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Abstract

Available characterizations of the various notions of stochastic dominance concern continuous random variables. Yet, discrete random variables are often used either in pedagogical presentations of stochastic dominance or in experimental tests of this notion. This note provides complete characterizations of the various notions of stochastic dominance for discrete random variables.

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1   Introduction

The notion of stochastic dominance is a key concept in social sciences. It is especially relevant in the economic theory of risk bearing as well as in the analysis of inequalities. To the best of our knowledge, while the definition of stochastic dominance applies to general random variables, available characterizations make the assumptions of a differentiable utility function as well as a continuous density function.

There are two reasons why a characterization would be useful for the case of a discrete random variable. First, pedagogical presentations of stochastic dominance often rely on an informal discretization argument whereas a rigorous proof would be more satisfying. For a remarkable pedagogical presentation of first- and second-order stochastic dominance, see for instance Elton et al. (2003). Their presentation uses a discrete distribution (pages 241-247) while the proofs use a continuous distribution (page 254-255). Chavas (2004) is also a very good source on this issue. Chavas even provides expressions in the case of a discrete distribution from which one could characterize first- and second-order stochastic

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dominance (see Chavas (2004), pages 56-60 and page 65). However, for the case of third-order stochastic dominance he gives an expression of the difference of expected utilities which seems to be only an approximation of the continuous case (see Chavas (2004), page 60, "discrete implementation")). This prevents one from having a characterization of third-order stochastic dominance in the discrete case. Hence, a general argument seems to be needed[1].

Second, experimental tests of stochastic dominance make use of discrete random variables and discrete variants of characterizations available in the continuous case (see, e.g., Batalio et al. (1985)[2], Birnbaum (2005) and Birnbaum and Navarette (1998)). Here again, a proof that the analogy is indeed relevant seems to be missing.

This note presents characterizations of stochastic dominance for discrete random variables and provides a rigorous support for the usual practice of using a discrete random variables setting.

This note is organized as follows. Some definitions and notations are presented in section 2. We present a characterization of first- and second-order stochastic dominance (these are the notions which are usually considered) in section 3. A characterization of the kth-order stochastic dominance is offered in the appendix. The proof uses an induction argument.

2   Definitions and Notations

Let $u : \mathbb{R} \to \mathbb{R}$ be a utility function. Consider a list of real numbers such that: $x_1 < x_2 < ... < x_n$. We set $u_i = u(x_i)$, i=1,..., n, and $u = (u_1, ..., u_n)$.

Let U_1 denote the class of increasing utility functions on $\{x_1, x_2, ..., x_n\}$, i.e. functions u satisfying

$$\frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \geq 0 \ \ \text{for all} \ \ i, i = 1, .., n - 1.$$

Set

$$u_i^1 = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}; \ \ i = 1, ..., n - 1,$$

and $u^1 = (u_1^1, ..., u_{n-1}^1)$. $u^1 \neq 0$ means that there exists i such that $u_i^1 \neq 0$.

Let U_2 denote the class of utility functions u which are in U_1 and safisfy [3]:

$$\frac{(u_{i+2} - u_{i+1})}{(x_{i+2} - x_{i+1})} - \frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \leq 0 \ \ \text{for all} \ \ i, i = 1, .., n - 2.$$

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[1]Chambers and Quiggin (2004), page 113, have already stressed the interest of studying simple mean preserving spreads in the discrete case. As they put it, "..., most discussion of stochastic dominance revolves around continuous probability distributions. We, therefore, felt it important to develop these ideas in some depth for the discrete case...". Nevertheless, these authors do not address the case of stochastic dominance per se. The notion of increasing risk and mean preserving spread have also been presented for discrete random variables by Rothschild and Stiglitz (1970) (see also, Leshno, Levy and Spector (1997)).

[2]Batalio et al. (1985) contains an early test of second-order stochastic dominance.

[3]The functions defined on an interval of $\mathbb{R}$ satisfying this property are the concave ones. See e.g, Aliprantis and Border (1999), page 179.

We consider two probability functions $f(x)$ and $g(x)$ where the random variable x can take the n possible values $x_1, ..., x_n$. We define $f_i = f(x_i)$ and $g_i = g(x_i)$. We assume that $f \neq g$.

Let F and G be their distribution functions respectively. We set $F_i = F(x_i) = \sum_{j=1}^{i} f(x_i)$ and $G_i = G(x_i) = \sum_{j=1}^{i} g(x_i)$, $i = 1, ..., n$. Notice that $F_1 = f_1$, $F_n = 1$, $G_1 = g_1$, $G_n = 1$. Finally, we let E_f and E_g be the expectation operators based on f and g respectively.

