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From Velocities to Fluxions

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Newton¹ reached the main results that would later constitute his theory of fluxions between the end of 1663 and the Fall of 1666. Many notes dating back to this period have been conserved, and D. T. Whiteside has published them in the first volume of Newton's *Mathematical Papers* ([22], I). They can be used to reconstruct the evolution of Newton's ideas at the very beginning of his mathematical researches and his progressive achievements².

In none of these notes does the term 'fluxions' appear. Newton used it for the first time in the *De Methodis*, which he probably composed in the winter of 1670-71 ([22], III, pp. 3-372) but never published during his life³. The role that this term plays in this treatise and in the later presentations of Newton's theory is, *mutatis mutandis*, played in his first notes by several other terms like 'motion', 'determination of motion', and 'velocity'.

Though the *De Methodis* results, for its essential structure and content, from a re-elaboration of a previous unfinished treatise composed in the Fall of 1666—now known, after Whiteside, as *The October 1666 tract on fluxions* ([22], I, pp. 400-448)—, the introduction of the term 'fluxion' goes together with an important conceptual change concerned with Newton's understanding of his own achievements. I shall argue that this change marks a crucial

¹I thank Annalisa Coliva, Mary Domski, Massimo Galuzzi, Michael Friedman, Christian Houzel, Andrew Janiak, Vincent Jullien, Sébastien Maronne, Eric Schliesser, George Smith and Josu Zabaleta for valuable comments and/or suggestions.

²This is what I have done in [25]. The present paper develops some points I have made in this book.

³The *De Methodis* first appeared, in an English translation by J. Colson, in 1736: cf. [21].

step in the origins of analysis, conceived as an autonomous mathematical theory.

In section 1, I shall distinguish three different senses in which the term ‘analysis’ can be used in historical contexts concerned with classic and early-modern mathematics. This will allow me to clarify what I mean by speaking of the origins of analysis conceived as an autonomous mathematical theory. This is what I suggest we call ‘Eulerian analysis’, a term I shall clarify by contrasting it with ‘Aristotelian analysis’ and ‘Vietian analysis’.

In section 2, I shall compare, in the light of the distinctions introduced in section 1, the senses in which Newton speaks of analysis in the *De analysis* (presumably written in 1669) and in the *De methodis*, and argue that what he calls, in the latter, ‘field of analysis’ is much more extended than the domain of application of the analytical techniques described in the former.

In section 5, I shall argue for the main thesis of my paper, namely that Newton’s field of analysis is, in fact, the original kernel of Eulerian analysis. My main point will be that fluxions were conceived by Newton as abstract quantities related to other abstract quantities called ‘fluents’, whereas that which he called ‘motion’, ‘determination of motion’ or ‘velocity’ in his previous notes were understood as (scalar components of) punctual speeds of motions generating particular geometric magnitudes, typically segments.

In order to clarify this point, I shall reconstruct, in sections 3 and 4, some of Newton’s arguments and achievements concerning motion dating back to the years 1664-1666. This will allow us to appreciate the evolution of his ideas on this matter up to the *October 1666 tract*, and also make a comparison with the new approach of the *De methodis* possible.

Namely, in section 3, I shall consider Newton’s proof of a theorem showing an intrinsic link between the problems of tangents and normals and the problem of areas for curves referred to a system of Cartesian co-ordinates. This proof manifests a crucial idea that Newton will henceforth never abandon, that of considering related geometric magnitudes as generated by motions whose punctual speeds are mutually dependent on each other. But this theorem is also relevant in connection with a claim made in another note, according to which—when these motions are rectilinear and the generated segments are related by a polynomial equation and are taken as Cartesian co-ordinates of a curve—the problem of determining the ratio of (the scalar components of) their speeds is equivalent to the problems of tangents and normals for this curve. It follows that, for curves like these, these last problems and the problem of areas are connected with appropriate problems concerned with

motion.

In section 4, I shall show how Newton tackles and responds to the question of knowing whether this link holds also in general for any sort of curves. The (positive) response will come through his researches into Roberval's method of tangents. Newton succeeded in unifying this method in a unique, quite general proposition (proposition 6 of the *October 1666 tract*) concerned with the trajectory of the intersection point of two rigid curves that move separately from each other. This is a modality of composition of motion to which any other modality involved in Roberval's method can be reduced. Hence, Newton's theorem provides a recursive rule that can be applied to find tangents for any curve described by a composed motion. In the light of this proposition, the connection between the problems of tangents, normals and areas and appropriate problems concerned with motions—which Newton had shown to hold for curves expressed, with respect to a system of Cartesian co-ordinates, by a polynomial equation—appears to be a particular case of a more fundamental and general connection. This is the base of Newton's theory of fluxions. This theory appeared as such, when Newton, in the *De methodis*, replaced the motions of lines with the variation of fluents, conceived, as said, as abstract quantities.

Finally, in section 6, I shall address some conclusions by discussing, in quite general terms, the links of this theory with Newton's natural philosophy.

1 Analysis

The term 'analysis' is highly polysemic. In order to understand the point I would like to make, it is necessary to distinguish three different senses in which it is habitually used by historians of mathematics. These senses reflect three different ways in which this term and its translated forms and cognates have been used by mathematicians up to the 18th century. They do not of course exhaust the spectre of significations that it has taken and continues to take in mathematics and related fields.

In the first of these senses, 'analysis' refers to a pattern of argumentation largely used in Greek, Arabic and early modern mathematics—especially geometry—, often (but not always) in the context of the application of a twofold method, called 'the method of analysis and synthesis'. In order to avoid misunderstandings, call this pattern of argumentation 'Aristotelian analysis'. This appellation is justified, since Aristotle used '*ἀνάλυσις*' and

its cognates in this sense on different occasions⁴.

Aristotelian analysis is the common pattern of any argument which is based on the consideration of something that is not actually available as if it were available. Aristotle's clearest example (*Nicomachean Ethics*, III, 5) is deliberation: this is an argument that comes back from an imaginary situation that one aims to obtain to the actual one, so as to suggest a way for obtaining the former by operating on the latter.

Pappus' classical description of the method of analysis and synthesis and the corresponding distinction between theorematic and problematic analysis (*Mathematical Collection*, VII, 1-2) clearly refer to Aristotelian analysis.

According to Pappus, a theorematic analysis applies when a certain proposition has to be proved. It consists in deducing from it an accepted principle, a proved theorem, or their negations.

A problematic analysis applies, instead, when a geometrical problem asking for the construction of a geometric object satisfying certain spatial conditions relative to other given objects, is advanced⁵. One begins by supposing that this problem is solved and representing its solution through a diagram involving both the given and the sought after objects. Then, by reasoning about this diagram, and possibly by extending it through licensed constructions, one isolates a configuration of given objects and known data concerned with them, based on which the sought after objects can be constructed and thus the problem solved⁶.

⁴For example in: *Posterior Analytics*, 78a 6-8, 84a 8, 88b 15-20; *Sophistical Refutations*, 175a 26-28; *Metaphysics*, 1063b 15-19; *Nicomachean Ethics*, 1112b 20-24. For a discussion of these passages and a reconstruction of Aristotle's views on analysis, cf. [23], 370-383 and 395.

⁵An example is the following: suppose that two straight lines, two points on them and a third point outside them are given (in position); find a straight line from this point that intersects the given ones so as to cut on them—together with the given points on them—two segments that stand to each other in a given ratio. This is the problem considered in Apollonius' *Cutting-off of a Ratio*.

⁶To be a little bit more precise, consider the relevant problem as a configuration $\mathcal{C}_{g,?}$ constituted by a system \mathbf{O}_g of geometric objects which are taken as given (in the example mentioned in footnote (5), the two given straight lines and the three given points), an amount \mathbf{D} of data (in the example, the given ratio), and a characterisation $O_?$ of some objects to be constructed based on \mathbf{O}_g and \mathbf{D} (in the example, the sought after straight line, or better, the points at which it has to intersect the given ones). The analysis begins by supposing that the problem is solved. This is the same as supposing that some objects satisfying $O_?$ are given. The configuration $\mathcal{C}_{g,?}$ can thus be represented by a diagram representing both the objects included in \mathbf{O}_g and the objects satisfying $O_?$. Insofar as these

Both Pappus's theorematic and problematic analyses are reductions. A theorematic analysis in Pappus' sense can provide *ipso facto* a proof (by *reductio ad absurdum*) if that which is deduced is the negation of an accepted principle or a proved theorem. This case apart, both a theorematic and a problematic analysis, as described by Pappus, are preliminary arguments suggesting another and conclusive argument, generally called 'synthesis': a theorematic analysis suggests a valid proof; a problematic one suggests an admissible construction.

There is no doubt that Pappus' theorematic and problematic analyses are both forms of Aristotelian analysis. Still, they are not the only possible forms that Aristotelian analysis has actually taken in classical, medieval and early-modern mathematics. Another relevant form of Aristotelian analysis occurring in classical, medieval and early-modern mathematics applies when a certain geometrical problem, asking for the construction of a geometric object satisfying certain purely quantitative conditions, is advanced⁷. In this case, the analysis aims to transform this condition into another equivalent but different one capable of suggesting a way for constructing the sought after objects⁸. Also in this case, the analysis is a reduction. But it is now the reduction of a given problem to a new and equivalent, yet still distinct, one⁹.

last objects are not taken as given and the solution is only supposed, such a diagram cannot be completely obtained by applying the licensed constructive clauses, but is partially freely traced, so as to represent the relevant spatial relations between the relevant objects and to reflect the data. By reasoning about it and possibly by extending it through licensed constructions, one isolates a sub-configuration \mathcal{C}_g based on which the objects satisfying $O_?$ can be constructed.

