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Emmanuelle Taugourdeau, Abderrahmane Ziad. Further Consideration of the Existence of Nash Equilibria in an Asymmetric Tax Competition Game. 2010. halshs-00492098

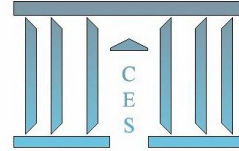
HAL Id: halshs-00492098

<https://shs.hal.science/halshs-00492098v1>

Submitted on 15 Jun 2010

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an Asymmetric Tax Competition Game**

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2010.37



Further Consideration of the Existence of Nash Equilibria in an Asymmetric Tax Competition Game

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January 2010

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[‡]We are indebted to Maurice Salles for his helpful comments.

Abstract

In this methodological paper we prove that the key tax competition game introduced by Zodrow and Mieszkowski (1986) and Wildasin (1988), extended to asymmetric regions, possesses a Nash equilibrium under several assumptions commonly adopted in the literature: goods are supposed to be normal; the public good is assumed to be a desired good; the demand for capital is concave; and the elasticity of the marginal product is bounded. The general framework we develop enables us to obtain very tractable results. By applying our method to several examples with standard production functions, we show that it is easy to use.

Keywords: Nash Equilibrium, Tax Competition.

JEL Classification: C72, H21, H42, R50

Résumé

Dans cet article méthodologique, nous déterminons les conditions suffisantes à l'existence d'un équilibre de Nash dans le modèle de concurrence fiscale développé Zodrow and Mieszkowski (1986) et Wildasin (1988) et généralisé au cas de régions asymétriques : les biens sont supposés normaux, le bien public est un bien « désiré », le demande de capital est concave ; et l'élasticité de la production marginale est bornée. Le cadre général que nous développons permet d'obtenir des résultats facilement exploitables. L'application de notre méthode à plusieurs fonctions de production usuellement utilisées dans la littérature économique montre qu'elle permet de vérifier aisément l'existence de l'équilibre de Nash dans le jeu de concurrence fiscale.

Mots-clef: Equilibre de Nash, Concurrence fiscale.

Classification JEL : C72, H21, H42, R50

1 Introduction

In the established literature on tax competition the existence of a Nash equilibrium is assumed (see Zodrow and Mieszkowski (1986), Wilson (1985, 1986) and Wildasin (1988)). These studies focus on the comparative statics of Nash equilibria, and demonstrate that public services are provided at inefficiently low levels in equilibrium. However, a little attention has been devoted to the question of whether such equilibria do exist. This is for the most part because the demonstration is very difficult, as noted by Laussel and Le Breton (1998): "Both the existence and uniqueness issues are difficult in general and have not been up to now dealt with in the literature. It seems however of primary interest to solve them in order to understand the comparative statics of the equilibrium." Interestingly, an example developed by Iritani and Fujii (2002) showed that there exists at least one pair of utility and production functions for which no Nash equilibrium exists in the Zodrow-Mieszkowski framework. This case emphasizes the importance of examining whether such an equilibrium exists in a tax competition game.

Some results have already been established in the literature for particular cases. Firstly, Bucovetsky (1991) demonstrated the existence of a Nash equilibrium in tax rates in the case of two regions and quadratic production functions. A second and important result was highlighted by Laussel and Le Breton (1998) who proved the existence of a symmetric Nash equilibrium when private and public goods are perfect substitutes and when capital is not owned by residents. In addition, this framework enables the authors to prove the uniqueness of the equilibrium, which is the primacy purpose of their paper. In a more recent paper, Bayindir-Upman and Ziad (2005) apply a weaker concept than the standard Nash equilibrium — the concept of a second-order locally consistent equilibrium (2-LCE) — which is a local Nash equilibrium (i.e., a small deviation is undesirable). With this tool, the authors are able to show both the existence and uniqueness of a symmetric equilibrium in tax rates when regions are homogeneous and when either (i) there are only two regions, (ii) capital demand curves are concave, or firms apply (iii) CES, (iv) Cobb–Douglas, or (v) logistic production functions. More recently, Dhillon, Wooders and Zissimos (2006) investigate the existence of a Nash equilibrium in a symmetric tax competition model where the public good enters the production function. Rothstein (2007) analyses the fiscal competition game as a game with discontinuous payoff and demonstrates the existence of a pure strategy Nash equilibrium for this kind of game under several assumptions respecting the production function. Rothstein moves away from the standard fiscal competition game “à la Wildasin” by assuming: first an ad valorem tax; and second that the aggregate amount of mobile capital is fixed in all regions. Finally, Petchey and Shapiro (2009) examine the problem of the existence of Nash equilibrium in a tax competition model when governments are no longer benevolent but only make constrained efficient choices.

A key point of this paper is that we deal with asymmetric regions. The liter-

ature on asymmetric tax competition is mainly based on two articles by Wilson (1991) and Bucovetsky (1991). Both assume that regions differ in their population and show that the "small" region may benefit from the tax competition by attracting capital from the "large" region thanks to the tax competition mechanism. In the present paper, we retain the methodological question regarding the existence of Nash equilibrium in the tax competition model "à la Wildasin" that we extend to regions that differ by their production functions. In doing so, we extend the existing literature by proving the existence of a Nash equilibrium in a more general framework. Our paper is in line with Laussel and Le Breton (1998) and Bayindir-Upman and Ziad (2005), but we basically depart from their analysis by relaxing the assumption of symmetry. We also depart from the paper by Rothstein firstly by considering proportional taxes, whereas Rothstein uses an ad valorem tax, secondly, by establishing a weaker condition of existence than the quasiconcavity condition of Rothstein, and thirdly, by deriving directly our result in the tax competition model. To prove our results, we use several assumptions: that goods are normal; that the public good is desired; that the demand for capital is concave; and that the elasticity of the marginal product of capital is bounded.

This paper is organized as follows. The second section outlines the tax competition model for a given number of regions, notation and description of the model being taken for the most part from Bayindir-Upmann and Ziad (2005). Section 3 studies the existence of the Nash equilibrium, first when there are no holders of capital in the jurisdictions, and second, when residents do hold capital. The final Section summarizes our conclusions.

2 The Model

Consider n ($n \geq 2$) jurisdictions inhabited by a given number of homogeneous residents that we normalize to one without loss of generality. A fixed number of competitive firms produce a homogeneous output in each jurisdiction using capital and some fixed factor(s) (land or labour). Aggregating production over all firms in each region allows us to treat the industry of one jurisdiction as one competitive firm. Let f_i be the production function of the firm in jurisdiction i , f_i is assumed to be monotonously increasing and strictly concave in capital, K_i . Fixed factors as explicit arguments of f_i are suppressed so that the production function is expressed in terms of capital only. The jurisdiction i firm's profit can be written as

$$\Pi_i = f_i(K_i) - p_i K_i, \quad i = 1, \dots, n \quad (1)$$

where p_i denotes the after-tax price of capital in jurisdiction i . Equating the price of capital to the value of its marginal product, $f'_i(K_i) = p_i$ determines the region i 's capital demand as a function of the corresponding after-tax price of capital, $\hat{K}_i(p_i)$.