3 Characterization of first- and second-order stochastic dominance

We first recall Abel's Lemma[4], which is a discrete version of the summation by parts formula.

Lemma 1 *(Abel's Lemma)*
Let $a_1, ..., a_n, b_1, ..., b_n$ be real numbers. Set $A_i = \sum_{j=1}^{i} a_j$ and $B_i = \sum_{j=1}^{i} b_j$. Then

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n.$$

We now give the definition of stochastic dominance.

Definition 1 *We say that f first-order stochastically dominates g (and we write $f \geq_{FSD} g$) if $E_f u(x) \geq E_g u(x)$ for all $u \in U_1$.*

Let $D^1 = G - F$, $D_i^1 = D^1(x_i) = G(x_i) - F(x_i) = G_i - F_i$. The following Proposition characterizes the first-order stochastic dominance property.

Proposition 1 *(First-order stochastic dominance).*
$f \geq_{FSD} g$ if and only if $D_i^1 \geq 0$ for all $i = 1, ..., n-1$.

Proof. By definition, $E_f u(x) - E_g u(x) = \sum_{i=1}^{n}(f_i - g_i)u_i$. Set $f_i - g_i = a_i$, $u_i = b_i$, $F_i - G_i = A_i$ and apply Lemma 1. So:

$$
\begin{aligned}
E_f u(x) - E_g u(x) &= \sum_{i=1}^{n-1}(F_i - G_i)(u_i - u_{i+1}) + (F_n - G_n)u_n \\
&= \sum_{i=1}^{n-1}(-D_i^1)(u_i - u_{i+1}) \\
&= \sum_{i=1}^{n-1} D_i^1(u_{i+1} - u_i) \quad\quad (1)
\end{aligned}
$$

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[4]See, e.g., http://planetmath.org/encyclopedia/AbelsLemma.html.

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since $F_n - G_n = 0$.

It is clear that if $D_i^1 \geq 0$ for all $i = 1, ..., n-1$ then $f \geq_{FSD} g$. Let us show the converse. Suppose that $f \geq_{FSD} g$ so $\sum_{i=1}^{n-1} D_i^1(u_{i+1} - u_i) = E_f u(x) - E_g u(x) \geq 0$ for all $u \in U_1$. If there exists j such that $D_j^1 < 0$, then consider u where $u_i = u_{i+1}$, $\forall i \neq j$, and $u_{j+1} > u_j$. We have $u \in U_1$ and $E_f u(x) - E_g u(x) = D_j^1(u_{j+1} - u_j) < 0$ which contradicts the hypothesis. Q.E.D.

We now turn to the case of second order stochastic dominance.

Definition 2 *We say that f second-order stochastically dominates g (and we write $f \geq_{SSD}$ g) if $E_f u(x) \geq E_g u(x)$ for all $u \in U_2$.*

The following Proposition characterizes the second order stochastic dominance property.

Proposition 2 *(Second-order stochastic dominance)*
$f \geq_{SSD} g$ if and only if $D_i^2 = \sum_{j=1}^{i} D_j^1(x_{j+1} - x_j) \geq 0$ for all $i = 1, ..., n-1$.

Proof. From Proposition 1,

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-1} D_i^1(u_{i+1} - u_i)$$

$$= \sum_{i=1}^{n-1} D_i^1(x_{i+1} - x_i) \frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \quad (2)$$

We shall use again Lemma 1 by setting

$$D_i^1(x_{i+1} - x_i) = a_i, \quad \frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} = b_i$$

so that $A_i = \sum_{j=1}^{i} a_j = \sum_{j=1}^{i} D_j^1(x_{j+1} - x_j)$.

