Stability and Index of the Meet Game on a Lattice

Joseph ABDOU

2010.50
Stability and Index of the Meet Game on a Lattice

Joseph M. Abdou *
†

June 8, 2010

Abstract. We study the stability and the stability index of the meet game form defined on a meet-semilattice. Given any active coalition structure, we show that the stability index relative to the equilibrium, to the beta core and to the exact core is a function of the Nakamura number, the depth of the semilattice and its gap function.

Keywords: Effectivity Function, Lattice, Stability Index, Equilibrium, Nakamura Number.

JEL Classification: C70, D71 AMS Classification: 91A44
Introduction

The object of this paper is to study the stability of the meet game form. Let \((A, \wedge)\) be a meet-semilattice. The \(n\)-player meet game form on \(A\) is defined as follows: each player chooses \(x_i \in A\), the outcome is given by \(\mu(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n\). Let \(\mathcal{M}\) be any subset of non-empty coalitions. Solutions that are considered in this paper are either the \(\beta\)-core, or the exact-core or Nash-like equilibrium where only coalitions in \(\mathcal{M}\) are active. Given a solution concept, stability means that for any preference profile, the game form admits at least one such solution, while the stability index is a measure of instability (see \([2]\) for an introduction to this notion). It turns out, that the stability and stability index depend on three parameters: On the side of the players the Nakamura number or \(\nu_M\), and on the side of the alternative set, the depth of \(A\) or \(\delta_A\) and the gap function or \(\gamma_A\).

1 Game forms

1.1 Notations

Throughout this paper we shall consider a finite set \(N = \{1, \ldots, n\}\) the elements of which are called players, and a finite set \(A = \{a_1, \ldots, a_p\}\) the elements of which are called alternatives. We make use of the following notational conventions: For any set \(X\), we denote by \(P(X)\) the set of all subsets of \(X\) and by \(P_0(X) = P(X) \setminus \{\emptyset\}\) the set of all non-empty subsets of \(X\). \(Q(X)\) (resp. \(L(X)\)) will denote the set of all preorders (resp. linear orders) on \(X\), that is all binary relations on \(X\) which are transitive and complete (resp. transitive, complete and antisymmetric). If \(R \in Q(X)\) we denote by \(R^s\) (resp. \(R^c\)) the strict binary relation (resp. the equivalence relation) induced by \(R\) on \(X\). Elements of \(P_0(N)\) are called coalitions. If \(S \in P_0(N)\) then \(N \setminus S\) is denoted \(S^c\). Similarly if \(B \in P(A)\), \(A \setminus B\) is denoted \(B^c\). A preference profile (over \(A\)) is a map from \(N\) to \(Q(A)\), so that a preference profile is an element of \(Q(A)^N\). For every preference profile \(R_N \in Q(A)^N\) and \(S \in P_0(N)\) we put

\[ P(a, S, R_N) = \{ b \in A \mid b R_i^c a, \forall i \in S \} \]

(so that \(P(a, S, R_N)\) consists of all the outcomes considered to be strictly better than \(a\) by all members of the coalition \(S\)), and \(P^c(a, S, R_N) = A \setminus P(a, S, R_N)\).

1.2 Game forms and solutions

Let \(G = (X_1, \ldots, X_n, A, g)\) be a strategic game form. The set of players is \(N = \{1, \ldots, n\}\), \(X_i\) is the strategy set of players \(i\), \(g : \prod_{i \in N} X_i \rightarrow A\) is the outcome function, assumed to be surjective. For any \(S \in P_0(N)\) the product \(\prod_{i \in S} X_i\) will be denoted \(X_S\). An element \((x_i)_{i \in N} \in X_N\) will be denoted simply \(x_N\) and its projection on \(X_S\) will be denoted \(x_S\). Given any preference profile \(R_N \in Q(A)^N\), the game form \(G\) induces a game \((X_1, \ldots, X_n; Q_1, \ldots, Q_n)\) with the same strategy spaces and where \(Q_i\) is the preorder on \(X_N\) defined by: \(x_N Q_i y_N\) if and only if \(g(x_N) R_i g(y_N)\) for \(x_N, y_N \in X_N\). We denote this game by \(G(R_N)\).
For our solution concepts we shall assume that only some coalitions can form. Any $M \subset P_0(N)$ is called an active coalition structure. The first solution concept is similar to Nash equilibrium. It has been introduced in [6] (definition 5.1.6):

- A strategy array $x_N \in X_N$ is an $M$-equilibrium of the game $G(R_N)$ if there is no coalition $S \in M$ and $y_S \in X_S$ such that $g(y_S, x_{S^c}) R_i^s g(x_N)$ for all $i \in S$. An alternative $a$ is an $M$-equilibrium outcome of $G$ at $R_N$ if there exists some equilibrium $x_N \in X_N$ of $G(R_N)$ such that $g(x_N) = a$. We denote by $EO(M)(G, R_N)$ the set of all $M$-equilibrium outcomes of $(G, R_N)$. In particular, when $M = N \equiv \{1, \ldots, n\}$, an $M$-equilibrium is a Nash equilibrium. Similarly, when $M = P_0(N)$, an $M$-equilibrium is a strong Nash equilibrium.

The following solutions have been defined respectively in [1] and [4]:

- An alternative $a$ is in the $M$-exact core of $(G, R_N)$ if there is no coalition $S \in M$ with the following property: for any $z_N \in X_N$ such that $g(z_N) = a$ there exists $y_S \in X_S$ such that $g(y_S, z_{S^c}) R_i^s g(z_N)$ for all $i \in S$. Denote by $C_{1,M}(G, R_N)$ the $M$-exact core of $(G, R_N)$.