⁷An example is the classical problem of finding two mean-proportional segments between two other given ones.

⁸To be a little bit more precise, consider the relevant problem as a configuration $\mathcal{C}_{g,?}$ constituted by a system \mathbf{Q}_g of given quantities (in the example mentioned in footnote (7), the two given segments), and a characterisation $Q_?$ of some other quantities to be determined (that is, calculated or constructed) based on \mathbf{Q}_g (in the example the sought after mean-proportional segments). In this case, the analysis needs no diagram and, rather than isolating a sub-configuration \mathcal{C}_g of $\mathcal{C}_{g,?}$, transforms the latter in a new configuration $\mathcal{C}'_{g,?}$ constituted by a system \mathbf{Q}'_g of given quantities that can be determined based on the quantities included in \mathbf{Q}_g , and a new characterisation $Q'_?$ of the same quantity characterised by $Q_?$ (in the example, the condition $a : x = x : y = y = b$ is possibly transformed in the system of proportions $a : x = x : y$ and $x : y = y = b$ providing the symptomata of two parabolas an intersection of which determines the sought after segments).

⁹For a more comprehensive description of these two forms of Aristotelian analysis applied to mathematical problems, cf. [27].

This last form of Aristotelian analysis may also apply if the relevant problems are not stated using the symbolic language introduced by Viète and Descartes and the related formalism. The possibility of appealing to some crucial theorems included in the *Elements* (especially in books II, V and VI) is enough for allowing the required transformations¹⁰. Still, this form of Aristotelian analysis naturally applies to the solution of problems stated by means of equations using this formalism. In this case, it consists in appropriate transformations of these equations according to the rules of such a formalism. Viète’s *Zeticorum Libri* ([33])¹¹ contains many examples of this form of Aristotelian analysis. This is because, after Viète, it became quite usual to employ the term ‘analysis’ to refer—rather than to a pattern of argumentation—to the formalism or family of techniques that these transformations depend on. In order to avoid misunderstandings, call this formalism ‘Vietian analysis’.

Under this meaning, the term ‘analysis’ is often used in early modern mathematics as a synonym for ‘algebra’, another highly polysemic term. For the sake of simplicity, I shall not use this term in the present paper, and I shall use the adjective ‘algebraic’ in a modern sense, as opposed to ‘transcendent’.

Newton’s theory of functions and Leibniz’s differential calculus are largely dependent on Vietian analysis, which occurs in them under the form that it takes in Descartes’ *Geometry* ([5]). They can even be viewed as appropriate extensions of it. The development of these theories went together with other, and partially independent, extensions of Vietian analysis, for example those connected with power series expansions. From this process, the crucial notion of function emerged and acquired a quite central role in mathematics. In his *Introductio in analysin infinitorum* ([10]), Euler launched a foundational program aimed at a reformulation of any mathematical theory within the general frame of a theory of functions, defined as appropriate expressions

¹⁰Another nice example of this possibility is found in Thābit ibn Qurra’s treatise on the “restoring of the problems of algebra through geometrical demonstrations” [cf. [17]; a French translation of Thābit’s treatise is provided by the conjunction of the three quotations inserted in [15], 33-34, 37-38 and 41]. The first of the three second-order equations of al-Khwārizmī is here understood as the problem of looking for a segment x such that $S(x) + R(a, x) = S(b)$, where a and b are two given segments, $S(x)$ and $S(b)$ are the squares constructed on them, and $R(a, x)$ is the rectangle constructed on a and x . The appeal to proposition II.6 of the *Elements* is enough for allowing Thābit to transform this problem into that of looking for the segment x such that $S(b) + S\left(\frac{a}{2}\right) = S\left(x + \frac{a}{2}\right)$, which can be easily solved using the Pythagorean theorem.

¹¹A recent very comprehensive study of Viète’s treatise is [11].

expressing abstract quantities¹². The following part of my paper will be devoted to a partial clarification of this notion of abstract quantity through the reconstruction of the intellectual path that led Newton to the connected notion of fluxion. For the time being, it is enough to say that in the first half of the 18th-century, the term ‘analysis’ and its cognates begun to be used to refer to a general theory of functions conceived as abstract quantities, and to some of its features and connected developments. In order to avoid misunderstandings, call this theory ‘Eulerian analysis’. It is to this form of analysis that I refer when I claim that the conceptual change that goes together with Newton’s introduction of the term ‘fluxion’ in the *De Methodis* is a crucial step in the origins of analysis, conceived as an autonomous mathematical theory.

2 From the *De analysis* to the *De methodis*¹³

On June 20, 1669, Isaac Barrow, at that time Lucasian Professor of Mathematics at Cambridge, replied to Collins, who had sent to him a copy of Mercator’s *Logarithmotechnia* ([18]), with these words ([13], I, 13; cf. also [22], II, 166 (footnote 11), and [34], 243):

A friend of mine here, that hath a very excellent genius to those things, brought me the other day some papers wherein he hath sett downe methods of calculating the dimension of magnitudes like that of Mr. Mercator concerning the hyperbola, but very generall; as also of resolving æquations; which I suppose will please you.

Ten days later, Barrow sent to Collins an example of this genius: a short treatise that is today known as the *De analysis per Æquationes Numero Terminorum Infinitas* ([22], II, 206-247). Collins made a copy of it circulated it. As a result, the young Newton and some of his early results became known in the English scientific community, though he did not allow the publication of his treatise before 1711, when it appeared, in fact, as a piece of history

¹²For a clarification of Euler’s notion of function as I understand it, and some related bibliographical references, cf. [26].

¹³For the factual pieces of information contained in this section, cf. [34] and the critical apparatus of [22], vols. II and III.

([20]), in order to support the thesis of Newton's priority in the famous *querelle* with Leibniz.

Because of its circulation among the members of Collins's circle, the *De analysis* is often considered as the first public presentation of Newton's theory of fluxions. This is not properly correct, however. It is rather a sort of instant book, which Newton wrote to expound only some of his results: those equivalent or similar to Mercator's.

After presenting two rules ([22], II, 206-210) for squaring curves expressed by equations of the form

$$y = ax^\lambda + bx^\mu + cx^\nu + \&c. \quad (1)$$

where λ, μ, ν, \dots are rational exponents and '&c.' means that the right-side member is either a finite or an infinite sum, he devotes the main part of his treatise to the detailed exemplification of a third rule ([22], II, 211-213; for the examples, cf. *ibid.*, 212-242)¹⁴:

If the value of y or of some of its terms is more composed than the previous ones, it should be reduced to simpler terms by operating on letters in the same way as the arithmeticians get decimal numbers by division, extract roots and solve equations.

To say it more explicitly, Newton supposes that curves be expressed by algebraic equations $F(x, y) = 0$ of different forms, and shows how to operate on these equations so as to transform all of them into equations of the form (1), by applying to literal expressions procedures derived, by generalisation or infinitary extension, from the arithmetic rules used for calculating with numbers.

Finally, he considers some mechanical curves, like the cycloid, and shows how to express also these curves by means of (infinitary) equations of the form (1), through the application of some appropriate yet peculiar tricks.

The term 'analysis' and its cognates occur quite seldom in Newton's treatise ([22], II, 206, 222, 240, 242), and always to refer to, or to speak of Vietian

¹⁴I quote Whiteside's translation. Here is Newton's original: [[22], II, 210-212]: "Sin valor ipsius y vel aliquis ejus terminus sit præcedentibus magis compositus, in terminos simpliciores reducendus est, operando in literis ad eundem modum quo Arithmetici in numeris decimalibus dividunt, radices extrahunt, vel Æquationes solvunt."

analysis¹⁵. One could say, however, that the *De analysis* includes several examples of Aristotelian analysis performed through Vietian analysis. They allow for the expression of different families of algebraic curves and some transcendent curves by equations of the form (1). This makes it possible to apply to these curves the following rules of quadrature:

$$y = ax^\lambda \Rightarrow \mathcal{A}(y) = \frac{a}{\lambda+1}x^{\lambda+1} \quad ; \quad \mathcal{A}(y+z) = \mathcal{A}(y) + \mathcal{A}(z)$$

(where ‘ $\mathcal{A}(w)$ ’ denotes the area of the trapezoid delimited by the curve of Cartesian orthogonal ordinate w , which, in modern terms, corresponds to $\int_0^x w(t) dt$, supposing that $w(0) = 0$).

This is merely a small fragment of the huge amount of mathematical results that Newton had obtained between 1663 and 1666. Despite that, on October 29, 1669, based on the samples of his competence offered in the *De analysis* and in some other short notes that he had probably showed to Barrow, Newton was appointed as Lucasian Professor of Mathematics at the University of Cambridge, to replace Barrow himself.