Applying now Lemma 1 gives:

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-2} \left(- \sum_{j=1}^{i} D_j^1(x_{j+1} - x_j) \right) \left(\frac{(u_{i+2} - u_{i+1})}{(x_{i+2} - x_{i+1})} - \frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \right)$$

$$+ \left(\sum_{j=1}^{n-1} D_j^1(x_{j+1} - x_j) \right) \frac{(u_n - u_{n-1})}{(x_n - x_{n-1})}$$

$$= \sum_{i=1}^{n-2} D_i^2 \left(\frac{(u_{i+2} - u_{i+1})}{(x_{i+2} - x_{i+1})} - \frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \right) + D_{n-1}^2 \frac{(u_n - u_{n-1})}{(x_n - x_{n-1})} \quad (3)$$

It is clear that if $D_i^2 \geq 0$ for all $i = 1, ..., n-1$ then $f \geq_{SSD} g$. Let us show the converse. Suppose that $f \geq_{SSD} g$ that is $E_f u(x) - E_g u(x) \geq 0$ for all $u \in U_2$. If there exists j such that $D_j^2 < 0$, then consider u where $u_i = u_{i+1}$, $\forall i \geq j+1$, $u_{j+1} > u_j$ and $u_{i+1}^1 = u_i^1$, $\forall i \leq j-1$

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that is $\frac{(u_{i+2}-u_{i+1})}{(x_{i+2}-x_{i+1})} = \frac{(u_{i+1}-u_i)}{(x_{i+1}-x_i)}$, $\forall i \leq j-1$. We have $u \in U_2$ and $E_f u(x) - E_g u(x) = -D_j^2 \left(\frac{-(u_{j+1}-u_j)}{(x_{j+1}-x_j)} \right) < 0$ which contradicts the hypothesis.
Q.E.D.

A characterization of the kth-order stochastic dominance is offered in the appendix.

4 Concluding Remarks

In this section we shall give two remarks concerning characterization of stochastic dominance with alternative definitions.

Let U_1^s denote the class of strictly increasing utility functions i.e. functions u satisfying

$$u_i^1 = \frac{(u_{i+1}-u_i)}{(x_{i+1}-x_i)} > 0 \ \ \text{for} \ \ \text{all} \ \ i, i = 1,..,n-1.$$

Remark 1. Some authors (for instance, Chavas (2004), Elton et al. (2003)) use the following definition:

Definition 3 *We say that f first-order-s- stochastically dominates g (and we write $f \geq_{FsSD} g$) if and only if $E_f u(x) \geq E_g u(x)$ for all $u \in U_1^s$.*

The following Lemma, whose proof is in this appendix, shows that definitions 1 and 3 are essentially similar.

Lemma 2 *$f \geq_{FsSD} g$ if and only if $f \geq_{FSD} g$.*

The next Proposition characterizes definition 3 of stochastic dominance.

Proposition 3 *(First-order-s stochastic dominance).*
$f \geq_{FsSD} g$ if and only if $D_i^1 \geq 0$ for all $i = 1,...,n-1$.

Proof. This is an immediate consequence of Lemma 2 and Proposition 1. Q.E.D.

The main reason for using definition 1 instead of definition 3 is that the proofs of the characterizations in the high-order case are simpler[5].

Remark 2. One could also propose to strengthen the definition of stochastic dominance in requiring that the inequality be strict. More precisely, we could use the following definition:

Definition 4 *We say that f strictly first-order stochastically dominates g (and we write $f >_{FSD} g$) if and only if $E_f u(x) > E_g u(x)$ for all $u \in U_1^s$.*

Using an argument similar to that of Proposition 3 we can show that
$f >_{FSD} g$ if and only if $D_i^1 \geq 0$ for all $i = 1,...,n-1$ with $D_j^1 > 0$ for some $j \in i,...,n-1$.

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[5]Moreover, definition 1 is implicit in some classic textbooks, e.g. Huang and Litzenberger (1988) (page 40), and Ingersoll (1987) (page 137). For instance, in order to show the only if part of the characterization in Proposition 3, these authors make use of a non-decreasing function.

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Appendix

A1) Characterization of kth-order stochastic dominance.

Set

$$u_i^k = \frac{u_{i+1}^{k-1} - u_i^{k-1}}{x_{i+1} - x_i} \quad k = 1, ..., n-1 \; ; \quad i = 1, ..., n-k,$$

$$u_i^0 = u_i, \quad i = 1, ..., n$$

We let U_k denote the class of functions u such that $(-1)^{r+1} u_i^r \geq 0$, $r = 1, ..., k$, $i = 1, ..., n-k$.
let

$$D_i^k = \sum_{j=1}^{i} D_j^{k-1}(x_{j+1} - x_j) \quad i = 1, ..., n-1; \quad k = 2, ..., n,$$

$$D_i^1 = G_i - F_i \quad i = 1, ..., n$$

Definition 5 *We say that f k-th-order stochastically dominates g (and we write $f \geq_{kSD} g$) if and only if $E_f u(x) \geq E_g u(x)$ for all $u \in U_k$.*

We shall need the following Lemma.

