles either in a right or in a left neighborhood of ρ^*.

[11]The proof of the results presented in this section can be found in Nishimura and Venditti [21].

$$e_c(\bar{K}_c, \bar{L}_c) = a_1 \bar{K}_c^{-\rho_c} + a_2 \bar{L}_c^{-\rho_c}, \quad e_y(\bar{K}_c, \bar{L}_c) = b_1 \bar{K}_y^{-\rho_y} + b_2 \bar{L}_y^{-\rho_y}$$

with $a_i, b_i \geq 0$, $i = 1, 2$. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. Denoting $\hat{\alpha}_i = \alpha_i + a_i$, $\hat{\beta}_i = \beta_i + b_i$, the *social production functions* are defined as

$$c = \left(\hat{\alpha}_1 K_c^{-\rho_c} + \hat{\alpha}_2 L_c^{-\rho_c}\right)^{-1/\rho_c} \text{ and } y = \left(\hat{\beta}_1 K_y^{-\rho_y} + \hat{\beta}_2 L_y^{-\rho_y}\right)^{-1/\rho_y}$$

The returns to scale are again constant at the social level, and decreasing at the private level. We will assume in the following that $\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\beta}_1 + \hat{\beta}_2 = 1$ so that the production functions collapse to Cobb-Douglas in the particular case $\rho_c = \rho_y = 0$.

We follow the same methodology as in the previous section with Cobb-Douglas technologies. We need however to assume the following restriction:

Assumption 4 . $\hat{\beta}_1 < (\delta\beta_1)^{\rho_y/(1+\rho_y)}$

For some given β_1, $\hat{\beta}_1$ and δ, Assumption 4 provides an upper bound $\hat{\rho}_y > 0$ for ρ_y. We have indeed

$$\rho_y < \frac{ln\hat{\beta}_1}{ln(\delta\beta_1) - ln\hat{\beta}_1} \equiv \hat{\rho}_y \tag{5}$$

Such a restriction is quite standard when CES technologies are considered. It is well-known indeed that when the elasticity of capital/labor substitution is less than 1, Inada conditions are not satisfied and corner solutions cannot be a priori ruled out. Assumption 4 precisely ensures positiveness and interiority of all the steady state values for input demand functions K_c, K_y, L_c and L_y. Throughout the paper we will therefore consider that $\rho_y \in (-1, \hat{\rho}_y)$.

Remark: In the Cobb-Douglas case with $\rho_y = 0$, the Inada conditions are satisfed and Assumption 4 becomes $\hat{\beta}_1 < 1$ which always holds.

Under this restriction we then obtain existence and uniqueness of the steady state k^*:

Proposition 2 . *Under Assumption 4, there exists a unique stationary capital stock $k^* > 0$, such that:*

$$k^* = \frac{\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{1+\rho_y}{\rho_y(1+\rho_c)}}}{1 - (\delta\beta_1)^{\frac{1}{1+\rho_y}} \left[1 - \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{\rho_y - \rho_c}{\rho_y(1+\rho_c)}}\right]}$$

4.1 Symmetric elasticities of substitution

In order to start with the simplest case, we will assume that both sectors are characterized by the same elasticity of substitution:[12]

Assumption 5 . $\rho_c = \rho_y = \rho$

We may now provide expressions of the characteristic roots.

Theorem 3 . *Under Assumption 4-5, the characteristic roots are given by*

$$\begin{aligned} x_1 &= \left\{ (\delta\beta_1)^{\frac{1}{1+\rho}} \left[1 - \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1} \right)^{\frac{1}{1+\rho}} \right] \right\}^{-1} \\ x_2 &= (\delta\beta_1)^{\frac{-\rho}{1+\rho}} \hat{\beta}_1 \left[1 - \frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1} \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2} \right)^{\frac{\rho}{1+\rho}} \right] \end{aligned}$$

Remark: If both technologies are Cobb-Douglas with $\rho = 0$, the roots given in Theorem 1 are recovered.

Theorem 3 shows that the stability properties of the steady state will depend, among all the parameters, on the sign of the following differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. As in the Cobb-Douglas case, it can be easily shown around the steady state that if the elasticities of capital/labor substitution are identical across sectors, the consumption good is capital intensive at the private level if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$ while it is capital intensive at the social level if and only if $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 = \hat{\alpha}_1 - \hat{\beta}_1 > 0$.

As in the Cobb-Douglas framework, local indeterminacy will require the consumption good to be capital intensive at the private level.[13] The following theorem extends Theorem 2 to technologies with non unitary elasticities of capital-labor substitution. It shows that under symmetric substitutability, local indeterminacy may still occur for elasticities significantly different from unity. The only consequence of such a restriction is that extreme values for ρ are excluded:

[12] A continuous-time version of this model extended to n sector is studied in Nishimura and Venditti [20].