- An alternative $a$ is in the $M$-β-core of $(G, R_N)$ if there is no coalition $S \in M$ with the following property: for any $z_N \in X_N$, there exists $y_S \in X_S$ such that $g(y_S, z_{S^c}) R_i^s a$ for all $i \in S$. Denote by $C_{\beta,M}(G, R_N)$, the $M$-β-core of $(G, R_N)$

Let $\Pi_r$ denote the set of all partitions of $A$ with $r$ elements (classes). If $\pi \in \Pi_r$ and $a \in A$ we denote by $\pi(a)$ the class of the partition that contains $a$. Let $Q_\pi(\pi)$ be the set of all $R \in Q(A)$ such that whenever $\pi(a) = \pi(b)$ then $a R^* b$. We say that $G$ is $r$-$M$-solvable if $EO(M)(G, R_N) \neq \emptyset$ for all $R_N \in Q(\pi)^N$ and all $\pi \in \Pi_r$. $G$ is $r$-$M$-exactly stable if $C_{1,M}(G, R_N) \neq \emptyset$ for all $R_N \in Q_\pi(\pi)^N$ and all $\pi \in \Pi_r$. $G$ is $r$-$M$-$\beta$-stable if $C_{\beta,M}(G, R_N) \neq \emptyset$ for all $R_N \in Q_\pi(\pi)^N$ and all $\pi \in \Pi_r$. We say that $G$ is $M$-solvable if $G$ is $r$-$M$-solvable for all $r \geq 1$. Similar definitions can be made for the $M$-exact core and the $M$-$\beta$-core.

**Definition 1.1** The stability index of $G$ relatively to the $M$-equilibrium (resp. $M$-exact core, resp. $M$-$\beta$-core) is the smallest integer $r \geq 1$ such $G$ is not $r$-$M$-solvable (resp. $r$-$M$-exactly stable, $r$-$M$-$\beta$-stable) (with the convention that the index is $+\infty$ if no such integer exists).

The object of this paper is to give necessary and sufficient conditions for stability and determine the stability index of the following game form, called the meet game form $\Gamma = \langle X_1, \ldots, X_n, A, \mu \rangle$, where $X_1 = \cdots = X_n = A$, $A$ is a meet-semilattice (precise definitions are given below), and $\mu$ is the meet function that is:

$$\mu(x_1, \ldots, x_n) = x_1 \land \cdots \land x_n \quad (x_1 \in A, \ldots, x_n \in A).$$

### 2 Definitions related to binary relations

For $q \in \mathbb{N}^+$ the set $\{1, \ldots, q\}$ will be denoted $\mathbb{I}_q$. An interval of $\mathbb{Z}/q\mathbb{Z}$ is any sequence $(k_1, \ldots, k_r)$ in $\mathbb{Z}/q\mathbb{Z}$, where $r \in \mathbb{I}_{q+1}$ and $k_{s+1} = k_s + 1$ ($s = 1, \ldots, r - 1$). Thus if $r < q + 1$, the elements of an interval $(k_1, \ldots, k_r)$ are distinct. When $k = q + 1$ we have $k_1 = k_r$ and the interval is said to be closed. A directed graph or digraph is an ordered pair $(A, \rhd)$ where $\rhd$ is a binary relation on $A$. A couple $(a, b) \in A \times A$ such that
$a \setminus b$ will be called a step. Let $q \in \mathbb{N}^*$. A $q$-enumeration of $A$ is an injective mapping $e : \mathbb{Z}/q\mathbb{Z} \to A$. Let $e$ be a $q$-enumeration of $A$. An $e$-edge is any ordered pair of the form $v = (e_k, e_{k+1})$ where $k \in \mathbb{Z}/q\mathbb{Z}$. Thus a 1-enumeration $e$ has only one edge $(e_1, e_1)$. Two $e$-edges $v$ and $w$ are said to be adjacent if $v = (e_k, e_{k+1})$ and $w = (e_\ell, e_{\ell+1})$ and $k + 1 = \ell$. An $e$-chain is any sequence $c = (v_1, \ldots, v_r)$ of distinct $e$-edges such that $v_k$ and $v_{k+1}$ are adjacent ($k = 1, \ldots, r - 1$). The length of $c$ is the number of its $e$-edges. It is denoted $|c|$. Since there are no repetition of edges in a chain: $|c| \leq q$. Alternatively, an $e$-chain is the image by $e$ of some interval of $\mathbb{Z}/q\mathbb{Z}$, with the order induced by $e$. There are exactly $q$ $e$-chains with length $q$, where only the initial vertex differ; we shall identify them all with $e$. An $e$-edge is an $e$-step if it is a step. We usually use the same notation for an $e$-chain (a sequence of $e$-edges) and the set of its edges. Thus $c \cap c' = \emptyset$ means that $c$ and $c'$ do not have common edges. Let $c$ and $c'$ be two $e$-chains such that $c' \subseteq c$. We say that $c'$ is a $c$-gap if, if $c'$ contains no steps and if it is maximal for inclusion in $c$ for this property. If $c$ is an $e$-chain, we denote by $d(c)$ the number of $e$-steps in $c$, and $g(c)$ the number of $c$-gaps. It is easy to see that $d(c) + g(c) \leq |c|$. For $k \geq 1$, let $C^k_c$ be the set of all $e$-chains such that $d(c) = k$. We introduce the following numbers related to the graph structure:

$$
\delta_A = \max_e d(e) \text{ where } e \text{ describes all the set of } p\text{-enumerations}.
$$

$$
\gamma_e(k) = \min_{c \in C^k_c} g(c) \text{ with the convention } \gamma_e(k) = +\infty \text{ if } C^k_c = \emptyset.
$$

$$
\gamma_A(k) = \min_e \gamma_e(k) \text{ where } e \text{ describes all the set of } p\text{-enumerations}.
$$

$\delta_A$ will be called the depth of $A$, $\gamma_e(.)$ will be called the gap function of $e$ and $\gamma_A(.)$ will be call the gap function of $A$. Remark that $\gamma_e$ and $\gamma_A$ are increasing functions. By convention $\gamma_e(+\infty) = \gamma_A(+\infty) = +\infty$.