Lemma 3 *For every $k = 1, ..., n-1$, we have:*

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k} (-1)^{k+1} D_i^k (x_{i+1} - x_i) u_i^k + \sum_{r=2}^{k} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}.$$

Proof. The proof is by induction. We know from the previous propositions that it is true till $k = 2$.(Recall also that $D_n^1 = 0$, this is why r starts at 2.) Suppose it is true for k and let us show it is true for $k + 1$ provided $k \leq n - 2$:

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k} (-1)^{k+1} D_i^k (x_{i+1} - x_i) u_i^k + \sum_{r=2}^{k} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}$$

Apply Lemma 1 by setting:

$$(-1)^{k+1} D_i^k (x_{i+1} - x_i) = a_i, \quad u_i^k = b_i,$$

so

$$A_i = \sum_{j=1}^{i} a_j = \sum_{j=1}^{i} (-1)^{k+1} D_j^k (x_{j+1} - x_j) = (-1)^{k+1} D_i^{k+1}.$$

This gives:

$$\sum_{i=1}^{n-k} a_i b_i = \sum_{i=1}^{n-k-1} A_i (b_i - b_{i+1}) + A_{n-k} b_{n-k},$$

so:

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k-1} (-1)^{k+1} D_i^{k+1} (u_i^k - u_{i+1}^k) + (-1)^{k+1} D_{n-k}^{k+1} u_{n-k}^k + \sum_{r=2}^{k} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}.$$

and

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k-1} (-1)^{k+1} D_i^{k+1}(x_{i+1} - x_i) \frac{(u_i^k - u_{i+1}^k)}{(x_{i+1} - x_i)} + \sum_{r=2}^{k+1} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}.$$

Therefore:

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k-1} (-1)^{k+2} D_i^{k+1}(x_{i+1} - x_i) u_i^{k+1} + \sum_{r=2}^{k+1} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}.$$

and the proof is complete. Q.E.D.

The characterization of the k-th-order stochastic dominance property relied on the following remark. From now on, we assume that $k \geq 2$.

Remark. Let $z \geq 2$ and j be fixed. Consider a function $u : \mathbb{R} \to \mathbb{R}$ such that

$$u_{i+1} = u_i \text{ for } i \geq j+1$$
$$u_{j+1} > u_j$$
$$u_i^{z-1} = u_{i+1}^{z-1} \text{ for } i \leq j-1$$

We claim that this function is in U_z.

Indeed we have $u_i^m = 0$ for $1 \leq m \leq z$, $j+1 \leq i \leq n-m$.

We also have: $u_j^1 > 0$, $u_j^2 = \frac{u_{j+1}^1 - u_j^1}{x_{j+1} - x_j} = \frac{-u_j^1}{x_{j+1} - x_j} < 0$ and for $m \leq z$, $u_j^m = \frac{-u_j^{m-1}}{x_{j+1} - x_j} = \frac{(-1)^{m-1}(u_{j+1} - u_j)}{(x_{j+1} - x_j)^m}$ so $(-1)^{m+1} u_j^m = (-1)^{m+1} \frac{(-1)^{m-1}(u_{j+1} - u_j)}{(x_{j+1} - x_j)^m} > 0$.

Moreover for $i \leq j-1$, $(-1)^z u_i^{z-1} > 0$ since $u_i^{z-1} = u_j^{z-1}$ for $i \leq j-1$.

Now we can show that for $1 \leq i \leq j-1$, $(-1)^{z-1} u_i^{z-2} > 0$. Since $u_{j-1}^{z-2} = u_j^{z-2} - u_j^{z-1}(x_j - x_{j-1})$, we have $(-1)^{z-1} u_{j-1}^{z-2} = (-1)^{z-1} u_j^{z-2} - (-1)^{z-1} u_j^{z-1}(x_j - x_{j-1}) > 0$ and we can compute recursively u_i^{z-2} for $1 \leq i \leq j-2$.

In a similar way we can compute recursively u_i^m for $i \leq j-1$, $2 \leq m \leq z-3$ to show that $(-1)^{m+1} u_i^m > 0$.

Proposition 4 *(k-th-order stochastic dominance)*
$f \geq_{kSD} g$ *if and only if* $D_i^k = \sum_{j=1}^{i} D_j^{k-1}(x_{j+1} - x_j) \geq 0$ *for all* $i = 1, ..., n-k+1$ *and* $D_{n-r+1}^r \geq 0$ *for all* $r = 2, ..., k-1$.