[13] If the investment good is capital intensive at the private level, it is easy to show that $x_1 > 1$ and the steady state is locally determinate.

Theorem 4 . *Under Assumptions 3-5, consider $\hat{\rho}_y$ as defined by equation (5) and let $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$. There exist $\underline{\rho} \in (-1, 0)$ and $\bar{\rho} \in (0, \hat{\rho}_y)$ such that the steady state is locally indeterminate for any $\rho \in (\underline{\rho}, \bar{\rho})$ if one of the two following conditions is satisfied:*

i) the investment good is capital intensive at the social level;

ii) the investment good is labor intensive at the social level and $\hat{\beta}_1 - \hat{\alpha}_1 > -\hat{\alpha}_2$.

4.2 Asymmetric elasticities of substitution

We may consider now the general formulation with asymmetric elasticities of capital-labor substitution.

Theorem 5 . *Under Assumption 4, the characteristic roots are given by*

$$\begin{aligned}
x_1 &= \left\{ (\delta\beta_1)^{\frac{1}{1+\rho_y}} \left[1 - \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2} \right)^{\frac{\rho_y - \rho_c}{\rho_y(1+\rho_c)}} \right] \right\}^{-1} \\
x_2 &= (\delta\beta_1)^{\frac{-\rho_y}{1+\rho_y}} \hat{\beta}_1 \left[1 - \frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1} \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\frac{\rho_c}{1+\rho_c}} \left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2} \right)^{\frac{\rho_y - \rho_c}{\rho_y(1+\rho_c)}} \right]
\end{aligned}$$

Remark: If both technologies have the same elasticity of substitution, i.e. $\rho_c = \rho_y = \rho$, the roots given in Theorem 3 are recovered.

Contrary to the case with symmetric elasticities of substitution, the capital intensity differences at the private and social levels are not easily captured by the differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. They also depend on prices and the parameters ρ_c and ρ_y. We may however obtain the following characterization at the steady state:

Proposition 3 . *Under Assumption 4, at the steady state:*

i) the consumption (investment) good sector is capital intensive from the private perspective if and only if

$$\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}} - \hat{\beta}_1}{\hat{\beta}_2} \right)^{\frac{\rho_c - \rho_y}{\rho_y(1+\rho_c)}} < (>) \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} \tag{6}$$

ii) the consumption (investment) good sector is capital intensive from the social perspective if and only if

$$\left(\frac{(\delta\beta_1)^{\frac{\rho_y}{1+\rho_y}}-\hat{\beta}_1}{\hat{\beta}_2}\right)^{\frac{\rho_c-\rho_y}{\rho_y(1+\rho_c)}} < (>) \left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right)^{\frac{1}{1+\rho_c}} \left(\frac{\hat{\beta}_2\beta_1}{\hat{\beta}_1\beta_2}\right)^{\frac{\rho_c-\rho_y}{(1+\rho_y)(1+\rho_c)}} \tag{7}$$

Remark: If $\rho_c = \rho_y = \rho$, condition (6) becomes $\alpha_1\beta_2 - \alpha_2\beta_1 > (<)0$ and condition (7) becomes $\hat{\alpha}_1 - \hat{\beta}_1 > (<)0$. Notice also from Theorem 1 and (6) that as in the Cobb-Douglas formulation, the root x_1 is positive if and only if the investment good is capital intensive at the private level. On the contrary, when $\rho_c \neq \rho_y \neq 0$, the sign of the second root x_2 does not directly depend on the sign of the capital intensity difference across sectors at the social level.

In order to simplify the exposition, we will discuss the local stability properties of the steady state depending on the sign of the differences $\alpha_1\beta_2 - \alpha_2\beta_1$ and $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$, and the values of the elasticities of substitution in both sectors. We will only refer to capital intensities when the results are economically interpreted.

We have now to give conditions for local indeterminacy when the consumption good is capital intensive at the private level. We first consider the case $\alpha_1\beta_2 - \alpha_2\beta_1 < 0$ which, as we have shown previously, is known in the Cobb-Douglas framework to imply local determinacy of the steady state.[14] The following theorem shows on the contrary that with asymmetric elasticities of substitution, there is room for local indeterminacy.

Theorem 6 . *Under Assumptions 3-4, consider $\hat{\rho}_y$ as defined by equation (5). Let $\alpha_1\beta_2 < \alpha_2\beta_1$ and $(\hat{\alpha}_1/\hat{\alpha}_2)/(\alpha_1/\alpha_2) < \delta\beta_2$. Then there exist $\underline{\rho}_c > 0$ and $\bar{\rho}_y \in (-1, 0)$ such that the steady state is locally indeterminate if $\rho_c > \underline{\rho}_c$ and $\rho_y \in (-1, \bar{\rho}_y)$.*

Theorem 6 proves that even in the unusual situation with $\alpha_1\beta_2 < \alpha_2\beta_1$, local indeterminacy may occur provided the consumption good sector has

[14]When $\rho_y = \rho_c = 0$, such a restriction implies indeed that the investment good is capital intensive at the private level. Local indeterminacy is thus ruled out. As shown in sub-section 4.1, the same result actually holds when $\rho_y = \rho_c = \rho \neq 0$.

a technology close to a Leontief function while the investment good sector has a technology close to a linear function. A direct inspection of inequality (6) from Proposition 3 shows that when ρ_c is high enough while ρ_y is close to -1, the consumption good is capital intensive at the private level.