**Examples 2.1**

(a) Let $A = \{1, \ldots, p\}$ and $a \setminus b$ if and only if $a = b + 1$ (addition in $\mathbb{N}$). Let $e$ be the $p$-enumeration $e(k) = p - k + 1 \mod p$ then $\delta_A = d(e) = p - 1$. $\gamma_e(k) = 0$ if $1 \leq k < p$, $\gamma_e(k) = +\infty$ if $k \geq p$. $\gamma_A = \gamma_e$.

(b) Let $A = \{1, \ldots, p\}$ ($p \geq 2$) and $a \setminus b$ if and only if $a = b + 1$ (addition in $\mathbb{Z}/p\mathbb{Z}$). Let $e$ be the $p$-enumeration $e(k) = p - k + 1$ then $\delta_A = d(e) = p$. $\gamma_e(k) = 0$ if $1 \leq k \leq p$, $\gamma_e(k) = +\infty$ if $k > p$. $\gamma_A = \gamma_e$.

(c) Let $A = \{1, \ldots, p\}$ ($p \geq 2$) and $a \setminus b$ if and only if $a = p$, $b \neq p$. Let $e$ be the $p$-enumeration $e(k) = p - k + 1$ then $\delta_A = d(e) = 1$. $\gamma_e(k) = 1$ if $k = 1$, $\gamma_e(k) = +\infty$ if $k > 1$. $\gamma_A = \gamma_e$.

A digraph $(A, \setminus)$ is said to be acyclic if for any $q \in \mathbb{N}^*$, any $q$-enumeration $e$ contains at least one $e$-gap. A partially ordered set, or poset, is a pair $(A, \geq)$ where $\geq$ is a binary relation on $A$ that is reflexive, transitive and antisymmetric. To a poset $(A, \geq)$ we shall associate the digraph $(A, >)$ where $x > y$ if and only if $x \geq y$ and $x \neq y$. $(A, >)$ is then an acyclic digraph. A poset is a meet-semilattice if any pair $\{x, y\} \subseteq A$ has an infimum, that is a greatest lower bound, denoted $x \wedge y$. The infimum of any family $(x_1, \ldots, x_k)$ will be denoted $x_1 \wedge \cdots \wedge x_k$.

**Definition 2.2** Let $(A, \setminus)$ be a digraph, let $e$ be a $q$-enumeration and let $\tilde{e}$ be a $p$-enumeration. $\tilde{e}$ is an extension of $e$ if there exists $k \in \mathbb{N}_q$ such that $(e_{k+1}, e_{k+2}, \ldots, e_{k+q})$ (mod $q$) is an $\tilde{e}$-chain, or equivalently if there is a bijection $j$ from $\mathbb{Z}/q\mathbb{Z}$ onto some interval of $\mathbb{Z}/p\mathbb{Z}$ such that $e = \tilde{e} \circ j$.
Lemma 2.3 Let \((A, \prec)\) be a digraph. Let \(e\) be a \(q\)-enumeration containing some gap \(h\) and some chain \(c\), and let \(k = d(c)\). There exists some \(e\)-chain \(c'\) such that \(d(c') = k\) and \(c' \cap h = \emptyset\). Any \(e\)-chain \(c'\) such that \(g(c') = \gamma_e(k)\) leaves some gap in its complement. In particular: \(g(c') < g(e)\).

Proof. All the \(e\)-steps are in \(\bar{h}\), therefore the first assertion. In particular: \(g(e) < g(e)\).

Lemma 2.4 Let \((A, \prec)\) be a digraph. Let \(e\) be a \(q\)-enumeration containing some gap \(h\) and some chain \(c\) that do not intersect. Then there exists an extension \(\tilde{e}\) of \(e\) such that \(c\) is an \(\tilde{e}\)-chain.

Proof. Let \(B = A \setminus \{e_1, \ldots, e_q\}\). Then \(|B| = p - q\). Let \(f\) be any bijection of \(\mathbb{I}_{p-q}\) onto \(B\). Without loss of generality let \((e_q, e_1)\) be some \(e\)-edge of \(h\). One can define \(\tilde{e}\) as follows: \(\tilde{e}(\ell) = e(\ell)\) for \(\ell \in \mathbb{I}_q\) and \(\tilde{e}(q + k) = f(k)\) for any \(k \in \mathbb{I}_{p-q}\). It is clear that \(c\) is an \(\tilde{e}\)-chain.

\[\square\]

Proposition 2.5 Let \((A, \prec)\) be an acyclic digraph, and let \(k \geq 1\). Then:

(i) \(\delta_A\) is the maximum of \(d(e)\) where \(e\) describes the union of all \(q\)-enumerations \((q = 1, \ldots, p)\).

(ii) \(\gamma_A(k)\) is the minimum of \(\gamma_e(k)\) where \(e\) describes the union of all \(q\)-enumerations \((q = 1, \ldots, p)\).