Let $U(X_i, P_i)$ be the utility that the representative household of jurisdiction i derives from the provision of the public good, P_i , and from the consumption of the private good, X_i , produced by the firms. The utility function $U(X_i, P_i)$ is twice-continuously differentiable and monotonously increasing. The source of the households' income is twofold: one part from the provision of the fixed factor, which is exclusively owned by local residents; and one part from their initial capital endowment. Let $\theta_i \in [0, 1]$ denote region i 's share of the fixed national capital stock \bar{K} , and ρ , the net return of capital in region i . The private budget constraint of the consumer in region i amounts to

$$X_i = f_i(K_i) - p_i K_i + \theta_i \rho_i \bar{K} \quad (2)$$

for each jurisdiction $i = 1, \dots, n$.

Each local government provides a public good that it finances by taxing the mobile capital at a tax rate t_i . The budget constraint of jurisdiction i is given by

$$P_i = t_i K_i \quad (3)$$

When choosing the level of the tax rate, each local government acts as a benevolent one and aims to maximize its representative resident's utility. In doing so, each local authority behaves non-cooperatively and treats its specific tax on capital t_i , $i = 1, \dots, n$, as the strategic variable. This leads to a tax competition game between jurisdictions.

The capital market clearing condition implies that aggregate demand for capital must equal capital supply:

$$\sum_{i=1}^n K_i = \bar{K}, \quad (4)$$

for some exogenously given capital supply \bar{K} .

Capital being freely mobile across regions, the arbitrage condition equals the net return of capital in each jurisdiction:

$$\rho = f'_i(K_i) - t_i (= \rho_i), \quad \forall i = 1, \dots, n. \quad (5)$$

Let $t := (t_1, \dots, t_n)$ be the profile of tax rates, $t_{-i} := (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ be the profile of all tax rates except t_i ; whereas (t_{-i}, t_i^0) stands for $(t_1, \dots, t_{i-1}, t_i^0, t_{i+1}, \dots, t_n)$. Both equations (4) and (5) define the equilibrium allocation of capital and the equilibrium of the net return of capital, i.e. $K_1(t), \dots, K_n(t)$ and $\rho(t)$.

The capital market equilibrium enables us to determine the response of the net return of capital to an increase in jurisdiction i 's tax rate:

$$\frac{\partial \rho(\mathbf{t})}{\partial t_j} = - \frac{\hat{K}'_j(p_j)}{\sum_i \hat{K}'_i(p_i)}.$$

and since each region's capital demand monotonously decreases with the price of capital $\left(\hat{K}'_j(p_j) = \frac{1}{f_{kk}^j} < 0\right)$, we have $\frac{\partial \rho(\mathbf{t})}{\partial t_j} \in (-1, 0)$, which is equivalent to $\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_j}\right) \in (0, 1)$ ¹.

Differentiating $K_j(t)$ and $K_i(t)$ with respect to t_j , we obtain the overall impact of a variation of the tax rate of region j on capital demand:

$$\begin{aligned}\frac{\partial K_j(\mathbf{t})}{\partial t_j} &= \hat{K}'_j(p_j) \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_j}\right) < 0, \\ \frac{\partial K_i(\mathbf{t})}{\partial t_j} &= \hat{K}'_i(p_i) \frac{\partial \rho(\mathbf{t})}{\partial t_j} > 0, \quad \forall i \neq j,\end{aligned}$$

with $p_j = \rho(t) + t_j, \forall j$.

Using $p_i = \rho + t_i$ and substituting $K_i(t)$ and $\rho(t)$ into the private and the public budget constraint (2) and (3), we obtain

$$X_i(\mathbf{t}) = f_i(K_i(\mathbf{t})) - (t_i + \rho(\mathbf{t})) K_i(\mathbf{t}) + \theta_i \bar{K} \rho(\mathbf{t}), \quad (6)$$

$$P_i(\mathbf{t}) = t_i K_i(\mathbf{t}). \quad (7)$$

Differentiating equations (6) and (7) with respect to t_i ² yields:

$$\frac{\partial X_i}{\partial t_i} = -K_i \left(1 + \frac{\partial \rho}{\partial t_i}\right) + \theta_i \bar{K} \frac{\partial \rho}{\partial t_i} < 0, \quad (8)$$

$$\frac{\partial^2 X_i}{\partial t_i^2} = -\frac{\partial K_i}{\partial t_i} \left(1 + \frac{\partial \rho}{\partial t_i}\right) - (K_i - \theta_i \bar{K}) \frac{\partial^2 \rho}{\partial t_i^2}, \quad (9)$$

$$\frac{\partial X_i}{\partial t_j} = (\theta_i \bar{K} - K_i) \frac{\partial \rho}{\partial t_j} \quad \forall i \neq j, \quad (10)$$

$$\frac{\partial P_i}{\partial t_i} = K_i + \frac{\partial K_i}{\partial t_i} t_i, \quad (11)$$

$$\frac{\partial^2 P_i}{\partial t_i^2} = 2 \frac{\partial K_i}{\partial t_i} + t_i \frac{\partial^2 K_i}{\partial t_i^2}, \quad (12)$$

$$\frac{\partial P_i}{\partial t_j} = t_i \hat{K}'_i \frac{\partial \rho}{\partial t_j} > 0 \quad \forall i \neq j, \quad (13)$$

Equation (8) states that the private good is clearly decreasing with the local tax rate whereas nothing can be said about the reaction of the public good. Both equations (6) and (7) enable us to write an indirect utility function $V_i(t) := U_i(X_i(t), P_i(t))$, which directly relates tax policy to welfare. Maximizing the indirect utility function of jurisdiction i with respect to the tax rate t_i yields the following first-order condition:

$$\frac{\partial V_i(\mathbf{t})}{\partial t_i} = \frac{\partial U_i}{\partial X_i} \frac{\partial X_i}{\partial t_i}(\mathbf{t}) + \frac{\partial U_i}{\partial P_i} \frac{\partial P_i}{\partial t_i}(\mathbf{t}) = 0,$$

¹These expressions are perfectly in line with the standard results of the tax competition litterature such as Keen and Kotsogiannis (2002) for instance.