We will consider now the converse configuration with $\alpha_1\beta_2 > \alpha_2\beta_1$. As this was already the case in a Cobb-Douglas framework, local indeterminacy requires a slightly stronger restriction concerning these parameters:

Assumption 6 . $\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho_c}} > 1 + (\delta\beta_1)^{-1}$

If technologies are either Cobb-Douglas or with identical elasticities of substitution, Assumption 6 implies Assumption 3. Notice also that if $\rho_c = 0$, we get the condition $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ in Theorem 2. Assumption 6 actually ensures that $x_1 \in (-1, 0)$ when $\rho_y = 0$.

Let us start with some conditions that cover the case in which the investment good sector has a Cobb-Douglas technology with $\rho_y = 0$. Under Assumption 6 we will introduce additional restrictions to get $x_2 \in (-1, 1)$

Theorem 7 . *Under Assumptions 3-4 and 6, consider $\hat{\rho}_y$ as defined by equation (5). If the following condition holds for some given $\rho_c > -1$*

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) < \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho_c}{1+\rho_c}} \tag{8}$$

then there exist $\underline{\rho}_y \in (-1, 0)$ and $\bar{\rho}_y \in (0, \hat{\rho}_y)$ such that the steady state is locally indeterminate for any $\rho_y \in (\underline{\rho}_y, \bar{\rho}_y)$. Moreover the lower bound $\underline{\rho}_y$ is equal to -1 if the following additional restrictions hold:

$$1 \leq \frac{\hat{\beta}_2^{1+\rho_c}}{\beta_2} < \delta\frac{\alpha_1}{\alpha_2} \text{ and } \left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)^{\frac{\rho_c}{1+\rho_c}} < (\delta\beta_1)^{\frac{\rho_c}{1+\rho_c}}\frac{\hat{\alpha}_2}{\hat{\alpha}_1} \tag{9}$$

When the additional conditions (9) hold, Theorem 7 shows that for some given $\rho_c > -1$, local indeterminacy is compatible with arbitrarily large elasticities of capital/labor substitution in the investment good sector. Notice that this cannot be the case with symmetric elasticities of substitution.

We may discuss Theorem 7 depending on the sign of the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. As in the Cobb-Douglas case, local indeterminacy with CES technologies does not require a capital intensity reversal. We derive indeed from Theorem 1 and Proposition 3 that the characteristic root x_2 may be either less than 1 when positive or greater than -1 when negative for any sign of the capital intensity difference at the social level.

Consider first the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < 0$. It is then easy to see that condition (8) holds for any $\rho_c \geq 0$, i.e. for any consumption good technology having an elasticity of capital/labor substitution less than unity. It follows from Assumption 6 that if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$ local indeterminacy will hold for any $\rho_c \geq 0$. It is worth noticing that this covers the case $\rho_c = +\infty$ of a Leontief technology for the consumption good. As already mentioned previously, local indeterminacy also occurs for $\rho_c < 0$ but far enough from -1. The robustness of this result will indeed depend on the CES coefficients at the social level.

Consider now the converse case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 > 0$. Condition (8) may still hold but no clear restriction on the parameter ρ_c can be derived. If the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$ is significantly greater than zero, i.e. for instance if

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) > 1, \tag{10}$$

then local indeterminacy cannot hold when ρ_c is close to zero and will require much lower elasticities of capital/labor substitution in the consumption good sector.

Notice also that since $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$ and $\lim_{\rho_c \to -1} \rho_c/(1+\rho_c) = -\infty$, condition (8) cannot hold when ρ_c is close enough to -1. It follows that under the Assumptions of Theorem 7 there exists $\underline{\rho}_c \in (-1, 0)$ such that local indeterminacy occurs when $\rho_c > \underline{\rho}_c$.

We may finally give conditions which cannot be satisfied when the technology of the investment good is Cobb-Douglas. When condition (8) does not hold, local indeterminacy appears while the elasticity of substitution in the investment good sector is less than unity.