Proof of (i). If \(e\) is a \(q\)-enumeration, such that \(d(e)\) achieves the maximum defined in the statement, then, in view of the acyclicity of the digraph and lemma 2.3, there exists some \(e\)-chain \(c\) of \(e\) and some \(e\)-gap \(h\) such that \(d(c) = d(e)\) and \(h\) does not intersect \(c\). In view of lemma 2.4, there is an extension \(\tilde{e}\) of \(e\) such that \(c\) is a chain of \(\tilde{e}\). Since the number of steps of \(c\) is the same in \(e\) and \(\tilde{e}\). This proves (i).

Proof of (ii). If \(c\) is some \(e\)-chain where \(e\) is a \(q\)-enumeration, such that \(g(c)\) achieves the minimum defined in the statement, then, in view of lemma 2.3, there exists some \(e\)-gap \(h\) that does not intersect \(c\). In view of lemma 2.4, there is an extension \(\tilde{e}\) of \(e\) such that \(c\) is a chain of \(\tilde{e}\). Since the number of steps and gaps in \(c\) remain the same, (ii) is proved.

\[\square\]

3 Effectivity structures

Definition 3.1 A local effectivity function on \((N, A)\) is a family \(E \equiv (E[U], U \in \mathcal{P}_0(A))\) where for any \(U \in \mathcal{P}_0(A)\), \(E[U] : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A))\) and such that the following conditions are satisfied:

(i) \(E[U](\emptyset) = \emptyset\).

(ii) \(B \in E[U](S), B \subset B' \Rightarrow B' \in E[U](S)\).

(iii) \(U \subset V \Rightarrow E[V](S) \subset E[U](S)\).

A local effectivity function is an effectivity function if it does not depend on \(U\). The formula \(B \in E[U](S)\) is interpreted as follows: When the current state is in \(U\), coalition
S can adapt its response in order to achieve some state in B. Let \( R_N \in Q(A)^N \). An alternative \( a \in A \) is dominated at \( R_N \) if there exists \( U \in \mathcal{P}_0(A) \), \( S \in \mathcal{P}_0(N) \) such that \( a \in U \) and \( P(a, S, R_N) \in E[U](S) \). The core of \( E \) at \( R_N \) is the set of undominated alternatives. It is denoted \( C(E, R_N) \). We say that \( E \) is \( r \)-stable if \( C(E, R_N) \neq \emptyset \) for all \( R_N \in Q_\pi(r)^N \) and all \( \pi \in \Pi_r \). We say that \( E \) is stable if \( E \) is \( r \)-stable for all \( r \geq 1 \). The stability index of \( E \) is the minimal integer \( r \) such that \( E \) is not \( r \)-stable (with the convention that this index is \( +\infty \) if \( E \) is stable). It will be denoted \( \sigma(E) \).

Let \( G \) be a strategic game form. The local effectivity function \( E^G_{1,M} \) associated to \((G, \mathcal{M})\) is defined as follows: For \( U \in \mathcal{P}_0(A) \), \( S \notin \mathcal{M} \): \( E^G_{1,M}[U](S) = \emptyset \), and for \( S \in \mathcal{M} \):

\[
E^G_{1,M}[U](S) = \{ B \in \mathcal{P}_0(A)| \forall x_N \in g^{-1}(U), \exists y_S \in X_S : g(x_S, y_S) \in B \}
\]

The \( \beta \)-effectivity function associated to \((G, \mathcal{M})\) is defined by \( E^G_{0,M}(S) = E^G_{1,M}[A](S) \) \((S \in \mathcal{P}(N))\).

**Lemma 3.2** The \( \mathcal{M} \)-exact core (resp. \( \mathcal{M} \)-\( \beta \)-core) of \((G, R_N)\) coincides with the core of \( E^G_{1,M} \) (resp. \( E^G_{0,M} \)) at \( R_N \). Therefore \( G \) is \( r \)-\( \mathcal{M} \)-exactly stable (resp. \( r \)-\( \mathcal{M} \)-\( \beta \)-stable) if and only if \( E^G_{1,M} \) (resp. \( E^G_{0,M} \)) is \( r \)-stable. In particular the stability index relatively to the \( \mathcal{M} \)-exact core of \( G \) is equal to the stability index of \( E^G_{1,M} \).

**Proof.** Straightforward. 

**Definition 3.3** Let \( E \) be a local effectivity function. An \( r \)-tuple \( ((C_1, B_1, S_1), \ldots, (C_r, B_r, S_r)) \) where \( r \geq 1 \), \( C_k \in \mathcal{P}_0(A) \), \( B_k \in \mathcal{P}_0(A) \), \( S_k \in \mathcal{P}_0(N) \) \((k = 1, \ldots, r)\) is a dominance configuration of \( E \) if:

(i) \( B_k \in E[C_k](S_k) \) \((k = 1, \ldots, r)\).

(ii) \( (C_1, \ldots, C_r) \) is a partition of \( E \).

\((C_1, \ldots, C_r)\) is said to be the basis of the dominance configuration and \( r \) its length or order.

A dominance configuration \((((C_1, B_1, S_1), \ldots, (C_r, B_r, S_r))\) is a cycle of \( E \) if it satisfies the following property:

(C) For any \( \emptyset \neq J \subset \{1, \ldots, r\} \) such that \( \cap_{k \in J} S_k \neq \emptyset \), there exists \( k \in J \) such that for all \( l \in J \): \( B_k \cap C_l = \emptyset \).

In the context of effectivity functions cycles have been introduced in [3]. They generalize the Condorcet cycle and play a fundamental role in studying stability. In view of Abdou [2] Theorems 4.1, 4.2, and 4.4, we have:

**Theorem 3.4** The stability index of a local effectivity function \( E \) is equal to the minimal length of a cycle of \( E \) (with the convention that this number is \( +\infty \) if \( E \) has no cycle).

