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Ordering operations in square root extractions

Analyzing some early medieval Sanskrit mathematical texts with the help of Speech Act Theory*

Agathe Keller

Abstract

Rules for extracting square roots written in Sanskrit in two treatises and their commentaries (from the Vth to the Xth century) are approached with questions inspired by textology and Speech Act Theory. An analysis of the number and order of steps spelled out is used to show that their aims were not limited to describing how to carry out the algorithm. The intentions of authors of mathematical texts in Sanskrit are questioned by lending an ear to the expressivity of relations established between world and text.

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July 29, 2010

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1 Introduction

The Sanskrit scholarly tradition of composing compact procedural *sūtras* with hairsplitting prose commentaries offers a fertile field to reflect, as Speech Acts Theory (SAT) does, on how prescriptive discourses relate to the real world. Studying the construction and composition of hermetic algorithmic statements and the way they are unravelled in commentaries sheds light on how diverse the relations can be between algorithmical texts and the algorithms they refer to. In other words the way a procedure is stated and the way a procedure is carried out are two different realities whose relations are studied here in a very specific case.

Most available Sanskrit sources on mathematics provide procedures to extract square and cube roots (*vargamūla*, *ghanamūla*). Such rules were part of the set of elementary operations (*parikarma*, *vidha*) that formed the basis of arithmetic and algebra. The square root procedure has remained unchanged, small details of inner working set aside, from the end of the Vth century to at least the XIIth century and probably later¹.

Two, tightly linked, kinds of texts have handed down these rules : treatises and commentaries. This study then will bring to light different ways in which a treatise and its commentary worked out the tension of how a procedure should be stated versus how a procedure should be executed. This analysis is part of a larger endeavour, aiming at studying descriptive practices in Sanskrit mathematical texts, while focussing on how commentaries relate to their treatises².

In the following, the focus will be on how different steps of a square root extraction are spelled out both by treatises and commentaries. Attention will be paid to how different actions of the procedure are explained and stated, in order to unravel the intentions with which these rules were composed.

1.1 Corpus

Five Sanskrit mathematical texts serve as a basis for this study, as illustrated in Figure 1. First, an astronomical *siddhānta* (that is a theoretical text)³ of the Vth century, the

¹Starting with the procedure given by Āryabhaṭa (499), remaining still similar in Bhāskarācārya's (fl. 1114) *Līlāvati* and *Bījagaṇita*, and thus still alive in later commentaries of these texts.

²(Keller, 2010)

³(Pingree, 1981, 13)

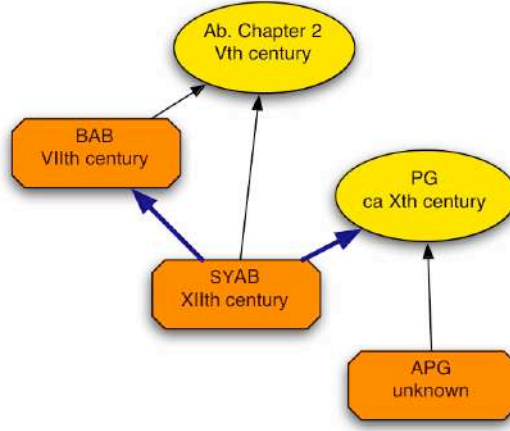


Figure 1: A connected set of five Sanskrit texts on mathematics

Āryabhaṭīya (Ab)⁴, and two of its prose and rather prolix commentators: Bhāskara's, seventh century, *Āryabhaṭīyabhāṣya*⁵ (BAB) and Sūryadeva Yajvan's (SYAb) twelfth century *Bhaṭaparakāśikā*⁶. Then, the “practical” *Pāṭīgaṇita* of Śrīdhara (ca. Vth-Xth century) and its anonymous and undated commentary (APG)⁷. These texts belong to the early medieval period of Indian mathematics: after the antique tradition of ritual geometry stated in the *sūlbasūtras*, before the popular synthetical accomplishment of Bhāskara II (XIIth century) in the *Līlāvatī* and the *Bījagaṇita*.

The corpus consists in a connected set of texts, although they have been composed in different times and places. As seen in Figure 1, commentaries, naturally, are linked to the text they comment. Furthermore, one commentary of the Ab, SYAb, has read PG, and quotes it. It also often paraphrases BAB. Considered together, these texts belong and testify of the Sanskrit cosmopolitan culture in mathematics of the Vth-XIIth century⁸. However the two treatises examined are of different nature: as stated previously the Ab is an astronomical theoretical text, with one chapter devoted to mathematics (*gaṇita*), while PG is solely a mathematical text, devoted to wordly (eg. everyday) practice (*lokavyavahāra*)⁹. Practices of procedure stating changed from one kind of a text to

⁴(K. V. Shukla & Sharma, 1976).

⁵(K. S. Shukla, 1976, 52-53). A translation of Bhāskara's commentary to Āryabhaṭa's verse on root extraction can be found in (Keller, 2006, Volume 1, p. 20-21), and an explanation of the process in (Keller, 2006, Volume 2, p. 15-18).

⁶(Sarma, 1976) A translation of his commentary on Āryabhaṭa's verse for square root extractions is given in Appendix C.

⁷(K. S. Shukla, 1959). A translation of the anonymous and undated commentary on *Pāṭīgaṇita*'s rules for extracting square roots is given in Appendix D.

⁸(Pollock, 2006, Part I)

⁹Thus Śrīdhara starts his treatise by the following statement, (K. S. Shukla, 1959, Sanskrit: i, English: 1):

another. The aim of this study then is to forge tools to better describe and understand such differences.

The rules given by Āryabhaṭa and Śrīdhara are collected with their Sanskrit transliteration in Table 1. In the following, different analysis of these rules implicitly suppose that the reader has this table at hand, and can compare and analyze graphics, lists etc. with the texts presented here.

Table 1: Two Rules to extract Square Roots

Treatise	Sanskrit Transliteration	English Translation ^a
Ab.2.4.ab ^b	<i>bhāgaṃ hared avargān nityaṃ dviguṇena vargamūlena </i>	One should divide, constantly, the non-square [place] by twice the square-root
Ab.2.4.cd	<i>vargād varge śuddhe labdhaṃ sthānāntare mūlam </i>	When the square has been subtracted from the square [place], the result is a root in a different place
PG.25.abcd ^c	<i>viśamāt padas tyaktyvā vargaṃ sthānacyutena mūlena dviguṇena bhajec cheṣaṃ labdhaṃ viniveśayet pañktau </i>	Having removed the square from the odd term, one should divide the remainder by twice the root that has dropped down to a place [and] insert the quotient on a line
PG.26.abcd	<i>tadvargaṃ saṃśodhya dviguṇaṃ kurvīt purvaval labdham utsārya tato vib- hajec śeṣaṃ dviguṇīkṛtaṃ dalayet </i>	Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. [Finally] one should halve what has been doubled.

^a [] indicate my own completions.

^b (K. S. Shukla, 1976, 52).

^c (K. S. Shukla, 1959, Sanskrit text, 18; English Translation, 9)

PG.1cd (aham) lokavyavahārārthaṃ gaṇitaṃ saṃkṣepato vakṣye||

I will briefly state mathematics aiming at wordly practice

1.2 The underlying mathematical idea of square root extraction procedures

The process to extract square root relies on the decompositional nature of the decimal place value notation: the number whose square root should be extracted, is considered to be a square. Extracting its square root means recovering the different elements of a developed square. In other words, if we take the numerical example of APG, 186 624, the process will try to uncover the different b_i (that is both the values of b and i , i giving the powers of ten concerned) such as

$$186624 = \left(\sum_{i=0}^p b_i 10^i \right)^2$$

To do so, the process will consider the decimal development of 186 624 as a sum of squares, $\sum_{i=0}^p b_i^2 10^{2i}$, and of double products of the type $2b_i b_j 10^{i+j}$, for $0 \leq i, j \leq p$ and $i + j \leq 2p$. Consequently, the process of extracting square roots, an iterative process, will be emphasized as repeated subtractions of squares, and divisions by a doubled number. The repeated division by a doubled number explains the difference between the process provided by Ab and the one given by PG: PG will arrive at a doubled root (useful during the process), while Ab describes a process that enables one to obtain immediately the square root.

1.3 The procedure to extract square roots in Sanskrit texts: a difficult question of description

Unearthing intentions and meanings of Sanskrit mathematical texts is made difficult by the fact that as historians, we know little of the context in which mathematical texts were produced and used. Furthermore, I do not possess native knowledge of Sanskrit. To put it with Austin's words, the accompaniments and circumstances of the utterance of *sūtras* are largely lost to us as readers today¹¹. Or, to state the difficulties inherent to the historian's trade according to Searle's categories¹², and as described by Virbel in this volume, condition 1 of Searle's "how to promise" (e.g. in this case, being able to carry out an algorithm) involves native knowledge of the language. Furthermore ignorance of the context means that we cannot know Searle's conditions 4, 6, 9. Indeed, we are not

¹¹(Austin, 1962, 76)

¹²(Searle, 1969, 57-61)

sure of the author's aim (6), or of his imagined reader or hearer (4) and are ill judges of then of the correctness with which the texts that transmit the author's intentions in this respect (9). As pointed out by Virbel then, certain conditons on the possibility of communication (1 and 9) and of making promisses (4 and 6) are not fulfilled. Nonetheless, in the following, treading carefully, intentions of Sanskrit authors of statements on mathematical algorithms will be discussed. To do so, the light shed on such authors by their own commentators will be used.

The procedure to extract square root has consistently attracted attention of historians of mathematics working on India. It testifies of an early use of the decimal place value notation. Furthermore, the process found here is very similar to the one taught until the middle of the XXth century in primary schools in Europe, the United States and probably elsewhere in the world. However, variations in time mingled with the concision of given rules has questioned how the procedure was carried out practically, step by step, for different authors. Different reconstructions have been offered by secondary sources, from Singh¹³ to more recently Plofker¹⁴. The practical performance of such processes is however but little discussed and justified by a close quotations of sources.

Indeed, several difficulties are layered in such reconstructions.

Even as we set aside the difficulties inherent to the historian's trade, the reconstruction of a procedural text is made difficult because of what one may term, following K. Chemla in this volume, the *granularity* of steps. This problem is certainly familiar to any one who has had to describe an algorithm: a stated step can often hide several others. For instance when "one should subtract the square from the square", given that the numbers and place where the subtraction should be carried out are given, this operation will be considered as a unique step, although a procedure for subtraction or an operation for squaring might describe such operations in many steps. Thus "elementary" operations in a more complex algorithm are stated without being described.