Theorem 8 . *Under Assumptions 3-4 and 6, consider $\hat{\rho}_y$ as defined by equation (5). If the following condition holds for some given $\rho_c \in (-1, \hat{\rho}_y]$*

$$\frac{\hat{\beta}_1}{1+\hat{\beta}_1}\left(\frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1}\right) > \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho_c}{1+\rho_c}} \tag{11}$$

then there exist $\underline{\rho}_y \in (0, \hat{\rho}_y)$ *and* $\bar{\rho}_y \in (0, \hat{\rho}_y)$ *with* $\underline{\rho}_y < \bar{\rho}_y$ *such that the steady state is locally indeterminate for any* $\rho_y \in (\underline{\rho}_y, \bar{\rho}_y)$*.*

Notice that contrary to condition (8) in Theorem 7, condition (11) is now compatible with ρ_c close to -1, i.e. with an arbitrarily large elasticity of capital/labor substitution in the consumption good sector.

We may again discuss Theorem 8 depending on the sign of the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$. Consider first the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 < 0$. Condition (11) shows that local indeterminacy cannot occur for any $\rho_c \geq 0$. The same conclusion holds also for $\rho_c < 0$ but close to 0. Local indeterminacy indeed requires a strong enough elasticity of capital/labor substitution in the consumption good sector while that elasticity in the investment good sector is restricted to be less than unity.

Consider finally the case $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1 > 0$. Local indeterminacy now becomes compatible with positive values for ρ_c provided the difference $\hat{\alpha}_1\hat{\beta}_2 - \hat{\alpha}_2\hat{\beta}_1$ is significantly greater than zero, i.e. if equation (10) holds.

5 Extensions with Cobb-Douglas technologies

5.1 Partial depreciation

Until now we have assumed that capital fully depreciates every period. This much criticized assumption has been proved to be quite particular by Baierl, Nishimura and Yano [2] in two-sector optimal growth models. Unlike continuous-time models, introducing depreciation of capital indeed creates additional difficulty in studying dynamical properties of equilibrium paths in discrete time models.[15]

We now extend Baierl, Nishimura and Yano [2] to the case with externalities.[16] We thus assume partial depreciation of capital so that the capital

[15] Baierl, Nishimura and Yano [2] show indeed that around the steady-state, optimal paths become less likely to oscillate in the case of partial depreciation than in that of full depreciation.

[16] The proof of the results presented in this subsection can be found in Benhabib, Nishimura and Venditti [17].

accumulation equation becomes $y_t = k_{t+1} - (1-\mu)k_t$, with $\mu \in [0,1]$. In this case, the envelope theorem provides the following equilibrium prices:

$$\begin{aligned} p_t &= -T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) \\ \omega_t &= T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) + (1-\mu)T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) \end{aligned}$$

and the Euler equation becomes

$$-p_t + \delta[\omega_{t+1} + (1-\mu)p_{t+1}] = 0$$

Existence and uniqueness of the steady state still hold but it now depends on the parameter μ. As in Section 2, let $\theta = \delta(1-\mu) \in [0,1]$:

Corollary 1 . *There exists a unique stationary capital stock k^* satisfying:*

$$k^* = \frac{\alpha_1\beta_2(1-\theta)}{\beta_1[\alpha_2(1-\theta)+(\alpha_1\beta_2-\alpha_2\beta_1)\delta\mu]}\left(\frac{\delta\beta_1}{1-\theta}\right)^{\frac{1}{\beta_2}}$$

The characteristic roots depend also on the rate of capital depreciation:

Theorem 9 . *The characteristic roots are given by:*

$$\begin{aligned} x_1 &= \frac{\alpha_2(1-\theta)+\theta(\alpha_2\beta_1-\alpha_1\beta_2)}{\delta(\alpha_2\beta_1-\alpha_1\beta_2)} \\ x_2 &= \frac{\hat{\alpha}_2\hat{\beta}_1-\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2(1-\theta)+\theta(\hat{\alpha}_2\hat{\beta}_1-\hat{\alpha}_1\hat{\beta}_2)} \end{aligned}$$

When capital depreciates slowly, local indeterminacy still requires a capital intensive consumption good at the private level.[17] Moreover, as in the case with full depreciation, a capital intensity reversal is not necessary. Assume first that the investment good is capital intensive at the social level.

Theorem 10 . *Under Assumption 3, let $\hat{\beta}_1 > \hat{\alpha}_1$ and $\bar{\theta} = \alpha_2/[\alpha_2(1-\beta_1)+\alpha_1\beta_2] < 1$. Then the following cases hold:*

i) if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$, there exists $\hat{\theta} \in]\bar{\theta}, 1[$ such that the steady state is locally indeterminate for any $\theta \in [0, \hat{\theta}[\backslash\{\bar{\theta}\}$;

ii) if $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2/\delta$, there exist $\tilde{\theta}, \hat{\theta} \in]0,1[$, with $\tilde{\theta} < \bar{\theta} < \hat{\theta}$, such that the steady state is locally indeterminate for any $\theta \in]\tilde{\theta}, \hat{\theta}[\backslash\{\bar{\theta}\}$.