Finally we need to recall from [5] a classical definition. Let \( \mathcal{M} \) be an active coalition structure. A nonempty subset \( T \subset \mathcal{M} \) has the empty intersection property if \( \cap_{S \in T} S = \emptyset \). The Nakamura Number of \( \mathcal{M} \), denoted \( \nu_{\mathcal{M}} \), is the minimum of the cardinality of \( T \) where \( T \) describes all the subsets of \( \mathcal{M} \) with the empty intersection property (with the convention that this number is \( +\infty \) if no subset of \( \mathcal{M} \) has the empty intersection property).
4 The meet game form

In this section $(A, \geq)$ is a meet-semilattice and $\Gamma = (X_1, \ldots, X_n, A, \mu)$ is the meet game form (1) defined on $A$. $\Gamma$ has the following remarkable property:

**Proposition 4.1** For any $R_N \in Q(A)^N$, an outcome is an $M$-equilibrium outcome of $\Gamma$ if and only if it is in the $M$-exact core of $\Gamma$ that is:

$$EO(M)(\Gamma, R_N) = C_{1, M}(\Gamma, R_N)$$

**Proof.** $EO(M)(\Gamma, R_N) \subset C_{1, M}(\Gamma, R_N)$ for any game form. In order to prove the opposite inclusion, let $a \notin EO(M)(\Gamma, R_N)$. For any $x = (x_1, \ldots, x_n)$ such that $\mu(x) = a$, there exists some $S_x \in M$ and $y_{S_x}$ such that $\mu(x_{S_x}, y_{S_x}) R^a a$ for all $i \in S_x$. The main point is to prove that one can choose some “deviation” $(S_x, y_{S_x})$ that do not depend on $x$. Let $S$ be the coalition corresponding to $\bar{x} = (a, \ldots, a)$ and let $b = \mu(\bar{x}_{S'}, y_S)$. One has $bR_i^a a$ for all $i \in S$. Let $c = \wedge_{i \in S} y_i$. Then $a \wedge c = b$. Clearly $b \neq a$. If $S = N$ then for any $x$ such that $\mu(x) = a, \mu(y_N) = b = c$ thus $a \notin C_{1, M}(\Gamma, R_N)$. If $S \neq N$ then $b < a$. Let $b_S \in A_S$ with all components equal to $b$. For any $x$ such that $\mu(x) = a$ one has: $b \leq x \wedge \wedge_{j \in S'} x_j$. It follows that $\mu(x_{S'}, b_S) = (\wedge_{j \in S'} x_j) \wedge b = b$. Again $a \notin C_{1, M}(\Gamma, R_N)$. \hfill \Box

**Corollary 4.2** The meet game form $\Gamma$ is $M$-solvable if and only if it is $M$-exactly stable. The stability index of $\Gamma$ is the same whether we consider the $M$-exact core or the $M$-equilibrium.

Thus studying stability of the local effectivity function is sufficient not only for $M$-exact stability of $\Gamma$, but also for its $M$-solvability. Here is its precise description for any $M$:

**Proposition 4.3** For $a \in A$, one has:

$$E_{1, M}^f[U](S) = \begin{cases} \{B \in P_0(A) | \forall a \in U, \exists b \in B : a \geq b\} & \text{if } S \in M, S \neq N \\ P_0(A) & \text{if } S \in M, S = N \\ \emptyset & \text{if } S \notin M \end{cases}$$

**Proof.** Since $E_{1, M}^f[U](S) = \cap_{a \in U} E_{1, M}^f[a](S)$, it is enough to prove the formula for $E_{1, M}^f[a](S)$ ($a \in A$). That $E_{1, M}^f[a](N) = P_0(A)$ is straightforward. Let $S \in P_0(N)$, $S \neq N$ and let $B \in P_0(A)$ and $b \in A$ such that $b \in B$ and $b \leq a$. Let $b_S \in A_S$ with all components equal to $b$. For any $x$ such that $\mu(x) = a, b \leq a \leq (\wedge_{j \in S'} x_j)$. Thus $\mu(x_{S'}, b_S) = (\wedge_{j \in S'} x_j) \wedge b = b$. Therefore $B \in E_{1, M}^f[a](S)$. Conversely if $B \in E_{1, M}^f[a](S)$, then in particular taking $x = (a, \ldots, a) \in A^N$ there exists $y_S \in A_S$ such that $\mu(x, y_S) \in B$. Since $\mu(x, y_S) \leq a$, the proof is complete. \hfill \Box

We conclude this section, by a statement of the main results of the paper:

**Theorem 4.4** For any $M$, the meet game form $\Gamma$ is $M$-$\beta$-stable. $\Gamma$ is $M$-exactly stable (and therefore $M$-solvable) if and only if either $N \notin M$ or $\delta_A < \nu_M$. 

7
In particular the meet game form is Nash solvable. If \( n \geq 2 \) and \( \mathcal{M} = \mathcal{P}_0(\mathcal{N}) \), then \( \nu_{\mathcal{M}} = 2 \). Thus the meet game form is strongly solvable if and only if \( \delta_A = 1 \).

**Theorem 4.5** If \( N \in \mathcal{M} \), the stability index of the meet game form relatively to the \( \mathcal{M} \)-exact core (and therefore \( \mathcal{M} \)-equilibrium) is equal to: \( \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1 \).

In particular, if \( n \geq 2 \) and \( \mathcal{M} = \mathcal{P}_0(\mathcal{N}) \), the strong Nash stability index of the meet game form is equal to \( \gamma_A(\nu_{\mathcal{M}}) + 3 \). In the next section, we give a proof of both theorems in a more general framework.