Of course, part of this granularity may have to do with "tacit knowledge". Thus some steps may have been considered so obvious that they did not need to be stated. For instance, none of the authors considered here specify that the remainder of the division should replace the initial dividend, or that the remainder of the subtraction replaces the subtrahend. Similarly, they do not state explicitly that after each arithmetical operation (division, subtraction) one needs to move to one slot to the right. Since all texts silence these steps that however are required, they may thus be considered tacit, as illustrated

¹³(Singh, 1927)

¹⁴(Plofker, 2009, 123-125)

in Figure 2. Furthermore, commentators of theoretical *siddhāntas* may have considered, tacitly, that “practical” steps of the process were not to be specified. This may explain why so little layouts are indeed provided in handed down texts. The following study will focus on the steps that are actually stated leaving the tacit in the shade.

However, concentrating on the corpus’s “actually stated”, or “spelled out” actions only helps pinpointing the difficulties of defining and specifying what “spelling out an algorithm’s step” actually means. Indeed, obstacles in recovering “the algorithm” may be inherent to the complexity of the object itself; more so while exploring the difference between saying and doing. A symptom of this difficulty has pervaded the writing of this article: each new approaches to the (texts on) square root extraction, induced a new representation of the algorithm. Each new representation never exactly coincided with others. Of course, I could endlessly try to coordinate each different depiction: checking that they all keep the same number of steps, respect the same singled out actions, or the hierarchies between different operations. But I’ve finally decided on the contrary, to leave each description its singular expressivity: none should be wrong or faulty in respect to the text it seeks to illustrates, or the algorithm it refers to. But none coincides exactly either: each representation only gives part of the information. No two representations coincide with each other. Indeed these multiple depictions demonstrate and illustrate how complex the relations are between stated and performed algorithm. The heuristic representations we forge to explain one or the other but add a subsequent layer to this complexity. In other words, there is no single absolute way of describing the algorithms to extract square roots, nor the different way they are stated.

The analysis in this article will be restricted to three elements that are usually associated with algorithms. If a procedure is thought of as an ordered list of actions to be carried out, the kind of actions, their listing and ordering will be discussed here. More specifically, in the following, first the kind of statement the rules of Āryabhaṭa and Śrīdhara provide will be discussed. In order to understand how and why an author “states” the steps of an algorithm, the focus needs to be on the kind of texts that transmits the procedure. What kind of statements about procedures are produced in Sanskrit mathematical texts: Descriptions? Incetives to actually realise the procedure?

2 The prescriptive paradox of compact procedures

Procedures are transmitted through rules (*sūtras*) and their commentaries. A *sūtra*, as it has been noted by Louis Renou with some detail, is a complex linguistic object used

in a great diversity of communication acts¹⁵. In the following focus will be on how this complexity is given an additional twist as mathematical algorithmic *sūtras* are evoked¹⁶. Often *sūtras* are described as being cryptic, let us look closely at what this means in the case of *sūtras* providing a procedure to extract square roots.

2.1 Being cryptic

As seen in Table 1, the rules given by Āryabhaṭa and Śrīdhara are difficult to understand¹⁷: Ab and PG, do not specify what is produced if the rule is followed, nor do they specify what is an odd term/square place. None of the verses¹⁸ indicate how to start the procedure, nor how to end it. Only part of the steps allowing the procedure to be carried out are given. This ellipse is illustrated in Figure 3 and in Figure 4.

In both Figures, spelled out steps are contrasted against the steps required to have the process carried out, such as it is analyzed in the heuristic representation of Figure 2. The “tacit” steps, in green, in Figure 2 are left out. The specificities of Śrīdhara’s extraction of a double square root are not represented in the Figure on Āryabhaṭa’s rule. Āryabhaṭa’s *sūtra* appears as giving but a piece of the process, its heart: reduced to four steps, the algorithm is given in an uncertain order and seems restricted to a succession of divisions and subtractions around which other steps gravitate. Śrīdhara rule, although more detailed, appears also as giving but part of the process: reduced to seven steps, of uncertain order (how does one go from step A to step B?), the emphasis is less on the heart of the iteration than on the details of what is done to the number “inserted on a line”.

Thus, a first level of reading immediately reveals the ellipses of the rules in respect to the algorithm.

Another way to state the same fact consists in listing spelled out steps. With arbitrariness and limitations in mind, the steps of the procedure given in Āryabhaṭa’s verse can be listed as follows:

¹⁵(Renou, 1963)

¹⁶Incidentally, this study shows that mathematical rules do not correspond either to Group A or Group B as defined by Renou in (Renou, 1963, part C). Features of group A such as the use of the optative, are mingled here with the prescriptive norms of group B.

¹⁷The cryptic character of this rule has been analyzed in (Keller, 2006, xvii), (Keller, 2010, 235-236) and is noted in (Plofker, 2009, 123-125). Some of its characteristics are described in (Singh, 1927).

¹⁸Ab and PG provide rules of extraction with a verse that counts the number of syllabic units, the *āryā*. This is a very common verse for prescriptive texts.

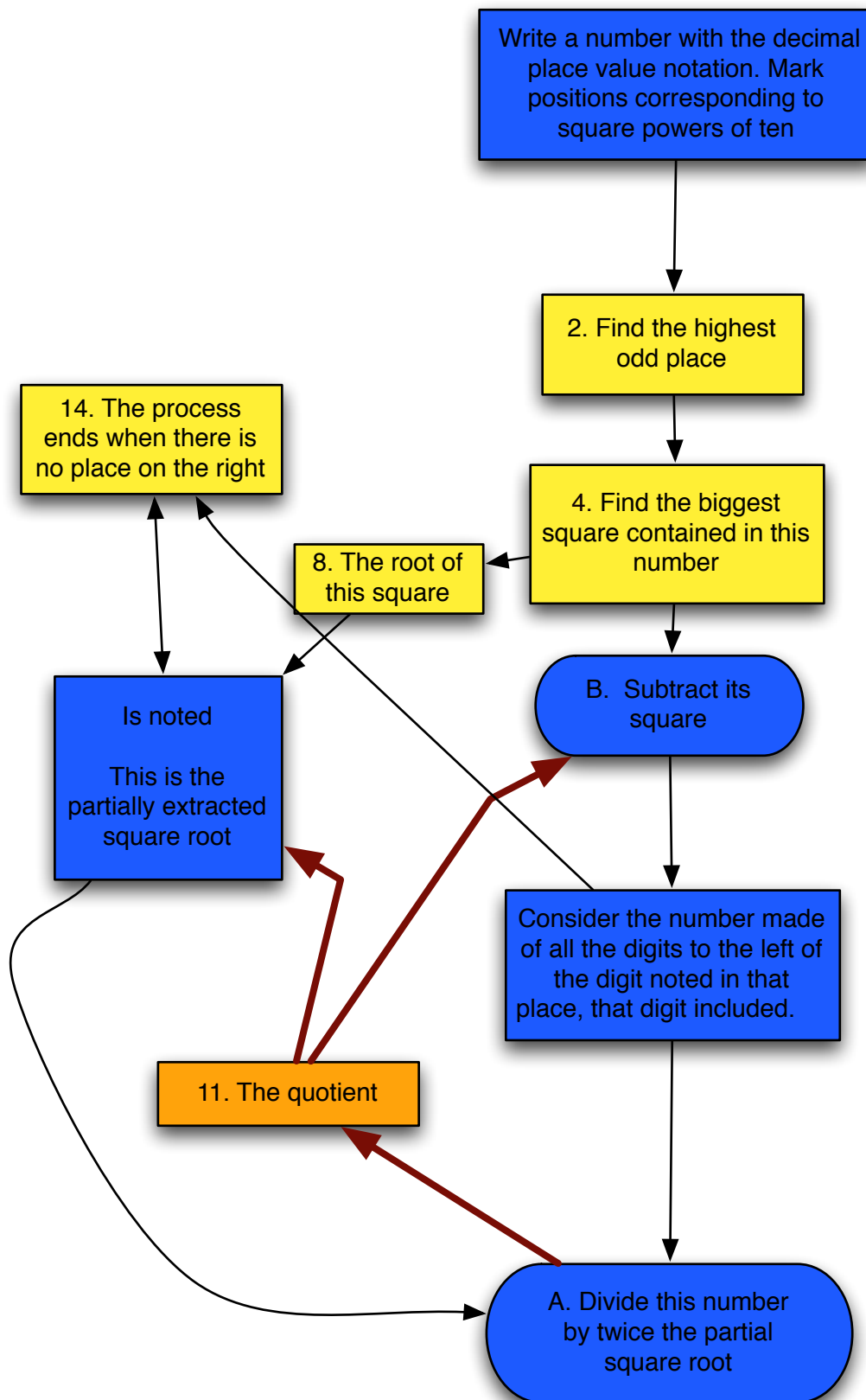


Figure 3: Steps provided by Āryabhaṭa in blue

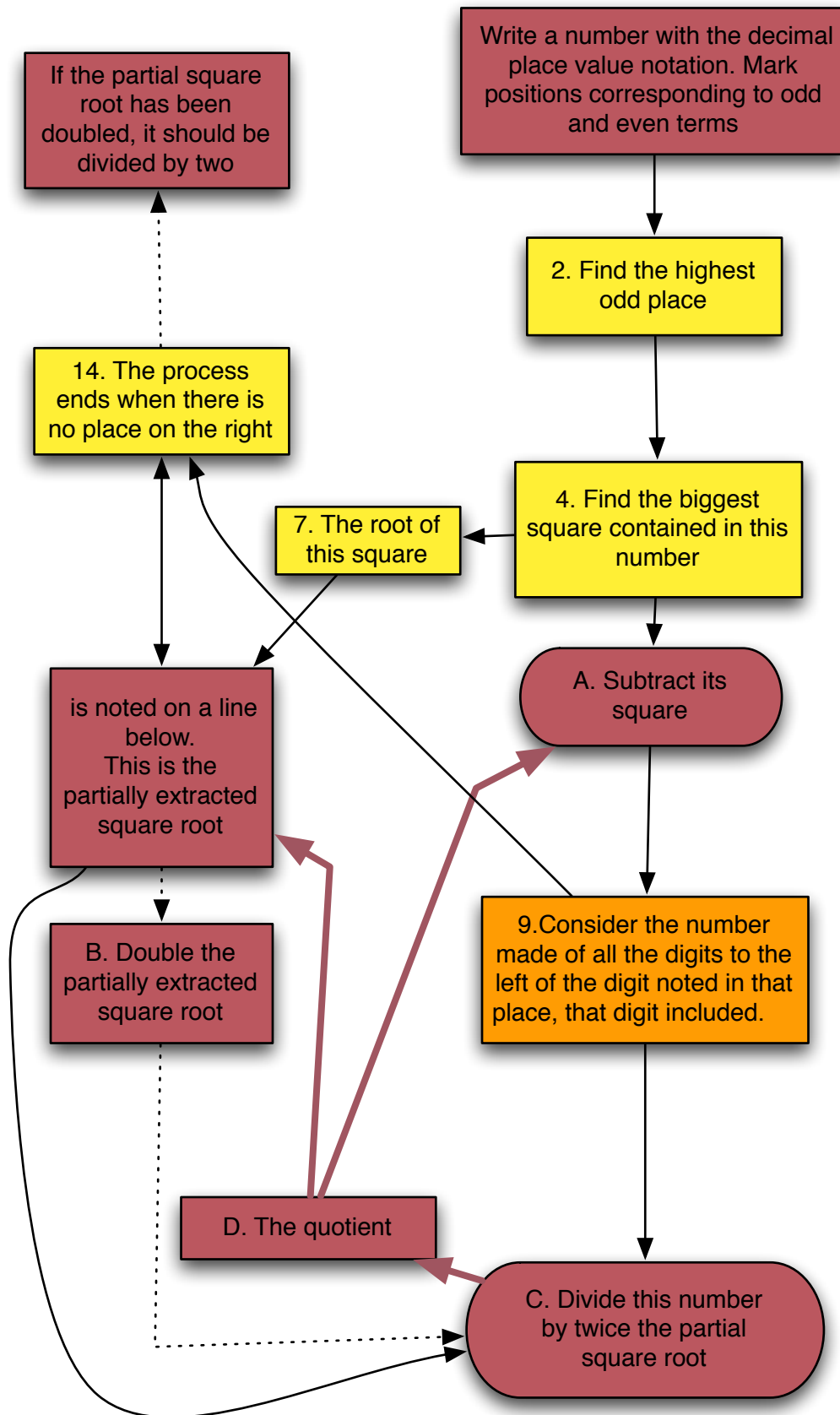


Figure 4: Steps provided by Śrīdhara in pink

- i. Divide the non square place by twice the square root
- ii. Iterate (“constantly”)
- iii. Subtract the square from the square place
- iv. The result, a quotient, noted in a “different place” is a (square) root

