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# Ordering operations in square root extractions 

Analyzing some early medieval Sanskrit mathematical texts with the help of Speach Act Theory*

Agathe Keller


#### Abstract

Rules for extracting square roots written in Sanskrit in two treatises and their commentaries (from the Vth to the Xth century) are approached with questions inspired by textology and Speach Act Theory. An analysis of the number and order of steps spelled out is used to show that their aims were not limited to describing how to carry out the algorithm. The intentions of authors of mathematical texts in Sanskrit are questioned by lending an ear to the expressivity of relations established between world and text.


[^0]
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## 1 Introduction

The Sanskrit scholarly tradition of composing compact procedural sūtras with hairsplitting prose commentaries offers a fertile field to reflect, as Speach Acts Theory (SAT) does, on how prescriptive discourses relate to the real world. Studying the construction and composition of hermetic algorithmic statements and the way they are unravelled in commentaries sheds light on how diverse the relations can be between algorithmical texts and the algorithms they refer to. In other words the way a procedure is stated and the way a procedure is carried out are two different realities whose relations are studied here in a very specific case.

Most available Sanskrit sources on mathematics provide procedures to extract square and cube roots (vargamūla, ghanamūla). Such rules were part of the set of elementary operations (parikarma, vidha) that formed the basis of arithmetic and algebra. The square root procedure has remained unchanged, small details of inner working set aside, from the end of the Vth century to at least the XIIth century and probably later ${ }^{1}$.

Two, tightly linked, kinds of texts have handed down these rules : treatises and commentaries. This study then will bring to light different ways in which a treatise and its commentary worked out the tension of how a procedure should be stated versus how a procedure should be executed. This analysis is part of a larger endeavour, aiming at studying descriptive practices in Sanskrit mathematical texts, while focussing on how commentaries relate to their treatises ${ }^{2}$.

In the following, the focus will be on how different steps of a square root extraction are spelled out both by treatises and commentaries. Attention will be paid to how different actions of the procedure are explained and stated, in order to unravel the intentions with which these rules were composed.

### 1.1 Corpus

Five Sanskrit mathematical texts serve as a basis for this study, as illustrated in Figure 1. First, an astronomical siddhānta (that is a theoretical text) ${ }^{3}$ of the Vth century, the

[^1]

Figure 1: A connected set of five Sanskrit texts on mathematics

Āryabhatīya $(\mathrm{Ab})^{4}$, and two of its prose and rather prolixe commentators: Bhāskara 's, seventh century, Āryabhaṭ̄̄yabhāṣa ${ }^{5}$ (BAB) and Sūryadeva Yajvan's (SYAb) twelfth century Bhaṭaprakāsikā". Then, the "practical" Pāṭ̄̄gaṇita of Śrīdhara (ca. Vth-Xth century) and its anonymous and undated commentary (APG) ${ }^{7}$. These texts belong to the early medieval period of Indian mathematics: after the antique tradition of ritual geometry stated in the sūlbasūtras, before the popular synthetical accomplishment of Bhāskara II (XIIth century) in the Līlāvatı̄ and the Bījagaṇita .

The corpus consists in a connected set of texts, although they have been composed in different times and places. As seen in Figure 1, commentaries, naturally, are linked to the text they comment. Furthermore, one commentary of the Ab, SYAb, has read PG, and quotes it. It also often paraphrases BAB. Considered together, these texts belong and testify of the Sanskrit cosmopolitan culture in mathematics of the Vth-XIIth century ${ }^{8}$. However the two treatises examined are of different nature: as stated previously the Ab is an astronomical theoretical text, with one chapter devoted to mathematics (ganita ), while PG is solely a mathematical text, devoted to wordly (eg. everyday) practice (lokavyavahāra) ${ }^{9}$. Practices of procedure stating changed from one kind of a text to

[^2]another. The aim of this study then is to forge tools to better describe and understand such differences.

The rules given by Āryabhaṭa and Śrīdhara are collected with their Sanskrit transliteration in Table 1. In the following, different analysis of these rules implicitely suppose that the reader has this table at hand, and can compare and analyze graphics, lists etc. with the texts presented here.

Table 1: Two Rules to extract Square Roots

\begin{tabular}{|c|c|c|}
\hline Treatise \& Sanskrit Transliteration \& English Translation ${ }^{\text {a }}$ <br>
\hline Ab.2.4.ab

Ab.2.4.cd \& \begin{tabular}{lr}

$\quad$ bhāgaṃ \& hared | avargān |
| ---: |
| nityaṃ | <br>


| dviguṇena |
| :--- | ---: |
| vargamūlenal | <br>

vargād \& varge <br>
labdhaṃ \& śuddhe <br>
mūlam\| \& sthānāntare

 \& 

One should divide, constantly, the non-square [place] by twice the squareroot| <br>
When the square has been subtracted from the square [place], the result is a root in a different place\|
\end{tabular} <br>

\hline  \& | viṣamāt padas tyaktyvā vargaṃ sthānacyutena mūlena\| dvigunena bhajec cheṣaṃ labdhaṃ viniveśayet paniktau || |
| :--- |
| tadvargaṃ saṃśodhya dviguṇaṃ kurvīt purvaval labdham\| utsārya tato vibhajec śeṣaṃ dviguṇīkrtaṃ dalayet|| | \& | Having removed the square from the odd term, one should divide the remainder by twice the root that has dropped down to a place [and] insert the quotient on a line\\| |
| :--- |
| Having subtracted the square of that, having moved the previous result that has been doubled, then, one should divide the remainder. [Finally] one should halve what has been doubled.\|| | <br>

\hline
\end{tabular}

[^3]The procedure followed to extract a square root will not be discussed here. Appendix A lists steps to extract a square root, Appendix B illustrates it with an extraction of 186 624, an example carried out by APG. A visualisation of the process, is given in Figure $2^{10}$.

Figure 2: Different steps of a square root extraction


| Red arrows: Heart of the iteration. | Yellow: Begining and End. |
| :--- | :--- |
| Orange: Main iteration. | Green : Tacit |
| Pink: Singled out by PG and APG. |  |

[^4]
### 1.2 The underlying mathematical idea of square root extraction procedures

The process to extract square root relies on the decompositional nature of the decimal place value notation: the number whose square root should be extracted, is considered to be a square. Extracting its square root means recovering the different elements of a developed square. In other words, if we take the numerical example of APG, 186 624, the process will try to uncover the different $b_{i}$ (that is both the values of $b$ and $i, i$ giving the powers of ten concerned) such as

$$
186624=\left(\sum_{i=0}^{p} b_{i} 10^{i}\right)^{2}
$$

To do so, the process will consider the decimal development of 186624 as a sum of squares, $\sum_{i=0}^{p} b_{i}^{2} 10^{2 i}$, and of double products of the type $2 b_{i} b_{j} 10^{i+j}$, for $0 \leq i, j \leq p$ and $i+j \leq 2 p$. Consequently, the process of extracting square roots, an iterative process, will be emphasized as repeated subtractions of squares, and divisions by a doubled number. The repeated division by a doubled number explains the difference between the process provided by Ab and the one given by PG: PG will arrive at a doubled root (useful during the process), while Ab describes a process that enables one to obtain immediately the square root.

### 1.3 The procedure to extract square roots in Sanskrit texts: a difficult question of description

Unearthing intentions and meanings of Sanskrit mathematical texts is made difficult by the fact that as historians, we know little of the context in which mathematical texts were produced and used. Furthemore, I do not possess native knowledge of Sanskrit. To put it with Austin's words, the accompaniments and circumstances of the utterance of sūtras are largely lost to us as readers today ${ }^{11}$. Or, to state the difficulties inherent to the historian's trade according to Searle's categories ${ }^{12}$, and as described by Virbel in this volume, condition 1 of Searle's "how to promise" (e.g. in this case, being able to carry out an algorithm) involves native knowledge of the language. Furthermore ignorance of the context means that we cannot know Searle's conditions 4, 6, 9. Indeed, we are not

[^5]sure of the author's aim (6), or of his imagined reader or hearer (4) and are ill judges of then of the correctness with which the texts that transmit the author's intentions in this respect (9). As pointed out by Virbel then, certain conditons on the possibility of communication (1 and 9) and of making promisses (4 and 6) are not fulfilled. Nonetheless, in the following, treading carefully, intentions of Sanskrit authors of statements on mathematical algorithms will be discussed. To do so, the light shed on such authors by their own commentators will be used.

The procedure to extract square root has consistently attracted attention of historians of mathematics working on India. It testifies of an early use of the decimal place value notation. Furthermore, the process found here is very similar to the one taught until the middle of the XXth century in primary schools in Europe, the United States and probably elsewhere in the world. However, variations in time mingled with the concision of given rules has questioned how the procedure was carried out practically, step by step, for different authors. Different reconstructions have been offered by secondary sources, from Singh ${ }^{13}$ to more recently Plofker ${ }^{14}$. The practical performance of such processes is however but little discussed and justified by a close quotations of sources.

Indeed, several difficulties are layered in such reconstructions.