## 5 Stability and Index of the meet game form

In this section we assume that \((A, \prec)\) is an acyclic digraph. We shall write \((a \prec b)\) if \((a, b) \in E\) or \((a = b)\). For any \( \emptyset \neq \mathcal{M} \subset \mathcal{P}_0(\mathcal{N}) \), we consider the local effectivity function \( E_{\mathcal{M}} \) defined as follows: For \( U \in \mathcal{P}_0(A) \):

\[
E_{\mathcal{M}}[U](S) = \begin{cases} 
\{B \in \mathcal{P}_0(A) \mid \forall a \in U, \exists b \in B : a \prec b\} & \text{if } S \in \mathcal{M}, S \neq N \\
\mathcal{P}_0(A) & \text{if } S \in \mathcal{M}, S = N \\
\emptyset & \text{if } S \notin \mathcal{M}
\end{cases}
\]

The corresponding effectivity function is defined by \( E_{0,\mathcal{M}}[U](S) = E_{0,\mathcal{M}}(S) = E_{\mathcal{M}}[A](S) \) \((S \in \mathcal{P}(\mathcal{N}))\). Let \( A_0 \) be the set of minimal elements of \((A, \prec)\): \( x \in A_0 \) if and only if there is no \( y \in A \) such that \( x \prec y \). Since \( A \) is finite and \((A, \prec)\) acyclic, \( A_0 \neq \emptyset \). It is then easy to see that, for any \( S \in \mathcal{M}, S \neq N \) any \( B \in E_{0,\mathcal{M}}(S) \) contains \( A_0 \). In the case where \( \prec \) is a poset, the converse is also true: \( B \in E_{0,\mathcal{M}}(S) \) if and only if \( A_0 \subseteq B \).

**Lemma 5.1** Let \((U_1, \ldots, U_r)\) be a partition of \( A \) and let \((B_1, \ldots, B_r)\) be a family of nonempty subsets of \( A \). Then there exists a subset \( I = \{k_1, \ldots, k_s\} \) where \( 1 \leq s \leq r \) such that \( B_{k_j} \cap U_{k_{j+1}} \neq \emptyset \) \((j = 1, \ldots, s) \) (mod s).

**Proof.** Let \( I \) be the set of nonempty subsets \( I \subseteq \mathbb{I}_r \) such that for any \( k \in I \) there exists \( \ell \in I \) such that \( B_k \cap C_\ell \neq \emptyset \). Clearly \( \mathbb{I}_r \in I \). Let \( I_0 \) be a minimal set for inclusion in \( I \). For any \( k \in I_0 \) put \( \theta(k) \) one of the indices \( l \in I_0 \) such that \( B_k \cap U_l \neq \emptyset \). Take \( k_1 \in I_0 \) arbitrary and put \( k_{j+1} = \theta(k_j) j = 1, 2, \ldots \). By minimality of \( I_0 \), the sequence \((k_1, \ldots, k_s)\) is composed of distinct indices and \( k_{s+1} = k_1 \). \( \square \)

**Theorem 5.2** \( E_{0,\mathcal{M}} \) is stable for any \( \mathcal{M} \).

**Proof.** Assume that \( C(E_{0,\mathcal{M}}, R_N) = \emptyset \) for some \( R_N \in Q(A)^N \). Let \( x_0 \in A_0 \). Then \( P(x_0, S, R_N) \in E_{0,\mathcal{M}}(S) \) for some \( S \in \mathcal{M} \). In view of the remark preceding Lemma 5.1, we cannot have \( S \neq N \); indeed \( x_0 \in A_0 \) and \( x_0 \notin P(x_0, S, R_N) \). it follows that \( S = N \). Therefore \( N \in \mathcal{M} \). Moreover, one can construct by induction a sequence \( x_0, \ldots, x_{t+1} \) such that \( x_k \) is Pareto dominated by \( x_{k+1} \) for \( k = 0, \ldots, t-1 \) and \( x_{t+1} \) not Pareto dominated. Two consequences follow: (1) \( x_{t+1} \in P(x_0, N, R_N) \) and (2): there exists some \( S \in \mathcal{M}, S \neq N \) such that \( P(x_{t+1}, S, R_N) \in E_{0,\mathcal{M}}(S) \). Since \( x_0 \in A_0 \subset P(x_{t+1}, S, R_N) \) we have \( x_0 \in P(x_{t+1}, S, R_N) \). The latter contradicts (1). \( \square \)