[17] If the investment good is capital intensive at the private level, we easily show that $x_1 > 1$ and the steady state is locally determinate.

Case i) provides an extension to partial depreciation of Theorem 2*i)* which has been established under $\theta = 0$. Moreover we show that given production functions and a discount factor close to 1, equilibrium paths become less likely to be locally indeterminate in the case of partial depreciation (θ close enough to 1) than in that of full depreciation ($\theta = 0$). Baierl, Nishimura and Yano [2] have obtained a similar result concerning the occurrence of period-two cycles in an optimal growth model.

In case ii), a similar result is obtained. We provide however some new conditions for local indeterminacy that cannot arise under full depreciation. For intermediary values of the depreciation rate, local indeterminacy arises under mild conditions on the capital intensity difference at the private level: the consumption good needs to be only slightly more capital intensive than the investment good.[18]

Assume now that the consumption good is also capital intensive at the social level.

Theorem 11 . *Under Assumption 3, let* $\hat{\alpha}_1 > \hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$, $\theta^* = 2\hat{\alpha}_2/\hat{\beta}_2 - 1 \in (0,1)$ *and* $\bar{\theta} = \alpha_2/[\alpha_2(1-\beta_1) + \alpha_1\beta_2] < 1$. *Then the following cases hold:*

i) if $\alpha_1\beta_2 - \alpha_2\beta_1 > \alpha_2/\delta$, *the steady state is locally indeterminate for any* $\theta \in [0, \theta^*[\setminus \{\bar{\theta}\}$*;*

ii) Let $\tilde{\theta} = [\alpha_2(1+\delta\beta_1) - \delta\alpha_1\beta_2]/[\alpha_2(1-\beta_1) + \alpha_1\beta_2] < 1$. *If* $\alpha_1\beta_2 - \alpha_2\beta_1 < \alpha_2/\delta$ *and* $\tilde{\theta} < \theta^*$, *the steady state is locally indeterminate for any* $\theta \in]\tilde{\theta}, \theta^*[\setminus \{\bar{\theta}\}$.

Case i) provides an extension to partial depreciation of Theorem 2*ii)* which has been derived under $\theta = 0$. As in the previous case, we show that given production functions and a discount factor close to 1, equilibrium paths become less likely to be locally indeterminate in the case of partial depreciation (θ close enough to 1) than in that of full depreciation ($\theta = 0$).[19]

[18] When θ crosses $\tilde{\theta}$ from above the steady state becomes saddle-point stable, a flip bifurcation occurs and there exist equilibrium period-two cycles either in a left or in a right neighborhood of $\tilde{\theta}$.

[19] Notice that a flip bifurcation occurs when θ crosses $\hat{\theta}$ from below and the steady state

In case ii), local indeterminacy is more difficult to obtain than under a capital intensity reversal between the private and the social level. The condition on the capital intensity difference at the private level is less demanding than in case i), but the restriction $\tilde{\theta} < \theta^*$ does not have a precise economic interpretation.[20]

Remark: Nishimura and Venditti [18] have also extended to partial depreciation the CES formulation with symmetric elasticities of substitution considered in Section 4.1. Depending on the value of the elasticity of capital-labor substitution, local indeterminacy may arise for any value $\mu \in [0, 1]$ of the rate of depreciation, for low depreciation with μ close to zero, or for high depreciation with μ close to one. The conclusion that local indeterminacy is less likely in the case of partial depreciation than in that of full depreciation is therefore specific to the Cobb-Douglas formulation.

5.2 Intersectoral externalities

Up to now we have considered sector-specific external effects. Although equilibria become sub-optimal, such a formulation remains quite close to standard optimal growth models since no direct additional intersectoral mechanisms are introduced. This is not the case if we consider the initial formulation of Romer [24] in which the aggregate capital stock is used as global technological externalities.

In order to introduce these additional mechanisms in a simple Cobb-Douglas framework, we assume now that the consumption good production function contains positive intersectoral externalities given by a convex combination of the capital stocks of the two sectors.[21] The production functions

then becomes sadle-point stable while there exist equilibrium period-two cycles either in a right or in a left neighborhood of $\hat{\theta}$.

[20]If these conditions hold, endogenous fluctuations again appear through a flip bifurcation: when θ crosses $\tilde{\theta}$ (θ^*) from above (below) the steady state becomes saddle-point stable, and there exist equilibrium period-two cycles either in a left or in a right neighborhood of $\tilde{\theta}$ (θ^*).