Hence, though attracted by other topics, like natural philosophy, spectral colours and alchemy, he could not refuse Barrow and Collins’ invitation to prepare some additions to be annexed to the Latin edition of Kinckhuysen’s *Algebra* ([16]), which Mercator had just translated from Dutch. On July 11, 1670 Newton was convinced that he had finished his work and sent it to Collins. But Collins had the bad idea of sending it back to Newton with the request of some further clarifications about the roots of binomials. He never received back either of these clarifications or the old version of Newton’s additions.

On September 27, 1669, Newton informed Collins that he had decided to replace his additions with a new treatise, which he wrote, in fact, but did not finish before 1683 ([22], V, 54-532). Then he kept silent until July, 20th, 1670 when he sent a letter to Collins including the following passage ([13], I, 66; cf. also: [22], II, 288, and III, 5 and 32 (footnote 1), and [34], 268):

¹⁵Cf., for example, the following quotation ([22], II, 242 and 240): “And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method [the method of quadrature previously expounded] may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.” [“Et quicquid Vulgaris Analysis per æquationes ex finito terminorum numero constantes (quando id sit possibile) perficit, hæc per æquationes infinitas semper perficiat: Ut nil dubitaverim nomen Analysis etiam huic tribuere.”]

The last winter [...] partly upon Dr Barrow’s instigation I began to new methodiz the discourse of infinite series, designing to illustrate it with such problems as many (some of them perhaps) be more acceptable then the invention it selfe of working by such series. But [...] I have not yet had leisure to returne to those thoughts, & I feare I shall not before winter. But since you informe me there needs no hast, I hope I may get into the hummour of completing them before the impression of the introduction, because if I must helpe to fill up its title page, I had rather annex something which I may call my owne & which may bee acceptable to Artist as well as the other to Tyros.

This “something acceptable to Artist” that Newton was planning to annex to Kinckhuysen’s *Algebra* was just the *De Methodis*: a treatise quite different from the *De analysis* in which he aimed to expound his own new theory and all of its extensions.

Here is the how the treatise begins ([22], III, 33)¹⁶:

Observing that the majority of geometers, with an almost neglect of the ancient’s synthetic methods, now for the most part apply themselves to the cultivation of analysis and with its aid have overcome so many formidable difficulties that they seems to have exhausted virtually everything apart from the squaring of curves and certain topics of like nature not yet elucidated: I found it not amiss, for the satisfaction of learners, to draw up the following short tract in which I might at once widen the boundaries of the field of analysis and advance the doctrine of curves.

Though he seems reluctant to admit that the “neglect of the ancient’s synthetic methods” is a symptom of progress, Newton clearly inscribes his own results into the “field of analysis”. But it seems to me that he is no longer speaking of Vietian analysis, as in the *De Analysis*: his aim is no longer to show how the problem of quadratures can be solved by series for any algebraic

¹⁶I quote Whiteside’s translation. Here is Newton’s original: [[22], III, 32]: “Animadvertenti plerosque Geometras, posthabitâ fere Veterum syntheticâ methodo, Analyticæ excolendæ plurimum incumbere, et ejus ope tot tantasque difficultates superasse ut pene omnia extra curvarum quadraturas et similia quædam nondum penitùs enodata videantur exhaustisse: placuit sequentia quibus campi analytici terminos expandere juxta ac curvarum doctrinam promovere possem in gratiam discentium breviter compingere.”

curve and for some mechanical ones, based on a preliminary transformation. As a matter of fact, the methods expounded in the *De analysis* are also expounded in the *De methodis*, and a new quite powerful one of the same kind—the so-called method of Newton’s parallelogram—is added to them. But these methods are now conceived as nothing but preliminary material only concerned with some “modis computandi” ([22], III, 70).

After having expounded them, Newton writes ([22], III, 71)¹⁷:

It now remains, in illustration of this analytical art, to deliver some typical problems and such especially as the nature of curves will present.

It seems thus that, for him, the “analytic art” does not merely consist in some appropriate techniques to be used in preparing the solution of some problems, but is rather concerned with these problems as such, and thus also with their solutions. It has taken on a peculiar form, and at the least is no longer Aristotelian or Vietian analysis.

This extension of the “field of analysis” is not independent of an appropriate reduction of these problems to other ones. But this reduction is no longer the mere reduction of a certain configuration of given and ungiven quantities to a new and more suitable one. It is rather a transformation of the very nature of these problems.

This is in fact a double reduction. Firstly, problems concerned with curves are reduced to problems concerned with motion. Secondly, problems concerned with motions are reduced to problems concerned with fluxions. The former reduction was already at work in the *October 1666 tract*. The theory expounded in that treatise is, indeed, a theory of motions and speeds to be used to solve geometrical problems: the aim of this treatise is of showing how to solve geometrical problems by motion. The latter reduction, however, is new and constitutes the essential novelty of the *De methodis*, which I would like to emphasise.

Let us clarify this matter.

Here is how Newton describes the former reduction ([22], III, 71)¹⁸:

¹⁷I quote Whiteside’s translation. Here is Newton’s original: [[22], III, 70]: “Jam restat ut in illustrationis hujus Artis Analyticæ tradam aliquot Problematum specimina qualia præsertim natura curvarum ministrabit”

¹⁸I quote Whiteside’s translation. Here is Newton’s original: [[22], III, 70]: “Sed imprimis observandum venit quod hujusmodi difficultates possunt omnes ad hæc duo tantùm

But first off all I would observe that difficulties of this sort may all be reduced to these two problems alone, which I may be permitted to propose with regard to the space traversed by any local motion however accelerated or retarded:

1. Given the length of the space continuously (that is, at every [instant of] time), to find the speed of motion at any time, proposed.
2. Given the speed of motion continuously, to find the length of the space described at any time proposed.

As a matter of fact, the language used by Newton to state these problems is more general than that used in the *October 1666 tract*. Here is, indeed, how these problems are stated in the propositions 7 and 8 of this treatise ([22], I, 402-403):

7. Having an Equation expressing the relation twixt two or more lines x , y , z &c: described in the same time by two or more moving bodys A , B , C , &c: the relation of their velocities p , q , r , &c may bee thus found, viz: [...]¹⁹

8 If two Bodys A & B , by their velocitys p & q describe the line x & y . & an Equation bee given expressing the relation twixt one of the lines x , & the ratio $\frac{p}{q}$ of their motion p & q ; To find the other line y .

problemata reduci quæ circa spatium motu locali utcunque accelerato vel retardato descriptum proponere licebit. 1. Spatij longitudine continuò (sive ad omne tempus) data, celeritatem motûs ad tempus propositum invenire. 2. Celeritate motûs continuò datâ longitudinem descripti spatij ad tempus propositum invenire.”

¹⁹Newton supposes that the equation expressing the relation between x , y , z , &c. is polynomial; the suspension points stand, thus, for the description (in fact for three equivalent but different descriptions) of the well-known algorithm that, in the simplest case of two variables, leads from

$$\sum_{i=0}^n \sum_{j=0}^i A_{i-j,j} x^{i-j} j^j = 0$$

to

$$\sum_{i=0}^n \sum_{j=0}^i (i-j) A_{i-j,j} x^{i-j-1} j^j p + \sum_{i=0}^n \sum_{j=0}^i j A_{i-j,j} x^{i-j} j^{j-1} q = 0.$$

The difference between these statements and those of the *De Methodis* seems to be quite relevant: by avoiding the supposition that the spaces described are to each other in a relation expressed by a polynomial equation²⁰, Newton seems to transform two problems concerned with the transformation of polynomial equations—that is, two algorithmic problems belonging to Vietian analysis—into two genuinely geometrico-mechanical problems. The comparative consideration of propositions 1-6 of the *October 1666 tract* ([22], I, 400-402) and the second reduction that Newton performs in the *De methodis* suggests, however, a quite different picture. Before considering this second reduction, and in order to understand its real meaning, it is thus necessary to consider these propositions more carefully.

They aim to provide a quite general theory of composition of motions that is completely independent of the possibility of expressing the relation of the spaces described by means of algebraic equations. When looked at in light of this theory, the algorithms involved in the subsequent propositions 7 and 8 thus appear as local tools to be used in this theory in some particular situations for determining appropriate ratios or relations. The purpose of the next two sections is to reconstruct the essential aspects of this theory and the evolution in thought that led Newton to it.

3 Motions and Geometry

Newton's first appeal to motions and their properties for proving geometrical theorems and solving geometrical problems occurs in a note composed in the Summer of 1664 ([22], I, 219-233; for the dating of this note, cf. [25], 183-184), after his reading of the second Latin edition of Descartes' *Géométrie* ([8]).

In this edition, Descartes' treatise is supplemented by a large number of commentaries, other treatises on connected topics, and notes. Among this material, there is a letter of H. van Heuraet ([8], I, 517-520), containing an important theorem about quadratures and rectifications: If *AML* (fig. 1) and *END* are such that, for every point *P* taken on their common

²⁰Cf. footnote (19), above.