**Theorem 5.3** \( E_{\mathcal{M}} \) is stable if and only if either \( N \notin \mathcal{M} \) or \( \delta_A < \nu_{\mathcal{M}} \).
Proof. Assume that $E_M$ is not stable. Let $R_N \in Q(A)^N$ be such that $C(E_M, R_N)$ is empty. Put $A = \{a_1, \ldots, a_p\}$. For any $k \in \mathbb{I}_p$, there exists $b_k \in A$ and $S_k \in \mathcal{M}$ such that $\{b_k\} \in E_M(S_k)$, and $b_k R_i a_k$ for all $i \in S_k$. Let $U_k = \{a_k\}$ and $B_k = \{b_k\}$. By Lemma 5.1, there exists a subset $I = \{k_1, \ldots, k_s\}$ where $1 \leq s \leq p$ such that $b_{k_j} = a_{k_{j+1}}$ $(j = 1, \ldots, s)$ $(\mod s)$. Let $e(j) = b_{k_j}$ for all $j \in \mathbb{I}_s$. Let $J = \{k \in \mathbb{I}_p \mid S_k \neq N\}$. For all $k \in J$, $a_k \backslash b_k$. Then $e$ is an $s$-enumeration, such that $e(j-1) \backslash e(j)$ if $k_j \in J$. It follows first that $I$ is not a subset of $J$, for otherwise $e$ would be a cycle for the binary relation $\backslash$, and the latter is acyclic. Therefore there exists $i \in I$ such that $S_i = N$, hence $N \in \mathcal{M}$. It follows also that $|J \cap I| \leq d(e) \leq \delta_A$. On the other hand since $e(j) R_i e(j-1)$ for all $i \in S_{k_j} \ (\text{mod} s)$ we have $\cap_{k \in I \cap S_k} \in \mathbb{I}_k \neq \emptyset$, so that $\nu_M \leq |J \cap I|$. We conclude that $\nu_M \leq \delta_A$. Conversely, assume that $N \in \mathcal{M}$ and $\nu_M \leq \delta_A$. Let $e$ be a $p$-enumeration such that $d(e) = \delta_A$. Let $J$ be the set of indices $k \in \{1, \ldots, p\}$ such that $e_k \backslash e_{k+1}$ $(\text{mod} \ p)$. Then $|J| = \delta_A$. Let $J \subset I$ such that $|I| = \nu_M$. Let $(T_k, k \in I)$ be a family of elements of $\mathcal{M}$ such that $\cap_{k \in I} T_k = \emptyset$. We consider the $n$-tuple $F = ((U_1, B_1, S_1), \ldots, (U_p, B_p, S_p))$ defined as follows: For $k \in \{1, \ldots, p\}$, $U_k := \{e_k\}$, $B_k := \{e_{k+1}\}$ $(\text{mod} \ p)$; if $k \in I$, $S_k := T_k$ and if $k \notin I$, $S_k := N$. Since $B_k \in E[U_k](S_k)$ for all $k \in \mathbb{I}_p$, $F$ is a dominance configuration. We now show that this configuration verifies property (C) of definition 3.3. Let $K$ be any subset of $\{1, \ldots, p\}$ such that $\cap_{k \in K} S_k \neq \emptyset$. There exists some $k_0 \in I$ such that $k_0 \notin K$. Let $k_1$ be the first index in $K$ that appears after $k_0$ $(\text{mod} \ p)$. $U_{k_1} = \{e_{k_1}\}$ is such that $U_{k_1} \cap (\cup_{k \in K} B_k) = \emptyset$. Thus $F$ is a cycle. \[ \square \]

**Theorem 5.4** Assume $N \in \mathcal{M}$. We have the equality: $\sigma(E_M) = \nu_M + \gamma_A(\nu_M) + 1$.

Proof. We first consider the particular case where $\nu_M > \delta_A$. In view of Theorem 5.3, $E_M$ is stable, so that $\sigma = +\infty$. If $\nu_M = +\infty$ then the equality is verified. If $\nu_M < +\infty$ then by definition, since $\nu_M > \delta_A$, one has $\gamma_A(\nu_M) = +\infty$. Again the equality is verified.

Assume that $\nu_M \leq \delta_A$. Let $F = ((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ be a cycle. We are going to prove $r \geq \nu_M + \gamma_A(\nu_M) + 1$. In view of the structure of $E_M$ there exists $\varphi: A \rightarrow A$ with the following properties: (1) $x \in U_k \Rightarrow \varphi(x) \in B_k$ and (2) $T_k \neq N \Rightarrow x \backslash \varphi(x)$. Since the set of minimal elements $A_0$ is non empty we choose $x_1 \in A_0$. We construct a sequence $(x_k)$ in $A$ by induction by putting $x_{k+1} = \varphi(x_k)$ $k = 1, 2, \ldots$, and a sequence $(t_k)$ in $\mathbb{I}_p$ by defining $t_k$ as the unique element in $\mathbb{I}_r$ such that $x_k \in U_{t_k}$. Let $k_1 \in N^*$ be the smallest integer such there exists $k_2 \in N^*$, $k_2 > k_1$ and $t_k = t_{k_2}$ for all $k_1, k_2 \in N$. Clearly $(t_1, \ldots, t_{k-1})$ are all distinct. Therefore $k_2 - 1 \leq r$. We distinguish 4 cases:

Case 1. $k_1 > 1$ and $x_{k_2} \neq x_{k_1}$. We put $c = (x_{k_1}, \ldots, x_{k_2})$ and $e = (x_{k_1}, \ldots, x_{k_2}, x_{k_1})$. $e$ is a $q$-enumeration, $c$ is an $e$-chain and $q = k_2 - k_1 + 1$. Thus $q \leq (r + 1) - 2 + 1 = r$. 

Case 2. $k_1 > 1$ and $x_{k_2} = x_{k_1}$. We put $c = e = (x_{k_1}, \ldots, x_{k_2-1}, x_{k_2})$ this is a $q$-enumeration with $q \leq r - 1$.

Case 3. $k_1 = 1$ and $x_{k_2} \neq x_{k_1}$. We put $c = (x_{k_1}, \ldots, x_{k_2})$ and $e = (x_{k_1}, \ldots, x_{k_2}, x_{k_1})$. $e$ is a $q$-enumeration, $c$ is an $e$-chain and $q \leq (r + 1) - 1 + 1 = r + 1$. Since $x_k = x_1 \in A_0$, $(x_{k_1}, x_{k_2}, x_{k_1})$ is not a step.

Case 4. $k_1 = 1$ and $x_{k_2} = x_{k_1}$. We put $c = e = (x_{k_1}, \ldots, x_{k_2-1}, x_{k_2})$ this is a $q$-enumeration with $q \leq r$. 