One may or may not add an implicit step, the one which marks the number to be extracted within a grid which identifies even powers of ten. This step can also be considered included in the subtraction step. Similarly step ii and step ii could also be interpreted as including in fact each two steps...¹⁹

Whatever the nuances we might want to add, this enumeration highlights how compact Āryabhaṭa’s verse is. Indeed the square root extraction as reconstructed in Appendix A in order to carry it out includes 16/17 steps, while Āryabhaṭa states maybe 3 steps, maybe 8.

Although, PG compared to AB is less concise- indeed Śrīdhara states the process in two verses while Āryabhaṭa uses only one- the process given in *Pāṭīganīta* is also compact.

Śrīdhara’s rule evokes the following steps:

- i. Remove the square from an odd place
- ii. Divide the remainder by twice the root
- iii. The digits of the partial root are placed below on a line
- iv. The square of the quotient is subtracted
- v. Double the quotient and place it on a line
- vi. Divide the remainder as in step 2, that is: Iterate
- vii. The final result is divided by two

Comparing with Āryabhaṭa’s compact verse, as can also be seen in Figure 7, doubling and dividing by two adds two steps to the process. Śrīdhara furthermore indicates more explicitly how the process ends.

¹⁹The difficulty of actually singling out steps in Āryabhaṭa’s verse, treated in the next section, can be seen when this enumeration is compared with Figure 3. With less contrast, the same can be seen for Śrīdhara’s rule as well.

This first analysis of the different steps provided by authors, shows that the *sūtras* considered here - whether overtly short as in Ab, or more explicit as in PG- do not state a complete algorithm. Obviously information is missing to actually carry out the procedure. If these rules aim at describing the process, then steps are missing. If these rules do not have such an aim, then the intention of the text is among the missing information. Therefore these rules are sufficiently compact to be difficult to understand alone: they are cryptic.

Elliptic formulations are often understood by indologists as recalling the oral sphere. The enigma of cryptic *sūtras* would have a mnemonic value unravelled by oral explanations. For instance, part of Ab.2.4's difficulty is rooted in a pun on the word 'square' (*varga*). Ab gives this name both to the square of a number and to the places of the place value notation having an even power of ten. Such places have the value of a square power of ten. It is also from such 'square places' that the 'square numbers' the process tries to uncover are found. Thus the pun recalls the main mathematical idea behind the procedure while simultaneously giving rhythm to the verse and making its understanding confusing. Ab.2.4 can thus be understood as a mnemotechnical "chimera"²⁰. Other commonly advanced reasons for using short forms include secrecy, or the desire to emphasize the difficulty of a given technical knowledge to add prestige for a profession living off patronage.

If these rules are compact to the point of being cryptic they nonetheless prescribe an action to be taken. This prescription is voiced by an optative.

2.2 Using optatives

Sanskrit makes a great use of nominal forms. Therefore, the use of conjugated forms is in itself an expressive statement. Conjugated verbs in mathematical *sūtras* indicate a prescription. Indeed, most *sūtras* of *jyotiṣa* texts (astral science including mathematics) use the optative²¹. Theoretically it is an equivalent of our conditional: it expresses doubt. However, it should be understood here as expressing requirement.²²

²⁰To apply to this context the concept that (Severi, 2007) uses to denote pictorial mnemotechnical artefacts: the important idea, as a knot to a kerchief is stamped into the artefact by relating two things that normally should not be related. In this case: a spot were a digit is noted and a square quantity.

²¹Note that in grammar (*vyākaraṇa*), according to an oral communication by Jan Houben, the optative belongs essentially to the commentary.

²²(Renou, 1984, §292):

L'optatif exprime les nuances variées d'un optatif propre- souhait, hortatif, délibératif, éventualité, prescriptif, hypothétiques (...). La coexistence de ces divers emplois n'est rel-

In Ab.2.4, “one should divide”, is the translation of an expression which uses an optative: *bhāgam hared*, ‘one should withdraw the share’- the usual expression of a division. The verb to withdraw (*hr*) is in the optative voice. PG 25-26, is a succession of optatives: one should divide (*bhājet*), place (*viniveśayet*), make (*kurvāt*) the double, divide (*vibhājet*), then halve (*dalayet*).

Commentaries follow closely the optatives given in treatises. Thus Bhāskara comments Āryabhaṭa’s optative by providing a synonym (*grh-*), conjugated as an optative²³: “One should remove the part, that is, one should divide”. Sūryadeva (whose commentary is translated in Appendix C) does not comment upon Āryabhaṭa’s terms for division but repeats the verb in the optative form, while commenting on what a non-square is²⁴: “One should divide by the (last) non-square place”. APG (the translation of the commentary is given in Appendix D) preserves Śrīdhara’s optative sometimes supplying a synonym for an other. Thus it uses *bhāgam apaharet* for *bhājet* (one should divide). APG provides optatives for a number of actions: the first subtraction of an arbitrary square (*tyājet*), the placement of the root of this first arbitrary square (*sthāpayet*), the subtraction of the square of the quotient (*śodhayet*), the fact that results should be considered as a unique quantity (*jñāyet*), etc.

Commentaries use also other modes to voice prescriptions: Imperatives while inviting one to solve a problem, obligational verbal adjectives while describing the steps to be taken. In APG the optative is only used while commenting directly Śrīdhara’s verse. When solving the problem, actions are given with absolutives (which give precise temporal orders), such as *śodhayitvā* (“having subtracted”) used twice, and by verbal obligation adjectives (such as *kartavya*, “on should carry out”).

Therefore, algorithmical Sanskrit texts are prescriptive. Their prescription is first voiced in treatises by an optative. These optatives are taken up and declined also in other prescriptive forms in commentaries. A hermetic *sūtra* prescribing a procedure to be carried out is a paradox: Indeed, why elaborate short hermetic prescriptions, if the aim is to have them followed? In other words, if the aim is to have a procedure applied, the directive character of an algorithmic rule is contradicted here by its cryptic form. What

evable que dans la poésie littéraire; dans les textes techniques prédomine la valeur prescriptive.

That is in English (my translation):

The optative expresses diverse shades of a proper optative: wish, hortative, deliberative, possibility, prescriptive, hypothetical (...) voices. The coexistence of these various uses are only found in literary poetry; in technical texts a prescriptive value prevails.

²³*tam bhāgam haret grhṇīyāt*

²⁴*avargasthānād bhāgam haret*

then were the intentions of the authors of such rules? As this question cannot be replied to directly, how commentators read author's intentions will be observed. But to do so requires to further unravel the complexity of mathematical *sūtra* 's statements.

2.3 Stating a procedure

Commentators are quite explicit on how they understand the kind of statement treatises provide. All comentators consider first the rule as a linguistic assertion: a text whose language is the first subject of the commentary (what kind of verb it uses, what it means, how is it syntactically constructed). In this respect, all three comentators refer to the text they explicate as a *sūtra*. They also sometimes refer to it as a verse, *kārikā*²⁵.

Commentaries further use another vocabulary. One that relates to procedures. Thus, *ānāyana*, “computation”, derived from the verb *ā-Nī*, to lead towards- is used by our three commentators to refer to the mathematical contents of the rule. Bhāskara writes as an introductory sentence²⁶: “In order to compute (*ānāyana*) square roots, he says:” He uses later in the commentary the word *gaṇitakarma* “mathematical process”. And Sūryadeva, with almost the same words, but different declentions introduces the verse in this way²⁷: “He states with an *ārya* a square root computation”. The anonymous and undated commentary of *Pāṭīgaṇita* starts by specifying²⁸: “A two *āryas* algorithmic rule (*karaṇasūtra*) concerning square roots”. He later refers to the process by using the expression *ānāyana*²⁹.

Thus commentaries understand the rule as being primarily about a mathematical procedure. Because the rules are also prescriptive, they contain a “promise”, that of stating an algorithm that provides a correct answer to a given problem. Note that commentary's first movement provides indeed the result the procedure aims at: authors of *sūtras*, expecting commentaries, may not have needed then to specify procedure's results in the rules they composed.

²⁵This crossreference may refer to the mingling of both forms as referred to by (Renou, 1963) who considers that real *sūtras* are non versified. The etymology of *kārikā*, derived form the verb *kṛ-*, “to make”, can maybe be heard in this context as “(verse) for action”.

²⁶*vargamūlānāyanāya āha*

²⁷*vargamūlānāyanam āryayāha*

²⁸*vargamūle karaṇasūtram āryadvayam*

²⁹Standard vocabulary is used throughout Sanskrit mathematical texts to name computations, methods, algorithms. We do not know if there was any difference between these different words, if their meaning changed overtime, according to authors. We have adopted the following translations here: “computation” for words derived from *ānī*; “method” for *karman*; algorithm for *karaṇa*.

Commentators then refer to the rule on both levels: as a statement (on whose language one may comment) and as a procedure (on the steps of which one may comment). How do commentators deal with this double level of *sūtras*? And if the rules to extract square roots are thus understood as prescribing a process that should be carried out, does this mean that they provide a list of steps to apply?

3 Spelling out steps to extract a square root

As noted earlier, the speech act “spelling out of an algorithm’s step” is complex. Two aspects of this act, the isolation of a certain number of steps and their subsequent ordering are studied now.