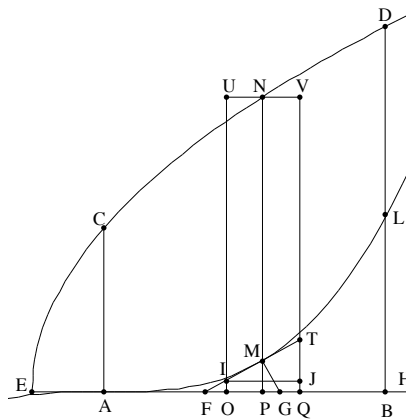


Figure 1

axis EH, their ordinates PM and PN comply with the following proportion

$$PM : MG = K : PN, \quad (2)$$

where MG is the normal to AML in M, and K any constant segment, then the trapezoid ABDNC is equal to the rectangle constructed on K and another segment equal to the arc AML.

In his note, Newton applies a slightly modified version of this theorem: if PM and PN are such that

$$PM : PG = K : PN, \quad (3)$$

where PG is the sub-normal to AML in M, then the trapezoid ABDNC is equal to a rectangle constructed on K and BL.

Van Heuraet's theorem allows one to rectify some curve provided that the areas of some other appropriate curve is known. Newton's modified version allows one to square some curve, provided that the normal or tangent of some other appropriate curve is known. More generally, it provides an intrinsic link between the problem of tangents and the problem of quadratures.

The two theorems can be proved in the same way, by a simple application of the method of indivisibles. Suppose that $OQ = IJ$ is an indivisible portion of the base AB and remark that

$$PM : MG = IJ : IT \quad \text{and} \quad PM : PG = IJ : JT.$$

Then, compare these proportions with the proportions (2) and (3) respectively, so as to derive that

$$R(IJ, PN) = QVUO = R(IT, K) \quad \text{and} \quad R(IJ, PN) = QVUO = R(JT, K),$$

where, for any pair of segments α and β , $R(\alpha, \beta)$ is the rectangle constructed on these same segments. Finally, sum up all the rectangles such as QVUO, $R(IT, K)$, and $R(JT, K)$, and get the theorems. It is essentially in this way that Van Heuraet proves his theorem.

Newton's argument for proving the second theorem ([22], I, 222-229) is quite different. He refers to another figure (fig. 2), where the segment K is identified with the constant base $DB = AQ$ of the rectangle DBCE and the ordinates PM and PN of the curves YV and ZW are, as before, such that the

proportion (3) holds, supposing that PG is the sub-normal to YV relative to M . Then he remarks ([22], I, 228-229²¹):

[...] supposing the line PN always moves over the same superficies in the same time, it will increase in motion from QL in the same proportion that it decreaseth in length and the line DB will move uniformly from EC , soe that the space $ECBD = NPQL$ ²².

The statement ‘it will increase in motion from QL in the same proportion that it decreaseth in length’ makes manifest that Newton is here understanding motions as scalar quantities, that is, as (scalar components of) punctual speeds. He seems to take for granted that which is the main object of the previous argument through indivisibles, namely, the equality of the elements of the rectangle $ECBD$ and the trapezoid $NPQL$. Then, he appeals to motions for proving that which is taken for granted in this argument, namely, that the equalities of elements entails the equality of the whole figures. Instead of appealing to an infinite sum of indivisibles or infinitely small elements, he considers figures as generated by motion (in the usual sense of this term) and admits that the relations of these figures depends on the relations of the punctual speeds of these motions.

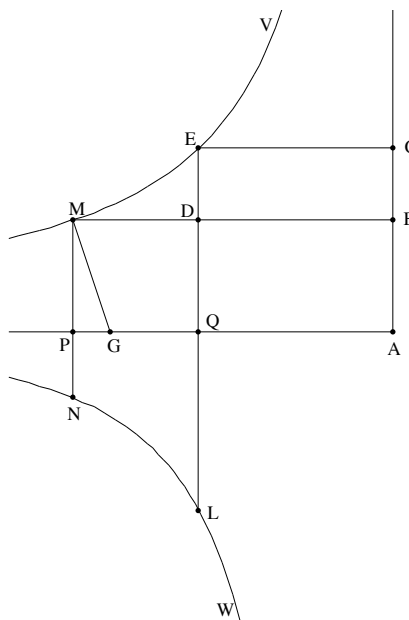


Figure 2

²¹For reasons of uniformity, I change the letters used by Newton to refer to the points in the diagram.

²²Note that $ECBD$ is the rectangle constructed on $AQ = K$ and the difference of the ordinates QE and PM is relative to the limit points of the trapezoid $NPQL$. It follows that the equality $ECBD = NPQL$ expresses, with respect to the curves represented in figure 2, the same result that, with respect to the curves represented in figure 1, is expressed by the claim that the trapezoid $ABDNC$ is equal to a rectangle constructed on K and BL .

This style of argument will not often be repeated by Newton in his later notes. Still, the central idea will never be abandoned: that of considering related geometric magnitudes as generated by motions whose punctual speeds mutually depend on each other and vary according to an appropriate rate corresponding to the geometric relations of these magnitudes.

As this example shows quite well, Newton uses the term ‘motion’ and its cognates (overall the verb ‘to move’) in two distinct senses: to refer both to motions of points or lines in our sense of this term, and to the punctual “determination” of these motions. The term ‘determination’ as related to motions had already been used by Descartes, Fermat and Hobbes, in different senses ([5], 17-18; [6], II, 55-58; [7], II, 41, 99; [9], letters XCVI, CXI, CCXX, CCXXX, CCXXXIV, DXXI), and Newton will use it later on different occasions (cf. [22], I, 372, for the first occurrence). When motions are rectilinear, their determination, in Newton’s sense, reduces to the scalar component of the punctual speed, since their directional component is constant and there is no need to take it into account. But things go in a quite different way when these motions are not rectilinear.

For the time being, let us consider only the simplest case, that of rectilinear motions. I shall come back on the case of curvilinear motions in section 4.

Let x and y be two variable segments generated at the same time by two points moving according to a rectilinear motion. The relation of these segments in any instant of time depends on the (scalar components of the) punctual speeds of these motions. But also the reciprocal is true: for these segments to be related by a certain relation, the (scalar components of the) punctual speeds of these motions have to satisfy some appropriate conditions. Hence, two problems arise quite naturally: *i*) Given the relation of x and y , to look for the (scalar components of the) punctual speeds of the motions that generate them; *ii*) Given the (scalar components of the) punctual speeds, to look for the relation of x and y . These are just the two problems that Newton states in the *De Methodis*. But why are they relevant for the solution of geometrical problems concerned with curves?

A first answer comes, implicitly, from a short note probably redacted at the beginning of the Fall of 1665 ([22], I, 343-347), where these problems are stated and the first of them is solved, in the particular case where both the relation between the segments and that between (the scalar components of) their punctual speeds are expressed by polynomial equations in two variables. Here is what Newton writes ([22], I, 344):

1. If two bodies **c**, **d** [fig. 3] describe the streight lines **ac**, **bd**, in the same time, (calling **ac** = x , **bd** = y , p = motion of **c**, q = motion of **d**) & if I have an equation expressing the relation of **ac** = x & **bd** = y whose termes are all put equal to nothing. I multiply each term of the equation by so many times py or $\frac{p}{x}$ as x hath dimensions in it. & also by soe many times qx or $\frac{q}{y}$ as y hath dimensions in it. the sumē of these products is an equation expressing the relation of the motions of **c** & **d**. [...]

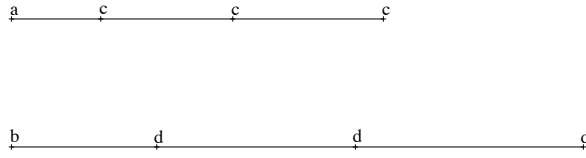


Figure 3

2. If an equation expressing the relation of their motions bee given, tis more difficult & sometimes Geometrically impossible, thereby to find the relation of the spaces described by these motions.

The algorithm described in the first proposition is the well-known direct algorithm that leads from any polynomial equation

$$\sum_{i=0}^n \sum_{j=0}^i A_{i-j,j} x^{i-j} j^j = 0 \quad (4)$$

to the equality

$$\frac{q}{p} = - \frac{\sum_{i=0}^n \sum_{j=0}^i (i-j) A_{i-j,j} x^{i-j-1} j^j}{\sum_{i=0}^n \sum_{j=0}^i j A_{i-j,j} x^{i-j} j^{j-1}}. \quad (5)$$

This is a particular case of the algorithm described in proposition 7 of the *October 1666 tract*²³. If we interpret it within the formalism of differential calculus, as we know it today, this algorithm allows us to pass from any polynomial equation $P(x, y) = 0$ to the equality

$$\frac{q}{p} = - \frac{\frac{\partial P}{\partial x}}{\frac{\partial P}{\partial y}}.$$

²³Cf. footnote (19), above.

Still, no compact and general notion equivalent to the partial derivatives of a polynomial is available for Newton at this stage of his researches. Hence, this algorithm is for him nothing but a rule to transform a polynomial $P(x, y)$ into an appropriate ratio of associated polynomials that is taken to express the ratio $\frac{q}{p}$ of (the scalar components of) the punctual speeds of the rectilinear motions generating the segments x and y .

In a note written about one year earlier ([22], I, 236-238), Newton had claimed that the product of y and the ratio of polynomials providing the right-hand side of the equality (5)—also described, of course, as the result of an appropriate transformation of an equation like (4)—provides the sub-normal on the x -axis and at the generic point (x, y) of the curves expressed, with respect to a Cartesian orthogonal system of co-ordinates, by this same equation. From this claim and the equality (5), it follows that

$$\frac{q}{p} = \frac{sn._x [P(x, y)]}{y}, \quad (6)$$

where $sn._x [P(x, y)]$ is just this sub-normal.