9
First we establish a lower bound on the depth of $c$. Precisely we prove:

Claim. $d(c) \geq \nu_M$.

Prove of the claim. Put $I = \{k_1, \ldots, k_2 - 1\}$, $J = \{k \in I \mid S_k \neq N\}$. We claim that:

$\cap_{k \in I} S_k = \emptyset$. The proof is by contradiction: Assume that $\cap_{k \in I} S_k = \cap_{k \in J} S_k \neq \emptyset$, then by property (C) of cycles there exists $\ell \in I$ such that for all $k \in I : U_{\ell} \cap B_k = \emptyset$. If $\ell \neq k_1$, $x_\ell \in U_{\ell}$ and $x_\ell \in B_{\ell-1}$, a contradiction. If $\ell = k_1$, then in cases 2 and 4, $x_{k_1} \in U_{k_1}$ and $x_{k_1} \in B_{k_1-1}$, a contradiction, and in cases 1 and 3, we have $U_{k_1} = U_{k_2}$, $x_{k_2} \in U_{k_2}$ and $x_{k_2} \in B_{k_2-1}$, again a contradiction. Thus we proved $\cap_{k \in I} S_k = \emptyset$. It follows that $|J| \geq \nu_M$. Put $v_k = (x_k, x_{k+1}) (k \in I)$. For any $k \in J$, $v_k$ is a step. Therefore $d(c) \geq |J|$. Thus $d(c) \geq \nu_M$, and our claim is proved.

Now, we establish a lower bound on the number of gaps in $e$ and conclude by the desired inequality.

Cases 2 and 4. Here $c = e$. Using lemma 2.3 for the first inequality and monotonicity of $\gamma_A$ for the third, one has:

$g(e) \geq \gamma_e(d(e)) + 1 \geq \gamma_A(d(e)) + 1 \geq \gamma_A(\nu_M) + 1$

$q = |e| \geq d(e) + g(e) \geq \nu_M + \gamma_A(\nu_M) + 1$

Moreover $r \geq q$, therefore: $r \geq \nu_M + \gamma_A(\nu_M) + 1$.

Cases 1 and 3. Here $c$ is an $e$-chain, $c = e \setminus (x_{k_2}, x_{k_1})$.

$g(c) \geq \gamma_e(d(c)) \geq \gamma_A(d(c)) \geq \gamma_A(\nu_M)$. 

Case 1. $|c| \geq d(c) + g(e) \geq \nu_M + \gamma_A(\nu_M)$

$r \geq q = |c| \geq \nu_M + \gamma_A(\nu_M) + 1$.

Case 3. Here $v_{k_1} = (x_{k_1}, x_{k_1+1})$ is not a step. Let $c' = c \setminus \{v_{k_1}\}$. We have $d(c') = d(c)$.

$|c| - 1 = |c'| \geq d(c') + g(c') = d(c) + g(c') \geq d(c) + \gamma_e(d(c))$

$|c| \geq d(c) + \gamma_e(d(c)) + 1 \geq \nu_M + \gamma_A(\nu_M) + 1$.

Since $q = |c| + 1$ and $r \geq q - 1$, we have: $r \geq \nu_M + \gamma_A(\nu_M) + 1$.

In conclusion we have in all cases the inequality $r \geq \nu_M + \gamma_A(\nu_M) + 1$. Thus $\sigma(E_M) \geq \nu_M + \gamma_A(\nu_M) + 1$. This ends the first part of the proof.

Conversely let $r = \nu_M + \gamma_A(\nu_M) + 1$. We are going to construct a cycle of order $r$. Let $e$ be a $p$-enumeration and let $c$ be an $e$-chain such that $d(c) = \nu_M$ and $g(c) = \gamma_A(\nu_M)$. It follows that $\nu_M + \gamma_A(\nu_M) \leq |c|$. In view of lemma 2.3, $\bar{c}$, the complement of $c$ in $e$ contains at least an edge that is not a step. Without loss of generality let $c = (e_1, \ldots, e_q)$ where $q \leq p$. Since $r - 1 = d(c) + g(c)$, we can write $c$ as a sequence of $e$-steps and $e$-gaps $(h_1, \ldots, h_{r-1})$. Moreover we put $h_r = (e_q, \ldots, e_p, e_1)$. Let $J$ be the set of indices $k \in \{1, \ldots, r-1\}$ such that $h_k$ is a step. Let $(T_k, k \in J)$ a family of elements in $M \setminus \{N\}$ such that $\cap_{k \in J} T_k = \emptyset$. Let $f(h_k)$ (resp. $h_k$) be the final node (resp. the set of nodes) of $h_k$ ( $k \in \bar{I}_r$). In particular: $f(h_r) = e_1$. Let $U_k = [h_k] \setminus \{f(h_k)\}$, $B_k = \{f(h_k)\}$ ( $k \in \bar{I}_r$). Let $S_k = T_k$ for all $k \in J$ and $S_k = N$ for all $k \in \bar{I}_r \setminus J$. We claim that $F = ((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ is a cycle of $E_M$. The only point that we need to verify is property (C) of definition 3.3. Let $K \subseteq \bar{I}_r$ such that $\cap_{k \in K} \neq \emptyset$. There exists some $k_0 \in J \setminus K$. Let $\ell$ be the first index that comes after $k_0$ (mod $r$) such that $\ell \in K$. One has $U_\ell \cap B_\ell = \emptyset$ for all $k \in K$. This shows that we have a cycle of order $r$. This
shows that $\sigma(E_M) \leq \nu_M + \gamma_A(\nu_M) + 1$. The proof is complete. \qed

References