3.1 Expressing Actions and enumerating steps

Earlier in this article, in an attempt to show that rules provided both by Āryabhaṭa and Śrīdhara were compact, they were crudely restricted to a list of steps. Indeed, our contemporary representations of what a good prescription should be involves listing actions. But how then do the rules given here, fair in this respect? Are they lists of actions? And if not, does this imply that they do not describe an algorithm?

Recall Śrīdhara’s statement of the procedure, as given in Table 1. A certain number of steps are voiced out by a succession of optatives. The essential ordered backbone of operations to be carried out is expressed in this way: a division, insertion of a quotient on a line, doubling, a division again and a halving³⁰. If we understand this succession as being a list³¹, then we can understand each action as being on a same level of co-enumerability.

The impression that Śrīdhara provides a list of actions is emphasized by APG’s way of

³⁰The fact that PG’s process provides a doubled root that needs to be halved, is highlighted (by mistake?) in SYAB. Indeed, this commentator of Āryabhaṭa notes (Sarma, 1976):

labdhe mūlarāśau dviguṇā kṛtaṃ dalayet

When the root quantity has been obtained, having multiplied it by two, it should be halved.

However, Āryabhaṭa’s rule does not provide a double root and therefore does not request a halving in the end.

³¹Note that the Indian subcontinent’s diversity of manuscripts presents a great variety of material settings, its scholarly texts, a great amount of lists. However, there seems to have been no specific typographical set up for lists in mathematical manuscripts of the Indian subcontinent.

taking each optative and following it through twice. Thus in the general commentary³²:

And one should divide (*bhāgam apaharet*) from above by twice that, just there. The result should be inserted on a line (*viniveśayet*), one should subtract (*śodhayet*) the square of that from above that, and that should be doubled (*dviguṇīkuryāt*). If when this is doubled an additional place is created (*jāyet*)³³, then it should be used as before (*yojayet*) when it is a result. (...) One should repeat (*utsārayet*) this, thus one should divide (*vibhajet*), one should insert (*viniveśayet*) the result on a line, etc. as before in as much as the serpentine progression is possible, when finished one should halve (*dalayet*) all the result.

The optative is used while commenting directly Śrīdhara's verse. More, APG takes elements that Śrīdhara did not formulate with conjugated verbs, and transforms them in conjugated, optative forms. Thus the subtraction, expressed by Śrīdhara with an absolutive (*saṃśodhya*), becomes an optative in the commentary (*śodhayet*). The repetition, an absolutive in the verse (*utsārya*), is an optative in APG (*utsārayet*). In the resolution of the problem, APG uses a conjugated verb, to make a quantity sneak onto a line (*sarpatī*), were the *Pāṭiganīta* describes with a non-conjugated form a quantity that has been dropped (*cyuta*). Both Śrīdhara and his commentator seem to consider the rule provided as a list of steps, singled out by the use of verbal forms, conjugated or not. Consequently, unravelling here how authors “spell out steps” seems fairly simple and straight-forward: conjugated verbs give us the clue.

Āryabhaṭa and his commentators Bhāskara and Sūryadeva provide a strike contrast to this attitude. Indeed, Āryabhaṭa's verse alone cannot be reduced to a list of steps of actions to be carried out. It uses but one conjugated verb referring to a division. Furthermore, the last statement of the verse is a description of the fact that a result gains a new status by changing place. It has a reflexive character. This statement is part of what makes this rule non reduceable either to a prescription or to an enumeration. Bhāskara's and Sūryadeva's readings of the action's of the rule emphasizes on division. Thus, the only conjugated verb referring to an action of the algorithm used by Bhāskara, concerns this action.

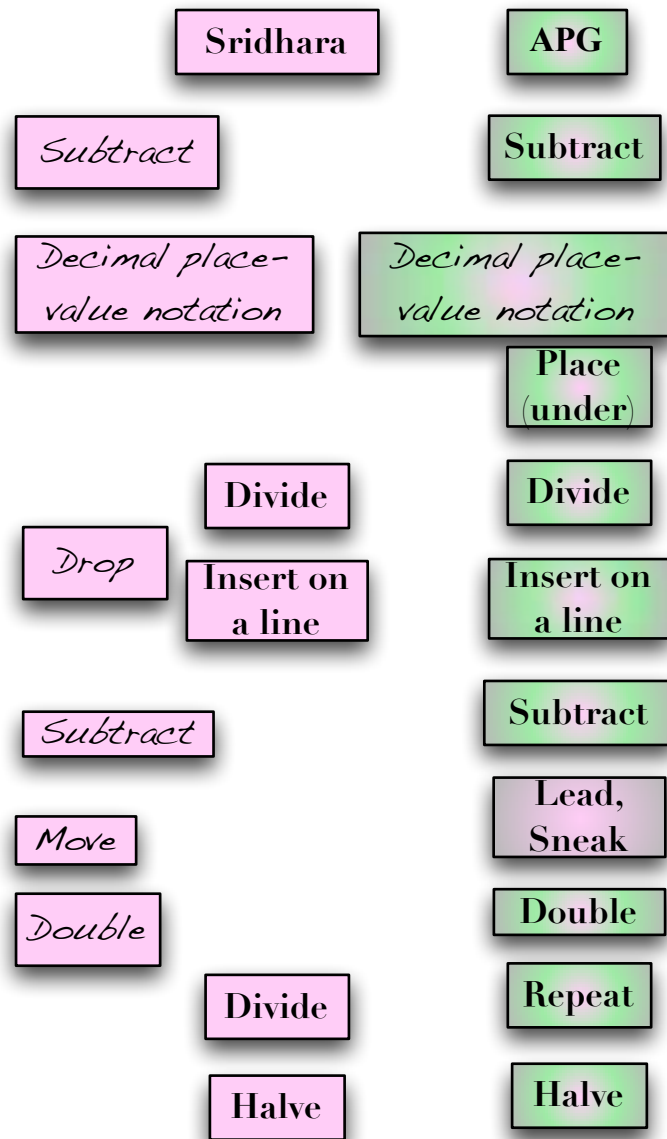
In both cases, in PG and in Ab, the presence of conjugated verbs on one side, and of

³² *dviguṇena ca tena tatraiva sthitena upariṣṭāt bhāgam apaharet labdhaṃ pañktau viniveśayet tatas tad vargam upariṣṭāc chodhayet tac ca dviguṇīkuryāt tasmin dviguṇe kṛte yadi sthānam adhikam jāyet tat prāglabdhe yojayet (...) tam utsārayet tato vibhajet labdhaṃ pañktau viniveśayed ity ādi pūrvavat yāvat utsarpanasambhavaḥ / samāptau sarvaṃ labdhaṃ dalayed/*

³³ In the corpus looked at here, this is the only use of an optative in a non-prescriptive form.

verbal non conjugated forms on the other, constructs a hierarchy among different steps of a stated procedure. In PG the actions, to subtract/remove (*tyaktvā, samśodhya*), to drop on a line (*cyuta*), and to evoke the past doubling (*dviguṇīkṛta*) are stated with verbal but non-conjugated forms (absolutives and verbal adjectives). They too seem to furnish a list of actions of less importance, thus creating a second level of co-enumerability. In other words, the two kinds of verbal forms draw a hierarchy in the actions to be carried out, as seen in Figure 5. APG, on the other hand, does not seem to reproduce this hierarchy³⁴.

Figure 5: Expressing Actions in Śrīdhara 's verse and in its anonymous commentary

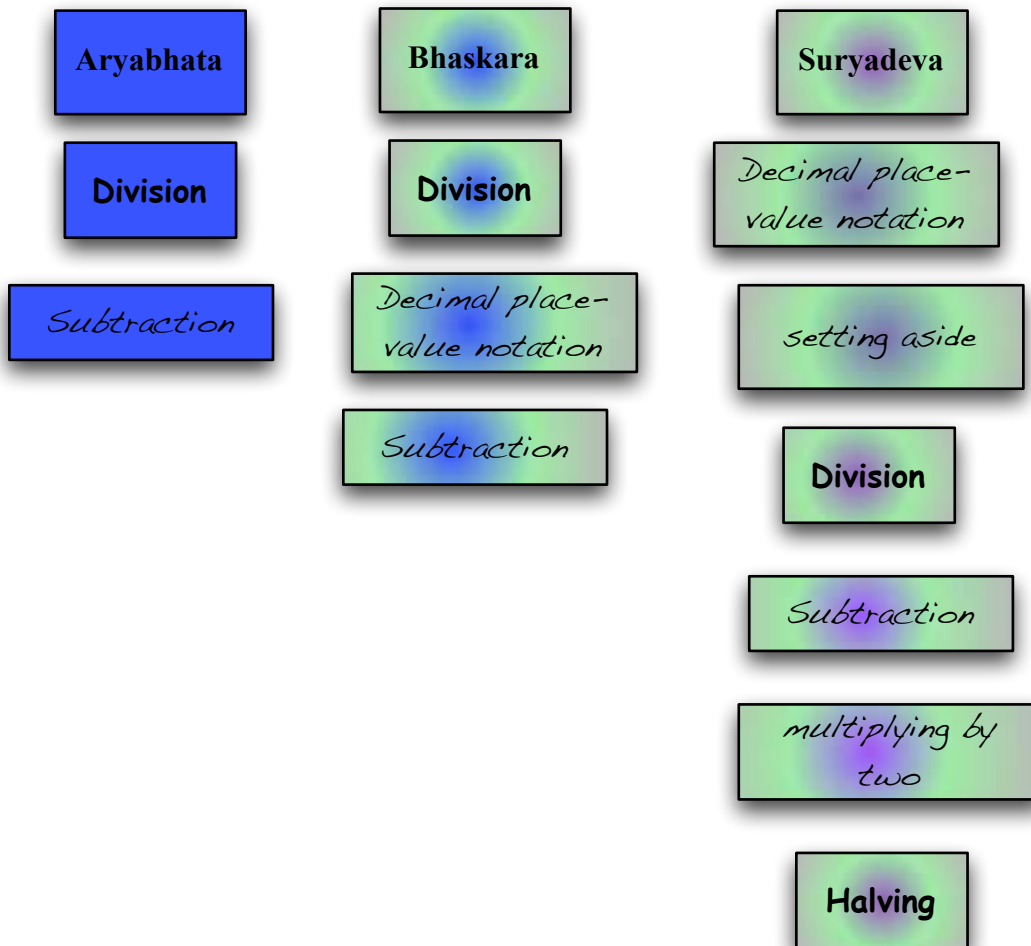


In Āryabhaṭa 's case, the voicing of steps to be carried out, cannot be restricted to

³⁴Two ambiguous moments are set aside here: the semi-tacit use of the decimal place value notation, what happens when a doubling results in a number with two digits.

verbal forms. Aside from the division another action, a subtraction (*śuddha*), is stated with a verbal non conjugated form, a verbal adjective. Other parts of the algorithm that could be expressed by actions, such as squaring, multiplying by two are not described in that way: the square of the number (*varga*) is considered directly, as if it was already computed. The multiplication by two is described with an adjective meaning “having two for multiplier” (*dviguṇa*). Among all the actions to be carried out to square a root, two main actions emerge from Āryabhaṭa’s verse, those given in verbal forms: the action of dividing (first in the verse, and by the fact that it is conjugated) and secondly, the action of subtracting, as illustrated in Figure 6. Both commentators of the *Āryabhaṭīya* further respect Āryabhaṭa’s use of an verbal adjective to evoke the subtraction.