Though Newton did not state explicitly this equality in his note of Fall of 1665, at this date he was certainly aware of it. Once compared with the result about the intrinsic link between the problem of tangents or normals and the problem of quadratures that Newton had obtained some months earlier by modifying the theorem of van Heuraet, this equality provides a way to connect these two geometric problems with the problems of speeds, in the case of curves referred to a system of Cartesian co-ordinates and expressed by polynomial equations.

Suppose that x and y are the orthogonal Cartesian co-ordinates of a curve, and that they stay to each other in a certain relation R . If this relation is expressed by a polynomial equation $P(x, y) = 0$, from the equality (6) it follows that the problems of tangents and normals can be solved by passing from this relation to the ratio $\frac{q}{p}$ according to the equality (5) and rewriting the right-hand side of this equality in terms of only one of the two variables x and y . Moreover if one sets $AP = x$, $PM = y$, $PN = z$ (fig. 1 or fig. 2), from the equality (6), it follows that the condition (3) transforms into $z = K \frac{q}{p}$. Hence, according to Newton's version of the theorem of van Heuraet, the problem of squaring the curve of orthogonal Cartesian co-ordinates x and z can be solved by passing from the relation R^* that links these co-ordinates to each other to a polynomial equation $P(x, y) = 0$ such that $\frac{q}{p} = \frac{z}{K}$. If this

is so, the trapezoid delimited by this curve, taken between the abscises $x = \xi$ and $x = \kappa$, is indeed equal to²⁴ $K |y_\kappa - y_\xi|$.

The only difficulty that possibly arises in to the solution of the former of these problems, when R is expressed by a polynomial equation $P(x, y) = 0$, is that of rewriting the right-hand side of the equality (5) in terms of only one of the two variables x and y . If the relation R^* is given somehow, the difficulty that possibly arises in to the solution of the latter problem is that of finding an appropriate polynomial $P(x, y)$, provided that there is one (which is of course not warranted, in general).

Two classical problems concerned with curves—the problem of tangents or normals and the problem of quadratures—are thus reduced, under appropriate restrictive conditions, to problems concerned with punctual speeds of rectilinear motions which are, in turn, equivalent to algorithmic problems belonging to the field of Vietian analysis. But if these conditions are not met, are the problem of tangents or normals and the problem of quadratures also connected in some ways with problems concerned with punctual speeds of rectilinear motions?

To answer this question, it is relevant to know how the equality (6) is obtained. Did Newton merely get, in two distinct ways, two coincident algorithms (the algorithm of the tangents or normals and the algorithms of speeds)? Or did he understand in general—based on geometrical-mechanical arguments, independent of any equation—that the ratio of the (scalar components of the) punctual speeds of the generative motions of two segments y and x is equal to the ratio of the sub-normal on the x -axis and the ordinate of the curve of orthogonal Cartesian co-ordinates x and y ? If the latter possibility obtained, then Newton knew, in the Fall of 1665, that the connection between the problems of tangents, normals and quadratures and problems concerned with punctual speeds of rectilinear motions—which is manifested by the equality (6) together with his version of the theorem of van Heuraet—does not depend on the way the relevant curves can be expressed with respect to a system of Cartesian co-ordinates. If the former possibility obtained, then he could not avoid to wonder if this connection also holds for curves that, though referred to such a system of co-ordinates, cannot be expressed by polynomial equations.

No direct evidence is available for deciding among these possibilities. Still, it is perhaps relevant to remark that, in his *Geometrical lectures* ([2]), Bar-

²⁴Cf. the footnote (22), above.

he was not only aware that equality (6) is only a particular case of the much more general equality 8, but he also knew a simple way to prove this last equality. In the printed version of Barrow's lecture this theorem is proved as follows. If it is considered as fixed, the tangent TM also results from the composition of two motions: the same motion of the straight line az from which the curve results, and the motion of the point t that moves uniformly on the same straight line az starting from T and so as to take the position M when az takes the position LN . The trajectory of a composed motion such as those that describe both the curve and its tangent depends only on the ratio of the (scalar components of the) punctual speeds of the motions that compose it, and is a straight line if and only if this ratio is constant ([2], 28 and [3], 49-50). Hence, one can suppose, without any loss of generality, that the motions of az and t are uniform. This being admitted, consider two positions of az : the position L^*N^* on one side of LN , such that the point m takes the position O^* which is between the position K^* of the point t and the point G^* at which az (in position L^*N^*) cuts PM ; and the position $L^{**}N^{**}$ on the other side of LN , such that the point m takes the position O^{**} which is beyond the position K^{**} of the point t which is in turn beyond the point G^{**} at which az (in position $L^{**}N^{**}$) cuts PM . Also admit that these positions are such that the curve does not change its concavity and has no extreme between them²⁷. When the straight line az is in the first of these positions, the (scalar components of the) punctual speed of the motion of m along it is smaller than that of the motion of the point t also along it, since the former speed is increasing whereas the latter is uniform, and the space O^*G^* covered by m in a certain time²⁸ is smaller than the space K^*G^* covered by t in the same time. For an analogous reason (considering the motions as going in the opposite directions²⁹), when the straight line az is in the second of these positions, the (scalar components of the) punctual speed of the point m along it is greater than that of the motion of the point t also along it. It follows

²⁷As a matter of fact, Barrow does not make this restrictive condition explicit. Still, such a condition is clearly required by his argument, and, as a consequence, this argument does not apply if M is an extreme or inflection point.

²⁸Barrow openly considers this time as being "represented" by the segment G^*M .

²⁹This condition is implicitly expressed by Barrow through the identification of the relevant time with the segment MG^{**} (as a matter of fact, Barrow, in setting out the second part of his argument, takes MG^{**} to be a time, rather than merely representing it), which is now described in the opposite direction than G^*M : cf. the previous footnote (28).

that, when \mathbf{az} is in the position LN, these speeds are equal, which is enough to prove the theorem³⁰.

4 Newton and Roberval's Method of Tangents

Though no similar proof occurs in Newton's notes, the way he succeeds in showing that the equality (6) is nothing but a particular case of a much more general result has a lot of affinities with Barrow's arguments. Still, Newton goes much farther than Barrow, since he also shows that the problem of tangents is intrinsically connected with some appropriate problems concerning motions (either rectilinear or not) and their punctual speeds even when the relevant curves are not referred to any system of Cartesian co-ordinates. This became possible when he became aware of Roberval's method of tangents³¹.

In 1665, this method was known in France by some mathematicians³², but had not yet been presented in any published text³³. This only happened in 1693, when a treatise written by a pupil of Roberval, François de Bonneau, Sieur de Verdus ([4]), appeared. This treatise certainly communicated notes taken from Roberval's lectures. Though Newton never mentions neither this treatise, nor the name of Roberval, the content of some of his notes leaves no doubt that he had somehow become acquainted with his method³⁴.

³⁰The constant ratio of the (scalar components of the) punctual speeds of the motions of \mathbf{az} and of \mathbf{t} on this last straight line is, indeed, equal to the ratio of PM and TP, so that, if the (scalar components of the) punctual speeds of \mathbf{m} at M is equal to the constant one of \mathbf{t} , the ratio of the (scalar components of the) punctual speeds of the motions of \mathbf{az} and of \mathbf{m} on this last straight line, when this last point is in M, is also equal to the ratio of PM and TP.

³¹On Newton and Roberval's method of tangents, cf. [35] which I did not yet know when I wrote my [25].

³²On Roberval's method and his diffusion, cf. [1], 58-77, [14], [28] and [29], 20-23.

³³A similar method had been, however, applied by Torricelli to find the tangent of a problem in [32], 119-121.

³⁴There is no evidence that speaks to the way this method became known to Newton. It was known by Barrow, who spoke of it in a letter to Collins ([30], 34) as a "method of finding the tangents to curved lines by composition of motions" that had been mentioned by Mersenne and Torricelli. This suggests that Barrow became acquainted with it through Mersenne's mention of it in the *Cogitata physico mathematica* ([19], 115-116). But it is also possible that he knew it on some other way, for instance through Hobbes, who was close to Verdus ([31]) and met Roberval himself in 1642 ([1], 72). It is highly plausible that Barrow mentioned this method in one of his lectures. The third of his *Geometrical Lectures* is, indeed, entirely devoted to the composition of motions which is then used, as

Here is how Verdus presents its “*principe d’invention*” ([4], 70)³⁵:

[...] in every [...] curve, the tangent at whatever point is the direction line of the motion of the movable that describes this same line. Hence, in composing some motions in different ways and in knowing the direction of the composed motion at whatever point of a curve, we shall know, at the same time, its tangent.

The problem with this principle is that it does not make clear how the composition of motions is understood, exactly. In fact in Verdus’ treatise at least three different sorts of compositions of motions are considered:

1. A point is submitted to a composed motion if it moves with respect to a system of reference that moves, in turn, with respect to another system of reference.
2. A point is submitted to a composed motion if it is the intersection point of two rigid curves and moves insofar as these curves move separately from each other.
3. A point is submitted to a composed motion if it moves insofar as its distances from two fixed poles, represented by two segments generated by two distinct motions, change at the same time.