Proof. From the previous Lemma we have:

$$E_f u(x) - E_g u(x) = \sum_{i=1}^{n-k} (-1)^{k+1} D_i^k(x_{i+1} - x_i) u_i^k + \sum_{r=2}^{k} (-1)^r D_{n-r+1}^r u_{n-r+1}^{r-1}.$$

It is clear that if $D_i^k \geq 0$ for all $i = 1, ..., n-k+1$ and $D_{n-r+1}^r \geq 0$, for all $r = 2, \cdots, k-1$, then $f \geq_{FSD} g$.

Let us show the converse. Assume then that $f \geq_{kSD} g$ that is $E_f u(x) - E_g u(x) \geq 0$ for all $u \in U_k$. Let us first show that $D_i^k \geq 0$, $\forall i \leq n-k$. If this is false, there exists j, with

$D_j^k < 0$, $1 \leq j \leq n - k$. We now use the function u presented in the above remark with $z = k$. We have:

$$E_f u(x) - E_g u(x) = D_j^k (-1)^{k+1} (x_{j+1} - x_j) u_j^k < 0$$

since $(-1)^{k+1} u_j^k > 0$ (u being in U_k). But this contradicts the assumption.

Now, let us show that $D_{n-r+1}^r \geq 0$, $\forall r = 2, \cdots, k$. If not, there exists p such that $D_{n-p+1}^p < 0$, $2 \leq p \leq k$. We use once again the function presented in the above remark with $z = p$, $j = n - p + 1$. So $u_{i+1} = u_i$ for $i \geq n - p + 2$, $u_{n-p+2} > u_{n-p+1}$ and $u_i^{p-1} = u_{i+1}^{p-1}$, $1 \leq i \leq n - p$.

Thus we obtain $u_i^p = 0$, for $i \leq n - p$, $(-1)^p u_{n-p+1}^{p-1} > 0$, and $u_i^m = 0$, $1 \leq m \leq p - 2$, $n - p + 2 \leq i \leq n - m$ (only if $p > 2$).So we have:

$$E_f u(x) - E_g u(x) = (-1)^p D_{n-p+1}^p u_{n-p+1}^{p-1} < 0$$

since $(-1)^p u_{n-p+1}^{p-1} > 0$ (u being in U_k). But this contradicts the assumption. Q.E.D.

A2) Proof of Lemma 2.

Proof. Recall that $E_f u(x) - E_g u(x) = \sum_{i=1}^{n-1} D_i^1(u_{i+1} - u_i)$. Set $m = n - 1$ and for every $i = 1, ..., m$, $y_i = u_{i+1} - u_i$ and $z_i = D_i^1$. Let $y = (y_1, ..., y_m) \in \mathbb{R}^m$, $z = (z_1, ..., z_m) \in \mathbb{R}^m$. $y > 0$ means $\forall i = 1, ..., m$, $y_i > 0$. With these notations it suffices to show that the following propositions are equivalent:

$$(i) \quad \forall y, \ y > 0 \Longrightarrow \sum_{i=1}^{m} y_i z_i \geq 0.$$

$$(ii) \quad \forall y, \ y \geq 0 \Longrightarrow \sum_{i=1}^{m} y_i z_i \geq 0.$$

We can write (i) and (ii) in the following way:

$$(i) \quad \mathbb{R}_{++}^m \subset \{y \in \mathbb{R}^m / \sum_{i=1}^{m} y_i z_i \geq 0\}$$

$$(ii) \quad \mathbb{R}_+^m \subset \{y \in \mathbb{R}^m / \sum_{i=1}^{m} y_i z_i \geq 0\}$$

- (ii) $\Longrightarrow$ (i) since $\mathbb{R}_{++}^m \subset \mathbb{R}_+^m$
- (i) $\Longrightarrow$ (ii). Indeed if $\mathbb{R}_{++}^m \subset \{y \in \mathbb{R}^m / \sum_{i=1}^{m} y_i z_i \geq 0\}$ then $\{y \in \mathbb{R}^m / \sum_{i=1}^{m} y_i z_i \geq 0\}$ being closed we have $\overline{\mathbb{R}_{++}^m} \subset \{y \in \mathbb{R}^m / \sum_{i=1}^{m} y_i z_i \geq 0\}$. But $\mathbb{R}_+^m = \overline{\mathbb{R}_{++}^m}$ so (ii) follows. Q.E.D