²Note that all functions are continuously differentiable with respect to the strategic variables.

which implies the relation

$$\frac{\partial U^i / \partial P_i}{\partial U^i / \partial X_i} = \frac{-\partial X_i / \partial t_i}{\partial P_i / \partial t_i}$$

and by equations (8) and (11) we have:

$$\text{MRS}_{X_i, P_i} := \frac{\partial U^i / \partial P_i}{\partial U^i / \partial X_i} = \frac{K_i + (K_i - \theta_i \bar{K}) \frac{\partial \rho}{\partial t_i}}{K_i + \left(1 + \frac{\partial \rho}{\partial t_i}\right) t_i \hat{K}'_i} = \frac{-\partial X_i / \partial t_i}{\partial P_i / \partial t_i} = \text{MRFT}_{X_i, P_i} \quad (14)$$

In what follows, we are interested in the existence of a Nash equilibrium, a profile of strategies t^* such that t_i^* maximizes $V_i(t_i, t_{-i}^*)$ with respect to t_i for each i .

3 The Existence of Nash Equilibrium

3.1 Absentee holders of capital

In this section we assume that $\theta_i = 0$ or is sufficiently small for each i , which is in line with Wildasin (1988). As explained by Laussel and Le Breton (1995), there is a dual interpretation of this assumption: one for a partial equilibrium model, and one for a general equilibrium model. In the partial equilibrium model, $\theta_i = 0$ means that capital is owned by agents outside the jurisdictions under consideration; in a general equilibrium model, even if all capital is located in the "nation", a majority of residents do not hold capital, and through the median voter argument, it can be simply stated that each variable is chosen to maximize the welfare of the residents who do not own capital (see Laussel and Le Breton (p285)). In this article we choose the partial equilibrium interpretation. In the following, we also assume that the net return is positive ($\rho \geq 0$)³, which immediately implies $t_i \leq f_k^i(K_i(t_1, \dots, t_n))$ for each t_i . f_k^i being a decreasing function of t_i , and that we also have $t_i \leq f_k^i(0)$. At this stage $f_k^i(0)$ may be finite or not.

So that we might obtain useful and clear results, we postulate the following assumptions:

(C₁): For each i and for all $X_i > 0$, $\text{MRS}_{X_i, 0} \geq 1$.

(C₂): For each i and for all $X_i, P_i > 0$: $\frac{\partial}{\partial X_i} \text{MRS}_{X_i, P_i} \geq 0$ and $\frac{\partial}{\partial P_i} \text{MRS}_{X_i, P_i} \leq 0$.

(C₃): The third derivative of the production functions is positive, $f_{kkk}^i \geq 0$ for each i .

³This assumption is commonly used in the literature (cf Bucovetsky (1991), Laussel and Lebreton (1995)), and logically implies that "capital owners cannot be forced to supply capital services at a loss" (Bucovetsky 1991, p171).

(C₄): For each i we have $f_k^i(f_{kk}^i + K_i f_{kkk}^i) < K_i (f_{kk}^i)^2$.

(C₅): Let $K := \bar{K}/n$. For each i , $\text{MRS}_{f_i(K)-Kf'_i(K), Kf'_i(K)}^i \leq \left[1 + \frac{n-1}{n} \frac{1}{\epsilon_k^i}\right]^{-1}$, where ϵ_k^i is the elasticity of the marginal product of capital.

Condition (C₁) stipulates that the public good is a desirable good and guarantees that jurisdictions will never select $t_i = 0$ as an optimal solution⁴. As we need the strategies subsets to be a convex compact of \mathbb{R} , we assume that there exists a bound \bar{t} such that $t_i \leq \bar{t}$ for each i . Both Bucovetsky (1991) and Bayindir-Upmann & Ziad (2005) also use this condition. Condition (C₂), already postulated by Bucovetsky (1991) and Bayindir-Upmann & Ziad (2005), requires that the marginal rate of substitution between the private and the public good is non-decreasing in the first argument and non-increasing in the second argument. This is equivalent to state that both the private and the public good are normal goods, which is regarded as standard in economics. Condition (C₃) was first introduced by Laussel and Le Breton (1998) and later used by Rothstein (2007) and Shapiro and Petchey (2009). It stipulates that the marginal product of capital is always a convex function of the amount of invested capital. Usual production functions such as Cobb-Douglass, CES and Quadratic functions satisfy (C₃). Laussel and Le Breton interpret this condition by stating that "for a jurisdiction the advantage of taxing capital is in the induced reduction of the equilibrium net rate of return of capital (i.e. of the equilibrium "price of capital" which must be paid to capital holders who invest in the jurisdiction)". Condition (C₄) is a knew condition that can be rewritten as $\epsilon_k^i < 1 + \frac{K_i f_{kkk}^i}{f_{kk}^i}$ where ϵ_k^i is the elasticity of the marginal product of capital, $\left(\frac{K_i f_{kk}^i}{f_k^i}\right)$. This condition states that the elasticity of the marginal product of capital cannot exceed some bound. Most of the standard production functions satisfy (C₄) (see examples latter). This condition is looser than conditions introduced by Rothstein (2007) who first assumes that the elasticity of the marginal product of capital ϵ_k^i must be higher than (-1) (assumption 8ii), and secondly, that $2(f_{kk}^i)^2 - f_k^i f_{kkk}^i \geq 0$ (assumption 9iii). Finally, (C₅) was introduced in Bayindir-Upmann & Ziad (2005) and states that when the capital is evenly distributed across regions and its return is fully taxed away, the resulting tax revenue is sufficient to provide a level of the public good which is, at least, close to the efficient level at which the $\text{MRS} = 1$. With condition (C₁), (C₅) guarantees the existence of a symmetric profile ($t_i = t_j$ for each i, j) such that in the symmetric case ($f_i = f_j$) the first derivative of each jurisdiction disappears (see Lemma 7 below).

In order to prove the existence of at least one Nash equilibrium, we consider the best response of each jurisdiction and prove that they are functions and not a

⁴The derivative $\frac{\partial V_i}{\partial t_i}$ evaluated at $t_i = 0$, is proportional to $\frac{\partial X_i}{\partial t_i} + \text{MRS}_{X_i,0} \frac{\partial P_i}{\partial t_i} = K_i(\text{MRS}_{X_i,0} - (1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}))$. According to $(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}) \in (0, 1)$, condition (C₁) ensures that $\frac{\partial V_i}{\partial t_i}(0, t_{-i}) > 0$.

correspondence. Then by continuity assumptions we can use a fixed point theorem to prove the existence of a Nash equilibrium. Let $t_{-i} := (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ be the profile of all tax rates except t_i , our first step is to look at the best response function of jurisdiction i . If the first order strategy is never satisfied, then the best response is unique (every continuous function reaches its maximum value on a compact subset) and the maximum is on the boundary. If the first order strategy is satisfied for some t_i , we have to consider the second derivative and prove that it is (at least locally) negative. Therefore we assume in the following that the first order strategy exists.