Verdus’ treatise expounds the method in general in a rather vague way, and then includes different examples each of which is concerned with one or more of these modalities of composition. In each example, we are told how to find the punctual direction of the composed motion, supposing that both the scalar and directional components of the punctual speeds of the two motions that compound it are known. Still, these modalities are not explicitly distinguished and no general procedure or construction is associated with each of them.

we have seen, to investigate tangents. It is possible that Newton was there, learned the fundamental ideas of this method and some of its paradigmatic examples (Newton’s notes include many examples occurring in Verdus’ treatise), and then elaborated on them by himself.

³⁵The translation is mine. Here is Verdus’ original text: “[...] en toutes les [...] lignes courbes qu’elles puissent estre, leur touchante, en quelque point que ce soit, est la ligne de direction du mouvement qu’a en ce mesme point le mobil qui la décrit. En sorte que composant des mouvemens en diverses façons, et venant à connoistre la direction du mouvement composé en quelque point que ce soit, d’une ligne courbe, nous connoistrions par mesme moyen sa touchante.”

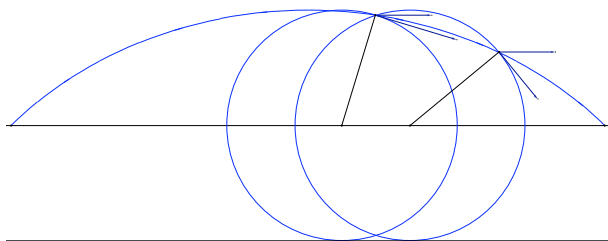


Figure 5

The first case is that of the motions that generate a cycloid and a spiral, provided that these motions are described respectively as the motion of a point on a wheel that advances by rotating on a straight line (fig. 5; this is the motion of a rotating point on a translating plane), and as the motion of a point advancing on a rotating ruler (fig. 6). In the first of these two examples, the second motion is rectilinear. In the second it is not. When it is rectilinear the situation is quite simple: the speed of the point moved according to the composed motion results from the application of the rule of parallelograms to the speeds of the composing motions (fig. 5a).

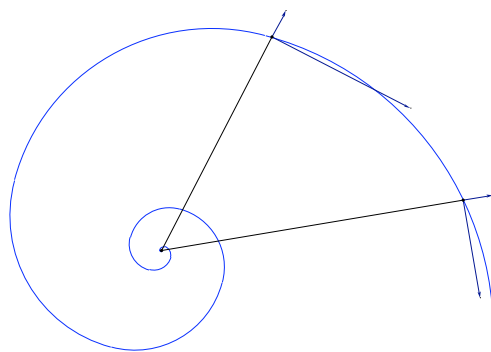


Figure 6

When the second motion is not rectilinear, there is no guaranty that the speed of the point moved according to the composed motion results from the application of the rule of parallelograms, at least if this rule is applied to the speeds of the composing motions. The reason is the following. Suppose that \mathbf{v}_1 and \mathbf{v}_2 are the punctual speeds of the first and the second motion, respectively. If the second motion is not rectilinear, we have no guarantee that \mathbf{v}_1 and \mathbf{v}_2 are also the components of the punctual speed \mathbf{v} of the composed motion along their own directions. The same is true also for the two other cases of composition of motions.

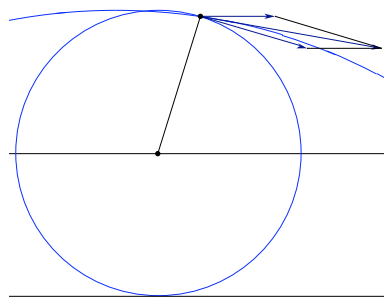


Figure 5a

An example of the second case is the quadratrix, described as the trajectory of the intersection point of two rules, one of which rotates around the vertex of a square while the other translates along the direction of a side

of this square by remaining perpendicular to it (fig. 7). An example of the

third case is the ellipses, described as the locus of the points such that the sum of their distances from two given points is fixed (fig. 8).

Suppose now that a curve \mathfrak{C} is the trajectory of a motion \mathcal{M} composed, in one of the previous three ways, by two other motions \mathcal{M}_1 and \mathcal{M}_2 . Suppose also that these two motions are either rectilinear or circular. In both cases the directions of their punctual speeds \mathbf{v}_1 and \mathbf{v}_2 are known (in the case of a rectilinear motion, this is the same trajectory of the motion; in the case of a circular motion, this is the perpendicular to the radius of this trajectory).

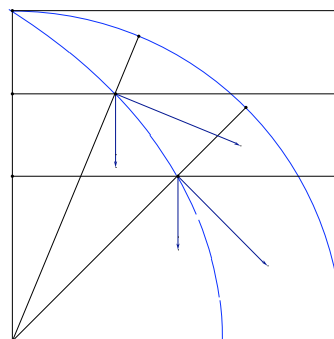


Figure 7

Suppose also that the ratio of the scalar components of these speeds is known as well: they can be represented by two segments s_1 and s_2 which are taken in the same directions of these speeds and that are in such a ratio to one other. To find the tangent of \mathfrak{C} , it is enough to determine the punctual direction of \mathcal{M} . The problem is thus to compose \mathbf{v}_1 and \mathbf{v}_2 in the right way, that is, to find a general construction to be applied to s_1 and s_2 so as to get a straight line that provides such a direction. Once the tangent of \mathfrak{C} is known, this curve can be added to straight lines and circles as a trajectory of motions of which other motions are composed so that the tangent of their trajectory can be found through the same method. And, of course, one can then continue in the same way up to other curves conceived as trajectories of motions composed by other, more and more complex motions.

Roberval treats different cases in different ways. Newton wants, instead, a general principle to be applied in any particular case. Many of the mathematical researches of Newton between the Fall of 1665 and the Spring of 1666 are locating just such a

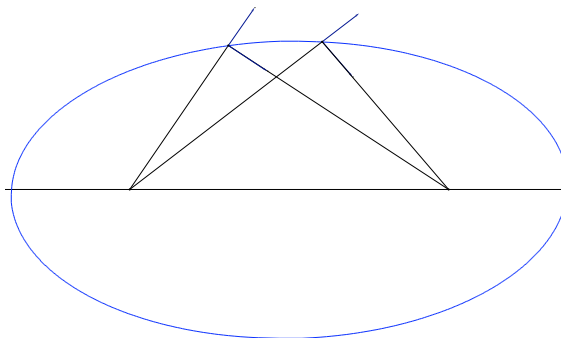


Figure 8

principle. It is finally found, in its definitive and general form, in May of 1665, and it is expounded in two notes (the second of which results from a revision of the first) written on May 14th and 16th ([22], I, 390-392 and 392-399). This same principle is also expounded in proposition 6 of the *October 1666 tract*. Propositions 1-5 of this treatise are merely used to provide the necessary ingredients of this exposition.

It seems that Newton understood that the first and third among the three previous modalities of composition of motions can be reduced to the second one, that is, that there is a way to pass, through appropriate constructions, from the two former cases to the latter (for details, cf. [25], 412-414). It follows that any composed motion can be viewed as the motion of the intersection point of two rigid curves that moves separately. If the tangents of these curves and the (ratio of the) punctual speeds of their respective motions are known, it is moreover easy to find the punctual direction of the composed motion, and thus the tangent of its trajectory, as follows.

Suppose that YM and ZM (fig. 9) are the moving curves and M is their intersection point. Suppose also that MU and MV are the tangents to these curves in the point M , and that the punctual speeds of the motions of these curves are represented (scalarly and directionally)

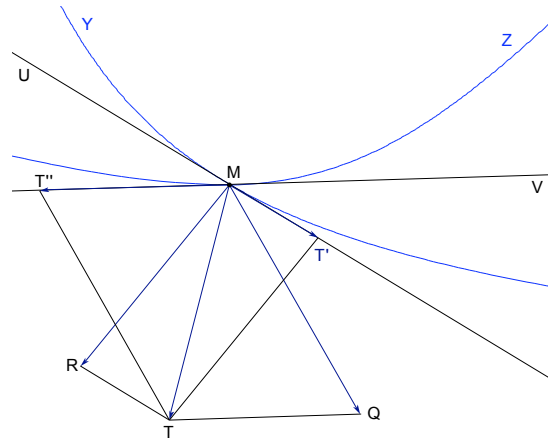


Figure 9

by the segments MR and MQ . It follows that the direction of M is provided by the diagonal MT of the quadrilateral $MRTQ$ which is constructed by drawing from R and Q two parallels lines to the tangents MU and MV , respectively.

The justification is easy. The point M is affected in fact by four motions: the two motions of the curves YM and ZM and the two motions that it has on these curves in order to continue to be their intersection point. The segments MR and MQ represent, respectively, the punctual speeds of the two former motions. The segments $RT = MT'$ and $TQ = MT''$ represent, respectively, the punctual speeds of the two latter ones. By composing these four motions two by two according to the rule of parallelograms, one gets exactly the

direction MT .