Even as we set aside the difficulties inherent to the historian's trade, the reconstruction of a procedural text is made difficult because of what one may term, following K. Chemla in this volume, the granularity of steps. This problem is certainly familiar to any one who has had to describe an algorithm: a stated step can often hide several others. For instance when "one should subtract the square from the square", given that the numbers and place were the subtraction should be carried out are given, this operation will be considered as a unique step, althought a procedure for subtraction or an operation for squaring might describe such operations in many steps. Thus "elementary" operations in a more complex algorithm are stated without being described.

Of course, part of this granularity may have to do with "tacit knowledge". Thus some steps may have been considered so obvious that they did not need to be stated. For instance, non of the authors considered here specify that the remainder of the division should replace the initial dividend, or that the remainder of the subtaction replaces the subtrahend. Similarly, they do not state explicitely that after each arithmetical operation (division, subtraction) one needs to move to one slot to the right. Since all texts silence these steps that however are required, they may thus be considered tacit, as illustrated

[^6]in Figure 2. Furthermore, commentators of theoretical siddhāntas may have considered, tacitly, that "practical" steps of the process were not to be specified. This may explain why so little layouts are indeed provided in handed down texts. The following study will focus on the steps that are actually stated leaving the tacit in the shade.

However, concentrating on the corpus's "actually stated", or "spelled out" actions only helps pinpointing the difficulties of defining and specifying what "spelling out an algorithm's step" actually means. Indeed, obstacles in recovering "the algorithm" may be inherent to the complexity of the object itself; more so while exploring the difference between saying and doing. A symptom of this difficulty has pervaded the writing of this article: each new approaches to the (texts on) square root extraction, induced a new representation of the algorithm. Each new representation never exactly coincided with others. Of course, I could endlessly try to coordinate each different depiction: checking that they all keep the same number of steps, respect the same singled out actions, or the hierarchies between different operations. But I've finally decided on the contrary, to leave each description its singular expressivity: none should be wrong or faulty in respect to the text it seeks to illustrates, or the algorithm it refers to. But none coincides exactly either: each representation only gives part of the information. No two representations coincide with each other. Indeed these multiple depictions demonstrate and illustrate how complex the relations are between stated and performed algorithm. The heuristic representations we forge to explain one or the other but add a subsequent layer to this complexity. In other words, there is no single absolute way of describing the algorithms to extract square roots, nor the different way they are stated.

The analysis in this article will be restricted to three elements that are usually associated with algorithms. If a procedure is though of as an ordered list of actions to be carried out, the kind of actions, their listing and ordering will be discussed here. More specifically, in the following, first the kind of statement the rules of Āryabhaṭa and Śrīdhara provide will be discussed. In order to understand how and why an author "states" the steps of an algorithm, the focus needs to be on the kind of texts that transmits the procedure. What kind of statements about procedures are produced in Sanskrit mathematical texts: Descriptions? Incetives to actually realise the procedure?

## 2 The prespcriptive paradox of compact procedures

Procedures are transmitted through rules (sūtras) and their commentaries. A sūtra, as it has been noted by Louis Renou with some detail, is a complex linguistic object used
in a great diversity of communication acts ${ }^{15}$. In the following focus will be on how this complexity is given an additional twist as mathematical algorithmic sūtras are evoked ${ }^{16}$. Often sūtras are described as being cryptic, let us look closely at was this means in the case of $s \bar{u} t r a s$ providing a procedure to extract square roots.

### 2.1 Being cryptic

As seen in Table 1, the rules given by Āryabhaṭa and Śrīdhara are difficult to understand ${ }^{17}$ : Ab and PG, do not specify what is produced if the rule is followed, nor do they specify what is an odd term/square place. None of the verses ${ }^{18}$ indicate how to start the procedure, nor how to end it. Only part of the steps alowing the procedure to be carried out are given. This ellipse is illustrated in Figure 3 and in Figure 4.

In both Figures, spelled out steps are contrasted against the steps required to have the process carried out, such as it is analyzed in the heuristic representation of Figure 2. The "tacit" steps, in green, in Figure 2 are left out. The specifities of Śrīdhara 's extraction of a double square root are not represented in the Figure on Āryabhaṭa 's rule. Āryabhaṭa 's sūtra appears as giving but a piece of the process, its heart: reduced to four steps, the algorithm is given in an uncertain order and seems restricted to a succession of divisions and subtractions around which other steps gravitate. Śrīdhara rule, althought more detailed, appears also as giving but part of the process: reduced to seven steps, of uncertain order (how does one go from step A to step B?), the emphasis is less on the heart of the iteration then on the details of what is done to the number "inserted on a line".

Thus, a first level of reading immediately reveals the ellipses of the rules in respect to the algorithm.

Another way to state the same fact consists in listing spelled out steps. With arbitrariness and limitations in mind, the steps of the procedure given in Āryabhaṭa 's verse can be listed as follows:

[^7]

Figure 3: Steps provided by Āryabhaṭa in blue


Figure 4: Steps provided by Śrīdhara in pink
i. Divide the non square place by twice the square root
ii. Iterate ("constantly")
iii. Subtract the square from the square place
iv. The result, a quotient, noted in a "different place" is a (square) root

One may or may not add an implicit step, the one which marks the number to be extracted within a grid which identifies even powers of ten. This step can also be considered included in the subtraction step. Similarly step ii and step ii could also be interpreted as including in fact each two steps... ${ }^{19}$

Whatever the nuances we might want to add, this enumeration highlights how compact Āryabhata's verse is. Indeed the square root extraction as reconstructed in Appendix A in order to carry it out includes $16 / 17$ steps, while A Aryabhaṭa states maybe 3 steps, maybe 8 .

Although, PG compared to AB is less concise- indeed Śrīdhara states the process in two verses while Āryabhaṭa uses only one- the process given in Pattūgaṇita is also compact.

Śrïdhara 's rule evokes the following steps:
i. Remove the square from an odd place
ii. Divide the remainder by twice the root
iii. The digits of the partial root are placed below on a line
iv. The square of the quotient is subtracted
v. Double the quotient and place it on a line
vi. Divide the remainder as in step 2, that is: Iterate
vii. The final result is divided by two

Comparing with Āryabhaṭa 's compact verse, as can also be seen in Figure 7, doubling and dividing by two adds two steps to the process. Śrīdhara furthermore indicates more explicitely how the process ends.

[^8]This first analysis of the different steps provided by authors, shows that the sütras considered here - wether overtly short as in Ab, or more explicit as in PG- do not state a complete algorithm. Obviously information is missing to actually carry out the procedure. If these rules aim at describing the process, then steps are missing. If these rules do not have such an aim, then the intention of the text is among the missing information. Therefore these rules are sufficiently compact to be difficult to understand alone: they are cryptic.

Elliptic formulations are often understood by indologists as recalling the oral sphere. The enigma of cryptic sūtras would have a mnemonic value unravelled by oral explanations. For instance, part of Ab.2.4's difficulty is rooted in a pun on the word 'square' (varga). Ab gives this name both to the square of a number and to the places of the place value notation having an even power of ten. Such places have the value of a square power of ten. It is also from such 'square places' that the 'square numbers' the process tries to uncover are found. Thus the pun recalls the main mathematical idea behind the procedure while simultaneously giving rythme to the verse and making its understanding confusing. Ab. 2.4 can thus be understood as a mnemotechnical "chimera" ${ }^{20}$. Other commonly advanced reasons for using short forms include secrecy, or the desire to emphasize the difficulty of a given technical knowledge to add prestige for a profession living off patronage.

If these rules are compact to the point of being cryptic they nonetheless prescribe an action to be taken. This prescription is voiced by an optative.

### 2.2 Using optatives

Sanskrit makes a great use of nominal forms. Therefore, the use of conjugated forms is in itself an expressive statement. Conjugated verbs in mathematical sūtras indicate a prescription. Indeed, most sūtras of jyotiṣa texts (astral science including mathematics) use the optative ${ }^{21}$. Theoretically it is an equivalent of our conditional: it expresses doubt. However, it should be understood here as expressing requirement. ${ }^{22}$

[^9]In Ab.2.4, "one should divide", is the translation of an expression which uses an optative: bhāgam hared, 'one should withdraw the share'- the usual expression of a division. The verb to withdraw ( $h r$ ) is in the optative voice. PG $25-26$, is a succession of optatives: one should divide (bhajed), place (viniveśayet), make (kurvāt) the double, divide (vibhajet), then halve (dalayet).

Commentaries follow closely the optatives given in treatises. Thus Bhāskara comments Āryabhaṭa 's optative by providing a synonym (grh-), conjugated as an optative ${ }^{23}$ : "One should remove the part, that is, one should divide". Sūryadeva (whose commentary is translated in Appendix C) does not comment upon Āryabhaṭa's terms for division but repeats the verb in the optative form, while commenting on what a non-square is ${ }^{24}$ : "One should divide by the (last) non-square place". APG (the translation of the commentary is given in Appendix D ) preserves Śrīdhara 's optative sometimes supplying a synonymym for an other. Thus it uses bhāgam apaharet for bhājet (one should divide). APG provides optatives for a number of actions: the first subtraction of an arbitrary square (tyajet), the placement of the root of this first arbitrary square (sthāpayed), the substraction of the square of the quotient (sodhayet), the fact that results should be considered as a unique quantity (jñāyet), etc.