Figure 6: Expressing Actions in Āryabhaṭa’s verse and subsequent commentaries



Bhāskara does not introduce new intermediary steps with conjugated verbs of action. For instance, he does not comment upon the subtraction, nor on the multiplication, but underlines how (by contrast with square places), the numerical square evoked in

Āryabhaṭa's verse, evokes an action³⁵: "When subtracting the square, a computed square is the meaning". But precisely the aim here is to insist on Āryabhaṭa's pun, while lifting its ambiguities: a square operation (*vargaganita*) is not to be confused with a square place (*vargasthāna*).

Because of the diverse ways of expressing algorithm's steps, rules do not appear directly in the form of a list- a form however quite current in Sanskrit technical texts (*śāstra*). However, loose enumerations of conjugated verbs, as those given by Śrīdhara, can quite easily be interpreted as a list of steps to be carried out³⁶. When a rule has but a unique conjugated verb, as in the case of Āryabhaṭa, then this interpretation although possible (as seen in our first section), distorts the statement of the rule itself.

Nonetheless, conjugated verbs by contrast with others tell us something of a hierachisation in the different steps of an algorithm. In the rule for extracting square roots studied here, the optative can be seen as first ordering the enumeration of steps that the algorithm carries. In this ordering, the optative provides the action around which the others will be structured. This feeling may emerge from commentaries, who always carefully preserve the different ranges of voices: they do not transform the optative, or nominal forms in conjugated ones- but when in APG the commentary focusses on describing evenly each effective action of the process.

Therefore, if rules do not necessarily provide lists, they do transmit a hierarchy of steps. The question then is what order this hierarchy reveals.

3.2 Ordering steps

One of the difficulties of algorithm reconstruction concerns the temporal order in which different steps of a procedure are to be carried out. Mathematical constraints might sometimes impose a temporal order, but not always. Thus in the procedure to extract square roots, once a first square is found by trial and error, two actions have to be carried out: the square is to be subtracted, the root of the square noted on a separate line. The order of performance is indifferent. Order is one of the problems raised by the granularity with which a process is stated: if several actions are lumped together in a same step, the order in which they should be carried out, if there are no mathematical or exterior constraints, remains ambiguous. Does the hierarchy of steps observed in the previous

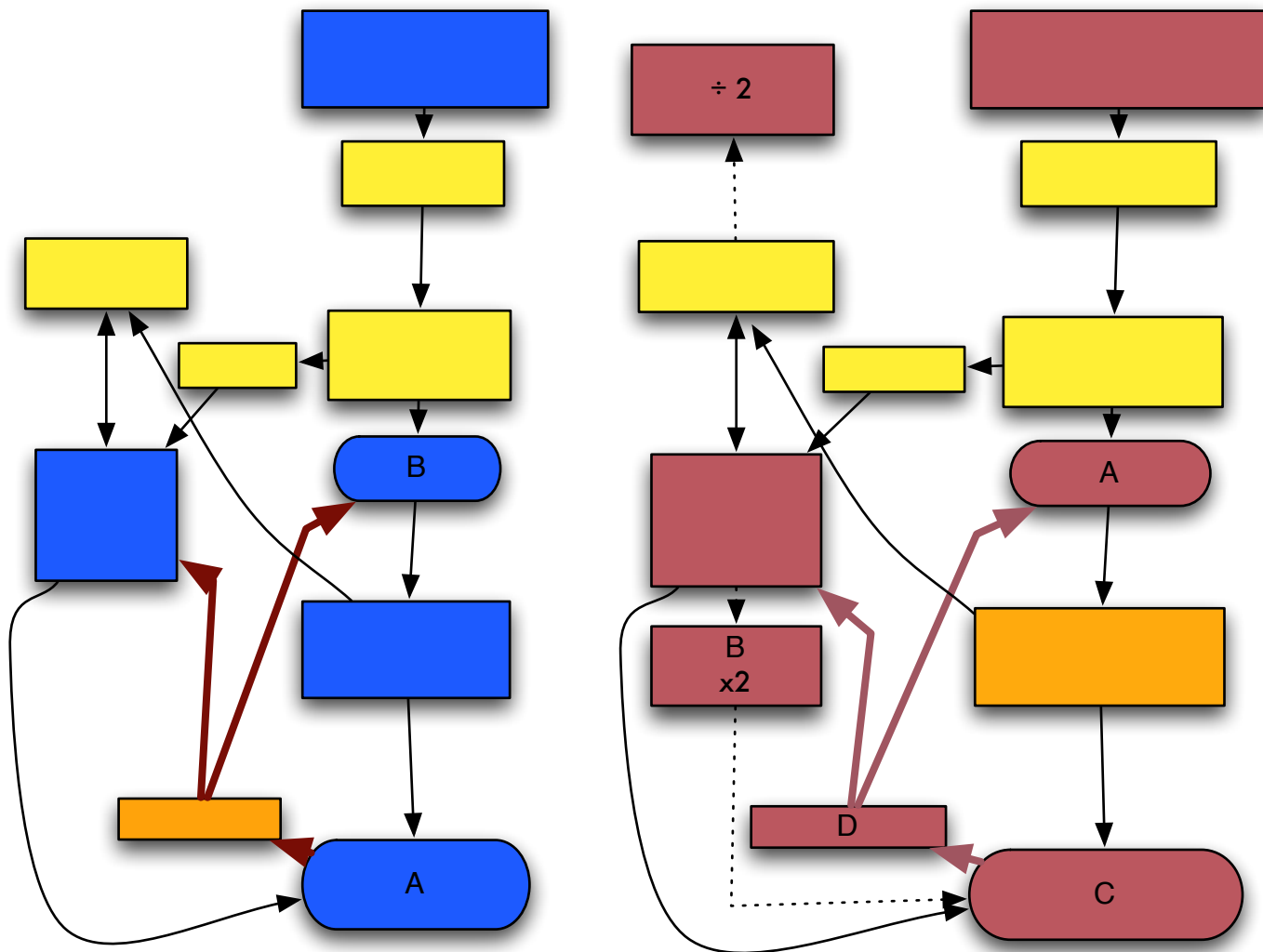
³⁵ *śuddhe varge vargaganite iti arthaḥ*

³⁶ Although, even in this case, there remains a great disparity between the representation of the action given in Figure 4 and in Figure 5.

section corresponds to a temporal order to carry out the algorithm?

A specific verbal mode is used in rules to order in time a set of actions. Absolutes are indeclinable. They are built on a verbal root and mean ‘having carried out the action of the verb concerned’. Absolutes thus indicate an action to be carried out before the main action indicated by a conjugated verb. Śrīdhara uses absolutes. In PG. 25 one must subtract (*tyaktvā, samśodhya*), before dividing or doubling; one should move before before dividing. This does not mean that all the different ordering of steps are elucidated in Śrīdhara’s formulation. As illustrated in Figure 4, the placing of the quotient which has dropped on a line is situated ambiguously in time during the process, as well as the doubling of its digits situated “after the subtraction”.

Figure 7: Comparing Ab and PG



APG describes in great detail the part of Śrīdhara's process which seemed ambiguous: To do so, he does not only use absolutes, but also the order in which conjugated verbs are enumerated, to which spatial modifiers (*upari* "above", *adhas*-below) are added. Actually, in his solved example, APG uses a great variety of verbal forms: absolutes (*apāśya*, "having subtracted"), verbal adjectives (*vyavasthita*, *sthita*, "placed"), and obligatoriness adjectives (*neya*, "one should lead") . Therefore not one device but all sorts of different ones are used in this commentary to express precisely the temporal order of execution.

Sūryadeva, at times uses absolutes. First when the place value notation and the grid of even-odd places is evoked, then when explaining how to extract the square root of fractions. In these parts of his commentary, he is very specific concerning the order in which different steps are to be carried out, using absolutes such as *cihnayitvā* (having marked), *dviguṇī kṛtya* (having multiplied by two), *vibhajya* (having divided), or *apāśya* (having removed). In both cases the order does not concern directly Āryabhaṭa's rule. In the first case Sūryadeva supplies actions that enables one to initialize Āryabhaṭa's rule. In the second case, Āryabhaṭa's is integrated into the rule to extract the square root of fractions. In the general commentary³⁷ Sūryadeva loosely follows the apparition of one, two and three digits of a square root being extracted, using a great diversity of verbal forms: verbal adjectives, absolutes, optatives. Furthermore this ordering of actions is certainly not the main part of his text. Sūryadeva can focus on an element of the process to detail its temporal order. Overall, Sūryadeva does not seem preoccupied by the kind of order implied within Āryabhaṭa's rule. The order he spells out, is rather that of cases where Āryabhaṭa's rule is used within other operations (extracting a square root for a fraction, and maybe obtaining a double square root).

Thus ambiguous temporal orders of actions are not always cleared by commentators. When they do so, no simple language act (such as the use of absolutes) is used.

However, sometimes, there is no temporal ambiguity. To extract a square root you need to start with a subtraction. Āryabhaṭa's verse starts with a division, the subtraction is stated with the end of the verse. That is, in the *Āryabhaṭīya* steps to carry out the algorithm are given in a reverted order. This is underlined in Figure 3, by the letters A, and B.

Note that Sanskrit being a language of declensions words need not be given in a strict order. A colloquial word order does exist; *sūtras* often scrambles it. Here, the two actions

³⁷And if my interpretation of the use of "both" and "three" in this text is correct, as noted in the footnotes of Appendix C.

are given in two successive verses: the action of division is highlighted by the fact that it starts the verse, while Sanskrit often keeps the conjugated verb at the end of the sentence. This disposition leaves no ambiguity on the order in which the actions are stated: first the division, then the subtraction.

Neither Āryabhaṭa nor Bhāskara use absolutes. In BAB, steps are spelled out in a succession of questions and answers. Bhāskara reads Āryabhaṭa's verse from the point of view of a division:³⁸:

One should take away [in other words] one should divide that [square].
 Beginning with what place?
 He says: 'From the non-square [place]' (...) In this computation the square is the odd place.
 (...)
 By what should one divide?
 He says: 'Constantly, by twice the square root'. (...)
 How, then, is this square-root obtained?
 He says: 'When the square has been subtracted from the square [place], the result is a root in a different place'.

The dialog argues that the order given by Āryabhaṭa is logic from this operative point of view. A place, a divisor and a dividend are required to carry out a division. By unravelling where the division is performed and what the divisor and dividend are, steps are thus re-ordered and specified. The use of this staged dialog simultaneously emphasizes that the verse's steps are disordered while arguing for its logic.