[21]The proof of the results presented in this subsection can be found in Nishimura and Venditti [19].

are thus:

$$y = K_y^{\beta_1} L_y^{\beta_2}, \quad c = K_c^{\alpha_1} L_c^{\alpha_2} e(\bar{K}_c, \bar{K}_y) \quad \text{with} \quad e = \left[\phi \bar{K}_c + (1-\phi)\bar{K}_y\right]^a$$

where $\phi \in [0,1]$, $a \geq 0$ and $\bar{K}_i$ denotes the average use of capital in sector $i = c, y$. Depending on the value of ϕ, our formulation therefore encompasses the usual assumptions of sector specific externalities ($\phi = 1$), and global external effects ($\phi = 1/2$). We will also consider the case with purely intersectoral externalities ($\phi = 0$). We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}_i = K_i$ and $\bar{K}_i = K_i$. The *social production function* for the consumption good is therefore

$$c = K_c^{\alpha_1} L_c^{\alpha_2} \left[\phi K_c + (1-\phi) K_y\right]^a$$

We will assume non-increasing returns to scale at the social level in the consumption good sector, i.e. $\alpha_1 + a + \alpha_2 \equiv \hat{\alpha}_1 + \alpha_2 \leq 1$, and constant returns to scale in the investment good sector, i.e. $\beta_1 + \beta_2 = 1$.

It can be easily shown that if $\alpha_1\beta_2 - \alpha_2\beta_1 < (>) 0$ the investment (consumption) good sector is capital intensive from the private perspective. Note that this definition is still valid with intersectoral external effects ($\phi < 1$). If the externalities are sector specific ($\phi = 1$), the condition $\beta_1 > (<) \hat{\alpha}_1$ implies that the investment (consumption) good sector is capital intensive from the social perspective.

We follow the same procedure as in the previous sections. Full depreciation of capital is again assumed for simplicity. The steady state is given by Proposition 1 with $\hat{\beta}_2 = \beta_2$. We start by assuming that there are only sector specific externalities in the consumption good sector. We show in this case that the steady state is always locally determinate.

Proposition 4 . *If the externalities are sector specific ($\phi = 1$), the steady state k^* is locally determinate.*

Theorem 2 establishes that if the consumption good is capital intensive from the private perspective, locally indeterminate equilibria may occur when the consumption good is either capital or labor intensive at the social level. However, they assume that there are external effects on capital and labor in both sectors. Proposition 4 shows however that when only the consumption good technology is affected by external effects, indeterminacy necessarily requires externalities coming from labor. Note that this result does not hold and indeterminacy is still possible if we assume that the investment good sector contains external effects on capital. Our formulation therefore will strongly enlighten the role of intersectoral external effects.

Consider indeed the case in which the externality in the consumption good technology comes only from the capital stock of the investment good sector ($\phi = 0$).

Assumption 7 . *min* $\left\{ \frac{\beta_1 \alpha_2}{\hat{\alpha}_1}, 1, \frac{a\alpha_2(1-\alpha_1)}{\alpha_1+\alpha_2} \right\} > \beta_2$.

Given arbitrary $a > 0$, Assumption 7 may be satisfied if β_2 is chosen to be sufficiently small. Assumption 7 also implies that the investment good is capital intensive at the private level since $\beta_1\alpha_2/\hat{\alpha}_1 > \beta_2$ implies $\beta_1 > \hat{\alpha}_1 > \alpha_1$.

Proposition 5 . *Let* $\phi = 0$. *Under Assumption 7, there exists* $\delta_1 < 1$ *such that the steady state is locally indeterminate for any* $\delta \in]\delta_1, 1]$.

Contrary to the sector-specific formulation in which local indeterminacy requires the consumption good to be capital intensive at the private level, we show that when pure intersectoral external effects are considered, a continuum of equilibria may arise under a capital intensive investment good at the private level.

From Proposition 5 it is straightforward to extend the indeterminacy result to intermediary cases with positive values of θ as in the following Theorem:

Theorem 12 . *Under Assumption 7, there exist* $0 < \delta_1 < 1$ *and a function* $\phi^* :]\delta_1, 1] \rightarrow]0, 1[$ *such that the steady state is locally indeterminate for each* $\delta \in]\delta_1, 1]$ *and* $\phi \in [0, \phi^*(\delta)[$.

Remark: It can be shown that if externalities coming from capital are also introduced in the investment good sector, Proposition 5 and Theorem

12 still hold with some more complicated sufficient conditions which will depend on the external effect parameters of the capital good. Indeterminacy under a capital intensive investment good at the private level is thus a robust property as soon as externalities are intersectoral.[22]