Commentaries use also other modes to voice prescriptions: Imperatives while inviting one to solve a problem, obligational verbal adjectives while describing the steps to be taken. In APG the optative is only used while commenting directly Śīdhara 's verse. When solving the problem, actions are given with absolutives (which give precise temporal orders), such as śodhayitvā ("having subtracted") used twice, and by verbal obligation adjectives (such as kartavya, "on should carry out").

Therefore, algorithmical Sanskrit texts are prescriptive. Their prescription is first voiced in treatises by an optative. These optatives are taken up and declined also in other prescriptive forms in commentaries. A hermetic sūtra prescribing a procedure to be carried out is a paradox: Indeed, why elaborate short hermetic prescriptions, if the aim is to have them followed? In other words, if the aim is to have a procedure applied, the directive character of an algorithmic rule is contradicted here by its cryptic form. What

[^10]then were the intentions of the authors of such rules? As this question cannot be replied to directly, how commentators read author's intentions will be observed. But to do so requires to further unravel the complexity of mathematical sūtra's statements.

### 2.3 Stating a procedure

Commentators are quite explicit on how they understand the kind of statement treatises provide. All comentators consider first the rule as a linguistic assertion: a text whose language is the first subject of the commentary (what kind of verb it uses, what it means, how is it syntaxically constructed). In this respect, all three comentators refer to the text they explicate as a sūtra. They also sometimes refer to it as a verse, $k \bar{a} r i k \bar{a}{ }^{25}$.

Commentaries further use another vocabulary. One that relates to procedures. Thus, $\bar{a} n a y a n a$, "computation", derived from the verb $\bar{a}-N \bar{l}$, to lead towards- is used by our three commentators to refer to the mathematical contents of the rule. Bhāskara writes as an introductory sentence ${ }^{26}$ : "In order to compute (ānayana) square roots, he says:" He uses later in the commentary the word ganitakarma "mathematical process". And Sūryadeva, with almost the same words, but different declentions introduces the verse in this way ${ }^{27}$ : "He states with an $\bar{a} r y a$ a square root computation". The anonymous and undated commentary of Pāṭ̄̄gaṇita starts by specifying ${ }^{28}$ : "A two $\bar{a} r y a s$ algorithmic rule (karanasūtra) concerning square roots". He later refers to the process by using the expression $\bar{a} n a y a n a^{29}$.

Thus commentaries understand the rule as being primarly about a mathematical procedure. Because the rules are also prescriptive, they contain a "promise", that of stating an algorithm that provides a correct answer to a given problem. Note that commentary's first movement provides indeed the result the procedure aims at: authors of sūtras, expecting commentaries, may not have needed then to specify procedure's results in the rules they composed.

[^11]Commentators then refer to the rule on both levels: as a statement (on whose language one may comment) and as a procedure (on the steps of which one may comment). How do commentators deal with this double level of sūtras? And if the rules to extract square roots are thus understood as prescibing a process that should be carried out, does this mean that they provide a list of steps to apply?

## 3 Spelling out steps to extract a square root

As noted earlier, the speach act "spelling out of an algorithm's step" is complex. Two aspects of this act, the isolation of a certain number of steps and their subsequent ordering are studied now.

### 3.1 Expressing Actions and enumerating steps

Earlier in this article, in an attempt to show that rules provided both by Āryabhata and Śrīdhara were compact, they were crudely restricted to a list of steps. Indeed, our contemporary representations of what a good prescription should be involves listing actions. But how then do the rules given here, fair in this respect? Are they lists of actions? And if not, does this imply that they do not describe an algorithm?

Recall Śrīdhara 's statement of the procedure, as given in Table 1. A certain number of steps are voiced out by a succession of optatives. The essential ordered backbone of operations to be carried out is expressed in this way: a division, insertion of a quotient on a line, doubling, a division again and a halving ${ }^{30}$. If we understand this succession as being a list ${ }^{31}$, then we can understand each action as being on a same level of co-enumerability.

The impression that Śrīdhara provides a list of actions is emphasized by APG's way of

[^12]taking each optative and following it through twice. Thus in the general commentary ${ }^{32}$ :

And one should divide (bhāgam apaharet) from above by twice that, just there. The result should be inserted on a line (viniveśayet), one should subtract (sodhayet) the square of that from above that, and that should be doubled (dviguṇīkuryāt). If when this is doubled an additional place is created $(j \bar{a} y e t)^{33}$, then it should be used as before (yojayet) when it is a result. (...) One should repeat (utsārayet) this, thus one should divide (vibhajet), one should insert (viniveśayet) the result on a line, etc. as before in as much as the serpentine progression is possible, when finished one should halve (dalayet) all the result.

The optative is used while commenting directly Śrīdhara 's verse. More, APG takes elements that Śrīdhara did not formulate with conjugated verbs, and transforms them in conjugated, optative forms. Thus the subtraction, expressed by Śrīdhara with an absolutive (saṃśodhya), becomes an optative in the commentary (śodhayet). The repetition, an absolutive in the verse (utsārya), is an optative in APG (utsārayet). In the resolution of the problem, APG uses a conjugated verb, to make a quantity sneak onto a line (sarpati), were the Pạt $\bar{\imath} g a n ̣ i t a ~ d e s c r i b e s ~ w i t h ~ a ~ n o n-c o n j u g a t e d ~ f o r m ~ a ~ q u a n t i t y ~ t h a t ~ h a s ~$ been dropped (cyuta). Both Śrīdhara and his commentator seem to consider the rule provided as a list of steps, singled out by the use of verbal forms, conjugated or not. Consequently, unravelling here how authors "spell out steps" seems fairly simple and straight-foward: conjugated verbs give us the clue.

Āryabhaṭa and his commentators Bhāskara and Sūryadeva provide a strike contrast to this attitude. Indeed, Āryabhaṭa 's verse alone cannot be reduced to a list of steps of actions to be carried out. It uses but one conjugated verb referring to a division. Furthermore, the last statement of the verse is a description of the fact that a result gains a new status by changing place. It has a reflexive character. This statement is part of what makes this rule non reduceable either to a prescription or to an enumeration. Bhāskara 's and Sūryadeva 's readings of the action's of the rule emphasizes on division. Thus, the only conjugated verb referring to an action of the algorithm used by Bhāskara , concerns this action.

In both cases, in PG and in Ab, the presence of conjugated verbs on one side, and of

[^13]verbal non conjugated forms on the other, constructs a hierarchy among different steps of a stated procedure. In PG the actions, to subtract/remove (tyaktvā, saṃśodhya), to drop on a line (cyuta), and to evoke the past doubling (dviguṇikrta) are stated with verbal but non-conjugated forms (absolutives and verbal adjectives). They too seem to furnish a list of actions of less importance, thus creating a second level of co-enumerability. In other words, the two kinds of verbal forms draw a hierarchy in the actions to be carried out, as seen in Figure 5. APG, on the other hand, does not seem to reproduce this hierarchy ${ }^{34}$.

Figure 5: Expressing Actions in Śrīdhara 's verse and in its anonymous commentary


In Āryabhaṭa 's case, the voicing of steps to be carried out, cannot be restricted to

[^14]verbal forms. Aside from the division another action, a subtraction (śuddha), is stated with a verbal non conjugated form, a verbal adjective. Other parts of the algorithm that could be expressed by actions, such as squaring, multiplying by two are not described in that way: the square of the number (varga) is considered directly, as if it was already computed. The multiplication by two is described with an adjective meaning "having two for multiplier" (dviguna). Among all the actions to be carried out to square a root, two main actions emerge from Āryabhaṭa 's verse, those given in verbal forms: the action of dividing (first in the verse, and by the fact that it is conjugated) and secondly, the action of subtracting, as illustrated in Figure 6. Both commentators of the Āryabhatīya further respect Āryabhata 's use of an verbal adjective to evoke the subtraction.

Figure 6: Expressing Actions in Āryabhaṭa 's verse and subsequent commentaries


Bhāskara does not introduce new intermediary steps with conjugated verbs of action. For instance, he does not comment upon the subtraction, nor on the multiplication, but underlines how (by contrast with square places), the numerical square evoked in

Āryabhaṭa 's verse, evokes an action ${ }^{35}$ : "When subtracting the square, a computed square is the meaning". But precisely the aim here is to instist on Āryabhaṭa 's pun, while lifting its ambiguities: a square operation (vargaganita) is not to be confused with a square place (vargasthāna).

Because of the diverse ways of expressing algorithm's steps, rules do not appear directly in the form of a list- a form however quite current in Sanskrit technical texts ( śāstra). However, loose enumerations of conjugated verbs, as those given by Śrīdhara , can quite easily be interpretated as a list of steps to be carried out ${ }^{36}$. When a rule has but a unique conjugated verb, as in the case of Āryabhata, then this interpretation although possible (as seen in our first section), distorts the statement of the rule itself.

Nonetheless, conjugated verbs by contrast with others tell us something of a hierachisation in the different steps of an algorithm. In the rule for extracting square roots studied here, the optative can be seen as first ordering the enumeration of steps that the algorithm carries. In this ordering, the optative provides the action around which the others will be structured. This feeling may emerge from commentaries, who always carefully preserve the different ranges of voices: they do not transform the optative, or nominal forms in conjugated ones- but when in APG the commentary focusses on describing evenly each effective action of the process.

Therefore, if rules do not necessarily provide lists, they do transmit a hierarchy of steps. The question then is what order this hierarchy reveals.