Different orders then can be layered in the statement of one rule: a temporal order, a logical one, or even an order of different cases in which a rule is applied. Furthermore, statements of procedures do not necessarily list, and not necessarily in a temporal order,

38

tam bhāgam, haret gṛhṇīyāt
kasmāt sthānāt prabhṛti iti
āha avargāt (...) atra gaṇite viṣamam sthānam vargaḥ
(...) kena bhāgam haret iti
āha nityam dviguṇena vargamūlena
(...)
katham punar tat vargamūlam labhyate iti

āha vargāt varge śuddhe labdham sthānāntare mūlam

actions that are to be performed. But what do they do then?

4 Back to the prescriptive paradox of procedural *sūtras*

As noted by Austin as a “descriptive fallacy”³⁹, it is after all difficult not to consider procedural *sūtras* as descriptions of the procedure they evoke. It is also tempting to understand literally the prescription they voice. However, Āryabhaṭa’s rule is a paradoxical act of communication: compact, scrambling the algorithm’s order. This is a symptom, that Āryabhaṭa’s verse contains an intention stated indirectly (an illocutionary force), which may be evading us.

In the following, an initial question will be raised again: What kind of statement does a *sūtra* provide when it evokes a procedure?

4.1 Promises and Iteration

Sūtras contain a rule that invites one to carry out a process (one or several operations): It thus contains a more or less implicit promise, that of obtaining a result (*labdha*, sometimes translated “quotient”, the result of a division.). At the heart of the paradox of prescriptive *sūtras* : an action to be carried out is suggest, but the promise the action will fulfill is evanescent. The result is literally constantly shifting. In Ab.2.4, *labdham sthānāntare mūlam*, “the quotient/the result (a digit of the partial square-root) is the root in a different place”. In PG.2.25, *purvaval labdham*, “the previous result/quotient” is doubled and moved. In both case, implicitly, we are in the midst of a process in which a given result will produce another. The condition of success of the procedure, we understand, has less to do with “obtaining” a given result, than with repeating the process until it is exhausted⁴⁰.

Śrīdhara and Āryabhaṭa do not express the iteration in the same way. Āryabhaṭa states the procedure by beginning with the middle of the process. Moving the quantity from a line where it is a quotient, to a line where it is a digit of the square root, is what enables the procedure to get back on its feet. Śrīdhara repeats with different words the process twice: In the first verse it may seem that he indicates how the process starts (its

³⁹(Austin, 1962, 100)

⁴⁰The difficulty of pinpointing how exactly the iteration is given in the verses, explains why it appears and disappears in the previous representations we have given of the algorithm.

first round), while in the second verse, a next or final round is described. Both authors of *sūtras* use a literary device to explain how the procedure should be repeated: they do not so much describe the performance then imitate it.

Bhāskara and APG do not reproduce this imitation. Thus Bhāskara states (using *āvartate*, to revolve, turn) plainly⁴¹: “This very rule is repeated again and again, until the mathematical process is completed (*parisamāptam*).” As seen previously, APG takes up Śrīdhara’s expression using the verb *utsrj-* (to turn), and adds to it an ordered list of actions to be carried out. On the contrary, by repeating the process for several digits, and then describing how it ends, Sūryadeva actually seems to use *Pāṭīgaṇita*’s device.

Turning a rule upside down, repeating it twice: textual devices are used by treatises to imitate what should be going on a working surface, where the algorithm is carried out. For the iteration then, the treatises observed here, are intent in making their text and the world in which an algorithm is completed coincide not by describing *in words* what is going on, but by describing *with words* the repeated algorithm. Let us look more closely at how treatises and commentaries play with the world of text and that of algorithm performance.

4.2 World and Text

Although Sanskrit mathematical texts may not list or give temporal orders for actions to be carried out, *sūtras* and commentaries are nonetheless stating something about the algorithm. The kind of adjustments between world and text taken up in the corpus will be observed now.

Śrīdhara at times give a momentary description of what the computation should be at a given moment. Verbal adjectives seem here to be the clue to such descriptions where text and algorithm should coincide. Thus Śrīdhara uses the expression *sthānacyuta*, “that has trickled down to a place”, *dviguṇīkṛtam*, “that has been doubled”. An action is not spelled out but a description of the state of the working surface on which the process is performed is made. These descriptions enjoin one who performs the procedure to adjust the world to the statements in the rule.

Specifically because this is a part of the process, which requires a know-how that belongs to the world of algorithm execution, APG, is careful to describe, digit by digit

⁴¹ *etat eva sūtram punar punar āvartaye yāvat parisamāptam gaṇitakarma iti*

how things are moved around on the working surface. Certain expressions are invitations to verify that at a given moment, the result obtained coincides with the text. Like for instance, in the purely descriptive⁴²: “When twenty four is subtracted by three from below, above two remains.” In other cases, the text describes a temporary state of the working surface, followed by a disposition⁴³:

“Below, eighty six is produced. This quantity slides (*sarpati*) on a line. Below

two, there is six, below seven, eight. Setting down:

1	7	2	4
	8	6	

”

Thus *Pāṭīganīta* and its anonymous undated commentary are intent in making the statement of the procedure and its performance using tabular dispositions coincide.

Understanding conjugated verbs are an expressive device, reveals how Bhāskara's commentary on Āryabhaṭa's rule is mainly on the level of language. Bhāskara uses four times the expression "he says" (*āha*)⁴⁴. His answers to the questions of the dialog always refer to Āryabhaṭa's statements. Bhāskara then is not describing independently from Āryabhaṭa how the process is being carried out. He is not adjusting Āryabhaṭa's statements with how the process should be executed. Indeed, he is just modifying Āryabhaṭa's statements in an attempt to show the coherence of their arrangement. He explains that this arrangement makes sense if the division is taken as the core from which all other steps of the process derive.

In three instances Bhāskara explores the limits of the mathematical reality expressed by Āryabhaṭa . First, when he specifies evenness as the opposite of oddness, then when the process ends because no other space to carry it out can be found, and finally as he gives an example concerned with fractions in which he then introduces his own rule. In other words, the world of Bhāskara is not like APG one of algorithm execution but a world of mathematical objects.

The iteration of the process is further explained by a change of status, voiced by Āryabhata in the last part of the verse and repeated by Bhāskara: a quotient changes

⁴²*tribbhiḥ patanāt caturviṃśatau śuddhāyām upari dvau śeṣah/*

⁴³ *adhaḥ śāḍasūtrījayite / eśa rāṣiḥ sarpati, paṅktyāṃ dvayor adhaḥ (ṣāṭkaṃ) bhavati, aṣṭakaṃ sapṭādhāḥ*
/ *nyāsaḥ-* 1 7 2 4
 8 6

⁴⁴He also uses once each *labh* (to obtain), *bhū* (to be, have, produce), *vidyate* (to exist, discern). The three only other conjugated verse of this part of his commentary is the optative of the division, and expressions in the solved example given at the end of the commentary.

place, becomes a root, and enters in the process again. The end of the process precisely appears when this status changing becomes impossible. Bhāskara states it as follows⁴⁵:

The quotient here becomes in a different place what is technically called the root (*mūlasaṃjñā*). (...) In this different place, this quotient has the technical name root. However, precisely when a different place is not found, there, in that very place, that [result] has the technical name ‘root’.

The change of place which simultaneously is a change of status is acted out by a solemn action of text and world adjustment: a name change. This name change is even acted out as no movement is possible anymore, signalling the end of the process. Bhāskara insists on this point, repeating the compound *mūlasaṃjñā* (“technical name root”). This can seem at first glance confusing, together with the perpetual movement of the process. The insistence points to its importance. Indeed the name change explains how the process works: each repeated division provides digits of the square root. The centrality of the division is thus staged, the quotient is the root itself.

All three commentaries link the movement on the working surface, to a status change of the quotient. In the anonymous commentary on *Pāṭīgaṇita* a digression discusses the status of the quantity on a separate line. The double square root is called “the result” (*labdha*). After having inserted the result/quotient of the division on a line, having subtracted its square and having doubled it, APG examines the case when the doubling provides a number bigger than ten⁴⁶: “If when this is doubled an additional place is created, then it should be used as before when it is a result (*praglabdhe*). Both have the quality of being a unique quantity (*rāśitā*). This quantity has the name “result” (*labdhasaṃjñā*). ” And when one arrives at the end of the process, the “result” appears again, to be halved.

Sūryadeva starts by considering the first digit of the square root obtained by trial and error. He calls this quantity a “special number” (*saṃkhyāviśeṣa*) and notes⁴⁷: “that special number is referred to (in the rule) as a root”. Then commenting on Āryabhaṭa’s last quarter of verse, he adds⁴⁸: “that ⟨quotient⟩, in the next square place, becomes (*bhavati*) the root.” Although, the status change is the same, Sūryadeva does not use here

⁴⁵ *yat atra labdham tat sthānāntare mūlasaṃjñam bhavati/ (...) tasmin sthānāntare tasya labdhasya mūlasaṃjñā/ yatra punaḥ sthānāntaram eva na vidyate, tatra tasya tatra eva mūlasaṃjñā*

⁴⁶ *tasmin dviguṇe kṛte yadi sthānam adhikaṃ jāyeta tat praglabdhe yojayet, tayoṛ ubhayor ekarāśitājñeyā / tasya rāśerlabdhasaṃjñā*

⁴⁷ *taṃ saṃkhyāviśeṣaṃ mūlatvena gṛhṇīyāt /*

⁴⁸ *tat purve vargasthāne mūlaṃ bhavati*

a renaming. He does so before, when he establishes the equivalence between “square, non-square places” and “uneven, even” ones⁴⁹: “In places where numbers are set-down, the odd places have the technical name (*saṃjñā*) ‘square’. Even places have the technical name ‘non-square’. ”

Naming appears as the moment of commentarial activity when statements of the algorithm are adjusted to coincide with the world it refers to. This world can be on the level of performance (APG) but also of mathematical objects (Bhāskara), or a mingling of both (Sūryadeva).

Therefore, the way authors relate mathematical statements to the working surface on which a procedure is being carried out, gives us a clue to what is important to them: APG develops Śrīdhara ’s brief descriptions in order to give as much as possible a non-ambiguous algorithm to carry out. Bhāskara insists on the fact that when the quotient is moved to a separate line, its name changes: what happens on the working surface should always be coherent in the world of Āryabhaṭa ’s statements, the world that is important. Finally, Sūryadeva , not surprisingly for a commentator of Āryabhaṭa quoting Śrīdhara , seems to bridge his own way in between both approaches. He thus is intent on adjusting the decimal place value grid he uses to Āryabhaṭa ’s statements, and specifies how, practically, the process should start by trial and error. In other words, all the different analysis carried out so far, seem to shed lights on the intentions of the different authors of the corpus.

4.3 Intention

Compacting devices, expressing iteration, listing or not actions, relating language and practice, all these textology approaches can help us infer author’s intentions. For instance, by paying attention to what “essential” elements of the algorithm a rule states, how different hierarchies of actions are imbedded in a *sūtra* provides us with each author’s interpretations of the algorithm’s important points.