Provided that the tangents of straight lines and circles are known, one can easily find, in this way, the tangents of the trajectories of the intersection point of two moving straight lines, two moving circles, or a moving straight line and a moving circle. And again, once this is done, the tangents of the trajectories of the intersection point of two curves corresponding to these trajectories can be found in the same way, and so on.

But, for this to be possible, the ratio of the scalar components of certain speeds has to be determined. And, for that, the algorithm of speeds for segments related by a polynomial equation can be useful.

The simplest case obtains when the two curves YM and ZM reduce to two straight lines each of which translate along the direction of the other (fig. 10). This is just the configuration involved in Barrow's previous proof. But now it is nothing but a particular case of a more general configuration. In such a particular case, each of the two lines provides, then, the punctual direction of the motion of the other and is its own tangent.

Newton's general principle reduces, thus, to the rule of parallelograms (which is consistent with the fact that the motion of the point M can also, in this case, be described as the motion of a point with respect to a system of reference that moves rectilinearly with respect to another system of reference). Hence, if the punctual speeds of these straight lines are represented by the segments MR and MQ , to solve the problem it is enough to construct the rectangle $MRTQ$, for its diagonal MT is the sought after tangent.

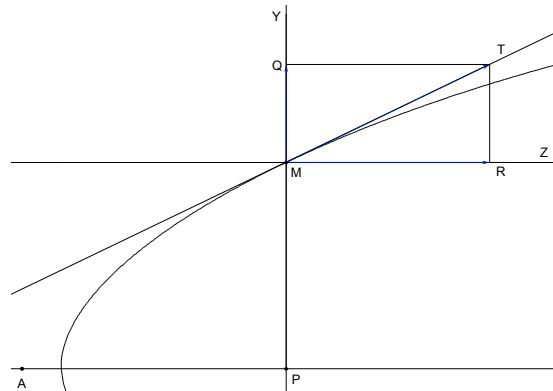


Figure 10

Barrow's result—that is, the equality (7)—is thus quite easily proved as a particular case of a more general result that concerns tangents of curves independently of any system of co-ordinates to which these curves might be referred.

5 Back to the *De Methodis*

With all this in mind, we can now come back to the first reduction of the *De Methodis*, which, recall, consists in reducing geometrical problems concerning curves to two quite general problems concerning motions (cf. p. 12 above). If we compare these problems with propositions 1-8 of the *October 1666 tract*, we find that Newton has eliminated both the general context provided by the theory of composition of motions (propositions 1-6) and the particular assumption that spaces (in the first problem) and speeds (in the second) are linked to each other by a polynomial equation (propositions 7-8: cf. p. 12). So, a question quite naturally arises: What are the spaces and speeds that Newton is speaking of? Are they merely segments generated by rectilinear motions of points and the punctual speeds of these motions (which are nothing but scalar quantities)? Or are they some sort of trajectories of rigid curves and points on these curves and the punctual speeds of them (which cannot be reduced to scalar quantities)?

There is no question that Newton's text is ambiguous. However, Newton offers some clarifications by presenting a quite simple example ([22], III, 73)³⁶:

So in the equation $x^2 = y$ if y designates the length of the space described in any time which is measured and represented by a second space x as it increases with uniform speed: then $2mx$ will designate the speed with which the space y at the same moment of time proceeds to be described.

The letter ' m ' replaces ' p ', here. This is a minor change, but it goes together with two other, more relevant ones. Newton openly supposes that: *i*) the space x is covered by a uniform motion; and *ii*) this space measures and represents time.

The first supposition is not new. It had been used by Barrow³⁷ and Newton himself had at times appealed to it in his earlier notes. And, once

³⁶I quote Whiteside's translation (but maintain the symbol ' m ' instead of replacing it with the symbol ' x ' as Whiteside does by using a notation that Newton will only introduce in 1691: cf. Whiteside's footnotes 83 and 86). Here is Newton's original: [[22], III, 72]: "Sic in æquatione $xx = y$ si y designat spatij longitudinem ad quodlibet tempus quod aliud spatium x uniformi celeritate increscendo mensurat et exhibet descriptam: tunc $2mx$ designabit celeritatem qua spatium y ad item temporis momentum describi pergit [...]."

³⁷Cf. the previous footnotes (28) and (29).

it is admitted, the second—also used by Barrow—seems quite natural. Still, the way Newton employs this supposition reveals that it is not merely a local trick for him. It is rather a symptom of a quite deep change in Newton’s conceptions. Time is not here understood as the real time in which motion takes place; it is merely the second term of an analogy ([12], 19-20). And this is also the case of space. The reason is simple: Newton is no longer referring to the motions of points or lines; he is no longer considering geometrical quantities generated by these motions. He is rather referring to variations of quantities conceived as pure variables. There is no need to insist on this point, since Newton himself is quite clear about it ([22], III, 73)³⁸:

And hence it is that in the sequel I consider quantities as though they were generated by continuous increase in the manner of a space which a moving object describes in its course.

Quantities are, thus, not spaces generated by motions, that is, segments generated by moving points or surfaces generated by moving lines. They are instead that which is “generated by continuous increase” in the same way as space is “generated by motion”. But things are even clearer in what follows³⁹:

However, as we have no estimate of time except in so far as it is expounded and measured by an equable motion, and as, furthermore, only quantities of the same kind may be compared one with another [as well as only] their speeds of increase and decrease [can be compared one with another], I shall, in what follow, have no regard to time, formally so considered, but, among the quantities

³⁸I quote Whiteside’s translation. Here is Newton’s original: [[22], III, 72]: “Et hinc est quod in sequentibus consideratem quantitates quasi generatæ essent per incrementum continuuum ad modo spatij quod mobile percurrendo describit.”

³⁹I slightly modify here Whiteside’s translation. Here is Newton’s original: [[22], III, 72]: “Cùm autem temporis nullam habeamus æstimatione nisi quatenus id per æquabilem motum localem exponitur et mensuratur, et præterea cùm quantitates ejusdem tantùm generis inter se conferri possint et earum incrementi et decrementi celeritates inter se, eapropter ad tempus formaliter spectatum in sequentibus haud respiciam, sed e propositis quantitativibus quæ sunt ejusdem generis aliquam æquabili fluxione augeri fingam cui cæteræ tanquam temporis referantur, adeoque cui nomen temporis analogicè tribui mereatur. Si quando itaque vocabulum temporis in sequentibus occurrat [...] eo nomine non tempus formaliter spectatum subintelligi debet sed illa alia quantitas cujus æquabili incremento sive fluxione tempus exponitur et mensuratur.”

propounded which are of the same kind, I shall suppose some one to increase with an equable flow, and the others to be referred to it as though it were time, so that the name ‘time’ may, by analogy, be conferred upon it. And so, whenever in the following the term ‘time’ occurs [...] by that name should be understood not time formally considered but that other quantity through whose equable increase or flow time is expounded and measured.

If Newton does not speak of trajectories and composed motions, it is, thus, because he does not want to refer to real motions, but rather to a more general kind of change. To say it in Aristotle’s language, he is no longer interested in displacement of points or local change (*φορά*) as such, but rather to a more general kind of change (*κίνησις*) that includes displacement of points as a particular case. Let us call it ‘quantitative variation’. What is this exactly?

From the beginning of 1664—while studying Wallis’ method of quadratures ([22], I, 91-95)—Newton had understood that it is enough, for a certain geometric quantity—typically a segment or a portion of space—to be able to be regarded as a variable, that another geometric quantity be available and be such that the value of the former depends on its value. This latter quantity works then as a parameter for the variation of the former. It is the crucial idea of principle variable.

For a quite long time, Newton seems to have been convinced that the fundamental way to express the relation between a geometric quantity and the parameter of its variation consists in writing an algebraic—typically a polynomial—equation interpreted on these quantities. His work on tangents and quadratures, especially that inspired by Roberval’s method, taught him that this same relation can also be expressed in a quite different, and more general and fundamental, way, by appealing to generative motions and their compositions.

The previous quotation manifests a new, crucial, achievement. It reveals that Newton is not dealing with the variation of geometric quantities—or of any other particular sort of quantities—and with the way of expressing their mutual relations. He is rather dealing with quantitative variation as such, understood as a special kind of change. This is the kind of change characterised by the fact that any particular example of it—let us say \mathfrak{X} —is univocally identified and completely determined insofar as the link that connects it to a principal change of this same kind, on which any other one depends, is determined by means of a law that establishes how this principal

change is reflected in \mathfrak{X} . Let \mathfrak{T} be the principal quantitative variation. This means that a particular quantitative variation \mathfrak{X} is univocally identified and completely determined insofar as an appropriate particular relation $R(\mathfrak{X}, \mathfrak{T})$ is determined. The subjects of \mathfrak{X} and \mathfrak{T} (the entities that are supposed to vary) are not relevant here, and the intrinsic nature of \mathfrak{T} is also not relevant, and, as a matter of fact, could not be determined. This is not the principal quantitative variation because it is uniform. Things go the other way around: \mathfrak{T} is (taken to be) uniform because it is the principal variation.