### 3.2 Ordering steps

One of the difficulties of algorithm reconstruction concerns the temporal order in which different steps of a procedure are to be carried out. Mathematical constraints might sometimes impose a temporal order, but not always. Thus in the procedure to extract square roots, once a fist square is found by trial and error, two actions have to be carried out: the square is to be subtracted, the root of the square noted on a separate line. The order of performance is indifferent. Order is one of the problems raised by the granularity whith which a process is stated: if several actions are lumped together in a same step, the order in which they should be carried out, if there are no mathematical or exterior constraints, remains ambiguious. Does the hierarchy of steps observed in the previous

[^15]section corresponds to a temporal order to carry out the algorithm?

A specific verbal mode is used in rules to order in time a set of actions. Absolutives are indeclinable. They are built on a verbal root and mean 'having carried out the action of the verb concerned'. Absolutives thus indicate an action to be carried out before the main action indicated by a conjugated verb. Śrīdhara uses absolutives. In PG. 25 one must subtract (tyaktvā, saṃśodhya), before dividing or doubling; one should move before before dividing. This does not mean that all the different ordering of steps are elucidated in Śrīdhara's formulation. As illustrated in Figure 4, the placing of the quotient which has dropped on a line is situated ambiguiously in time during the process, as well as the doubling of its digits situated "after the subtraction".

Figure 7: Comparing Ab and PG


APG describes in great detail the part of Śrīdhara 's process which seemed ambiguious: To do so, he does not only use absolutives, but also the order in which conjugated verbs are enumerated, to which spatial modifiers (upari "above", adhas-bellow) are added. Actually, in his solved example, APG uses a great variety of verbal forms: absolutives ( apāsya, "having subtracted"), verbal adjectives (vyavasthita, sthita, "placed"), and obligatationary adjectives (neya, "one should lead"). Therefore not one device but all sorts of different ones are used in this commentary to express precisely the temporal order of execution.

Sūryadeva, at times uses absolutives. First when the place value notation and the grid of even-odd places is evoked, then when explaining how to extract the square root of fractions. In these parts of his commentary, he is very specific concerning the order in which different steps are to be carried out, using absolutives such as cihnayitvā (having marked), dviguṇ̄ kṛtya (having multiplied by two), vibhajya (having divided), or apāsya (having removed). In both cases the order does not concern directly Āryabhaṭa 's rule. In the first case Sūryadeva supplies actions that enables one to initialize Āryabhaṭa 's rule. In the second case, Āryabhața 's is integrated into the rule to extract the square root of fractions. In the general commentary ${ }^{37}$ Sūryadeva loosely followes the apparition of one, two and three digits of a square root being extracted, using a great diversity of verbal forms: verbal adjectives, absolutives, optatives. Furthermore this ordering of actions is certainly not the main part of his text. Sūryadeva can focus on an element of the process to detail its temporal order. Overall, Sūryadeva does not seem preoccupied by the kind of order implied within Āryabhaṭa 's rule. The order he spells out, is rather that of cases where Āryabhaṭa's rule is used within other operations (extracting a square root for a fraction, and maybe obtaining a double square root).

Thus ambiguous temporal orders of actions are not always cleared by commentators. When they do so, no simple language act (such as the use of absolutives) is used.

However, sometimes, there is no temporal ambiguity. To extract a square root you need to start with a subtraction. Aryabhata 's verse starts with a division, the subtraction
 algorithm are given in a reverted order. This is underlined in Figure 3, by the letters A, and B .

Note that Sanskrit being a language of declensions words need not be given in a strict order. A colloquial word order does exist; sūtras often scrambles it. Here, the two actions

[^16]are given in two successive verses: the action of division is highlighted by the fact that it starts the verse, while Sanskrit often keeps the conjugated verb at the end of the sentence. This disposition leaves no ambiguity on the order in which the actions are stated: first the division, then the subtraction.

Neither Āryabhata nor Bhāskara use absolutives. In BAB, steps are spelled out in a succession of questions and answers. Bhāskara reads Āryabhaṭa 's verse from the point of view of a division: ${ }^{38}$ :

One should take away [in other words] on should divide that [square]. Beginning with what place?
He says: 'From the non-square [place]' (...) In this computation the square is the odd place.
(...)

By what should one divide?
He says:'Constantly, by twice the square root'. (...)
How, then, is this square-root obtained?
He says: 'When the square has been subtracted from the square [place], the result is a root in a different place'.

The dialog argues that the order given by A ryabhata is logic from this operative point of view. A place, a divisor and a dividend are required to carry out a division. By unravelling where the division is performed and what the divisor and dividend are, steps are thus re-ordered and specified. The use of this staged dialog simultaneously emphasizes that the verse's steps are disordered while arguing for its logic.

Different orders then can be layered in the statement of one rule: a temporal order, a logical one, or even an order of different cases in which a rule is applied. Furthermore, statements of procedures do not necessarily list, and not necessarily in a temporal order,

[^17]actions that are to be performed. But what do they do then?

## 4 Back to the prescriptive paradox of procedural sūtras

As noted by Austin as a "descriptive fallacy" ${ }^{39}$, it is after all difficult not to consider procedural sūtras as descriptions of the procedure they evoke. It is also tempting to understand literally the prescription they voice. However, Āryabhaṭa's rule is a paradoxical act of communication: compact, scrambling the algorithm's order. This is a symptom, that Āryabhaṭa 's verse contains an intention stated indirectly (an illocutionary force), which may be evading us.

In the following, an initial question will be raised again: What kind of statement does a sūtra provide when it evokes a procedure?

### 4.1 Promises and Iteration

Sūtras contain a rule that invites one to carry out a process (one or serveral operations): It thus contains a more or less implicit promise, that of obtaining a result ( labdha, sometimes translated "quotient", the result of a division.). At the heart of the paradox of prescriptive sūtras : an action to be carried out is suggest, but the promise the action will fulfill is evanescent. The result is litterally constantly shifting. In Ab.2.4, labdhaṃ sthānāntare mūlam, "the quotient/the result (a digit of the partial square-root) is the root in a different place". In PG.2.25, purvaval labdham, "the previous result/quotient" is doubled and moved. In both case, implicitely, we are in the midst of a process in which a given result will produce another. The condition of success of the procedure, we understand, has less to do with "obtaining" a given result, than with repeating the process until it is exhausted ${ }^{40}$.

Śrīdhara and Āryabhaṭa do not express the iteration in the same way. Āryabhaṭa states the procedure by beginning with the middle of the process. Moving the quantity from a line where it is a quotient, to a line where it is a digit of the square root, is what enables the procedure to get back on its feet. Śrīdhara repeats with different words the process twice: In the first verse it may seem that he indicates how the process starts (its

[^18]first round), while in the second verse, a next or final round is described. Both authors of sūtras use a litterary device to explain how the procedure should be repeated: they do not so much describe the performance then imitate it.

Bhāskara and APG do not reproduce this imitation. Thus Bhāskara states (using $\bar{a} v a r t a t e$, to revolve, turn) plainly ${ }^{41}$ :"This very rule is repeated again and again, until the mathematical process is completed (parisamāptaṃ)." As seen previously, APG takes up Śrïdhara 's expression using the verb utsṛj- (to turn), and adds to it an ordered list of actions to be carried out. On the contrary, by repeating the process for several digits, and then describing how it ends, Sūryadeva actually seems to use Pāṭ̂̄gaṇita 's device.

Turning a rule upside down, repeating it twice: textual devices are used by treatises to imitate what should be going on a working surface, where the algorithm is carried out. For the iteration then, the treatises observed here, are intent in making their text and the world in which an algorithm is completed coincide not by describing in words what is going on, but by describing with words the repeated algorithm. Let us look more closely at how treatises and commentaries play with the world of text and that of algorithm performance.

### 4.2 World and Text

Although Sanskrit mathematical texts may not list or give temporal orders for actions to be carried out, sūtras and commentaries are nonetheless stating something about the algorithm. The kind of adjustments between world and text taken up in the corpus will be observed now.

Śrīdhara at times give a momentary description of what the computation should be at a given moment. Verbal adjectives seem here to be the clue to such descriptions were text and algorithm should coincide. Thus Śrīdhara uses the expression sthānacyuta, "that has trickled down to a place", dvigunikrtam, "that has been doubled". An action is not spelled out but a description of the state of the working surface on which the process is performed is made.These descriptions enjoins one who performs the procedure to adjust the world to the statements in the rule.

Specifically because this is a part of the process, which requires a know-how that belongs to the world of algorithm execution, APG, is careful to describe, digit by digit

[^19]how things are moved around on the working surface. Certain expressions are invitations to verify that at a given moment, the result obtained coincides with the text. Like for instance, in the purely descriptive ${ }^{42}$ : "When twenty four is subtracted by three from below, above two remains." In other cases, the text describes a temporary state of the working surface, followed by a disposition ${ }^{43}$ :
"Below, eighty six is produced. This quantity slides (sarpati) on a line. Below

$\begin{array}{llll}1 & 7 & 2 & 4\end{array}$
two, there is six, below seven, eight. Setting down: 86

Thus Pātīganita and its anonymous undated commentary are intent in making the statement of the procedure and its performance using tabular dispositions coincide.