6 Other formulations

6.1 Variable capital utilization

In Section 5.1, we have introduced partial depreciation of capital and we have shown how the occurrence of local indeterminacy is affected by this parameter. Building on the fact that capacity utilization is potentially a powerful driving force behind business cycles, Wen [28] considers a discrete-time extension of the Benhabib and Farmer [3] model in which the speed of capital depreciation is endogenously determined. A representative consumer solves indeed

$$\begin{array}{ll} \max\limits_{\{c_t,l_t,u_t,k_{t+1}\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t \left(\log c_t - \dfrac{l_t^{1-\chi}}{1-\chi} \right) \\ s.t. & c_t + k_{t+1} - (1-\mu_t)k_t = (u_t k_t)^\alpha l_t^{1-\alpha} e_t(\bar{u}_t \bar{k}_t, \bar{l}_t) \\ & \mu_t = \tau u_t^\gamma \\ & k_0, \{e_t\}_{t=0}^{+\infty} \text{ given} \end{array}$$

with $\chi \leq 0$, $\alpha \in (0,1)$, $\tau \in (0,1)$, $u_t \in (0,1)$ the rate of capacity utilization, $\mu_t \in (0,1)$ the rate of capital depreciation defined as an increasing function of capacity utilization, i.e. $\gamma > 0$, and $e_t(\bar{u}_t, \bar{k}_t, \bar{l}_t)$ the externality expressed as a function of the average economy-wide levels of productive capacity and labor, i.e.

$$e_t(\bar{u}_t \bar{k}_t, \bar{l}_t) = (\bar{u}_t \bar{k}_t)^{\alpha\eta} \bar{l}_t^{(1-\alpha)\eta}$$

where $\eta \geq 0$. Variable capital utilization is ensured under $\gamma > 1$ while constant partial depreciation as in Benhabib and Farmer [3] follows from

[22] Under slight additional restrictions, it can also be proved that for any given ϕ close enough to zero, there exists a bound $\delta^*(\phi) \in (0,1)$ such that when δ crosses $\delta^*(\phi)$ from above the steady state becomes saddle-point stable and quasi-periodic cycles appear through a Hopf bifurcation. Notice that endogenous fluctuations are obtained under a capital intensive investment good while Benhabib and Nishimura [6] show in an optimal growth model that a capital intensive consumption good is necessary.

$\gamma \leq 1$. Standard linearization of the first order conditions around the steady state allows Wen to show that the product and sum of the characteristic roots satisfy

$$\begin{aligned}
\mathcal{D} &= \tfrac{1}{\delta}\left(1 - \tfrac{\eta(1-\chi)(1-\delta)\tau_l}{\delta(1-\alpha)(1+\eta)\tau_l-(1-\chi)}\right) \\
\mathcal{T} &= 1 + \mathcal{D} - \tfrac{(1-\chi)(1-\delta)(\gamma-\alpha)(1-\alpha(1-\chi)\tau_k)\mu/\alpha}{\delta(1-\alpha)(1+\eta)\tau_l-(1-\chi)}
\end{aligned}$$

where

$$\tau_k = \frac{\gamma - 1}{\gamma - \alpha(1+\eta)}, \ \tau_l = \frac{\gamma}{\gamma - \alpha(1+\eta)}$$

In a discrete-time framework, local indeterminacy requires $|\mathcal{D}| < 1$ and $|\mathcal{T}| < 1 + \mathcal{D}$. When compared with the corresponding expressions under constant partial depreciation given in Section 2, we easily derive that multiple equilibria become compatible with much lower increasing returns to scale and a downward sloping aggregate labor demand curve, i.e. $(1-\alpha)(1+\eta)-1 < 0$.

More recently, Guo and Harrison [13] provides an extension of the Wen's capacity utilization model to a discrete-time adaptation of the Benhabib and Farmer's [4] two-sector model with sector-specific externalities. Both sectors have the same Cobb-Douglas technology at the private level with constant returns to scale. Variable capital utilization is introduced into technologies as follows

$$c = (uK_c)L_c^{1-\alpha}e_c(\bar{u}\bar{K}_c, \bar{L}_c), \ y = (uK_y)^\alpha L_y^{1-\alpha}e_y(\bar{u}\bar{K}_y, \bar{L}_y)$$

The externalities $e_c(\bar{u}\bar{K}_c, \bar{L}_c)$ and $e_y(\bar{u}\bar{K}_y, \bar{L}_y)$ depend on the average use of capital and labor services and are equal to

$$e_c(\bar{u}\bar{K}_c, \bar{L}_c) = [(\bar{u}\bar{K}_c)^\alpha \bar{L}_c^{1-\alpha}]^\eta, \ e_y(\bar{u}\bar{K}_y, \bar{L}_y) = [(\bar{u}\bar{K}_y)^\alpha \bar{L}_y^{1-\alpha}]^\eta \tag{12}$$

with $\eta > 0$. Returns to scale are therefore increasing at the social level. Guo and Harrison show that local indeterminacy occurs under smaller externalities and thus lower increasing returns to scale than in the Benhabib and Farmer's [4] and Wen's [28] models.

6.2 Two-sector models with general technologies

In Section 5.2, we have considered a model with intersectoral externalities which is compatible with both sector-specific and global external effects. In order to provide simple conditions, we have assumed Cobb-Douglas technologies. Boldrin and Rustichini [9] also introduce Romer-type [24] global externalities in a two-sector discrete-time model but they consider general technologies.