The principal general result of the paper is as follows:

Theorem 1 *When consumers' preferences are represented by twice-continuously differentiable, monotonously increasing utility functions $U_i(\cdot, \cdot)$ in (X_i, P_i) and assumptions (C_1) , (C_2) , (C_3) and (C_4) then the asymmetrical fiscal game possesses a Nash equilibrium.*

To prove our result, we have to introduce several Lemmas, which are of great importance in understanding the tax competition mechanisms at work in the economy.

Lemma 1 *[Capital Allocation]*

For any given t , the capital allocation $K_1(t), \dots, K_n(t)$ and the net return of capital $\rho(t)$ are unique, with $\frac{\partial K_i(\mathbf{t})}{\partial t_i} < 0$, $\frac{\partial K_j(\mathbf{t})}{\partial t_i} > 0 \forall j \neq i$, and $\frac{\partial \rho(\mathbf{t})}{\partial t_i} < 0 \forall i$.

Lemma 2 *[Increasing branch of the Laffer curve]*

For any given tax vector t_{-i} , a utility local-maximizing (respectively local minimizing) strategy of region i requires $\frac{\partial P_i}{\partial t_i}(t) > 0$ on the left (respectively the right) of any local extremum point.

Proof: As the private good X_i is a decreasing function in t_i , and the utility function $U_i(\cdot, \cdot)$ is an increasing function in (X_i, P_i) , in the neighborhood of any local maximum (resp. minimum)⁵ t_i^0 , the public good must be an increasing function on the left (respectively on the right) of t_i^0 ■.

Lemma 3 *[Decreasing marginal rate of substitution]* *Let condition (C_2) hold. For any tax vector t , the marginal rate of substitution between the private and the public good is falling on the left or on the right of any local extremum.*

Proof: The derivative of the marginal rate of substitution is

$$\frac{\partial}{\partial t_i} \text{MRS}_{X_i, P_i}(\mathbf{t}) = \frac{\partial \text{MRS}_{X_i, P_i}}{\partial X_i} \frac{\partial X_i(\mathbf{t})}{\partial t_i} + \frac{\partial \text{MRS}_{X_i, P_i}}{\partial P_i} \frac{\partial P_i(\mathbf{t})}{\partial t_i}.$$

⁵Not on the boundary.

From condition (C_2) (goods are normal) we have $\frac{\partial}{\partial X_i} \text{MRS}_{X_i, P_i} \geq 0$ and $\frac{\partial}{\partial P_i} \text{MRS}_{X_i, P_i} \leq 0$. Since $\frac{\partial X_i(\mathbf{t})}{\partial t_i} < 0$ and $\frac{\partial P_i(\mathbf{t})}{\partial t_i} > 0$ (lemma 2), then $\frac{\partial}{\partial t_i} \text{MRS}_{X_i, P_i}(\mathbf{t}) < 0$ ■.

Lemma 4 [*Convexity of the net return to capital function*]

Condition C_3 is sufficient to ensure the convexity of the net return of capital $\rho(\mathbf{t})$.

Proof: The second derivative of the net return function is written as:

$$\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i^2} = \frac{\frac{\partial K_i(\mathbf{t})}{\partial t_i} \frac{f_{kkk}^i}{(f_{kk}^i)^2} \sum_j \frac{1}{f_{kk}^j} - \frac{1}{f_{kk}^i} \sum_j \frac{\frac{\partial K_j(\mathbf{t})}{\partial t_i} f_{kkk}^j}{(f_{kk}^j)^2}}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^2}.$$

But $\rho(\mathbf{t}) = f_k^j(K_j) - t_j$, then for $j \neq i$, we have $\frac{\partial \rho(\mathbf{t})}{\partial t_i} = \frac{\partial K_j(\mathbf{t})}{\partial t_i} f_{kk}^j(K_j)$. We obtain

$$\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i^2} = \frac{\frac{\partial K_i(\mathbf{t})}{\partial t_i} \frac{f_{kkk}^i}{(f_{kk}^i)^2} \sum_j \frac{1}{f_{kk}^j} - \frac{1}{f_{kk}^i} \sum_{j \neq i} \frac{\frac{\partial \rho(\mathbf{t})}{\partial t_i} f_{kkk}^j}{(f_{kk}^j)^3} - \frac{\frac{\partial K_i(\mathbf{t})}{\partial t_i} f_{kkk}^i}{(f_{kk}^i)^3}}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^2}$$

and

$$\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i} = \frac{\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right) \frac{f_{kkk}^i}{(f_{kk}^i)^3} \sum_{j \neq i} \frac{1}{f_{kk}^j} - \frac{\partial \rho(\mathbf{t})}{\partial t_i} \frac{1}{f_{kk}^i} \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3}}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^2}$$

which is positive as soon as $f_{kkk}^j \geq 0$ (condition C_3) ■.

Lemma 5 [*Second derivative of capital function*]

The second derivative of capital is given by:

$$\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2} = \frac{\left(\sum_{j \neq i} \frac{1}{f_{kk}^j}\right)^2 \cdot f_{kkk}^i + f_{kk}^i \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3} - f_{kkk}^i f_{kk}^i \left(\sum_j \frac{1}{f_{kk}^j}\right) \left(\sum_{j \neq i} \frac{1}{f_{kk}^j}\right)^2}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^3 \cdot (f_{kk}^i)^4}.$$

Proof : See Appendix 1.

Lemma 1 to 4 enable us to establish the Proof of Theorem 1:

Proof of Theorem 1: The set of first and second order strategies of government i are:

$$\frac{\partial V(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} = \frac{\partial U(X_i, P_i)}{\partial X_i} \frac{\partial P_i(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} \left[\frac{\partial X_i(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} + \text{MRS}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) \right] = 0.$$

$$\frac{\partial V(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} = \frac{\partial U(X_i, P_i)}{\partial X_i} \frac{\partial P_i(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} \left[-\text{MRFT}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) + \text{MRS}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) \right] = 0.$$

And

$$\frac{\partial^2 V(\mathbf{t}_{-i}, t_i^0)}{\partial t_i^2} = \frac{\partial U(X_i, P_i)}{\partial X_i} \frac{\partial P_i(\mathbf{t}_{-i}, t_i^0)}{\partial t_i} \left[-\frac{\partial}{\partial t_i} \text{MRFT}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) + \frac{\partial}{\partial t_i} \text{MRS}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) \right],$$

From Lemma 3:

$$\frac{\partial}{\partial t_i} \text{MRS}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) < 0.$$

We have to prove that

$$-\frac{\partial}{\partial t_i} \text{MRFT}_{X_i, P_i}(\mathbf{t}_{-i}, t_i^0) < 0.$$

Let

$$\frac{\partial}{\partial t_i} \left(\frac{\frac{\partial X_i}{\partial t_i}(\mathbf{t}^0)}{\frac{\partial P_i}{\partial t_i}(\mathbf{t}^0)} \right) = \left(\frac{\partial P_i}{\partial t_i} \right)^{-2} \left(\frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2} \right)$$