Understanding conjugated verbs are an expressive device, reveals how Bhāskara's commentary on Āryabhaṭa's rule is mainly on the level of language. Bhāskara uses four times the expression "he says" $(\bar{a} h a)^{44}$. His answers to the questions of the dialog always refer to Āryabhaṭa's statements. Bhāskara then is not describing independently from A Aryabhaṭa how the process is being carried out. He is not adjusting A Aryabhaṭa 's statements with how the process should be executed. Indeed, he is just modifying Āryabhaṭa 's statements in an attempt to show the coherence of their arrangement. He explains that this arrangement makes sense if the division is taken as the core from which all other steps of the process derive.

In three instances Bhāskara explores the limits of the mathematical reality expressed by Āryabhata. First, when he specifies evenness as the opposite of odness, then when the process ends because no other space to carry it out can be found, and finally as he gives an example concerned with fractions in which he then introduces his own rule. In other words, the world of Bhāskara is not like APG one of algorithm execution but a world of mathematical objects.

The iteration of the process is further explained by a change of status, voiced by Āryabhaṭa in the last part of the verse and repeated by Bhāskara: a quotient changes

[^20]place, becomes a root, and enters in the process again. The end of the process precisely appears when this status changing becomes impossible. Bhāskara states it as follows ${ }^{45}$ :

The quotient here becomes in a different place what is technically called the root ( $m \bar{u} l a s a m j \tilde{n} \tilde{a})$. (...) In this different place, this quotient has the technical name root. However, precisely when a different place is not found, there, in that very place, that [result] has the technical name 'root'.

The change of place which simultaneously is a change of status is acted out by a solemn action of text and world adjustment: a name change. This name change is even acted out as no movement is possible anymore, signalling the end of the process. Bhāskara insists on this point, repeating the compound mūlasamjñ $\bar{a}$ ("technical name root"). This can seem at first glance confusing, together with the perpetual movement of the process. The insistance points to its importance. Indeed the name change explains how the process works: each repeated division provides digits of the square root. The centrality of the division is thus staged, the quotient is the root itself.

All three commentaries link the movement on the working surface, to a status change of the quotient. In the anonymous commentary on Pāt $\bar{\imath} g a n i t a$ a disgression discusses the status of the quantity on a separate line. The double square root is called "the result" (labdha). After having inserted the result/quotient of the division on a line, having subtracted its square and having doubled it, APG examines the case when the doubling provides a number bigger then ten ${ }^{46}$ : "If when this is doubled an additional place is created, then it should be used as before when it is a result (praglabdhe). Both have the quality of being a unique quantity (rā́sitā). This quantity has the name "result" (labdhasamjjña). "And when one arrives at the end of the process, the "result" appears again, to be halved.

Sūryadeva starts by considering the first digit of the square root obtained by trial and error. He calls this quantity a "special number" (saṃkhyāviśesa) and notes ${ }^{47}$ : "that special number is referred to (in the rule) as a root". Then commenting on Āryabhaṭa 's last quarter of verse, he adds ${ }^{48}$ : "that 〈quotient〉, in the next square place, becomes (bhavati) the root." Although, the status change is the same, Sūryadeva does not use here

[^21]a renaming. He does so before, when he establishes the equivalence between "square, nonsquare places" and "uneven, even" ones ${ }^{49}$ : "In places where numbers are set-down, the odd places have the technical name (samjjña) 'square'. Even places have the technical name 'non-square'. "

Naming appears as the moment of commentarial activity when statements of the algorithm are adjusted to coincide with the world it refers to. This world can be on the level of performance (APG) but also of mathematical objects (Bhāskara) , or a mingling of both (Sūryadeva).

Therefore, the way authors relate mathematical statements to the working surface on which a procedure is being carried out, gives us a clue to what is important to them: APG develops Śrīdhara's brief descriptions in order to give as much as possible a nonambiguious algorithm to carry out. Bhāskara insists on the fact that when the quotient is moved to a separate line, its name changes: what hapens on the working surface should always be coherent in the world of Āryabhata 's statements, the world that is important. Finally, Sūryadeva, not surprisingly for a commentator of Āryabhaṭa quoting Śrīdhara, seems to bridge his own way in between both approaches. He thus is intent on adjusting the decimal place value grid he uses to Āryabhata's statements, and specifies how, practically, the process should start by trial and error. In other words, all the different analysis carried out so far, seem to shed lights on the intentions of the different authors of the corpus.

### 4.3 Intention

Compacting devices, expressing iteration, listing or not actions, relating language and practice, all these textology approaches can help us infer author's intentions. For instance, by paying attention to what "essential" elements of the algorithm a rule states, how different hierarchies of actions are imbedded in a sūtra provides us with each author's interpretations of the algorithm's important points.

We have thus read the use of conjugated verbs (especially the optative) in sūtras as an expressive device. Śrīdhara would single out a certain number of actions (division, inserting the number on la line) over others (subtracting, doubling). APG on the other hand does not follow such a hierarchy. This is a consistent attitude if the aim of the commentary is to treat each action on the even level of algorithm execution. Thus such a

[^22]hierarchy in Śrīdhara 's verse not undertaken by APG both shed's light on APG's aim and Śrīdhara 's objective. Śrīdhara aim would be to state (and maybe order) required actions. APG would be intent in describing how the algorithm is carried out on a working surface. Together with modifiers giving an orientation to the movements of the computation "above" (upari) and "bellow"- several verbs refer to the movement and dynamic of the process: the quotient even sneaks like a snake to the line where it becomes a root (sarpati).

Śrīdhara and his anonymous commentator present a strike contrast with the intellectual couple formed by Āryabhaṭa and Bhāskara, with their sparse number of conjugated verbs. Bhāskara's emphasis on the operative logic of Āryabhaṭa 's verse show that his aim is to comment the coherence of Āryabhata 's verse, not how it should be carried out. Similarly, Sūryadeva 's relative indiference to Āryabhaṭa 's scrambled orders points to us towards Sūryadeva 's own aim, which might be to make sure that the process covers all possible cases. Indeed, like in the processes described by Karine Chemla in this volume, Sūryadeva integrates Āryabhaṭa 's verse in other circumstances where the algorithm can be applied: root extractions which arrive at double roots and root extractions of fractions. Āryabhaṭa 's rule, then is a general rule, whose temporality or logic is not the question. The hierarchy of operations to be carried out uses Āryabhaṭa 's rule as a central nod from which further operations are assessed.

Finally, looking at how different authors treat the world of linguistic statements and the one in which a procedure is carried out on a working surface, confirms these unravelled intentions. Srīdhara at times seems to describe the working surface at specific tricky times, that are detailed by his anonymous commentator. On the contrary, Bhāskara does not comment on this level of action, but on the one of providing a name at the right time, for the right quantity: re-naming a quantity that has been moved, assures us that the process is coherent with the stated rule. Sūryadeva does a bit of both, renaming the grid of the decimal place value notation used in the process, and describing how it is marked.

Maybe the "descriptive fallacy" of algorithm statements is considering that all statements about them describe the way they should be carried out; and more specifically that all invitation to carry an algorithm out, contains a more or less explicit description of how to do so. Indeed, this study has showed that first an invitation to carry out an algorithm does not necessarily describe litterally how to carry it out. Second, that what the description of an algorithm can be, is very diverse.

For both BAB and SYAb, Āryabhaṭa gives the main mathematical idea behind this procedure. Furthermore, the square root process comes after the definition of a square. Bhāskara contrasts operations of increase which include squaring with operations of de-
crease which includes seeking square roots ${ }^{50}$. In a mathematical tradition where the correction of an algorithm is sometimes verified by inverting it, and finding the initial input, square root extractions may have been seen as inverting the squaring procedure. Solved examples in the commentaries to these rules, all revert the squarings that where illustrated in the squaring procedure. Āryabhata 's rule then would seem to exhort one to carry out a process whose steps are described, but his true aim (as seen through Bhāskara 's glasses) would be to transmit a reflection on the procedure's mathematical grounding.

Adopting Bhāskara 's reading, three hypothesis can be drawn about intentions laying behind Āryabhata 's way of stating this procedure: His first aim could be in establishing the procedure (both by explaining it, and being able to recall it), giving its gist. A second aim could be reflecting on what the procedure is about (how does one undo a squaring in the decimal place value notation), what does this tell us about numbers. Most probably the aim was of summing together all possible understandings, eg prescribing a procedure, giving its gist, and hinting that this is less about doing than reflecting.

## Conclusion

The hermetic algorithmic statements of mathematical sūtras are, to put it in Austin's words, neither unambiguious nor explicit. At first reading they can seem strangely vague and full of uncertain references. However this detailed study of rules for square root extractions has shown that verses stating algorithms are not neutral descriptions of how to carry an algorithm out. They indicate what authors wanted to transmit and emphasis concerning these rules. Statement of procedures then may also be read as litterature, with their expressivity, and choices of narrative highlights. Commentators readings these rules, show precisely that difficult expressions are a door leading to the highlight of the rule.