We have thus read the use of conjugated verbs (especially the optative) in *sūtras* as an expressive device. Śrīdhara would single out a certain number of actions (division, inserting the number on la line) over others (subtracting, doubling). APG on the other hand does not follow such a hierarchy. This is a consistent attitude if the aim of the commentary is to treat each action on the even level of algorithm execution. Thus such a

⁴⁹*saṃkhyāvinīyāsasthāneṣu viśamasthānāni vargasamjñāni / samasthānānyavargasamjñāni /*

hierarchy in Śrīdhara 's verse not undertaken by APG both shed's light on APG's aim and Śrīdhara 's objective. Śrīdhara aim would be to state (and maybe order) required actions. APG would be intent in describing how the algorithm is carried out on a working surface. Together with modifiers giving an orientation to the movements of the computation - "above" (*upari*) and "bellow"- several verbs refer to the movement and dynamic of the process: the quotient even sneaks like a snake to the line where it becomes a root (*sarpati*).

Śrīdhara and his anonymous commentator present a strike contrast with the intellectual couple formed by Āryabhaṭa and Bhāskara , with their sparse number of conjugated verbs. Bhāskara's emphasis on the operative logic of Āryabhaṭa 's verse show that his aim is to comment the coherence of Āryabhaṭa 's verse, not how it should be carried out. Similarly, Sūryadeva 's relative indifferenece to Āryabhaṭa 's scrambled orders points to us towards Sūryadeva 's own aim, which might be to make sure that the process covers all possible cases. Indeed, like in the processes described by Karine Chemla in this volume, Sūryadeva integrates Āryabhaṭa 's verse in other circumstances where the algorithm can be applied: root extractions which arrive at double roots and root extractions of fractions. Āryabhaṭa 's rule, then is a general rule, whose temporality or logic is not the question. The hierarchy of operations to be carried out uses Āryabhaṭa 's rule as a central nod from which further operations are assessed.

Finally, looking at how different authors treat the world of linguistic statements and the one in which a procedure is carried out on a working surface, confirms these unravelled intentions. Śrīdhara at times seems to describe the working surface at specific tricky times, that are detailed by his anonymous commentator. On the contrary, Bhāskara does not comment on this level of action, but on the one of providing a name at the right time, for the right quantity: re-naming a quantity that has been moved, assures us that the process is coherent with the stated rule. Sūryadeva does a bit of both, renaming the grid of the decimal place value notation used in the process, and describing how it is marked.

Maybe the "descriptive fallacy" of algorithm statements is considering that all statements about them describe the way they should be carried out; and more specifically that all invitation to carry an algorithm out, contains a more or less explicit description of how to do so. Indeed, this study has showed that first an invitation to carry out an algorithm does not necessarily describe literally how to carry it out. Second, that what the description of an algorithm can be, is very diverse.

For both BAB and SYAb, Āryabhaṭa gives the main mathematical idea behind this procedure. Furthermore, the square root process comes after the definition of a square. Bhāskara contrasts operations of increase which include squaring with operations of de-

crease which includes seeking square roots⁵⁰. In a mathematical tradition where the correction of an algorithm is sometimes verified by inverting it, and finding the initial input, square root extractions may have been seen as inverting the squaring procedure. Solved examples in the commentaries to these rules, all revert the squarings that were illustrated in the squaring procedure. Āryabhaṭa's rule then would seem to exhort one to carry out a process whose steps are described, but his true aim (as seen through Bhāskara's glasses) would be to transmit a reflection on the procedure's mathematical grounding.

Adopting Bhāskara's reading, three hypothesis can be drawn about intentions laying behind Āryabhaṭa's way of stating this procedure: His first aim could be in establishing the procedure (both by explaining it, and being able to recall it), giving its gist. A second aim could be reflecting on what the procedure is about (how does one undo a squaring in the decimal place value notation), what does this tell us about numbers. Most probably the aim was of summing together all possible understandings, eg prescribing a procedure, giving its gist, and hinting that this is less about doing than reflecting.

Conclusion

The hermetic algorithmic statements of mathematical *sūtras* are, to put it in Austin's words, neither unambiguous nor explicit. At first reading they can seem strangely vague and full of uncertain references. However this detailed study of rules for square root extractions has shown that verses stating algorithms are not neutral descriptions of how to carry an algorithm out. They indicate what authors wanted to transmit and emphasis concerning these rules. Statement of procedures then may also be read as literature, with their expressivity, and choices of narrative highlights. Commentators readings these rules, show precisely that difficult expressions are a door leading to the highlight of the rule.

Authors then did not necessarily list actions. The hierarchy of steps they provide does not always concern a temporal order. Conjugated verbs in the case examined here express something of the emphasis that each text puts on action or not: Āryabhaṭa's theoretical rule uses one, while Śrīdhara's practical rule gives several. Bhāskara is intent on commenting on statements, and thus abundantly conjugates the verb "to speak, state", while Sūryadeva who reflects on different forms of square-root extractions conjugates the verb *bhū*, "to produce, become, be". Finally the anonymous commentator of Śrīdhara's rule, intent on reworking and specifying different steps, supplies many optatives. The

⁵⁰(Keller, 2007)

different aims and practices of algorithmic statement appear here clearly according to the type of text (theoretical, practical).

This study has used all sorts of different tools to describe and understand how processes were compacted in *sūtras*. Whether by representing the algorithm as a tree, or as a list, Āryabhaṭa's verse does not appear as an arbitrary fragment of the algorithm. The study of how language related to the world of algorithm realisation, helped understand how the iteration was expressed by authors of *sūtras*. Śrīdhara and Sūryadeva figure it as a repetition, while it is signified by Āryabhaṭa by reversing the usual order of the procedure. In both cases, a literary device of imitation, is used to describe a reality which could take many words. Such processes, like the pun on *varga* used by Āryabhaṭa, can be seen as striking stylistic idiosyncrasies- specific to each rule and author- which may have had the role of “knot's to one's kerchief”, if such rules were made to be learned by heart.

The main aim of authors then would not have been to describe an algorithm, but rather comment, that is emphasize, a point of the procedure: its mathematical grounding for Āryabhaṭa, its coherence for Bhāskara, the fact that it was worked out on several connected horizontal lines for Śrīdhara as understood by his anonymous commentator, and finally as a fundamental operation which can be carried out both by integers and fractions for Sūryadeva. The compacting of *sūtras* then would less have to do with mnemonics and secrecy than with the expressive granularity of algorithm statement.

Finally, stating such rules for Āryabhaṭa and Bhāskara seems to have aimed at pointing to how the algorithm was made, and the mathematical properties they were grounded in. In other words, such a procedure gave insights into properties of numbers written with the decimal place value notation, into what made them or not perfect squares. Maybe the procedure itself was thought of as a reflexive algorithm.

The definition of the *sūtra* as recalled by (Renou, 1963), is a paradox: a self sufficient compact verse but also one of a series⁵¹:

“The word *sūtra*, or ‘string’ refers sometimes refers to a rule stated as a more or less short proposition (...), sometimes as a set of propositions forming a collection. The *sūtra* genre is defined by its relation rather than its contents, a

⁵¹English translation is mine: “Le genre du *sūtra* se définit par sa relation plutôt que par son contenu : un *sūtra* (au sens de “règle” ou “aphorisme”) est d’abord un élément dépendant du contexte, même s’il est grammaticalement autonome; il est déterminé par le système et (...) corrélatif au groupe qui l’environne.”

sūtra (understood as a ‘rule ’ or an ‘aphorism’) is foremost an element depending on its context, even if it is autonomous grammatically; it is determined by the system and is correlated with the group that surrounds it”

Indeed, this study has showed that mathematical *sūtras* and commentaries were intertwined. Commentaries were implicitly understood by *sūtras* as supplying mathematical context, result and details for practicing required. Looking at the statements and the way they are formulated and read, has underlined the technical reading a *sūtra* requires. If *sūtras* have an expressivity, the commentary is needed to unfold it and make it shine. There is a specific movement to reading a *sūtra* and its commentary: knowing the text of the *sūtra*, understanding it which means unfolding it and then folding it back, and understanding its expressivity. Nor the *sūtra*, nor maybe the commentary are intended to be read in a linear way once, but masticated in the way Nietzsche defines aphoristic reflexions. This is precisely the movement of an iterative algorithmic execution.

List of Abbreviations

Ab Āryabhaṭa I ’s *Āryabhaṭīya* (Vth century)

APG Anonymous and undated commentary on the *Pāṭīgaṇita* of Śrīdhara

BAB Bhāskara I ’s commentary on the *Āryabhaṭīyabhāṣya* : *Āryabhaṭīya* (VIIth century)

PG *Pāṭīgaṇita* of Śrīdhara (Xth century)

SYAB Sūryadeva Yajvan’s commentary on the *Āryabhaṭīya* : *Bhaṭaparakāśikā* (XIIth century)

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A Different steps of the algorithm to extract square root as spelled out in the corpus

In the following reconstruction, all the actins spelled out by the authors considered here (with the part of arbitrariness - the loops of the net may at times seem too small and at others too large- that has been underlined in paragraph 1.3) have been taken in account. Thirteen steps to extract a square root can be listed. Step 3, 6 and 12 state common tacit steps. The algorithm may be more efficiently illustrated in Figure 2.

1. The number whose square is extracted is noted within a decimal place value notation. Places are categorized with a grid that enables one to identify square powers of ten. Either square powers of ten are identified or the series enumerating positions, starting with the place with the lowest power of ten lists even and odd places.
2. The highest odd/square place is identified.
3. Consider (tacitly) the number made of all the digits to the left of the digit noted in that place, that digit included.
4. Find the biggest square contained in the number noted in the last/highest odd place.

From here, onwards, one could as well start by considering step 8 and 9, before turning to step 5 to 7.
5. Subtract the square of the considered number.
6. Replace (tacitly) by the remainder of the subtraction.
7. The root of the subtracted arbitrary square is the first digit of the square root being extracted.
8. The root of this square (Ab family)/ The double of the root of this square (PG family) is noted on the same line/ on a line below the line of the number whose root is being extracted⁵². In the PG family then, the doubling of the digit is a separate step of the process. The doubling does not necessarily need to take place immediately, one can note the digit, and then double it just before it enters the division discribed in Step 10. This is what APG recommends.

⁵²The “reconstruction” of these variants of the different steps of the process is not discussed here. Hopefully this issue will be tackled in a coming article.

9. Consider the number whose highest digit is the previously noted remainder and the next digit to its right.
10. Divide this number by (twice) the partial square root of Step 8. In the following, Step 11 can be carried out after Step 12.
11. The quotient is one of the digits of the partial square-root. It, or its double, is thus noted next to the previously found one, as in Step 8. Its square is what will be subtracted as the process is from here iterated from Step 5.
12. Replace the dividend by the remainder of the division. Then one should consider the next place on the right, which is a square/uneven place.
13. When there is no place on the right the algorithm is finished. Examples never consider a process that would not extract a perfect square, consequently, either the square-root is obtained, either its double is obtained, according to the procedure followed. If we are in the latter case, the obtained number is halved.