Then

$$\frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2} = -t \left(\frac{\partial K_i}{\partial t_i} \right)^3 f_{kk}^i - K_i^2 \frac{\partial^2 \rho}{\partial t_i^2} - t K_i \frac{\partial^2 \rho}{\partial t_i^2} \frac{\partial K_i}{\partial t_i} + K_i \left(\frac{\partial K_i}{\partial t_i} \right)^2 f_{kk}^i + t K_i \frac{\partial K_i}{\partial t_i} \frac{\partial^2 K_i}{\partial t_i^2} f_{kk}^i.$$

Replacing

$$\frac{\partial K_i(\mathbf{t})}{\partial t_i} = \frac{\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)}{f_{kk}^i}$$

and

$$\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2} = \frac{\left(\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i^2} \right) (f_{kk}^i)^2 - \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^2 f_{kkk}^i}{(f_{kk}^i)^3}$$

we get

$$\frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2} = \frac{-t_i (f_{kk}^i + K_i f_{kkk}^i) \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^3 + K_i (f_{kk}^i)^2 \left(\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^2 - K_i f_{kk}^i \frac{\partial^2 \rho}{\partial t_i^2} \right)}{(f_{kk}^i)^3}.$$

Then

$$\frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2}$$

is negative if and only if

$$t_i (f_{kk}^i + K_i f_{kkk}^i) \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^3 < K_i (f_{kk}^i)^2 \left(\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^2 - K_i f_{kk}^i \frac{\partial^2 \rho}{\partial t_i^2} \right) \quad (15)$$

With $f_{kk}^i < 0$, $f_{kkk}^i > 0$ from C_3 and $\frac{\partial^2 \rho}{\partial t_i^2} > 0$ from Lemma 4.

- If $f_{kk}^i + K_i f_{kkk}^i < 0$, (15) is always satisfied
- If $f_{kk}^i + K_i f_{kkk}^i \geq 0$, using $t_i \leq f_k^i(K_i)$, a sufficient condition to check (15) is then

$$f_k^i (f_{kk}^i + K_i f_{kkk}^i) \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^3 < K_i (f_{kk}^i)^2 \left(\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^2 - K_i f_{kk}^i \frac{\partial^2 \rho}{\partial t_i^2} \right),$$

$$\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^3 < \left(\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^2 - K_i f_{kk}^i \frac{\partial^2 \rho}{\partial t_i^2} \right), \quad (16)$$

As $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} \leq 1$ by condition (C_4) , $\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^3 < \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^2$ and $\frac{\partial^2 \rho}{\partial t_i^2} > 0$ by Lemma 4, we get the result. ■

Let summarize the role of the different conditions in the proof: (C_1) implies that the set of strategies is a convex closed interval and condition (C_2) implies that the marginal rate of substitution between the private and the public good is a non increasing function of the tax rate. Finally, both conditions (C_3) and (C_4) insures that the marginal rate of the fiscal transformation between the private and the public good is increasing with the tax rate. We crucially depart from the Rothstein's result by several aspects: firstly, we stipulate a weaker condition than those defined in Assumption 8 and 9 of Rothstein's paper, secondly, our proof is direct while Rothstein's proof is much more complicated and uses Reny's (1999) result, and, lastly, we consider proportional taxes whereas Rothstein uses an ad valorem tax.

3.1.1 Examples

In this Section, we introduce several examples so that we can check the tractability of our result with commonly-used production functions:

Remark 1 For a Cobb-Dougllass production function $f(K) = K^\alpha$, we have $f_k(K) = \alpha K^{\alpha-1}$, $f_{kk}(K) = \alpha(\alpha-1)K^{\alpha-2}$, $f_{kkk}(K) = \alpha(\alpha-1)(\alpha-2)K^{\alpha-3} > 0$, and finally, $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} = 1$.

For a Quadratic production function $f(K) = (a - bK)K$, with $\bar{K} < \frac{a}{2b}$ we have $f_k(K) = a - 2bK$, $f_{kk}(K) = -2b$, $f_{kkk}(K) = 0$, and finally, $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} = \frac{a-2bK}{-2bK} < 0$.

For a logarithm function $f(K) = \ln(1 + \beta K)$, we have $f_k(K) = \frac{\beta}{1+\beta K}$, $f_{kk}(K) = -\frac{\beta^2}{(1+\beta K)^2}$, and $f_{kkk}(K) = \frac{2\beta^3}{(1+\beta K)^3} > 0$. In addition, $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} = 1 - \frac{1}{\beta K} < 1$.

For an exponential function $f(K) = 1 - \exp(-\beta K)$, we have $f_k(K) = \beta \exp(-\beta K)$, $f_{kk}(K) = -\beta^2 \exp(-\beta K)$, and $f_{kkk}(K) = \beta^3 \exp(-\beta K) > 0$. We also have $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} = 1 - \frac{1}{\beta K} < 1$.

All the production functions presented above satisfy C_3 and C_4 .

Then for Cobb-Douglass, Quadratic, Logarithmic and Exponential production functions, Nash equilibrium exists under conditions C_1 and C_2 (from Theorem 1) even with asymmetries in technology. For instance, production functions can write $K^{\alpha_1}, K^{\alpha_2}, \dots, K^{\alpha_n}$ respectively with $\alpha_i \neq \alpha_j$ for each $i \neq j$.

Remark 2 For a Logistic production function $f(K) = \frac{1}{1+e^{-x}} - \frac{1}{2}$, with $x = x(K) := KR^\alpha$, then $f_k(K) = \frac{R^\alpha e^x}{(1+e^x)^2} > 0$, $f_{kk}(K) = \frac{R^{2\alpha}(1-e^x)e^x}{(1+e^x)^3}$ and $f_{kkk}(K) = \frac{R^{3\alpha}(1-4e^x+e^{2x})e^x}{(1+e^x)^4} > 0 \iff x > \ln[2 + \sqrt{3}]$.

$\frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} = \frac{(1+e^x)(1-e^x)+x(1-4e^x+e^{2x})}{(1+e^x)(1-e^x)}$ where $g(x) := x(1-4e^x+e^{2x}) + (1+e^x)(1-e^x)$ is non positive since $g(0) = 0$ and $g'_k(x) = (1-e^{2x}) - (4+2x)e^x < 0$. Therefore $f_{kk}^i + K_i f_{kkk}^i < 0$ and the marginal rate of fiscal transformation is increasing.

For a logistic production function with $x > \ln[2 + \sqrt{3}]$, Theorem 1 ensures that a Nash equilibrium exists under conditions C_1 and C_2 .