Authors then did not necessarily list actions. The hierarchy of steps they provide does not always concern a temporal order. Conjugated verbs in the case examined here express something of the emphasis that each text puts on action or not: Āryabhata 's theoretical rule uses one, while Śrīdhara 's practical rule gives several. Bhāskara is intent on commenting on statements, and thus abonduntly conjugates the verb "to speak, state", while Sūryadeva who reflects on different forms of square-root extractions conjugates the verb bhū, "to produce, become, be". Finally the anonymous commentator of Śrīdhara 's rule, intent on reworking and specifying different steps, supplies many optatives. The

[^23]different aims and practices of algorithmic statement appear here clearly according to the type of text (theoretical, practical).

This study has used all sorts of different tools to describe and understand how processes were compacted in sūtras. Whether by representing the algorithm as a tree, or as a list, Āryabhata 's verse does not appear as an arbitrary fragment of the algorithm. The study of how language related to the world of algorithm realisation, helped understand how the iteration was expressed by authors of sūtras. Srīdhara and Sūryadeva figure it as a repetion, while it is signified by A$r y a b h a t a ~ b y ~ r e v e r s i n g ~ t h e ~ u s u a l ~ o r d e r ~ o f ~ t h e ~ p r o-~$ cedure. In both cases, a litterary device of imitation, is used to describe a reality which could take many words. Such processes, like the pun on varga used by Āryabhaṭa, can be seen as striking stylistic indiosyncrasies- specific to each rule and author- which may have had the role of "knot's to one's kercheif", if such rules were made to be learned by heart.

The main aim of authors then would not have been to describe an algorithm, but rather comment, that is emphasize, a point of the procedure: its mathematical grounding for Āryabhaṭa, its coherence for Bhāskara, the fact that it was worked out on several connected horizontal lines for Śrīdhara as understood by his anonymous commentator, and finally as a fundamental operation which can be carried out both by integers and fractions for Sūryadeva. The compacting of sūtras then would less have to do with mnemonics and secrecy than with the expressive granularity of algorithm statement.

Finally, stating such rules for Āryabhaṭa and Bhāskara seems to have aimed at pointing to how the algorithm was made, and the mathematical properties they were grounded in. In other words, such a procedure gave insights into properties of numbers written with the decimal place value notation, into what made them or not perfect squares. Maybe the procedure itself was thought of as a reflexive algorithm.

The definition of the sūtra as recalled by (Renou, 1963), is a paradox: a self sufficient compact verse but also one of a series ${ }^{51}$ :
"The word sūtra, or 'string' refers sometimes refers to a rule stated as a more or less short proposition (...), sometimes as a set of propositions forming a collection. The sūtra genre is defined by its relation rather than its contents, a

[^24]sūtra (understood as a 'rule ' or an 'aphorism') is foremost an element depending on its context, even if it is autonomous grammatically; it is determined by the system and is correlated with the group that surrounds it"

Indeed, this study has showed that mathematical sūtras and commentaries were intertwined. Commentaries were implicitely understood by sūtras as supplying mathematical context, result and details for practicing required. Looking at the statements and the way they are formulated and read, has underlined the technical reading a sütra requires. If sūtras have an expressivity, the commentary is needed to unfold it and make it shine. There is a specific movement to reading a sūtra and its commentary: knowing the text of the sūtra, understanding it which means unfolding it and then folding it back, and understanding its expressivity. Nor the sūtra, nor maybe the commentary are intented to be read in a linear way once, but masticated in the way Nietszche defines aphoristic reflexions. This is precisely the movement of an iterative algorithmic execution.

## List of Abbreviations

Ab Āryabhaṭa I 's Āryabhaṭ̂̄ya (Vth century)
APG Anonymous and undated commentary on the Pātūgaṇita of Śrīdhara
BAB Bhāskara I's commentary on the Āryabhaṭ̄̄yabhāṣya: Āryabhaṭ̄ya (VIIth century)

PG Pāt $\bar{\imath} g a n ̣ i t a$ of Śrīdhara (Xth century)
SYAB Sūryadeva Yajvan's commentary on the Āryabhaṭ̂̄ya: Bhaṭaprakāśikā (XIIth century)

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## A Different steps of the algorithm to extract square root as spelled out in the corpus

In the following reconstruction, all the actins spelled out by the authors considered here (with the part of arbitrareness - the loops of the net may at times seem too small and at others too large- that has been underlined in paragraph 1.3) have been taken in account. Thirteen steps to extract a square root can be listed. Step 3, 6 and 12 state common tacit steps. The algorithm may be more efficiently illustrated in Figure 2.

1. The number whose square is extracted is noted within a decimal place value notation. Places are categorized with a grid that enables one to identify square powers of ten. Either square powers of ten are identified or the series enumerating positions, starting with the place with the lowest power of ten lists even and odd places.
2. The highest odd/square place is identified.
3. Consider (tacitly) the number made of all the digits to the left of the digit noted in that place, that digit included.
4. Find the biggest square contained in the number noted in the last/highest odd place.

From here, onwards, one could as well start by considering step 8 and 9 , before turning to step 5 to 7 .
5. Subtract the square of the considered number.
6. Replace (tacitly) by the remainder of the subtraction.
7. The root of the subtracted arbitrary square is the first digit of the square root being extracted.
8. The root of this square (Ab family)/ The double of the root of this square (PG family) is noted on the same line/ on a line below the line of the number whose root is being extracted ${ }^{52}$. In the PG family then, the doubling of the digit is a separate step of the process. The doubling does not necessarily need to take place immediately, one can note the digit, and then double it just before it enters the division discribed in Step 10. This is what APG recomends.

[^25]9. Consider the number whose highest digit is the previously noted remainder and the next digit to its right.
10. Divide this number by (twice) the partial square root of Step 8. In the following, Step 11 can be carried out after Step 12.
11. The quotient is one of the digits of the partial square-root. It, or its double, is thus noted next to the previously found one, as in Step 8. Its square is what will be subtracted as the process is from here iterated from Step 5.
12. Replace the dividend by the remainder of the division. Then one should consider the next place on the right, which is a square/uneven place.
13. When there is no place on the right the algorithm is finished. Examples never consider a process that would not extract a perfect square, consequently, either the square-root is obtained, either its double is obtained, according to the procedure followed. If we are in the latter case, the obtained number is halved.

## B Extracting the square root of 186624.

This is a numerical example treated in APG. Footnotes and asterisks indicate non attested forms.

1. The number whose square is extracted is noted with the decimal place value system. Places are categorized with a grid: Square (varga), non-square (avarga) powers of ten are considered ( Ab ), or even (sama, abr. sa) and odd (visama, abr. vi) place ranks - counted starting with the lowest power of ten- are considered (BAB, PG, SYAB, APG).
avarga varga avarga varga avarga varga

2. Subtract the square from the odd place

The highest "odd" (vi) place or "square" (varga) place is $10^{4}$. The process starts by finding by trial and error the highest square number contained in the number noted to the left of this place: in other words one subtracts 16 , the square of four, from 18 .
3. Replace by (BAB) or place below (APG)
4. The remainder (2)
5. The root of the subtracted square (4) is the first digit of the square root being extracted. The root of this square ( Ab family)/ The double of the root of this square ( PG family) is noted on a seperate line.

Bhāskara might have written

$$
\begin{array}{cccccc}
* 10^{5} & 10^{4} & 10^{3} & 10^{2} & 10^{1} & 10^{0} \\
& 4 / 2 & 6 & 6 & 2 & 4
\end{array}
$$

While APG writes:

$$
\begin{array}{lllll}
2 & 6 & 6 & 2 & 4
\end{array}
$$

8
$186624=4^{2} \times\left(10^{2}\right)^{2}+2 \times 10^{4}+6 \times 10^{3}+6 \times 10^{2}+2 \times 10^{1}+4 \times 10^{0}$
6. One should divide by twice the root

In other words 26 is divided by 8 , the quotient is 3,2 is the remainder.
Bhāskara style

$$
\begin{array}{llllll}
* & 4 & 3 / 2 & 6 & 2 & 4
\end{array}
$$

APG style

$$
\left.\begin{array}{c}
* 2 \begin{array}{lll}
6 & 6 & 2
\end{array} \\
8 \\
8
\end{array}\right]
$$

7. Iterate, that is "subtracting the square" again. This time the square of the quotient, 9 , is subtracted from the next place on the right, that is from 26: the remainder is 17.

Bhāskara style:

$$
\begin{array}{llllll}
* & 4 & 38 \\
38 & 7 & 2 & 4
\end{array}
$$

APG style:

$$
\begin{array}{llll}
1 & 7 & 2 & 4 \\
& 8 & \mathbf{6} &
\end{array}
$$

$186624=4^{2} \times\left(10^{2}\right)^{2}+2 \times\left(4 \times 10^{2}\right)\left(3 \times 10^{1}\right)+3^{2} \times\left(10^{1}\right)^{2}+1 \times 10^{3}+7 \times 10^{2}+$ $2 \times 10^{1}+4 \times 10^{0}$
8. Dividing by the root, and subtracting the square again Considering the next place on the right, 172 , divide it by twice the partial square root, 86 : the quotient is 2 and there is no remainder. The square root is 432 .