One could argue that this idea is not new, since that which is described is (in the language of Scholastic) nothing but the change of intensive qualities. This is incorrect, however. Newton seems, indeed, to permute the *definiens* and the *definiendum*: quantitative variation is not defined by appealing to the notion of intensive quality; rather a quantity is conceived, in its abstract generality, as that which is submitted to a quantitative variation. Though quantities are designated by atomic symbols—like ‘ x ’ or ‘ y ’—, they are not the specific objects that these symbols stand for. They are rather that which varies according to the relations that are somehow expressed by appealing to these symbols, for example—but not only—through a polynomial equation. The following passage is quite explicit about this ([22], III, 73)⁴⁰:

But to distinguish the quantities which I consider as just perceptibly but indefinitely growing from others which in any equations are to be looked on as known and determined and are designated by the initial letters a, b, c , &c., I will hereafter call them ‘fluents’ and designate them by the final letters v, x, y and z . And the speeds with which they each flow and are increased by their generating motion (which I might more readily call ‘fluxions’ or simply ‘speeds’) I will designate by the letters l, m, n , and r .

Newton is thus ready for the second reduction. The two previous problems about spaces and speeds can now be re-stated as follows ([22], III, 75 and

⁴⁰I quote Whiteside’s translation, with some minor, essentially typographic changes (among which there is that already mentioned in footnote (36)). Here is Newton’s original: [[22], III, 72]: “Quantitates autem quas ut sensim crescentes indefinitè considero, quo distinguam ab alijs quantitatibus quæ in æquationibus quibuscunque pro determinatis et cognitis habendæ sunt ac initialibus literis a, b, c , &c designantur, posthac denominabo fluentes, ac designabo finalibus literis v, x, y , et z . Et celeritates quibus singulæ a motu generante fluunt et augentur (quas possim fluxiones vel simplicitate celeritates vocitare) designabo literis l, m, n et r .”

83)⁴¹:

Problem 1. Given the relation of the flowing quantities to one another, to determine the relation of the fluxions.

Problem 2. When an equation involving the fluxions of quantities is exhibited, to determine the relation of the quantities one to another.

The explicit reference to equations that occurs both in the previous passage where Newton introduces the terms ‘fluent’ and ‘fluxion’ and in the statement of the second problem is confirmed in the solution of these problems. Though in the statement of the first problem, Newton is speaking in general of a relation between fluents and fluxions, he solves the problem ([22], III, 74-82) under the condition that this relation is expressed through an appropriate equation: namely, either an algebraic equation (either polynomial or not) between the relevant fluents, or an algebraic equation including a variable expressing the area or the length of a curve expressed in terms of one of the relevant fluents⁴². Moreover, in solving the second problem ([22], III, 82-112), he supposes that one or more algebraic equations among the relevant fluents and fluxions are given and shows how to determine one of these fluents in terms of another one through an algebraic, possibly infinitary, algebraic expression.

It could seem, thus, that the generalisation involved in the passage from motions to quantitative variations is immediately thwarted by a new regression to the particularity of algebra merely extended through the appeal to specific geometric quantities as areas and lengths. Things are not so, however. Newton began by considering polynomial equations as the privileged way to express curves with respect to Cartesian co-ordinates, and showed that, if curves are so expressed, the problems of tangents or normals and of

⁴¹I quote Whiteside’s translation. Here is Newton’s original: [[22], III, 74 and 82]: “Prob. 1. Relatione quantitatum fluentium inter se datâ; fluxionum relationem determinare.” “Prob. 2. Exposita æquatione fluxiones quantitatum involvente, invenire relationem quantitatum inter se.”

⁴²Newton’s example ([22], III, 78) is the equation $z^2 + axz - y^4 = 0$, where z is supposed to be the area of the circle referred to a system of Cartesian orthogonal co-ordinates of equation $w = \sqrt{ax - x^2}$. This example is tractable, since Newton proves that $r = m\sqrt{ax - x^2}$ (where r and m are the fluxions of z and x , respectively), that is, according to the differential formalism: $\frac{dz}{dx} = \sqrt{ax - x^2}$ or $z = \int_0^x \sqrt{at - t^2} dt$.

quadratures can be solved through the consideration of rectilinear motions that are taken as the generative motions of these co-ordinates. He passed then from curves so expressed to curves considered as trajectories of composed motions, independently of any particular system of co-ordinates or any sort of equation expressing them, and showed that the possibility of solving the problems of tangents or normals and of quadratures for curves expressed by polynomial equations, through the consideration of rectilinear motions, is nothing but a particular consequence of a more general relation between these trajectories and the components of the relative motions. Finally, he replaced motions with quantitative variations, rectilinear trajectories with fluents, and punctual speeds with fluxions. Still, in this new quite general context, the specification of any particular variation depends on the specification of relations between fluents and fluxions. And, insofar as fluents are not particular sorts of quantities—being rather quantities insofar as they are related to each other—, there is no way to specify these relations by considering particular geometrical or mechanical configurations. Hence, the formalism of Vietian analysis—that is, algebraic equations—returns to take a central role as a privileged way for specifying these relations and thus identifying particular fluents and fluxions. Even the appeal to area or the length appears, in this context, as an easy way to introduce a purely algebraic relation between fluents and fluxions⁴³. Fluent and fluxions are thus, so to say, abstract quantities: quantities conceived as nothing but the subjects of quantitative variations, and Vietian analysis is the tool used to specify these variations.

Of course, the intrinsic limitations of this tool affect the extension of the domain of quantitative variations. Still, through his double reduction, Newton has opened a new field for mathematical investigations. This is what, at the very beginning of the *De methodis*, he calls ‘field of analysis’⁴⁴, that is, the general doctrine of abstract quantities, conceived as I have just said. Though the *De methodis* comes back, after the solution of the two previous

⁴³Cf. the footnote (42), above. The appeal to the area of a circle, in the example considered in this last footnote, is only useful to introduce the following system of equations:

$$\begin{cases} z^2 + axz - y^4 = 0 \\ w = \sqrt{ax - x^2} \\ r = mw \end{cases} .$$

⁴⁴Cf. the quotation appended to footnote (16).

general problems, to the usual geometric problems concerned with curves, this field is largely extended after this first definition, and a large part of the history of mathematics, after Newton's *De methodis*, consisted in efforts to enlarge it, by extending the formalism of Vietian analysis (using, among others things, two crucial ingredients which are already part of the toolbox used in this treatise, namely, infinite series and fluxional—or better, in the language that later became common, differential and integral—equations). Eulerian analysis is just the result of the efforts made to structure this field and to absorb in it other branches of mathematics. Newton's field of analysis can thus be viewed as the original kernel of it.

6 After the *De Methodis* (or Concluding Remarks)

As is well known, Newton will quickly change his mind and devote his mathematical energy to classical geometry and the possibility of extending it without modifying its intrinsic nature by using any extraneous formalism⁴⁵. There are many reasons for this change, some of which are certainly not based on mathematical concerns. Still, the previous story teaches us something that may help us to understand this change, which, as far as I know, has gone unnoticed by commentators.

The theory of composition of motions that Newton elaborated based on his understanding and development of Roberval's method of tangents is a theory according to which punctual speeds are considered as proto-vectorial magnitudes: they have both a scalar and a directional component, which both are relevant for composition. Once real motions are abandoned in favour of quantitative variations, and punctual speeds are replaced by fluxions, only the scalar component is conserved, since, in Newton's theory of composition of motions, the directional one was accounted for only through the positional relations of the motions involved, which were represented by diagrams. Still, the problem of considering directions of motions and speeds in the description of physical phenomena could not be avoided.

Hence, Newton's field of analysis could appear to be the original kernel of an autonomous mathematical theory—like Eulerian analysis will later be—only under the condition that this theory be conceived as a theory of pure

⁴⁵Among many other possible references about this matter, cf. [12], 101-104

scalar relations, capable, at most, of providing a framework for accounting for the relations that physical bodies have to each other because of their intensive qualities. This theory could not provide, as such, a language for describing physical reality, by idealisation, but only a tool for calculating intensive relations of magnitudes whose particular nature and other sorts of relations had to be independently specified. Briefly speaking, the interpretation of the relations between abstract quantities as relations between particular quantities could not develop without a crucial addition of information that this autonomous mathematical theory could not account for. The further development of differential calculus, which allowed for the possibility of changing the principal variable by passing from some differential ratios to others, captured at least part of this information. Together with the introduction of appropriate differential and variational principles, this allowed, during the 18th century, the growth of analytical mechanics ([24]). But in Newton's theory of fluxions, these developments were blocked by the presence of a unique independent variable understood in analogy with time. The appeal to classical geometry—on which his theory of composition of motions was ultimately founded—should thus have appeared to Newton as a condition for using mathematics to speak of the physical world up to a sufficient degree of accuracy. This could, perhaps, partially explain the absence of the theory of fluxions in the *Principia*: this could have, at most, provided a local tool to be used there, but it could not have been, as such, the basic principles of a new natural philosophy⁴⁶.

Still, Newton's field of analysis, became—mostly thanks to mathematicians who did not share Newton's peculiar geometrical outlook—the nucleus of a new form of pure mathematics, whose applications depended on modalities quite different than those proper to classical geometry. This was just 18th-century analytic mathematics. My main aim has been to suggest that Newton has to be considered as one of the main fathers of this form of mathematics—better, as its original, first father.

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⁴⁶To avoid misunderstanding, I repeat myself: this could be, at most, a partial explanation; other reasons for which I cannot account here, are certainly also relevant.

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