Remark 3 For a CES production function $f(K) = (\alpha K^\psi + (1-\alpha)R^\psi)^{\frac{1}{\psi}}$ where r stands for the quantity of the fixed factor, $0 < \alpha < 1$, $\psi \leq 1$ and $\psi \neq 0$, we have $f_k(K) = \frac{\alpha K^\psi}{\alpha K^\psi + (1-\alpha)R^\psi} \frac{f(K)}{K}$, $f_{kk}(K) = -\frac{\alpha(1-\alpha)(1-\psi)(KR)^\psi}{(\alpha K^\psi + (1-\alpha)R^\psi)^2} \frac{f(K)}{K^2}$, and $f_{kkk}(K) = \frac{\alpha(1-\alpha)(1-\psi)(KR)^\psi}{(\alpha K^\psi + (1-\alpha)R^\psi)^3} \frac{f(K)}{K^3} (\alpha(1+\psi)K^\psi + (1-\alpha)(2-\psi)R^\psi) > 0$. In addition, $f_k^i(f_{kk}^i + K_i f_{kkk}^i) - K_i (f_{kk}^i)^2 = \frac{\alpha\psi K^\psi}{(1-\alpha)R^\psi + \alpha K^\psi}$, which implies $f_k^i(f_{kk}^i + K_i f_{kkk}^i) - K_i (f_{kk}^i)^2 < 0$ for $\psi < 0$ and $f_k^i(f_{kk}^i + K_i f_{kkk}^i) - K_i (f_{kk}^i)^2 > 0$ for $\psi > 0$. In the last case, the sufficient condition of theorem 1 is not satisfied. However a closer examination of inequality (15) in the proof of Theorem 1 shows that the term $\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i}$ may be higher than 1 and still respect (15). This is the case for some sufficiently low values for α or for ψ , for instance.

As a corollary of Remarks 2, 3 and 4, we have

Corollary 1 Assume that consumers' preferences are represented by twice-continuously differentiable, monotonously increasing utility functions $U_i(\dots)$ in (X_i, P_i) and that conditions (C_1) , and (C_2) are satisfied. If for each jurisdiction i , the production function $f_i \in \{\text{Cobb-Douglass, Quadratic, Logarithmic, Exponential, Logistic, CES}\}$ under further conditions for the parameters for the logistic and CES production functions, then the fiscal game possesses a Nash equilibrium.

3.1.2 Symmetric Jurisdictions

So that our results might be more comparable to those of Wildasin (1988), Bucovetsky (1991) and Laussel and Le Breton (1998), we now study the particular case of symmetric jurisdictions ($f_i = f_j = f$).

Lemma 6 [*Convex capital demand function in the symmetric case*] Assume homogeneous jurisdictions and let condition C_3 hold, then $\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2}$ is positive when $n \geq 3$ and vanishes for $n = 2$.

Proof: The sign of $\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2}$ is the opposite of

$$\left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2 \cdot f_{kkk}^i + f_{kk}^i \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3} - f_{kkk}^i f_{kk}^i \left(\sum_j \frac{1}{f_{kk}^j} \right) \left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2 \quad (17)$$

The first two terms are positive under condition C_3 ($f_{kkk}^i > 0$) while the last term is negative. In the symmetric case ($f_i = f_j = f$), expression (17) evaluated at the diagonal becomes:

$$\begin{aligned} & \frac{(n-1)^2}{f_{kk}^2} f_{kkk} + f_{kk} \frac{(n-1) f_{kkk}}{(f_{kk})^3} - f_{kkk} f_{kk} \frac{n}{f_{kk}} \frac{(n-1)^2}{f_{kk}^2} \\ = & \frac{(n-1)^2 f_{kkk} + (n-1) f_{kkk} - n(n-1)^2 f_{kkk}}{f_{kk}^2} \\ = & \frac{n(n-1)(2-n) f_{kkk}}{f_{kk}^2} \end{aligned}$$

Which is clearly negative for $n \geq 3$ and null for $n = 2$ ■.

Lemma 7 [*Existence of an extremum along the diagonal*] Assume homogeneous jurisdictions and let conditions (C_1) and (C_5) hold. Along the diagonal ($t_i = t_j$, for each i) each region's payoff function V_i reaches an extremum at some symmetric tax vector $0 < t^0 < f'(\bar{K}/n) = t_M$.

Proof: See Bayindir-Upman and Ziad (2005) p21.

Corollary 2 In the symmetric fiscal competition game, assume that conditions (C_1) , (C_2) and (C_5) hold.

1. If $n = 2$, or

2. If $n \geq 3$ and the production functions satisfy the following inequality

$$\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} < \left(\frac{n}{n-1} \right) \left(1 - \frac{K_i f_{kkk}^i}{f_{kk}^i} \right) \quad (18)$$

for each i and each t_i ,

Then a local Nash equilibrium exists.

Proof: See Appendix 2

Note that the case $n = 2$ corresponds to the Laussel and Le Breton's framework and that condition (18) is less restrictive than condition C_4

$$\left(\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} < 1 \right) \text{ since } \left(\frac{n}{n-1} \right) \left(1 - \frac{K_i f_{kkk}^i}{f_{kk}^i} \right) > 1.$$

3.2 Residents hold capital

In this section we relax the assumption of absentee capital owners and we consider $\theta_i \neq 0$. Even if our conclusions are not obvious, we are able to state several results and intuitions:

Remark 4 When residents hold capital, condition (15) which ensures the concavity of the indirect utility function and then the existence of the Nash equilibrium, writes:

$$\frac{f_k^i}{K_i f_{kk}^i} \frac{(f_{kk}^i + K_i f_{kkk}^i)}{f_{kk}^i} \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^3 < \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^2 - \left((K_i - \theta_i \bar{K}) f_{kk}^i - \frac{t_i \theta_i \bar{K}}{K_i} (1 + \frac{\partial \rho}{\partial t_i}) \right) \frac{\partial^2 \rho}{\partial t_i^2}.$$

At this stage we cannot say that condition (C_4) is sufficient because the sign of $h(\theta_i) := \left((K_i - \theta_i \bar{K}) f_{kk}^i - \frac{t_i \theta_i \bar{K}}{K_i} (1 + \frac{\partial \rho}{\partial t_i}) \right)$ is not determined *a priori* when $\theta_i \neq 0$ whereas it is negative when $\theta_i = 0$. By continuity of the function $h(\cdot)$ we can say that for a sufficiently small θ_i , $h(\theta_i)$ is negative, and therefore Corollary 1 remains true⁶.

As a corollary we have.

⁶Note that the derivative of function h : $h_k(\theta_i) = -\frac{\bar{K}}{K_i} f_{kk}^i \frac{\partial P_i}{\partial t_i}$ is positive in the neighbourhood of the optimal tax rate, whereas nothing can be said *a priori* about the level of $h(1) = f_{kk}^i (K_i - \frac{\bar{K}}{K_i} \frac{\partial P_i}{\partial t_i})$.