B Extracting the square root of 186 624.

This is a numerical example treated in APG. Footnotes and asterisks indicate non attested forms.

1. The number whose square is extracted is noted with the decimal place value system. Places are categorized with a grid : Square (*varga*), non-square (*avarga*) powers of ten are considered (Ab), or even (*sama*, abr. *sa*) and odd (*viṣama*, abr. *vi*) place ranks - counted starting with the lowest power of ten- are considered (BAB, PG, SYAB, APG).

avarga varga avarga varga avarga varga

sa vi sa vi sa vi

$10^5 \quad 10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0$

1 8 6 6 2 4

$$186624 = 1 \times 10^5 + 8 \times 10^4 + 6 \times 10^3 + 6 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

2. Subtract the square from the odd place

The highest “odd” (*vi*) place or “square” (*varga*) place is 10^4 . The process starts by finding by trial and error the highest square number contained in the number noted to the left of this place: in other words one subtracts 16, the square of four, from 18.

3. Replace by (BAB) or place below (APG)

4. The remainder (2)

5. The root of the subtracted square (4) is the first digit of the square root being extracted. The root of this square (Ab family)/ The double of the root of this square (PG family) is noted on a separate line.

Bhāskara might have written

$$\begin{array}{cccccc} *10^5 & 10^4 & 10^3 & 10^2 & 10^1 & 10^0 \\ & 4/2 & 6 & 6 & 2 & 4 \end{array}$$

While APG writes:

$$\begin{array}{cccccc} 2 & 6 & 6 & 2 & 4 \\ 8 \end{array}$$

$$186624 = 4^2 \times (10^2)^2 + 2 \times 10^4 + 6 \times 10^3 + 6 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

6. One should divide by twice the root

In other words 26 is divided by 8, the quotient is 3, 2 is the remainder.

Bhāskara style

$$* \quad 4 \quad \mathbf{3}/2 \quad 6 \quad 2 \quad 4$$

APG style

$$\begin{array}{cccccc} *2 & 6 & 6 & 2 & 4 \\ 8 & 3 \end{array}$$

$$186624 = 4^2 \times (10^2)^2 + 2 \times (4 \times 10^2)(3 \times 10^1) + 2 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

7. Iterate, that is “subtracting the square” again. This time the square of the quotient, 9, is subtracted from the next place on the right, that is from 26: the remainder is 17.

Bhāskara style:

$$* \quad 4 \quad \mathbf{3}/1 \quad 7 \quad 2 \quad 4$$

APG style:

$$\begin{array}{cccc} 1 & 7 & 2 & 4 \\ & 8 & 6 & \end{array}$$

$$186624 = 4^2 \times (10^2)^2 + 2 \times (4 \times 10^2)(3 \times 10^1) + 3^2 \times (10^1)^2 + 1 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

8. Dividing by the root, and subtracting the square again Considering the next place on the right, 172, divide it by twice the partial square root, 86: the quotient is 2 and there is no remainder. The square root is 432.

Bhāskara style:

$$* \quad 4 \quad 3 \quad 2 \quad 4$$

APG style:

$$\begin{array}{ccc} * & & 4 \\ & 8 & 6 & 2 \end{array}$$

$$186624 = 4^2 \times (10^2)^2 + 2 \times (4 \times 10^2)(3 \times 10^1) + 3^2 \times (10^1)^2 + 2 \times (3 \times 10^0)(4 \times 10^2 + 3 \times 10^1) + 4 \times 10^0 = (4 \times 10^2 + 3 \times 10^1 + 2 \times 10^0)^2$$

9. End

Bhāskara style:

$$4 \quad 3 \quad 2$$

APG style:

$$8 \quad 6 \quad 4$$

$$186624 = (432)^2$$

C SYAB.2.4.

He states (*āha*) with an *ārya* a square root computation (*vargamūlānāyana*)⁵³:

⁵³(Sarma, 1976, 36-37)

One should divide, constantly, the non-square ⟨place⟩ by twice the square root |

When one has removed the square from the square ⟨place⟩, the result is a root in a different place ||

In places where numbers are set-down (*vinyāsa*), the odd places have the technical name (*saṃjñā*) “square”. Even places have the technical name “non-square”. In this verse, when a square quantity is chosen (*uddiṣṭa*), having at first started by marking (*cihṇayitvā*) the square and non-square places, when one is able to subtract (*śodhayitum śakyate*) from the last square place the square of a special number- among those ⟨squares of the digits⟩ beginning with one and ending with nine, having subtracted (*apāśya*) that square; that special number is referred to (in the rule) as a root *mūlatvena grhṇīyāt*). Consequently, here, the result which is a square root (*vargamūlaphala*) has been mentioned (*ucyate*). One should divide (*bhāgaṃ haret*) the next contiguous non-square place by twice that ⟨root⟩. In this verse (*atra*), when the square of that quotient has been subtracted (*śuddhe*) from the next contiguous square place, that quotient from the non-square place, in a different place, in the next square place, that ⟨quotient⟩ becomes (*bhavati*) the root⁵⁴. Also, when one has multiplied it (the quotient) by two (*dviguṇīkṛtya*), dividing (*bhāgaḥāraṇa*) in due order both ⟨digits⟩ from its contiguous non-square, as before, the computation of the third root ⟨is accomplished⟩. Once again with three ⟨digit numbers, the process is carried out⟩⁵⁵. In this way, one should perform (*kuryād*) ⟨the process⟩ until no square and non square ⟨place⟩ remain (*bhavanti*). When the root quantity has been obtained (*labdhe*), having multiplied it by two (*dviguṇī kṛtaṃ*), it should be halved (*dalayet*). Concerning fractions also, having divided (*vibhajya*) the square root of the numerator by the square root of the denominator the quotient⁵⁶ becomes (*bhavanti*) the root. One states (*āha*) in this way:

When the square root of the numerator has been extracted, and the root born from the denominator ⟨also⟩ the root ⟨is obtained⟩(PG 34)|

In order to obtain the roots of the square which were previously explained (in the commentary of verse 3 which is on squares), setting down: 15 625. The result is the square

⁵⁴This long sentence has an equivocal expression: is *sthānātare* (in a different place) glossed into *pūrve vargaṣṭhāne* (in the next square place), or should one understand that two actions are prescribed, first setting aside the quotient as a digit of the root on the one hand, and then that its square enters an operation in the next square place ?

⁵⁵This is a cryptic mysterious expression, it is thus my interpretation that the three here, as the “both” (*ubhaya*) used in the sentence before refer to the number of digits of the square root being extracted

⁵⁶Reading *labdham* instead of the misprinted *ladhḍham*.

root 125. Setting down the second : $\frac{4}{9}$. The root of the numerator 2, the root of the denominator 3, having divided (*vibhajya*) the numerator by that, the result is the square root of the fraction: $\frac{2}{3}$. Thus the fourth rule (has been explained).

D APG

An algorithmic rule (*karaṇasūtra*) of two *āryas* for square roots⁵⁷:

PG.24. Having removed the square from the odd term, one should divide the remainder by twice the root that has trickled down to the place|

⟨And⟩ dispose the remainder on a line||

PG.25. Having subtracted the square of that, one should divide the previous result|

That has been doubled thus again and again one should halve twice the square.||

What is the root of an indicated quantity whose nature is a square? This is the aim of that procedure. One should subtract (*tyajet*) a possible square, from the *viṣama* (place) of the square quantity, (in other words) from what is called odd (*oja*), that is from the first, third, fifth, or seventh etc., (place); the places for one, one hundred, ten thousand, or one million, etc.; the *pada*, that is from the last among other places. This should provide (*syāt*) the root of that square which one should place (*sthāpayet*) beneath the place of decrease, (under) the place (where) the possible square is subtracted (*śodhita*) from that, (the place for) one, a hundred, ten or thousands, etc., the last among the other places. And one should divide (*bhāgam apaharet*) from above (*upariṣṭāt*) by twice that, just there. The result should be inserted (*viniveśayet*) on a line, one should subtract (*śodhayet*) the square of that from above that, and that should be doubled (*dviguṇīkūryāt*). If when this is doubled (*dviguṇe kṛte*) an additional place is created (*jāyet*), then it should be used (*yojayet*) as before when it is a result. Both have the quality of being a unique quantity (*rāśitā*). This quantity has the name “result”. One should repeat (*utsārayet*) this, thus one should divide (*vibhajet*), one should insert (*viniveśayet*) the result on a line, etc. as before in as much as the serpentine progression is possible (*sarpaṇasambhava*), when finished (*samāpta*) one should halve (*dalayet*) all the result, thus obtaining the square root.

⁵⁷(K. S. Shukla, 1959, 18-19)

Thus for 186624, of which quantity is this a square?

In due order starting from the first place which consists of four, making (*karaṇa*) the names: “odd (*viṣama*), even (*sama*), odd (*viṣama*), even (*sama*)”.

Setting down:

sa	vi	sa	vi	sa	vi
1	8	6	6	2	4

In this case, the odd terms which are the places one, a hundred, ten thousands, consist of four, six and eight. Therefore the last odd term is the ten thousand place which consists of eight. Then, the first quantity is eighteen. Having subtracted (*apāśya*) sixteen since it is a possible square for these quantities, the result is two. That last quantity is placed (*vyavatiṣṭhate*) separately above. Thus, where it is placed (*sthite sati*) the root of sixteen, four, with two for multiplier, eight, is to be led (*neyaḥ*) below the place where the square was subtracted (*vargaśuddhi*), which consists of six for the place of decrease. And then division (*bhāgāpahāraḥ*) of twenty six led above (*uparitanyā*). Setting down:

2	6	6	2	4
	8			

When twenty four is subtracted (*śuddha*) by three from below, above two remains. Below, the quotient which is three should be inserted (*niveśya*) on a line, they (eg these three units) should be placed (*sthāpya*) under ⟨the place⟩ consisting of six. Its square is nine. Having subtracted (*śodhayitvā*) that from above, these (1724) ⟨are placed above⟩, three is multiplied by two, six is to be made (*kartavya*). Below, eighty six is produced. This quantity slides (*sarpati*) on a line. Below two, there is six, below seven, eight. Setting down:

1	7	2	4
8	6		

Division above of a hundred increased by seventy two by that eighty six. Decreasing from above the dividend without remainder by two, the result is two, having inserted (*niveśya*) that on a line, having placed (*sthāpyau*) the two ⟨units⟩ below four, its square is four; having subtracted (*śodhayitvā*) from above, those two multiplied by two should be made (*kartavyā*) four, therefore eight hundred increased by sixty four is produced (*jayite*).

Since above the quantity subtracted has no remainder, there is no sliding like a snake etc. method, remains just to halve the quantity obtained. Thus, when that is done (*kṛta*), the result is 432. Its square is 186624.