Bhāskara style:

$$
\text { * } 4324
$$

APG style:

$$
\begin{array}{llll}
* & & & 4 \\
& \mathbf{8} & \mathbf{6} & 2
\end{array}
$$

$186624=4^{2} \times\left(10^{2}\right)^{2}+2 \times\left(4 \times 10^{2}\right)\left(3 \times 10^{1}\right)+3^{2} \times\left(10^{1}\right)^{2}+2 \times\left(3 \times 10^{0}\right)\left(4 \times 10^{2}+\right.$ $\left.3 \times 10^{1}\right)+4 \times 10^{0}=\left(4 \times 10^{2}+3 \times 10^{1}+2 \times 10^{0}\right)^{2}$
9. End

Bhāskara style:

## 432

APG style:

## $8 \quad 6 \quad 4$

$186624=(432)^{2}$

## C SYAB.2.4.

He states ( $\bar{a} h a)$ with an $\bar{a} r y a$ a square root computation (vargamūlānayana) ${ }^{53}$ :

[^26]
# One should divide，constantly，the non－square 〈place〉 by twice the square root <br> When one has removed the square from the square 〈place〉，the result is a root in a different place \｜ 

In places where numbers are set－down（vinyāsa），the odd places have the technical
 verse，when a square quantity is chosen（uddista），having at first started by marking （cihnayitvā）the square and non－square places，when one is able to subtract（śodhayituṃ śakyate）from the last square place the square of a special number－among those 〈squares of the digits〉 beginning with one and ending with nine，having subtracted（apāsya）that square；that special number is referred to（in the rule）as a root mūlatvena gṛhṇīyāt）．Con－ sequently，here，the result which is a square root（vargamūlaphala）has been mentionned （ucyate）．One should divide（bhāgam haret）the next contiguous non－square place by twice that $\langle$ root $\rangle$ ．In this verse（atra），when the square of that quotient has been sub－ tracted（śuddhe）from the next contiguous square place，that quotient from the non－square place，in a different place，in the next square place，that 〈quotient〉 becomes（bhavati）the root ${ }^{54}$ ．Also，when one has multiplied it（the quotient）by two（dviguṇ̂̄krtya），dividing （bhāgahārana）in due order both 〈digits〉 from its contiguous non－square，as before，the computation of the third root 〈is accomplished〉．Once again with three 〈digit numbers， the process is caried out $\rangle^{55}$ ．In this way，one should perform（kuryād）〈the process〉 until no square and non square 〈place〉 remain（bhavanti）．When the root quantity has been obtained（labdhe），having multiplied it by two（dviguṇ̄ k k！tamp），it should be halved （dalayet）．Concerning fractions also，having divided（vibhajya）the square root of the numerator by the square root of the denominator the quotient ${ }^{56}$ becomes（bhavanti）the root．One states（ $\bar{a} h a$ ）in this way：

When the square root of the numerator has been extracted，and the root born from the denominator $\langle$ also $\rangle$ the root $\langle$ is obtained $\rangle(\mathrm{PG}$ 34）｜

In order to obtain the roots of the square which were previously explained（in the com－ mentary of verse 3 which is on squares），setting down： 15625 ．The result is the square

[^27]root 125 ．Setting down the second $: \frac{4}{9}$ ．The root of the numerator 2 ，the root of the denominator 3 ，having divided（vibhajya）the numerator by that，the result is the square root of the fraction：$\frac{2}{3}$ ．Thus the fourth rule 〈has been explained〉．

## D APG

An algorithmic rule（karaṇasūtra）of two $\bar{a} r y a s$ for square roots ${ }^{57}$ ：

# PG．24．Having removed the square from the odd term，one should divide the remainder by twice the root that has trickled down to the place <br> 〈And〉 dispose the remainder on a line\｜ <br> PG．25．Having subtracted the square of that，one should divide the previous result｜ <br> That has been doubled thus again and again one should halve twice the square．｜｜ 

What is the root of an indicated quantity whose nature is a square？This is the aim of that procedure．One should subtract（tyajet）a possible square，from the visama 〈place〉 of the square quantity，〈in other words〉 from what is called odd（oja），that is from the first， third，fifth，or seventh etc．，〈place〉；the places for one，one hundred，ten thousand，or one million，etc．；the pada，that is from the last among other places．This should provide （syāt）the root of that square which one should place（sthāpayed）beneath the place of decrease，$\langle$ under〉 the place 〈where〉 the possible square is subtracted（sodhita）from that，〈the place for〉 one，a hundred，ten or thousands，etc．，the last among the other places． And one should divide（bhāgam apaharet）from above（upariṣtāt）by twice that，just there． The result should be inserted（viniveśayet）on a line，one should subtract（śodhayet）the square of that from above that，and that should be doubled（dviguṇ $\bar{\imath} k \bar{u} r y \bar{a} t)$ ．If when this is doubled（dvigune krte）an additional place is created（jāyet），then it should be used （yojayet）as before when it is a result．Both have the quality of being a unique quantity （rāsitā）．This quantity has the name＂result＂．One should repeat（utsārayet）this，thus one should divide（vibhajet），one should insert（viniveśayet）the result on a line，etc． as before in as much as the serpentine progression is possible（sarpanasambhava），when finished（samāpta）one should halve（dalayet）all the result，thus obtaining the square root．

[^28]Thus for 186624 ，of which quantity is this a square？

In due order starting from the first place which consists of four，making（karana）the names：＂odd（visama），even（sama），odd（visama），even（sama）＂．

## Setting down：

| sa | vi | sa | vi | sa | vi |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 8 | 6 | 6 | 2 | 4 |

In this case，the odd terms which are the places one，a hundred，ten thousands，consist of four，six and eight．Therefore the last odd term is the ten thousand place which consists of eight．Then，the first quantity is eighteen．Having subtracted（apāsya）sixteen since it is a possible square for these quantites，the result is two．That last quantity is placed （vyavatisthate）seperately above．Thus，where it is placed（sthite sati）the root of sixteen， four，with two for multiplier，eight，is to be led（neyah）below the place where the square was subtracted（vargaśuddhi），which consists of six for the place of decrease．And then division（bhāgāpahārah）of twenty six led above（uparitanyā）．Setting down：

```
2 6 6 6 2 4
    8
```

When twenty four is subtracted（śuddha）by three from below，above two remains． Below，the quotient which is three should be inserted（niveśya）on a line，they（eg these three units）should be placed（sthāpya）under 〈the place〉 consisting of six．Its square is nine．Having subtracted（śodhayitvā）that from above，these（1724）〈are placed above〉， three is multiplied by two，six is to be made（kartavya）．Below，eighty six is produced． This quantity slides（sarpati）on a line．Below two，there is six，below seven，eight．Setting down：

| 1 | 7 | 2 | 4 |
| :--- | :--- | :--- | :--- |
|  | 8 | 6 |  |

Division above of a hundred increased by seventy two by that eighty six．Decreasing from above the dividend without remainder by two，the result is two，having inserted （niveśya）that on a line，having placed（sthāpyau）the two 〈units〉 below four，its square is four；having subtracted（sodhayitvā）from above，those two multiplied by two should be made（kartavyā）four，therefore eight hundred increased by sixty four is produced（jayite）．

Since above the quantity subtracted has no remainder, there is no sliding like a snake etc. method, remains just to halve the quantity obtained. Thus, when that is done (krta), the result is 432. Its square is 186624 .


[^0]:    *This study was undertaken within the History of Science, History of Text Seminar in Rehseis in 2007. It was finished with the help of the algo-ANR. I would like to thank J. Virbel, K. Chemla, C. Proust, F. Bretelle-Establet, J. Ritter, C. Singh, A. Breard, and K. Vermeir: their thoughtful comments and encouragements have been weaved into this article and have brought it into existence.

[^1]:    ${ }^{1}$ Starting with the procedure given by Āryabhaṭa (499), remaining still simillar in Bhāskarācārya 's (fl. 1114) Līlāvatī and Bījaganita, and thus still alive in later commentaries of these texts.
    ${ }^{2}$ (Keller, 2010)
    ${ }^{3}$ (Pingree, 1981, 13)

[^2]:    ${ }^{4}$ (K. V. Shukla \& Sharma, 1976).
    ${ }^{5}$ (K. S. Shukla, 1976, 52-53). A translation of Bhāskara 's commentary to Āryabhaṭa 's verse on root extraction can be found in (Keller, 2006, Volume 1, p. 20-21), and an explanation of the process in (Keller, 2006, Volume 2, p. 15-18).
    ${ }^{6}$ (Sarma, 1976) A translation of his commentary on Āryabhata 's verse for square root extractions is given in Appendix C.
    ${ }^{7}$ (K. S. Shukla, 1959). A translation of the anonymous and undated commentary on Pātūgaṇita 's rules for extracting square roots is given in Appendix D.
    ${ }^{8}$ (Pollock, 2006, Part I)
    ${ }^{9}$ Thus Śrīdhara starts his treatise by the following statement, (K. S. Shukla, 1959, Sanskrit: i, English: 1):

[^3]:    PG.1cd (aham) lokavyavahārārthaṃ gaṇitaṃ saṃkṣepato vakṣye\|
    I will briefly state mathematics aiming at wordly practice

[^4]:    ${ }^{10}$ This diagram should not be seen as an attempt to formalise the algorithm: it is but a heuristic illustration.