Corollary 3 *Assume that consumers' preferences are represented by twice-continuously differentiable, monotonously increasing utility functions $U_i(.,.)$ in (X_i, P_i) , that conditions (C_1) , (C_2) , (C_3) , and (C_4) are satisfied, and that θ_i is sufficiently small. If for each jurisdiction i , the production function $f_i \in \{\text{Cobb-Dougllass, Quadratic, Logarithmic and Exponential, Logistic, CES}\}$, under further conditions on the parameters for the logistic and CES production functions, then the fiscal game possesses a Nash equilibrium.*

The particular case of symmetric jurisdictions enables us to obtain a clear result:

Corollary 4 *In the particular case of symmetric jurisdictions with $\theta_i = \frac{1}{n}$ for all i , if conditions (C_1) , (C_2) , (C_3) , and (C_4) are satisfied, then the fiscal game possesses a Nash equilibrium.*

Proof: Recall $h(\theta_i) := \left((K_i - \theta_i \bar{K}) f_{kk}^i - \frac{t_i \theta_i \bar{K}}{K_i} (1 + \frac{\partial \rho}{\partial t_i}) \right)$, then $h(\frac{1}{n}) = -\frac{t_i \theta_i \bar{K}}{K_i} (1 + \frac{\partial \rho}{\partial t_i}) < 0$ which completes the proof.

Corollary 2 can be compared to the Bayindir-Upmann & Ziad's result whereas Corollary 4 generalizes the Bayindir-Upmann & Ziad's result. Indeed, we focus on a global Nash equilibrium (that is any unilateral deviation (small or not) is not profitable) whereas the Nash equilibrium analyzed by Bayindir-Upmann & Ziad remains local. Moreover, in their paper, $n = 2$ constitutes a sufficient condition which allows the authors to prove the existence of a local Nash equilibrium for symmetric jurisdictions. In our paper, apart from conditions that are also used in Bayindir-Upmann and Ziad (C_1 and C_2) our result arises from two crucial conditions: the demand for capital is concave, and the elasticity of the marginal product of capital is bounded. Once these conditions are checked, our result is still available for more than two jurisdictions.

4 Conclusion

In this methodological paper we have determined conditions that enable the existence of a Nash equilibrium in a standard tax competition game. When there are no capital owners in the 'Nation', our result is general and simple to apply. Indeed, some different examples with standard production functions show that our method is tractable. For the case of resident owners, we are able to state that a Nash equilibrium exists for the particular case of capital equally owned among jurisdictions. Our result extend the existing literature by considering more than two jurisdictions and assuming that these jurisdictions may have asymmetric production functions. A further step for this analysis would be to concentrate on the uniqueness of the equilibrium in such a general framework. This would

be a challenging task. In parallel, a secondary objective is to conduct the same exercise (proof of the existence of a Nash equilibrium) in a model of trade with interactions between fiscal policies "à la Turnovsky" (1988). Our final objective is to combine both frameworks which would form a further interesting challenge. In addition, this framework will enable us to test the impact of trade upon public good provision in a tax competition model. A initial result of this ongoing work shows that the underprovision of the public good is reduced when a trade channel is introduced into the tax competition model. Our intuition is that an optimal public good provision may be restored in a general model for which many interdependent channels of fiscal policies transmission coexist.

5 Appendix

5.1 Appendix 1: Proof of lemma 5

$$\frac{\partial K_i(\mathbf{t})}{\partial t_i} = \frac{\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)}{f_{kk}^i}$$

The second derivative is:

$$\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2} = \frac{\left(\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i^2}\right) f_{kk}^i - \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right) \frac{\partial K_i(\mathbf{t})}{\partial t_i} f_{kkk}^i}{(f_{kk}^i)^2}.$$

replacing $\frac{\partial K_i(\mathbf{t})}{\partial t_i} = \frac{\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)}{f_{kk}^i}$, we obtain

$$\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2} = \frac{\left(\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i^2}\right) (f_{kk}^i)^2 - \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^2 f_{kkk}^i}{(f_{kk}^i)^3}.$$

From lemma 4,

$$\left(\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i}\right) (f_{kk}^i)^2 = \frac{\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right) \frac{f_{kkk}^i}{f_{kk}^i} \sum_{j \neq i} \frac{1}{f_{kk}^j} - f_{kk}^i \frac{\partial \rho(\mathbf{t})}{\partial t_i} \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3}}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^2}.$$

And

$$\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right)^2 f_{kkk}^i = \frac{f_{kkk}^i \left(\sum_{j \neq i} \frac{1}{f_{kk}^j}\right)^2}{\left(\sum_j \frac{1}{f_{kk}^j}\right)^2}.$$

Replacing $\left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i}\right) = \frac{\sum_{j \neq i} \frac{1}{f_{kk}^j}}{\sum_j \frac{1}{f_{kk}^j}}$ and $\frac{\partial \rho(\mathbf{t})}{\partial t_i} = \frac{-1}{\sum_j \frac{1}{f_{kk}^j}}$

$$\begin{aligned}
& \left(\frac{\partial^2 \rho(\mathbf{t})}{\partial t_i} \right) (f_{kk}^i)^2 - \left(1 + \frac{\partial \rho(\mathbf{t})}{\partial t_i} \right)^2 f_{kkk}^i = \\
& \frac{\sum_{j \neq i} \frac{1}{f_{kk}^j} \frac{f_{kkk}^i}{f_{kk}^i} \sum_{j \neq i} \frac{1}{f_{kk}^j} - f_{kk}^i \frac{-1}{f_{kk}^j} \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3}}{\left(\sum_j \frac{1}{f_{kk}^j} \right)^2} - \frac{f_{kkk}^i \left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2}{\left(\sum_j \frac{1}{f_{kk}^j} \right)^2} = \\
& \frac{\left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2 \cdot f_{kkk}^i + f_{kk}^i \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3} - f_{kkk}^i f_{kk}^i \left(\sum_j \frac{1}{f_{kk}^j} \right) \left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2}{\left(\sum_j \frac{1}{f_{kk}^j} \right)^3} \cdot f_{kk}^i.
\end{aligned}$$

Finally

$$\frac{\partial^2 K_i(\mathbf{t})}{\partial t_i^2} = \frac{\left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2 \cdot f_{kkk}^i + f_{kk}^i \sum_{j \neq i} \frac{f_{kkk}^j}{(f_{kk}^j)^3} - f_{kkk}^i f_{kk}^i \left(\sum_j \frac{1}{f_{kk}^j} \right) \left(\sum_{j \neq i} \frac{1}{f_{kk}^j} \right)^2}{\left(\sum_j \frac{1}{f_{kk}^j} \right)^3} \cdot (f_{kk}^i)^4.$$

5.2 Appendix 2: Proof of Corollary 2

From Lemma 7 we know that there exists a symmetrical profile (t, t, \dots, t) such that the first derivative of the utility function disappears. Let us examine the second derivative at (t, t, \dots, t) in two cases: $n = 2$ and $n \geq 3$

1. Symmetric case with $n = 2$

$$\begin{aligned}
& \frac{\partial}{\partial t_i} \left(\frac{\frac{\partial X_i}{\partial t_i}}{\frac{\partial P_i}{\partial t_i}} \right) < 0 \\
\Leftrightarrow & \frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2} < 0 \tag{19}
\end{aligned}$$

- We know that $\frac{\partial X_i}{\partial t_i} < 0$, and $\frac{\partial P_i}{\partial t_i} > 0$ in the neighborhood of any interior extremum point.