The labor supply is inelastic with total labor normalised to 1, and the population is constant. The pure consumption good c and the capital good y are produced with constant private returns to scale technologies which also depends on an intersectoral externality A:

$$c = f^0(k_0, l_0, A), \quad y = f^1(k_1, l_1, A)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$. At the equilibrium, the externality A will equal the aggregate capital stock k.

Assumption 8 . *Each production function $f^i(k^i, l^i, A)$, $i = 0, 1$, is C^2, increasing in each argument and, for any $A > 0$, concave, homogeneous of degree one and such that for any $l_i > 0$, $f^i_{11}(., l_i, A) < 0$.*

Externalities are therefore positive and returns to scale are increasing at the social level. For any given (k, y, A), the production frontier $T(k, y, A)$ is defined as

$$\begin{aligned} T(k, y, A) = \quad & \max_{k_0, k_1, l_0, l_1} \quad f^0(k_0, l_0, A) \\ & \quad s.t. \quad\quad y \leq f^1(k_1, l_1, A) \\ & \quad\quad\quad\quad\quad k_0 + k_1 \leq k \\ & \quad\quad\quad\quad\quad l_0 + l_1 \leq 1 \\ & \quad\quad\quad\quad\quad k_0, k_1, l_0, l_1 \geq 0 \end{aligned}$$

Under Assumption 8, for any given $A > 0$, $T(k, y, A)$ is concave. Assuming a linear utility function and full depreciation of capital within one period of time, the maximisation program of the representative agent is

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t T(k_t, k_{t+1}, A_t) \\ s.t. \quad & (k_t, k_{t+1}) \in \mathcal{D}(A_t) \\ & k_0, \{A_t\}_{t=0}^{+\infty} \text{ given} \end{aligned}$$

with

$$\mathcal{D}(A_t) = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / 0 \leq k_{t+1} \leq f^1(k_t, 1, A_t) \right\}$$

the set of admissble paths for any given A_t. Along an equilibrium path with $A_t = k_t$, the Euler equation is

$$T_2(k_t, k_{t+1}, k_t) + \delta T_1(k_{t+1}, k_{t+2}, k_{t+1}) = 0$$

An equilibrium path also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, k_t) = 0$$

A steady state $k_{t+1} = k_t = k^*$ is a solution of

$$f_1^1(k_1(k, k,), l_1(k, k, k), k) = \delta^{-1}$$

Assuming the existence of a locally unique steady state k^* and linearizing the Euler equation around k^* easily shows that the sum and product of the characteristic roots satisfy

$$\begin{aligned} \mathcal{T} &= -[\delta(T_{11}^* + T_{13}^*) + T_{22}^*]/\delta T_{12}^* \\ \mathcal{D} &= \delta^{-1} + T_{23}^*/\delta T_{12}^* \end{aligned}$$

with $T_{ij}^* = T_{ij}(k^*, k^*, k^*)$ and $T_{12}^* \neq 0$. It follows easily that $T_{23}^*/T_{12}^* < 0$ is a necessary condition for the occurrence of local indeterminacy. As in two-sector optimal growth models, the sign of T_{12}^* is ruled by the capital intensity difference at the private level. However, the sign of T_{23}^* is difficult to establish. Since $-T_2^*$ is equal to the price p of the capital good in terms of the price of the consumption good, we only know that

$$T_{23}^* = -\partial p / \partial A$$

Boldrin and Rustichini [9] provide formal conditions for local indeterminacy but it remains difficult to interpret these conditions in terms of the fundamentals.[23] In particular, although one may conjecture that local indeterminacy is compatible with a capital intensive investment good at the private level, there is no clear picture concerning the requirements on the capital intensity difference.

More recently, Drugeon [11] considers a discrete-time two-sector model with general technologies containing sector-specific and intersectoral external effects. Contrary to Boldrin and Rustichini he assumes constant returns

[23]See also Venditti [27] for more detailled conditions on local indeterminacy and local bifurcation of periodic cycles.

at the private and social levels by using production functions which are linear homogeneous with respect to private factors and homogeneous of degree zero with respect to public factors. Moreover, developing a methodology based on the *equilibrium production frontier*,[24] he provides an expression for the characteristic polynomial in terms of elasticities of factor substitution in each sectors and shares of consumption, investment, wage and profits into national income. While local indeterminacy still requires a capital intensive consumption good at the private level, his main results are the following: with strong sector-specific external effects, local indeterminacy requires strong substitutability in the investment good sector and weak substitutability in the consumption good sector.[25] When strong intersectoral externalities are considered, a continuum of equilibria occurs if substitutability is high in the consumption good sector and low in the investment good sector.

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[24] This corresponds to the function $T(k_t, k_{t+1}, A_t)$ evaluated along an equilibrium path.

[25] This result is in some sense close to Theorem 6 in Section 4.2.

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