[^5]:    ${ }^{11}$ (Austin, 1962, 76)
    ${ }^{12}$ (Searle, 1969, 57-61)

[^6]:    ${ }^{13}$ (Singh, 1927)
    ${ }^{14}$ (Plofker, 2009, 123-125)

[^7]:    ${ }^{15}$ (Renou, 1963)
    ${ }^{16}$ Incidently, this study shows that mathematical rules do not correspond either to Group A or Group B as defined by Renou in (Renou, 1963, part C). Features of group A such as the use of the optative, are mingled here with the prescriptive norms of group B.
    ${ }^{17}$ The cryptic character of this rule has been analyzed in (Keller, 2006, xvii), (Keller, 2010, 235-236) and is noted in (Plofker, 2009, 123-125). Some of its characteristics are described in (Singh, 1927).
    ${ }^{18} \mathrm{Ab}$ and PG provide rules of extraction with a verse that counts the number of syllabic units, the $\bar{a} r y \bar{a}$. This is a very common verse for prescriptive texts.

[^8]:    ${ }^{19}$ The difficulty of actually singling out steps in Āryabhaṭa 's verse, treated in the next section, can be seen when this enumeration is compared with Figure 3. With less contrast, the same can be seen for Śī̀dhara 's rule as well.

[^9]:    ${ }^{20}$ To apply to this context the concept that (Severi, 2007) uses to denote pictorial mnemotechnical artefacts: the important idea, as a knot to a kercheif is stamped into the artefact by relating two things that normally should not be related. In this case: a spot were a digit is noted and a square quantity.
    ${ }^{21}$ Note that in grammar (vyakarana), according to an oral communication by Jan Houben, the optative belongs essentially to the commentary.
    ${ }^{22}$ (Renou, 1984, §292):
    L'optatif exprime les nuances variées d'un optatif propre- souhait, hortatif, délibératif, éventualité, prescriptif, hypothétiques (...). La coexistence de ces divers emplois n'est rel-

[^10]:    evable que dans la poésie littéraire; dans les textes techniques prédomine la valeur prescriptive.

    That is in English (my translation):
    The optative expresses diverse shades of a proper optative: wish, hortative, deliberative, possibility, prescriptive, hypothetical (...) voices. The coexistence of these various uses are only found in literary poetry; in technical texts a prescriptive value prevails.
    ${ }^{23}$ tam bhāgam haret grhṇ̣īyāt
    ${ }^{24}$ avargasthānād bhāgaṃ haret

[^11]:    ${ }^{25}$ This crossreference may refer to the mingling of both forms as referred to by (Renou, 1963) who considers that real sūtras are non versified. The etymology of $k \bar{a} r i k \bar{a}$, derived form the verb $k r$-. , "to make", can maybe be heard is this context as "(verse) for action".
    ${ }^{26}$ vargamūlānayanāya āha
    ${ }^{27}$ vargamūlānayanam āryayāha
    ${ }^{28}$ vargamūle karaṇasūtram āryadvayam
    ${ }^{29}$ Standard vocabulary is used throughout Sanskrit mathematical texts to name computations, methods, algorithms. We do not know if there was any difference between these different words, if their meaning changed overtime, according to authors. We have adopted the following translations here: "computation" for words derived from $\bar{a} n \bar{\imath}$, "method" for karman; algorithm for karana.

[^12]:    ${ }^{30}$ The fact that PG's process provides a doubled root that needs to be halved, is highlighted (by mistake?) in SYAB. Indeed, this commentator of Āryabhaṭa notes (Sarma, 1976):
    labdhe mūlarāśau dviguṇı̄ krtam dalayet
    When the root quantity has been obtained, having multiplied it by two, it should be halved.
    However, Āryabhaṭa 's rule does not provide a double root and therefore does not request a halving in the end.
    ${ }^{31}$ Note that the Indian subcontinent's diversity of manuscripts presents a great variety of material settings, its scholarly texts, a great amount of lists. However, there seems to have been no specific typographical set up for lists in mathematical manuscripts of the Indian subcontinent.

[^13]:    ${ }^{32}$ dviguṇena ca tena tatraiva sthitena upariṣtāt bhāgam apaharet labdham paṇktau viniveśayet tatas tad vargam upariṣtāc chodhayet tac ca dviguṇīkuryāt tasmin dvigune krte yadi sthānam adhikam jāyet tat prāglabdhe yojayet (...) tam utsārayet tato vibhajet labdham painktau viniveśayed ity ādi pūrvavat yāvat utsarpaṇasambhavah / samāptau sarvaṃ labdhaṃ dalayed/
    ${ }^{33}$ In the corpus looked at here, this is the only use of an optative in a non-prescriptive form.

[^14]:    ${ }^{34}$ Two ambiguious moments are set aside here: the semi-tacit use of the decimal place value notation, what happens when a doubling results in a number 20 with two digits.

[^15]:    ${ }^{35}$ śuddhe varge vargaganite iti arthah
    ${ }^{36}$ Although, even in this case, their remains a great disparity between the representation of the action given in Figure 4 and in Figure 5.

[^16]:    ${ }^{37}$ And if my interpretation of the use of "both" and "three" in this text is correct, as noted in the footnotes of Appendix C.

[^17]:    38
    tam bhāgam, haret grhṇ̄̄yāt
    kasmāt sthānāt prabhrti iti
    $\bar{a} h a \quad$ avargāt (...) atra gaṇite viṣamam sthānam vargah
    (...) kena bhāgam haret iti
    $\bar{a} h a$ nityam dviguṇena vargamūlena
    (...)
    katham punar tat vargamūlam labhyate iti
    $\bar{a} h a ~ v a r g a ̄ t ~ v a r g e ~ s ́ u d d h e ~ l a b d h a m ~ s t h a ̄ n a ̄ n t a r e ~ m u ̄ l a m ~$

[^18]:    ${ }^{39}$ (Austin, 1962, 100)
    ${ }^{40}$ The difficulty of pinpointing how exactly the iteration is given in the verses, explains why it appears and disapears in the previous representations we haye given of the algorithm.

[^19]:    ${ }^{41}$ etat eva sūtram punar punar $\bar{a} v a r t a y e ~ y a \bar{a} v a t ~ p a r i s a m \bar{a} p t a m ~ g a n ̣ i t a k a r m a ~ i t i ~$

[^20]:    ${ }^{42}$ tribhih patanāt caturviṃśatau śuddhāyạ̣̄ upari dvau śeṣah/
    ${ }^{43}$ adhah ṣāḍaṣ̂̄tijaȳ̄te / eśa rāśih sarpati, pañktyāṃ dvayor adhah (ṣātkaṃ) bhavati, aṣ̣takaṃ saptādhāh / nyāsaḥ- $\begin{array}{lllll}1 & 7 & 2 & 4 \\ 8 & 6\end{array}$
    ${ }^{44} \mathrm{He}$ also uses once each labh (to obtain), bh $\bar{u}$ (to be, have, produce), vidyate (to exist, discern). The three only other conjugated verse of this part of his commentary is the optative of the division, and expressions in the solved example given at the end of the commentary.

[^21]:    ${ }^{45}$ yat atra labdham tat sthānāntare mūlasaṃjñam bhavati/ (...) tasmin sthānāntare tasya labdhasya mūlasaṃjñ̄a / yatra punaḥ sth $\bar{a} n \bar{a} n t a r a m ~ e v a ~ n a ~ v i d y a t e, ~ t a t r a ~ t a s y a ~ t a t r a ~ e v a ~ m u ̄ l a s a m ̣ j n ̃ ~ \bar{a}$
    ${ }^{46}$ tasmin dviguṇe krte yadi sthānam adhikam jāyet tat prāglabdhe yojayet, tayor ubhayor ekarāśitājñeyā / tasya rāśerlabdhasaṃjñā
    ${ }^{47}$ taṃ saṃkhyāviśeṣaṃ mūlatvena g̣̣hn̄̄̄̄āt /
    ${ }^{48}$ tat purve vargasthāne mūlaṃ bhavati

[^22]:    ${ }^{49}$ saṃkhyāvinyāsasthāneṣu viṣamasthānāni vargasaṃjñāni / samasthānānyavargasaṃjñāni /

[^23]:    ${ }^{50}$ (Keller, 2007)

[^24]:    ${ }^{51}$ English translation is mine: "Le genre du sūtra se définit par sa relation plutôt que par son contenu : un sūtra (au sens de "règle" ou "aphorisme") est d'abord un élément dépendant du contexte, même s'il est grammaticalement autonome; il est déterminé par le système et (...) corrélatif au groupe qui l'environne."

[^25]:    ${ }^{52}$ The "reconstruction" of these variants of the different steps of the process is not discussed here. Hopefully this issue will be tackled in a coming article.

[^26]:    ${ }^{53}$ (Sarma, 1976, 36-37)

[^27]:    ${ }^{54}$ This long sentence has an equivocal expression：is sthānātare（in a different place）glossed into pūrve vargasthāne（in the next square place），or should one understand that two actions are prescribed，first setting aside the quotient as a digit of the root on the one hand，and then that its square enters an operation in the next square place？
    ${ }^{55}$ This is a cryptic mysterious expression，it is thus my interpretation that the three here，as the＂both＂ （ubhaya）used in the sentence before refer to the number of digits of the square root being extracted
    ${ }^{56}$ Reading labdham instead of the misprinted ladhdham．

[^28]:    ${ }^{57}$（K．S．Shukla，1959，18－19）