Let $\frac{\partial^2 P_i}{\partial t_i^2} = 2 \frac{\partial K_i}{\partial t_i} + t_i \frac{\partial^2 K_i}{\partial t_i^2}$, and from Lemma 6, $\frac{\partial^2 K_i}{\partial t_i^2} = 0$ which implies $\frac{\partial^2 P_i}{\partial t_i^2} < 0$.

Then $\frac{\partial^2 X_i}{\partial t_i^2} < 0$ is a sufficient condition to reach the inequality (19)

$$\begin{aligned}
\frac{\partial^2 X_i}{\partial t_i^2} &= -\frac{1}{4} \left(\frac{1}{f_{kk}^i} + K_i \frac{f_{kkk}^i}{(f_{kk}^i)^2} \right) < 0 \\
\iff & f_{kk}^i + K_i f_{kkk}^i > 0
\end{aligned}$$

- If $f_{kk}^i + K_i f_{kkk}^i < 0$, (19) writes

$$\begin{aligned} \frac{\partial^2 X_i}{\partial t_i^2} \frac{\partial P_i}{\partial t_i} - \frac{\partial X_i}{\partial t_i} \frac{\partial^2 P_i}{\partial t_i^2} &= -\frac{1}{4} \left(\frac{1}{f_{kk}^i} + K_i \frac{f_{kkk}^i}{(f_{kk}^i)^2} \right) \left(K_i + \frac{1}{2} t_i \frac{1}{f_{kk}^i} \right) + \frac{1}{2} K_i \frac{1}{f_{kk}^i} < 0 \\ &\iff \\ 2(f_{kk}^i - K_i f_{kkk}^i) &< (f_{kk}^i + K_i f_{kkk}^i) \frac{1}{K_i} t_i \frac{1}{f_{kk}^i} \end{aligned}$$

Since the left-hand side of the last inequality is negative, (19) is always checked with $f_{kk}^i + K_i f_{kkk}^i < 0$ since it implies a positive right-hand side

2. For $n \geq 3$, in the symmetric case ($f_i = f_j = f$), an evaluation on the extremum point given in Lemma 7 gives:

$$\begin{aligned} 1 + \frac{\partial \rho}{\partial t_i} &= 1 - \frac{1}{n} = \frac{n-1}{n} \\ \frac{\partial^2 \rho}{\partial t_i^2} &= \frac{n-1}{n^2} \frac{f_{kkk}^i}{(f_{kk}^i)^2} \end{aligned}$$

The inequality became:

$$t_i(f_{kk} + K_i f_{kkk}) \left(\frac{n-1}{n} \right)^3 < K_i (f_{kk})^2 \left(\left(\frac{n-1}{n} \right)^2 - \frac{n-1}{n^2} \frac{K_i f_{kkk}}{f_{kk}} \right).$$

A sufficient condition for the last inequality (knowing that $t_i \leq f_k(K_i)$) is :

$$f_k(K_i)(f_{kk} + K_i f_{kkk}) \left(\frac{n-1}{n} \right)^3 < K_i (f_{kk})^2 \left(\left(\frac{n-1}{n} \right)^2 - \frac{n-1}{n^2} \frac{K_i f_{kkk}}{f_{kk}} \right),$$

which is equivalent to

$$\frac{f_k(K_i)(f_{kk} + K_i f_{kkk})}{K_i (f_{kk})^2} \left(\frac{n-1}{n} \right)^3 < \left(\left(\frac{n-1}{n} \right)^2 - \frac{n-1}{n^2} \frac{K_i f_{kkk}}{f_{kk}} \right).$$

■.

References

- [1] T. Bayindir-Upmann, A. Ziad, 2005, "Existence of second-order locally consistent equilibria in fiscal competition models", *Regional Sciences and Urban Economics*, 35, 1–22.
- [2] Bucovetsky, S., 1991, "Asymmetric Tax Competition", *Journal of Urban Economics*, 30, 167–181.
- [3] Dhillon A. & M. Wooders & B. Zissimos, 2007. "Tax Competition Reconsidered," *Journal of Public Economic Theory*, vol. 9(3), 391-423, 06.

- [4] Fujii. Y & J. Iritani, 2002, "A Note on Non-Existence of Equilibria in Tax Competition Models", Kobe University Working Paper
- [5] Keen M. J. and C. Kotsogiannis, 2002, "Does Federalism Lead to Excessively High Taxes?", *American Economic Review* 92, pp 363-370.
- [6] Laussel, D. and Le Breton, M., 1998, "Existence of Nash Equilibria in Fiscal Competition Models", *Regional Science and Urban Economics*, 28 , 283–296.
- [7] Petchey, J. & P. Shapiro, 2009, "Equilibrium in fiscal competition games from the point of view of the dual", *Regional Sciences and Urban Economics*, 39, 97-108.
- [8] Reny, P. J., 1999, "On the existence of pure and mixed strategy Nash equilibria in discontinuous games", *Econometrica*, 67, 1029-1056.
- [9] Rothstein, 2007, "Discontinuous Payoffs, Shared Resources, and Games of Fiscal Competition: Existence of Pure Strategy Nash Equilibrium", *Journal of Public Economic Theory*, vol. 9(2), 335-368.
- [10] Turnovsky S. J., (1988), "The gains from fiscal cooperation in the two-commodity real trade model", *Journal of International Economics*, vol. 25, 111-121.
- [11] Wildasin, D. E., 1988, Nash Equilibria in Models of Fiscal Competition, *Journal of Public Economics*, 35, 229–240.
- [12] Wilson, J. D., 1985, "Optimal Property Taxation in the Presence of Interregional Capital Mobility", *Journal of Urban Economics*, 17, 73–89.
- [13] Wilson, J. D., 1986, "A Theory of Interregional Tax Competition", *Journal of Urban Economics*, 19, 296–315.
- [14] Zodrow, G. R. and P. Mieszkowski, 1986, "Pigou, Tiebout, Property Taxation, and the Underprovision of Local Public Goods", *Journal of Urban Economics*, 19, 356–70.